DYNAMICAL YANG-BAXTER MAPS

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Abstract. In this work, we propose and investigate dynamical Yang-Baxter maps, some of which produce solutions to the quantum dynamical Yang-Baxter equation. Suppose that L is a loop and a group. If their unit elements coincide, then L gives birth to a bijective dynamical Yang-Baxter map from $L \times L$ to $L \times L$ whose dynamical parameter belongs to L. The above group L is abelian if and only if the corresponding dynamical Yang-Baxter map satisfies the unitary condition.

1. Introduction

The (quantum) Yang-Baxter equation (YBE) [1, 2, 28, 29] has been recognized as a characteristic feature of the integrable systems. In addition to the YBE, the (quantum) dynamical Yang-Baxter equation (DYBE) and the Yang-Baxter map (YB map) have attracted much interest in recent years.

Baxter's corner transfer matrix method [1, 3] in solving the lattice models showed the importance of the YBE. This study and related investigations gave birth to the quantum group [7, 17], which is a powerful tool in constructing trigonometric solutions to the YBE. Instead of the YBE, the DYBE [11, 12, 14] was required in order to define the elliptic quantum groups associated with elliptic solutions to the YBE.

The solution to the YBE, which we call the R-matrix, is a linear mapping on the tensor product of the vector space, while the YB map [8, 10, 13, 21, 24, 26, 27] is defined on the Cartesian product of the set. For this reason, we also call it the set-theoretical solution to the YBE. The geometric crystals [4, 9] produce YB maps, and so do the crystals [18, 19, 20], which are closely related to the soliton cellular automata called box-ball systems [15, 16, 25].

To this time, no work has focused on set-theoretical solutions to the DYBE.

In this paper, we propose and investigate dynamical Yang-Baxter maps (dynamical YB maps), some of which produce solutions to the DYBE. Suppose that L is a loop [5, 6, 23] and a group. If their unit elements coincide, then L gives birth to a bijective dynamical YB map from $L\times L$ to $L\times L$ whose dynamical parameter belongs to L. This construction is a generalization of the works [10], [21], [24], and [27].

The organization of the article is as follows. Section 2 describes the definitions of the dynamical YB map and the dynamical braiding map, which are equivalent. Some of the dynamical YB maps induce solutions to the

DYBE. In Sections 3, 4, and 5, we construct dynamical YB maps, and investigate their properties. Let L be a loop, G a group, and $\pi : L \to G$ a set-theoretical bijection satisfying $\pi(e_L) = e_G$. Here e_L and e_G are the unit elements of L and G, respectively. We denote by $LG\pi$ the set of all such triplets (L, G, π). Define the equivalence relation in $LG\pi$ as follows: $(L, G, \pi) \in LG\pi$ is equivalent to $(L', G', \pi') \in LG\pi$ if

(1.1) $L = L'$ as loops, and the mapping $\pi'\pi^{-1} : G \to G'$ is an isomorphism of groups.

From (1.1), all loops in the representatives (L, G, π) of an equivalence class V coincide, and we denote by L_V the loop L. Let $[(L, G, \pi)]$ denote the equivalence class to which (L, G, π) belongs. Then every equivalence class $V = [(L_V, G, \pi)]$ produces a bijective dynamical YB map $R^V(\lambda) : L_V \times L_V \to L_V \times L_V$ $(\lambda \in L_V)$. In addition, we give a characterization of such dynamical YB maps $R^V(\lambda)$, methods to construct more general dynamical YB maps, and a sufficient condition for the dynamical YB map $R^V(\lambda)$ to be dependent on the dynamical parameter λ. The above group G is abelian if and only if the corresponding dynamical YB map $R^V(\lambda)$ satisfies the unitary condition. Finally Section 6 demonstrates several examples of the dynamical YB maps constructed in Section 3.

Dynamical L-maps associated with the dynamical YB maps, solutions to the RLL=LLR relation, will be discussed in a forthcoming paper.

2. Dynamical Yang-Baxter Maps

In this section, we introduce dynamical YB maps.

Let H and X be non-empty sets. Let ϕ be a mapping from $H \times X$ to H. A mapping $R(\lambda) : X \times X \to X \times X$ $(\lambda \in H)$ is a dynamical YB map associated with H, X, and ϕ, if, for every $\lambda \in H$, $R(\lambda)$ satisfies the following equation on $X \times X \times X$:

$$R_{23}(\lambda)R_{13}(\phi(\lambda, X^{(2)}))R_{12}(\lambda) = R_{12}(\phi(\lambda, X^{(3)}))R_{13}(\lambda)R_{23}(\phi(\lambda, X^{(1)})).$$

Here $R_{12}(\lambda)$, $R_{12}(\phi(\lambda, X^{(3)}))$, etc., are the mappings from $X \times X \times X$ to $X \times X \times X$ defined as follows:

$$R_{12}(\lambda)(u, v, w) = (R(\lambda)(u, v), w) \quad (u, v, w \in X);$$
$$R_{12}(\phi(\lambda, X^{(3)}))(u, v, w) = (R(\phi(\lambda, w))(u, v), w) \quad (u, v, w \in X).$$

If $R(\lambda)$ is independent of the dynamical parameter λ, then $R(\lambda)$ is just a YB map, i.e., a set-theoretical solution to the YBE [8].

Let V denote the $\mathbb{C}$-vector space generated by the set X. For a dynamical YB map $R(\lambda)$ associated with H, X, and ϕ, define the $\mathbb{C}$-linear mapping $\widetilde{R}(\lambda) : V \otimes V \to V \otimes V$ by

$$\widetilde{R}(\lambda)(u \otimes v) = u' \otimes v' \quad (u, v \in X).$$

Here $(u', v') = R(\lambda)(u, v)$. Because $R(\lambda)$ is a dynamical YB map, $\widetilde{R}(\lambda)$ satisfies a version of the DYBE (For the YB map, see [8].):

$$\widetilde{R}_{23}(\lambda)\widetilde{R}_{13}(\phi(\lambda, X^{(2)}))\widetilde{R}_{12}(\lambda) = \widetilde{R}_{12}(\phi(\lambda, X^{(3)}))\widetilde{R}_{13}(\lambda)\widetilde{R}_{23}(\phi(\lambda, X^{(1)})).$$

Here $\widetilde{R}_{12}(\lambda)$, $\widetilde{R}_{12}(\phi(\lambda, X^{(3)}))$, etc., are the $\mathbb{C}$-linear mappings from $V \otimes V \otimes V$ to $V \otimes V \otimes V$ defined as follows:

$$\widetilde{R}_{12}(\lambda)(u \otimes v \otimes w) = \widetilde{R}(\lambda)(u \otimes v) \otimes w \quad (u, v, w \in X);$$
$$\widetilde{R}_{12}(\phi(\lambda, X^{(3)}))(u \otimes v \otimes w) = \widetilde{R}(\phi(\lambda, w))(u \otimes v) \otimes w \quad (u, v, w \in X).$$

Let h be a finite-dimensional abelian Lie algebra over $\mathbb{C}$. Let us suppose that $H = h^*$, and that there exists a mapping $wt : X \to h^*$ satisfying $\phi(\lambda, u) = \lambda - wt(u)$ $(\lambda \in h^*, u \in X)$. In addition, we assume that, for all $\lambda \in h^*$ and all $u, v \in X$, $wt(u') + wt(v') = wt(u) + wt(v)$, where $(u', v') = R(\lambda)(u, v)$. Then the vector space V is an h-module with respect to the action

$$a \cdot u = wt(u)(a)u \qquad (a \in h, u \in X).$$

This means that the element u of the basis X is of weight $wt(u) \in h^*$. The linear mapping $\widetilde{R}(\lambda)$ is thus an h-invariant solution to the DYBE [11, 12, 14]:

$$\widetilde{R}_{23}(\lambda)\widetilde{R}_{13}(\lambda - h^{(2)})\widetilde{R}_{12}(\lambda) = \widetilde{R}_{12}(\lambda - h^{(3)})\widetilde{R}_{13}(\lambda)\widetilde{R}_{23}(\lambda - h^{(1)}).$$

If $R(\lambda)$ is bijective, $\widetilde{R}(\lambda)$ is a (quantum) dynamical R-matrix [11] (See Remark 6.1.).

A mapping $\sigma(\lambda) : X \times X \to X \times X$ $(\lambda \in H)$ is a dynamical braiding map associated with H, X, and ϕ, if, for every $\lambda \in H$, $\sigma(\lambda)$ satisfies the following equation on $X \times X \times X$:

$$\sigma(\lambda)_{12}\sigma(\phi(\lambda, X^{(1)}))_{23}\sigma(\lambda)_{12} = \sigma(\phi(\lambda, X^{(1)}))_{23}\sigma(\lambda)_{12}\sigma(\phi(\lambda, X^{(1)}))_{23}.$$

Proposition 2.1. *Let us define the mapping $P : X \times X \to X \times X$ by $P(u, v) = (v, u)$ $(u, v \in X)$. Suppose that mappings $R(\lambda)$ and $\sigma(\lambda)$ $(\lambda \in H)$ from $X \times X$ to $X \times X$ satisfy $\sigma(\lambda) = PR(\lambda)$. The following conditions are equivalent*:

(1) $R(\lambda)$ *is a dynamical YB map associated with H, X, and ϕ*;

(2) $\sigma(\lambda)$ *is a dynamical braiding map associated with H, X, and ϕ.*

Proof. The proof is straightforward. □

3. Construction of Dynamical Yang-Baxter Maps

Our main aim in the present section is to show how to construct dynamical YB maps associated with a loop L, L, and $(\cdot)$. Here $(\cdot)$ is the binary operation on the loop L.

Let us introduce the quasigroup and the loop [5, 6, 23]. We say that $(Q, \cdot)$ is a quasigroup if Q is a non-empty set, together with a binary operation $(\cdot)$

having the properties below:

(3.1) For all u, $w \in Q$, there uniquely exists $v \in Q$ such that $u \cdot v = w$.

(3.2) For all v, $w \in Q$, there uniquely exists $u \in Q$ such that $u \cdot v = w$.

We will simply denote by Q a quasigroup $(Q, \cdot)$, and, for $u, v \in Q$, the symbol uv will be used in place of $u \cdot v$.

Remark 3.1. The binary operation on a quasigroup is not always associative (See Remark 6.2.).

It is clear that a quasigroup Q satisfies the cancellation laws:

$$\text{For } a,\ u,\ v \in Q,\ au = av \text{ implies that } u = v. \tag{3.3}$$

$$\text{For } a,\ u,\ v \in Q,\ ua = va \text{ implies that } u = v. \tag{3.4}$$

By virtue of (3.1), for elements u and w of a quasigroup Q, there uniquely exists the element $v \in Q$ such that $uv = w$. Let us define the binary operation $(\backslash)$ on Q by $u\backslash w = v$.

Let λ be an element of a quasigroup Q. Define the binary operation $m(\lambda)$ on Q by

$$m(\lambda)(u, v) = \lambda\backslash((\lambda u)v) \quad (u, v \in Q). \tag{3.5}$$

By taking Remark 3.1 into account, $m(\lambda)(u, v)$ does not always equal uv (See Remark 6.2.).

Lemma 3.2. (1) *For $\lambda \in Q$, $(Q, m(\lambda))$ is a quasigroup.*
(2) *For λ, u, v, $w \in Q$, $m(\lambda)(m(\lambda)(u, v), w) = m(\lambda)(u, m(\lambda u)(v, w))$.*

Proof. (1) Let u and w be elements of Q. Because of (3.1), there uniquely exists $v \in Q$ such that $(\lambda u)v = \lambda w$. From (3.5), $m(\lambda)(u, v) = \lambda\backslash(\lambda w) = w$. We omit the rest of the proof.
(2) By means of (3.5),

$$\begin{aligned} m(\lambda)(m(\lambda)(u, v), w) &= \lambda\backslash(((\lambda u)v)w) \\ &= \lambda\backslash((\lambda u)m(\lambda u)(v, w)) \\ &= m(\lambda)(u, m(\lambda u)(v, w)). \end{aligned}$$

We have thus proved (2). □

The following proposition is a generalization of Theorem 1 in [21].

Proposition 3.3. *Let Q be a quasigroup. For $\lambda, u, v \in Q$, let $\xi_\lambda(u)$ and $\eta_\lambda(v)$ be mappings from Q to Q having the properties below*:

$$\xi_\lambda(u)\xi_{\lambda u}(v) = \xi_\lambda(m(\lambda)(u, v)) \quad (\forall \lambda, u, v \in Q); \tag{3.6}$$

$$\eta_{\lambda\xi_\lambda(u)(v)}(w)(\eta_\lambda(v)(u)) = \eta_\lambda(m(\lambda u)(v, w))(u) \quad (\forall \lambda, u, v, w \in Q); \tag{3.7}$$

$$m(\lambda)(\xi_\lambda(u)(v), \eta_\lambda(v)(u)) = m(\lambda)(u, v) \quad (\forall \lambda, u, v \in Q). \tag{3.8}$$

For $\lambda \in Q$, define the mapping $\sigma(\lambda) : Q \times Q \to Q \times Q$ by

$$\sigma(\lambda)(u, v) = (\xi_\lambda(u)(v), \eta_\lambda(v)(u)) \quad (u, v \in Q).$$

Then $\sigma(\lambda)$ is a dynamical braiding map associated with Q, Q, and $(\cdot)$.

Proof. Let u, v, and w be elements of Q. Define the elements u_i, v_i, and w_i of Q $(i = 1, 2)$ by

$$\begin{aligned}(u_1, v_1, w_1) &= \sigma(\lambda)_{12}\sigma(\lambda Q^{(1)})_{23}\sigma(\lambda)_{12}(u, v, w),\\ (u_2, v_2, w_2) &= \sigma(\lambda Q^{(1)})_{23}\sigma(\lambda)_{12}\sigma(\lambda Q^{(1)})_{23}(u, v, w).\end{aligned}$$

The equations (3.6), (3.7), and (3.8) imply that $u_1 = u_2$ and that $w_1 = w_2$. By means of (3.8) and Lemma 3.2 (2),

$$\begin{aligned}m(\lambda)m(\lambda)_{12}(u_1, v_1, w_1) &= m(\lambda)m(\lambda)_{12}(u, v, w)\\ &= m(\lambda)m(\lambda u_2)_{23}(u_2, v_2, w_2)\\ &= m(\lambda)m(\lambda)_{12}(u_1, v_2, w_1).\end{aligned}$$

Here $m(\lambda)_{12} = m(\lambda) \times \mathrm{id}_Q$, and $m(\lambda)_{23} = \mathrm{id}_Q \times m(\lambda)$. The cancellation laws (3.3) and (3.4) of the quasigroup $(Q, m(\lambda))$ induce that $v_1 = v_2$. □

We say that $(L, \cdot, e)$ is a loop if $(L, \cdot)$ is a quasigroup satisfying that there exists an element $e \in L$ such that $ue = eu = u$ for all $u \in L$. Because the above element $e \in L$ is uniquely determined, we call e the unit element of the loop $(L, \cdot, e)$. We will simply denote by L a loop $(L, \cdot, e)$.

Remark 3.4. An associative loop is a group, and vice versa. More precisely, groups are associative quasigroups (See Theorem I.1.7 and Definition I.1.9 of [23].).

On account of Lemma 3.2 (1), the following lemma means that, for a loop $(L, \cdot, e)$, every $(L, m(\lambda), e)$ is also a loop.

Lemma 3.5. *Let $(L, \cdot, e)$ be a loop. For $\lambda, u \in L$, $m(\lambda)(u, e) = m(\lambda)(e, u) = u$.*

Lemma 3.6. *Let λ and u be elements of a loop $(L, \cdot, e)$, and let $u^{-1}_{\lambda,r}$ (resp. $u^{-1}_{\lambda,l}$) denote the right inverse (resp. the left inverse) of the element u with respect to $m(\lambda)$: $m(\lambda)(u, u^{-1}_{\lambda,r}) = m(\lambda)(u^{-1}_{\lambda,l}, u) = e$. Then $u^{-1}_{\lambda,r} = u^{-1}_{\lambda u,l}$.*

Proof. From Lemma 3.2 (2) and Lemma 3.5,

$$\begin{aligned}m(\lambda)(m(\lambda)(u, u^{-1}_{\lambda,r}), u) &= m(\lambda)(e, u)\\ &= m(\lambda)(u, e)\\ &= m(\lambda)(u, m(\lambda u)(u^{-1}_{\lambda u,l}, u))\\ &= m(\lambda)(m(\lambda)(u, u^{-1}_{\lambda u,l}), u).\end{aligned}$$

Lemma 3.2 (1) implies the cancellation laws (3.3) and (3.4) of $(L, m(\lambda))$, and consequently $u^{-1}_{\lambda,r} = u^{-1}_{\lambda u,l}$. □

For the main theorem, we need categories $\mathcal{A}$ and $\mathcal{D}$ [22].

Let $L = (L, \cdot, e_L)$ be a loop, $G = (G, *, e_G)$ a group, and $\pi : L \to G$ a set-theoretical bijection satisfying $\pi(e_L) = e_G$. We denote by $LG\pi$ the set of all such triplets (L, G, π). Define the equivalence relation in $LG\pi$ by (1.1). We write this relation in the form $(L, G, \pi) \sim (L', G', \pi')$. Let $[(L, G, \pi)]$ denote the equivalence class to which (L, G, π) belongs, $Ob(\mathcal{A})$ the class of all equivalence classes with respect to the above relation. On account of (1.1), all loops in the representatives (L, G, π) of $V \in Ob(\mathcal{A})$ coincide, and we denote by L_V the loop L.

For $V,\ V' \in Ob(\mathcal{A})$, define the class $\mathrm{Hom}_{\mathcal{A}}(V, V')$ as follows: f is an element of $\mathrm{Hom}_{\mathcal{A}}(V, V')$ if

$$f : L_V \to L_{V'} \text{ is a homomorphism of loops such that}$$
$$\pi' f \pi^{-1} : G \to G' \text{ is a homomorphism of groups for any}$$
$$\text{representatives } (L_V, G, \pi) \in V \text{ and } (L_{V'}, G', \pi') \in V'. \tag{3.9}$$

$\mathcal{A}$ is a category: its objects are the elements of $Ob(\mathcal{A})$ and its morphisms with a source V and a target V' are the elements of $\mathrm{Hom}_{\mathcal{A}}(V, V')$.

Let $L = (L, \cdot, e_L)$ be a loop, and $\xi_\lambda(u)$ and $\eta_\lambda(u)$ $(\lambda, u \in L)$ mappings from L to L satisfying the properties below:

$$\xi_\lambda(u)\xi_{\lambda u}(v) = \xi_\lambda(m(\lambda)(u, v)) \quad (\forall \lambda, u, v \in L); \tag{3.10}$$

$$\eta_{\lambda\xi_\lambda(u)(v)}(w)(\eta_\lambda(v)(u)) = \eta_\lambda(m(\lambda u)(v, w))(u) \quad (\forall \lambda, u, v, w \in L); \tag{3.11}$$

$$m(\lambda)(\xi_\lambda(u)(v), \eta_\lambda(v)(u)) = m(\lambda)(u, v) \quad (\forall \lambda, u, v \in L); \tag{3.12}$$

$$\xi_\lambda(e_L) = \eta_\lambda(e_L) = \mathrm{id}_L \quad (\forall \lambda \in L). \tag{3.13}$$

We denote by $Ob(\mathcal{D})$ the class of all such triplets (L, ξ, η), where $\xi = (\xi_\lambda(u))_{\lambda, u \in L}$ and $\eta = (\eta_\lambda(u))_{\lambda, u \in L}$.

For $V = (L, \xi, \eta), V' = (L', \xi', \eta') \in Ob(\mathcal{D})$, define the class $\mathrm{Hom}_{\mathcal{D}}(V, V')$ as follows: f is an element of $\mathrm{Hom}_{\mathcal{D}}(V, V')$ if

$$f : L \to L' \text{ is a homomorphism of loops satisfying}$$
$$f(\xi_\lambda(u)(v)) = \xi'_{f(\lambda)}(f(u))(f(v)) \text{ for all } \lambda, u, v \in L. \tag{3.14}$$

$\mathcal{D}$ is a category: its objects are the elements of $Ob(\mathcal{D})$ and its morphisms with a source V and a target V' are the elements of $\mathrm{Hom}_{\mathcal{D}}(V, V')$.

We are in a position to state the main theorem in this article (Cf. Theorem 2 of [21].).

Theorem 3.7. *The category $\mathcal{A}$ is isomorphic to the category $\mathcal{D}$.*

In the next section, for the proof of Theorem 3.7, we will explicitly construct functors $S : \mathcal{A} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{A}$ satisfying $TS = \mathrm{id}_{\mathcal{A}}$ and $ST = \mathrm{id}_{\mathcal{D}}$.

On account of Propositions 2.1, 3.3, and Theorem 3.7, every object $V \in Ob(\mathcal{A})$ gives birth to a dynamical YB map $R^V(\lambda)$ associated with L_V, L_V, and $(\cdot)$.

The following proposition gives methods to produce more general dynamical YB maps.

Proposition 3.8. (1) *Let H be a non-empty set, and $R'(\lambda)$ a dynamical YB map associated with H', X, and ϕ. Let us suppose that there exist mappings*

$\psi : H \to H'$ and $\rho : H' \to H$ satisfying $\psi\rho = \mathrm{id}_{H'}$. Define the mapping $R(\lambda) : X \times X \to X \times X$ ($\lambda \in H$) by $R(\lambda) = R'(\psi(\lambda))$. Then $R(\lambda)$ is a dynamical YB map associated with H, X, and $\rho\phi(\psi \times \mathrm{id}_X)$.

(2) *Let X be a non-empty set, and $R'(\lambda)$ a dynamical YB map associated with H, X', and ϕ. Let us suppose that there exist mappings $\rho : X' \to X$ and $\psi : X \to X'$ satisfying $\psi\rho = \mathrm{id}_{X'}$. Define the mapping $R(\lambda) : X \times X \to X \times X$ ($\lambda \in H$) by $R(\lambda) = (\rho \times \rho)R'(\lambda)(\psi \times \psi)$. Then $R(\lambda)$ is a dynamical YB map associated with H, X, and $\phi(\mathrm{id}_H \times \psi)$.*

4. Proof of Theorem 3.7

Let us introduce the lemma below before we define a functor $S : \mathcal{A} \to \mathcal{D}$.

Lemma 4.1. *Let $(Q, \cdot)$ be a quasigroup, and let us suppose that, for $\lambda, u, v \in Q$, there exist mappings $\xi_\lambda(u), \eta_\lambda(v) : Q \to Q$. For $\lambda \in Q$, define the mapping $\sigma(\lambda) : Q \times Q \to Q \times Q$ by*

$$\sigma(\lambda)(u, v) = (\xi_\lambda(u)(v), \eta_\lambda(v)(u)) \quad (u, v \in Q).$$

If $\sigma(\lambda)$ satisfies $m(\lambda)\sigma(\lambda) = m(\lambda)$ for all $\lambda \in Q$, then the following conditions are equivalent:

$$\eta_{\lambda\xi_\lambda(u)(v)}(w)(\eta_\lambda(v)(u)) = \eta_\lambda(m(\lambda u)(v, w))(u) \quad (\forall \lambda, u, v, w \in Q); \tag{4.1}$$

$$\xi_\lambda(u)(m(\lambda u)(v, w)) = m(\lambda)(\xi_\lambda(u)(v), \xi_{\lambda\xi_\lambda(u)(v)}(\eta_\lambda(v)(u))(w)) \quad (\forall \lambda, u, v, w \in Q); \tag{4.2}$$

$$\sigma(\lambda)m(\lambda Q^{(1)})_{23} = m(\lambda)_{12}\sigma(\lambda Q^{(1)})_{23}\sigma(\lambda)_{12} \quad (\forall \lambda \in Q). \tag{4.3}$$

Here $m(\lambda Q^{(1)})_{23} : Q \times Q \times Q \to Q \times Q \times Q$ is the mapping defined by $m(\lambda Q^{(1)})_{23}(u, v, w) = (u, m(\lambda u)(v, w))$ for u, v, $w \in Q$.

Proof. It follows immediately that the condition (4.3) implies (4.1) and (4.2). The rest of the proof is essentially the same as that of Proposition 3.3. □

Let (L, G, π) be an element of $LG\pi$. For $u \in L = (L, \cdot, e_L)$, define the mapping $\theta^{(L,G,\pi)}(u) : G \to G$ by

$$\theta^{(L,G,\pi)}(u)(x) = \pi(u)^{-1} * \pi(u\pi^{-1}(x)) \quad (x \in G = (G, *, e_G)). \tag{4.4}$$

Here we denote by $\pi(u)^{-1} \in G$ the inverse of the element $\pi(u)$.

Lemma 4.2. *The mapping $\theta^{(L,G,\pi)}(u)$ is bijective for every $u \in L$.*

Proof. Let us define the mapping $\theta^{(L,G,\pi)}(u)^{-1} : G \to G$ by

$$\theta^{(L,G,\pi)}(u)^{-1}(x) = \pi(u\backslash\pi^{-1}(\pi(u) * x)) \quad (x \in G). \tag{4.5}$$

This is the inverse of the mapping $\theta^{(L,G,\pi)}(u)$. □

For λ, $u \in L$, define the mappings $\xi_\lambda^{(L,G,\pi)}(u), \eta_\lambda^{(L,G,\pi)}(u) : L \to L$ by

$$\xi_\lambda^{(L,G,\pi)}(u) = \pi^{-1}\theta^{(L,G,\pi)}(\lambda)^{-1}\theta^{(L,G,\pi)}(\lambda u)\pi, \tag{4.6}$$

$$\eta_\lambda^{(L,G,\pi)}(u)(v) = (\lambda\xi_\lambda^{(L,G,\pi)}(v)(u))\backslash((\lambda v)u) \quad (v \in L). \tag{4.7}$$

Let $V = [(L_V, G, \pi)]$ be an object of the category $\mathcal{A}$. Define $S(V)$ by $S(V) = (L_V, \xi^{(L_V,G,\pi)}, \eta^{(L_V,G,\pi)})$. Here $\xi^{(L_V,G,\pi)}$ and $\eta^{(L_V,G,\pi)}$ are as follows:

$$\xi^{(L_V,G,\pi)} = (\xi_\lambda^{(L_V,G,\pi)}(u))_{\lambda,u\in L_V},\ \eta^{(L_V,G,\pi)} = (\eta_\lambda^{(L_V,G,\pi)}(u))_{\lambda,u\in L_V}.$$

Proposition 4.3 says that the definition of $S(V)$ does not depend on a choice of representatives of V, and that $S(V) \in Ob(\mathcal{D})$.

Proposition 4.3. (1) *Let (L, G, π) and (L', G', π') be elements of $LG\pi$. If $(L, G, \pi) \sim (L', G', \pi')$, then $L = L'$ as loops, $\xi^{(L,G,\pi)} = \xi^{(L',G',\pi')}$, and $\eta^{(L,G,\pi)} = \eta^{(L',G',\pi')}$.*
(2) *For $(L, G, \pi) \in LG\pi$, $(L, \xi^{(L,G,\pi)}, \eta^{(L,G,\pi)}) \in Ob(\mathcal{D})$.*

Proof of (1). From (1.1), it follows immediately that $L = L'$ as loops. Let λ and u be elements of L. Because $\pi'\pi^{-1} : G \to G'$ is an isomorphism of groups,

$$\theta^{(L',G',\pi')}(u) = \pi'\pi^{-1}\theta^{(L,G,\pi)}(u)(\pi'\pi^{-1})^{-1}$$

by means of (4.4). On account of (4.6), $\xi_\lambda^{(L',G',\pi')}(u) = \xi_\lambda^{(L,G,\pi)}(u)$, and consequently $\eta_\lambda^{(L',G',\pi')}(u) = \eta_\lambda^{(L,G,\pi)}(u)$ by virtue of (4.7). $\square$

Next we prove (2). In place of $\theta^{(L,G,\pi)}(u)$, $\xi_\lambda^{(L,G,\pi)}(u)$, and $\eta_\lambda^{(L,G,\pi)}(u)$, we will use the symbols $\theta(u)$, $\xi_\lambda(u)$, and $\eta_\lambda(u)$, respectively.

Lemma 4.4. (1) $\theta(e_L) = \mathrm{id}_G$.
(2) *For $u \in L$, $\theta(u)(e_G) = e_G$.*

For $\lambda, u \in L$, define the mapping $\theta_\lambda(u) : G \to G$ by

$$\theta_\lambda(u) = \theta(\lambda)^{-1}\theta(\lambda u). \tag{4.8}$$

Lemma 4.5. *Let λ, u, and v be elements of L.*
(1) $\theta_\lambda(e_L) = \mathrm{id}_G$.
(2) $\theta_\lambda(u)\theta_{\lambda u}(v) = \theta_\lambda(m(\lambda)(u, v))$.
(3) $\xi_\lambda(e_L) = \mathrm{id}_L$.
(4) $\xi_\lambda(u)\xi_{\lambda u}(v) = \xi_\lambda(m(\lambda)(u, v))$.
(5) *The mapping $\xi_\lambda(u)$ is bijective*: $\xi_\lambda(u)^{-1} = \xi_{\lambda u}(u^{-1}_{\lambda,r})$.

Lemma 4.6. *Let λ, u, v, and w be elements of L.*
(1) $\eta_\lambda(e_L) = \mathrm{id}_L$.
(2) $m(\lambda)(u, v) = (\pi^{-1}\theta(\lambda)^{-1})(\theta(\lambda)(\pi(u)) * \theta(\lambda u)(\pi(v)))$.
(3) *The mappings $\xi_\lambda(u)$ and $\eta_\lambda(u)$ satisfy* (4.2).

Proof. (1) Due to Lemma 4.4 (2), we deduce $\xi_\lambda(u)(e_L) = e_L$. By using this equation and the definition of $\eta_\lambda(e_L)(u)$, the rest of the proof is immediate.
(2) By the definition (4.4) of $\theta(\lambda)$,

$$\pi(\lambda m(\lambda)(u, v)) = \pi(\lambda) * \theta(\lambda)(\pi(m(\lambda)(u, v))).$$

In a similar fashion,

$$\pi((\lambda u)v) = \pi(\lambda u) * \theta(\lambda u)(\pi(v)) = \pi(\lambda) * \theta(\lambda)(\pi(u)) * \theta(\lambda u)(\pi(v)).$$

Because of (3.5), we get the desired result.

(3) With the aid of (2), (4.6), and (4.7),

$$\begin{aligned}
\text{R.H.S. of (4.2)} &= (\pi^{-1}\theta(\lambda)^{-1})(\theta(\lambda)(\pi(\xi_\lambda(u)(v))) \\
&\quad *\theta(\lambda\xi_\lambda(u)(v))(\pi(\xi_{\lambda\xi_\lambda(u)(v)}(\eta_\lambda(v)(u))(w)))) \\
&= (\pi^{-1}\theta(\lambda)^{-1})(\theta(\lambda)(\theta_\lambda(u)(\pi(v))) \\
&\quad *\theta(\lambda\xi_\lambda(u)(v))(\theta_{\lambda\xi_\lambda(u)(v)}(\eta_\lambda(v)(u))(\pi(w)))) \\
&= (\pi^{-1}\theta(\lambda)^{-1})(\theta(\lambda u)(\pi(v)) * \theta((\lambda u)v)(\pi(w))).
\end{aligned}$$

By using (2) and (4.6) again,

$$\begin{aligned}
\text{L.H.S. of (4.2)} &= (\pi^{-1}\theta(\lambda)^{-1}\theta(\lambda u))(\pi(m(\lambda u)(v,w))) \\
&= (\pi^{-1}\theta(\lambda)^{-1})(\theta(\lambda u)(\pi(v)) * \theta((\lambda u)v)(\pi(w))).
\end{aligned}$$

We have thus proved (3). □

Proof of Proposition 4.3 (2). The equation (3.12) is derived from (3.5) and (4.7). Lemmas 4.1, 4.6 (3), and (3.12) imply (3.11). The equations (3.10) and (3.13) have been proved in Lemmas 4.5 and 4.6. □

Lemma 4.7. *Let V and W be objects of the category $\mathcal{A}$. If $f \in \mathrm{Hom}_{\mathcal{A}}(V, W)$, then $f \in \mathrm{Hom}_{\mathcal{D}}(S(V), S(W))$.*

Proof. Let (L_V, G, π) and (L_W, G', π') be representatives of V and W, respectively. From (3.9) and (4.4),

$$\theta^{(L_W,G',\pi')}(f(u))(\pi' f \pi^{-1}) = (\pi' f \pi^{-1})\theta^{(L_V,G,\pi)}(u) \quad (\forall u \in L_V).$$

With the aid of (3.9) and (4.6), $\xi^{(L_W,G',\pi')}_{f(\lambda)}(f(u))(f(v)) = f(\xi^{(L_V,G,\pi)}(v))$ for all $\lambda, u, v \in L_V$. We have thus proved the lemma. □

For $f \in \mathrm{Hom}_{\mathcal{A}}(V, W)$ $(V, W \in Ob(\mathcal{A}))$, define $S(f) \in \mathrm{Hom}_{\mathcal{D}}(S(V), S(W))$ by $S(f) = f$.

Proposition 4.8. *S is a functor from $\mathcal{A}$ to $\mathcal{D}$.*

Remark 4.9. Let (L, G, π) be an element of $LG\pi$. From (4.4), $\pi(u(vw)) = \pi((uv)w)$ for $u, v, w \in L$ if and only if $\theta^{(L,G,\pi)}(u)(\pi(v) * \theta^{(L,G,\pi)}(v)(\pi(w))) = \theta^{(L,G,\pi)}(u)(\pi(v)) * \theta^{(L,G,\pi)}(uv)(\pi(w))$ for $u, v, w \in L$. On account of Lemma 4.2, if we assume one of the following conditions, then the others are equivalent:

(1) the loop L is associative;

(2) the mapping $\theta^{(L,G,\pi)}(u)$ satisfies

$$\theta^{(L,G,\pi)}(uv) = \theta^{(L,G,\pi)}(u)\theta^{(L,G,\pi)}(v) \quad (\forall u, v \in L); \tag{4.9}$$

(3) for every $u \in L$, $\theta^{(L,G,\pi)}(u)$ is a homomorphism of groups.

Let us assume any two conditions of (1), (2), and (3). Remark 3.4 induces that L is a group, and π is consequently a bijective 1-cocycle of L with coefficients in G [10, 21] (See Proposition 4.10.).

We describe a sufficient condition for $R^V(\lambda)$ to be dependent on the dynamical parameter λ.

Proposition 4.10. *Let $V = [(L_V, G, \pi)]$ be an object of the category $\mathcal{A}$. If the dynamical YB map $R^V(\lambda)$ is independent of the dynamical parameter λ, then the mapping $\theta^{(L_V,G,\pi)}(u)$ $(u \in L_V)$ satisfies* (4.9).

Proof. Because $R^V(\lambda)$ is independent of λ, all the mappings $\xi_\lambda^{(L_V,G,\pi)}(u)$ are also independent of λ, and, due to (4.6), $\theta^{(L_V,G,\pi)}(\lambda)^{-1}\theta^{(L_V,G,\pi)}(\lambda u) = \theta^{(L_V,G,\pi)}(e_L)^{-1}\theta^{(L_V,G,\pi)}(u)$ for all $\lambda, u \in L_V$. Lemma 4.4 (1) implies the desired result. $\square$

Next we present a functor $T : \mathcal{D} \to \mathcal{A}$.

Let $V = (L, \xi, \eta)$ be an object of the category $\mathcal{D}$. For $\lambda \in L$, define the binary operation $\star_\lambda : L \times L \to L$ by

$$u \star_\lambda v = \lambda\backslash((\lambda u)\xi_{\lambda u}(u^{-1}_{\lambda,r})(v)) = m(\lambda)(u, \xi_{\lambda u}(u^{-1}_{\lambda,r})(v)). \tag{4.10}$$

Proposition 4.11. *Every $(L, \star_\lambda, e_L)$ is a group.*

For the proof, we need the following.

Lemma 4.12. (1) *For all λ, $u \in L$, the mapping $\xi_\lambda(u)$ is bijective: the inverse is $\xi_{\lambda u}(u^{-1}_{\lambda,r})$.*
(2) *For all λ, $u \in L$, $\xi_\lambda(u)(e_L) = e_L$.*

Proof. (1) The proof is immediate from (3.10), (3.13), and Lemma 3.6.
(2) By taking the cancellation law (3.4) of $(L, m(\lambda))$ into account, it suffices to prove that $m(\lambda)(\xi_\lambda(u)(e_L), u) = m(\lambda)(e_L, u)$. Due to (3.12), (3.13), and Lemma 3.5, we get

$$m(\lambda)(\xi_\lambda(u)(e_L), u) = m(\lambda)(\xi_\lambda(u)(e_L), \eta_\lambda(e_L)(u)) = m(\lambda)(u, e_L) = m(\lambda)(e_L, u),$$

thereby completing the proof. $\square$

The lemma below induces Proposition 4.11.

Lemma 4.13. (1) *For $\lambda, u, v, w \in L$, $(u \star_\lambda v) \star_\lambda w = u \star_\lambda (v \star_\lambda w)$.*
(2) *The element e_L satisfies that $e_L \star_\lambda u = u \star_\lambda e_L = u$ for all λ, $u \in L$.*
(3) *For λ, $u \in L$, $u \star_\lambda (\xi_\lambda(u)(u^{-1}_{\lambda,r})) = (\xi_\lambda(u)(u^{-1}_{\lambda,r})) \star_\lambda u = e_L$.*

Proof. (1) The definition (4.10) of $\star_\lambda$, Lemmas 3.2 (2), and 4.1 imply that

$$\begin{aligned}
& u \star_\lambda (v \star_\lambda w) \\
= \;& m(\lambda)(u, \xi_{\lambda u}(u^{-1}_{\lambda,r})(m((\lambda u)u^{-1}_{\lambda,r})(v, \xi_{\lambda v}(v^{-1}_{\lambda,r})(w)))) \\
= \;& m(\lambda)(u, m(\lambda u)(\xi_{\lambda u}(u^{-1}_{\lambda,r})(v), \xi_{(\lambda u)\xi_{\lambda u}(u^{-1}_{\lambda,r})(v)}(\eta_{\lambda u}(v)(u^{-1}_{\lambda,r}))(\xi_{\lambda v}(v^{-1}_{\lambda,r})(w)))) \\
= \;& m(\lambda)(u \star_\lambda v, \xi_{(\lambda u)\xi_{\lambda u}(u^{-1}_{\lambda,r})(v)}(\eta_{\lambda u}(v)(u^{-1}_{\lambda,r}))(\xi_{\lambda v}(v^{-1}_{\lambda,r})(w))).
\end{aligned}$$

From this equation, it suffices to show that

$$\xi_{\lambda(u\star_\lambda v)}((u\star_\lambda v)^{-1}_{\lambda,r}) = \xi_{(\lambda u)\xi_{\lambda u}(u^{-1}_{\lambda,r})(v)}(\eta_{\lambda u}(v)(u^{-1}_{\lambda,r}))\xi_{\lambda v}(v^{-1}_{\lambda,r}). \tag{4.11}$$

By means of (3.12) and (4.10),

$$(u\star_\lambda v)^{-1}_{\lambda,r} = m(\lambda(u\star_\lambda v))(\eta_{\lambda u}(v)(u^{-1}_{\lambda,r}), v^{-1}_{\lambda,r}),$$

and

$$(\lambda(u\star_\lambda v))(\eta_{\lambda u}(v)(u^{-1}_{\lambda,r})) = ((\lambda u)\xi_{\lambda u}(u^{-1}_{\lambda,r})(v))\eta_{\lambda u}(v)(u^{-1}_{\lambda,r}) = \lambda v.$$

Hence (3.10) induces (4.11).

(2) Due to Lemma 4.12 (2), $u\star_\lambda e_L = u$. By using (3.13), we get

$$e_L\star_\lambda u = m(\lambda)(e_L, \xi_\lambda(e_L)(u)) = u,$$

thereby completing the proof.

(3) By virtue of Lemma 4.12 (1), $u\star_\lambda \xi_\lambda(u)(u^{-1}_{\lambda,r}) = e_L$. The equation (3.12) implies that $(\xi_\lambda(u)(u^{-1}_{\lambda,r}))^{-1}_{\lambda,r} = \eta_\lambda(u^{-1}_{\lambda,r})(u)$, and consequently

$$\begin{aligned}(\xi_\lambda(u)(u^{-1}_{\lambda,r}))\star_\lambda u &= m(\lambda)(\xi_\lambda(u)(u^{-1}_{\lambda,r}), \xi_{\lambda\xi_\lambda(u)(u^{-1}_{\lambda,r})}(\eta_\lambda(u^{-1}_{\lambda,r})(u))(u)) \\ &= \xi_\lambda(u)(m(\lambda u)(u^{-1}_{\lambda,r}, u))\end{aligned}$$

because of Lemma 4.1. Lemmas 3.6 and 4.12 (2) imply that $(\xi_\lambda(u)(u^{-1}_{\lambda,r}))\star_\lambda u = e_L$. □

Remark 4.14. For every $\lambda\in L$, the group $(L, \star_\lambda, e_L)$ is isomorphic to $(L, \star_{e_L}, e_L)$. For the proof, it suffices to show that, for λ, u, v, $w\in L$,

$$\xi_\lambda(u)(v\star_{\lambda u} w) = \xi_\lambda(u)(v)\star_\lambda \xi_\lambda(u)(w). \tag{4.12}$$

By means of (3.10), (3.12), and Lemma 4.1,

$$\begin{aligned}&\text{L.H.S. of (4.12)}\\ =\ & \xi_\lambda(u)(m(\lambda u)(v, \xi_{(\lambda u)v}(v^{-1}_{\lambda u,r})(w)))\\ =\ & m(\lambda)(\xi_\lambda(u)(v), \xi_{\lambda\xi_\lambda(u)(v)}(\eta_\lambda(v)(u))(\xi_{(\lambda u)v}(v^{-1}_{\lambda u,r})(w)))\\ =\ & m(\lambda)(\xi_\lambda(u)(v), \xi_{\lambda\xi_\lambda(u)(v)}(m(\lambda\xi_\lambda(u)(v))(\eta_\lambda(v)(u), v^{-1}_{\lambda u,r}))(w)).\end{aligned}$$

Because $m(\lambda\xi_\lambda(u)(v))(\eta_\lambda(v)(u), v^{-1}_{\lambda u,r}) = m(\lambda\xi_\lambda(u)(v))((\xi_\lambda(u)(v))^{-1}_{\lambda,r}, u)$, we get (4.12).

For $V = (L, \xi, \eta)\in Ob(\mathcal{D})$, let us define $T(V)\in Ob(\mathcal{A})$ by

$$T(V) = [(L, (L, \star_{e_L}, e_L), \mathrm{id}_L)].$$

Lemma 4.15. *Let $V = (L, \xi, \eta)$ and $W = (L', \xi', \eta')$ be objects of the category $\mathcal{D}$. If $f\in \mathrm{Hom}_{\mathcal{D}}(V, W)$, then $f\in \mathrm{Hom}_{\mathcal{A}}(T(V), T(W))$.*

Proof. Because f satisfies (3.14) and $m(e_L)$ is the binary operation $(\cdot)$,

$$\begin{aligned} f(u \star_{e_L} v) &= f(u\xi_u(u^{-1}_{e_L,r})(v)) \\ &= f(u)f(\xi_u(u^{-1}_{e_L,r})(v)) \\ &= f(u)\xi'_{f(u)}(f(u^{-1}_{e_L,r}))(f(v)) \end{aligned}$$

for $u, v \in L$. From (4.10),

$$f(u) \star_{e_{L'}} f(v) = f(u)\xi'_{f(u)}(f(u)^{-1}_{e_{L'},r})(f(v)).$$

Since $f : L \to L'$ is a homomorphism of loops, $f(u^{-1}_{e_L,r}) = f(u)^{-1}_{e_{L'},r}$, and $f : (L, \star_{e_L}, e_L) \to (L', \star_{e_{L'}}, e_{L'})$ is a homomorphism of groups as a result.

For $(L, G, \pi) \in T(V)$ and $(L', G', \pi') \in T(W)$, the mappings $\pi^{-1} = \mathrm{id}_L\pi^{-1} : G \to (L, \star_{e_L}, e_L)$ and $\pi' = \pi'\mathrm{id}^{-1}_{L'} : (L', \star_{e_{L'}}, e_{L'}) \to G'$ are all homomorphisms of groups because of (1.1). The mapping $\pi' f \pi^{-1} : G \to G'$ is hence a homomorphism of groups. This completes the proof. □

For $f \in \mathrm{Hom}_{\mathcal{D}}(V, W)$, define $T(f) \in \mathrm{Hom}_{\mathcal{A}}(T(V), T(W))$ by $T(f) = f$.

Proposition 4.16. *T is a functor from $\mathcal{D}$ to $\mathcal{A}$.*

Proof of Theorem 3.7. For completing the proof, we need to show: (1) $TS = \mathrm{id}_{\mathcal{A}}$; and (2) $ST = \mathrm{id}_{\mathcal{D}}$.

(1) We only show that $TS(V) = V$ for $V = [(L_V, G, \pi)] \in Ob(\mathcal{A})$.

Lemma 4.17. *The mapping $\pi : (L_V, \star_{e_{L_V}}, e_{L_V}) \to G$ is an isomorphism of groups. Here $\star_{e_{L_V}}$ is defined from $S(V) \in Ob(\mathcal{D})$ (See* (4.10)*.).*

Proof. For the proof, it suffices to show that π is a homomorphism of groups. Let u and v be elements of L_V. The equations (4.5), (4.6), and Lemma 4.4 (1) induce that

$$u \star_{e_{L_V}} v = u(\xi_u(u^{-1}_{e_{L_V},r})(v)) = u((\pi^{-1}\theta(u)^{-1})(\pi(v))) = \pi^{-1}(\pi(u) * \pi(v)),$$

because $m(e_{L_V})$ is exactly the binary operation $(\cdot)$. We have thus proved the lemma. □

From the above lemma, $(L_V, G, \pi) \sim (L_V, (L_V, \star_{e_{L_V}}, e_{L_V}), \mathrm{id}_{L_V})$, and we hence obtain $TS(V) = V$.

(2) Let $V = (L, (\xi_\lambda(u)), (\eta_\lambda(u)))$ be an object of the category $\mathcal{D}$. We only demonstrate that $ST(V) = V$. By the definition, the object $T(V)$ is the equivalence class to which $(L, G, \pi) = (L, (L, \star_{e_L}, e_L), \mathrm{id}_L)$ belongs. On account of the definition of $ST(V)$, we can denote $ST(V)$ by $(L, (\xi'_\lambda(u)), (\eta'_\lambda(u)))$. It suffices to prove that $\xi'_\lambda(u)(v) = \xi_\lambda(u)(v)$ and $\eta'_\lambda(u)(v) = \eta_\lambda(u)(v)$ for $\lambda, u, v \in L$.

Lemma 4.18. *For $u \in L$, let $\theta'(u)$ denote $\theta^{(L,G,\pi)}(u)$ defined by* (4.4)*. Then $\theta'(u) = \xi_{e_L}(u)$.*

Proof. Let v be an element of L. From (4.4), $\theta'(u)(v) = u^{-1} \star_{e_L} (uv)$, where u^{-1} is the inverse of u with respect to $\star_{e_L}$. With the aid of Lemma 4.13 (3) and (4.10),

$$u^{-1} \star_{e_L} (uv) = \xi_{e_L}(u)(u^{-1}_{e_L,r})\xi_{\xi_{e_L}(u)(u^{-1}_{e_L,r})}((\xi_{e_L}(u)(u^{-1}_{e_L,r}))^{-1}_{e_L,r})(uv). \tag{4.13}$$

By virtue of (3.12), $(\xi_{e_L}(u)(u^{-1}_{e_L,r}))^{-1}_{e_L,r} = \eta_{e_L}(u^{-1}_{e_L,r})(u)$, and, from Lemma 4.1,

$$\begin{aligned} & \text{R.H.S. of (4.13)} \\ = \;& m(e_L)(\xi_{e_L}(u)(u^{-1}_{e_L,r}), \xi_{\xi_{e_L}(u)(u^{-1}_{e_L,r})}(\eta_{e_L}(u^{-1}_{e_L,r})(u))(uv)) \\ = \;& \xi_{e_L}(u)(m(u)(u^{-1}_{e_L,r}, uv)) \\ = \;& \xi_{e_L}(u)(v) \end{aligned}$$

as a result. We have thus proved the lemma. □

By taking this lemma into account, $\theta'(u)^{-1}(v) = \xi_{e_L}(u)^{-1}(v)$, and consequently $\xi'_\lambda(u)(v) = \theta'(\lambda)^{-1}\theta'(\lambda u)(v) = \xi_\lambda(u)(v)$. From (4.7), it follows immediately that $\eta'_\lambda(u)(v) = \eta_\lambda(u)(v)$. □

5. Properties of $R^V(\lambda)$

Let V be an object of the category $\mathcal{A}$. This section is devoted to investigating properties of $R^V(\lambda)$, the dynamical YB map constructed in Theorem 3.7 (Cf. Theorem 1 and Proposition 4 of [21].).

Let (L, G, π) be a representative of V. In place of $\xi^{(L,G,\pi)}_\lambda(u)$ and $\eta^{(L,G,\pi)}_\lambda(u)$ defined by (4.6) and (4.7), we will use the symbols $\xi_\lambda(u)$ and $\eta_\lambda(u)$, respectively. They hence satisfy (3.10), (3.11), (3.12), and (3.13).

Proposition 5.1. *$R^V(\lambda)$ is bijective for all $\lambda \in L$.*

Denote by $\sigma^V(\lambda)$ the corresponding dynamical braiding map to $R^V(\lambda)$: $\sigma^V(\lambda) = PR^V(\lambda)$. The lemma below means that $\sigma^V(\lambda)$ is bijective, and implies Proposition 5.1 as a result.

Lemma 5.2. *For $\lambda \in L$, define the mapping $\iota_\lambda : L \times L \to L \times L$ by*

$$\iota_\lambda(u, v) = (u^{-1}_{\lambda v,r}, v^{-1}_{\lambda,r}) \quad (u, v \in L).$$

(1) *The mapping ι_λ is bijective: the inverse ι_λ^{-1} is as follows.*

$$\iota_\lambda^{-1}(u, v) = (u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l}, v^{-1}_{\lambda,l}) \quad (u, v \in L).$$

(2) $P\iota_\lambda^{-1}\sigma^V((\lambda(L^{(2)})^{-1}_{\lambda,l})(L^{(1)})^{-1}_{\lambda((L^{(2)})^{-1}_{\lambda,l}),l})\iota_\lambda P\sigma^V(\lambda) = \mathrm{id}_{L\times L}$.

(3) $\iota_\lambda P\sigma^V(\lambda)P\iota_\lambda^{-1}\sigma^V((\lambda(L^{(2)})^{-1}_{\lambda,l})(L^{(1)})^{-1}_{\lambda((L^{(2)})^{-1}_{\lambda,l}),l}) = \mathrm{id}_{L\times L}$.

Here the mapping $\sigma^V((\lambda(L^{(2)})^{-1}_{\lambda,l})(L^{(1)})^{-1}_{\lambda((L^{(2)})^{-1}_{\lambda,l}),l}) : L\times L \to L\times L$ is defined by

$$\sigma^V((\lambda(L^{(2)})^{-1}_{\lambda,l})(L^{(1)})^{-1}_{\lambda((L^{(2)})^{-1}_{\lambda,l}),l})(u, v) = \sigma^V((\lambda(v^{-1}_{\lambda,l}))u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l})(u, v)$$

for u, $v \in L$.

We give only the proof of (2) and (3).

Fix elements u and v of L, and let us define the elements x, $y \in L$ by $(y,x) = \sigma^V(\lambda)(u,v) = (\xi_\lambda(u)(v), \eta_\lambda(v)(u))$. The proof of Lemma 5.2 (2) is immediate from the lemma below.

Lemma 5.3. (1) $v^{-1}_{\lambda u,r} = \xi_{(\lambda y)x}(x^{-1}_{\lambda y,r})(y^{-1}_{\lambda,r})$.
(2) $u^{-1}_{\lambda,r} = \eta_{(\lambda y)x}(y^{-1}_{\lambda,r})(x^{-1}_{\lambda y,r})$.

Proof. (1) The equation (3.13) and Lemma 4.1 imply that

$$\begin{aligned}(5.1)\qquad & \xi_\lambda(u)(m(\lambda u)(v, \xi_{(\lambda y)x}(x^{-1}_{\lambda y,r})(y^{-1}_{\lambda,r}))) \\ = \ & m(\lambda)(\xi_\lambda(u)(v), \xi_{\lambda\xi_\lambda(u)(v)}(\eta_\lambda(v)(u))(\xi_{(\lambda y)x}(x^{-1}_{\lambda y,r})(y^{-1}_{\lambda,r}))) \\ = \ & m(\lambda)(x, \xi_{\lambda x}(y)(\xi_{(\lambda y)x}(x^{-1}_{\lambda y,r})(y^{-1}_{\lambda,r}))).\end{aligned}$$

From (3.10) and (3.13), R.H.S. of (5.1) $= e_L$. With the aid of Lemma 4.12, we get the desired result.

(2) By means of (3.12), $(\lambda y)x = (\lambda u)v$, and consequently

$$\begin{aligned} m((\lambda y)x)(v^{-1}_{\lambda u,r}, u^{-1}_{\lambda,r}) &= m((\lambda u)v)(v^{-1}_{\lambda u,r}, u^{-1}_{\lambda,r}) \\ &= ((\lambda u)v)\backslash\lambda \\ &= ((\lambda y)x)\backslash\lambda \\ &= m((\lambda y)x)(x^{-1}_{\lambda y,r}, y^{-1}_{\lambda,r}).\end{aligned}$$

From (1) and (3.12), we conclude that

$$\begin{aligned} & m((\lambda y)x)(v^{-1}_{\lambda u,r}, \eta_{(\lambda y)x}(y^{-1}_{\lambda,r})(x^{-1}_{\lambda y,r})) \\ = \ & m((\lambda y)x)(\xi_{(\lambda y)x}(x^{-1}_{\lambda y,r})(y^{-1}_{\lambda,r}), \eta_{(\lambda y)x}(y^{-1}_{\lambda,r})(x^{-1}_{\lambda y,r})) \\ = \ & m((\lambda y)x)(x^{-1}_{\lambda y,r}, y^{-1}_{\lambda,r}).\end{aligned}$$

This completes the proof by taking account of the cancellation law (3.3) of the quasigroup $(L, m((\lambda y)x))$ (See Lemma 3.2 (1).). □

Now we prove Lemma 5.2 (3). Fix elements u and v of L, and let us define the elements x, $y \in L$ by

$$\begin{aligned}(y,x) &= \sigma^V((\lambda(L^{(2)})^{-1}_{\lambda,l})(L^{(1)})^{-1}_{\lambda((L^{(2)})^{-1}_{\lambda,l}),l})(u,v) \\ &= (\xi_{(\lambda(v^{-1}_{\lambda,l}))(u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l})}(u)(v), \eta_{(\lambda(v^{-1}_{\lambda,l}))(u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l})}(v)(u)).\end{aligned}$$

For the proof, it suffices to show the following.

Lemma 5.4. (1) $m(\lambda)(x^{-1}_{\lambda,l}, y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}) = m(\lambda)(v^{-1}_{\lambda,l}, u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l})$.
(2) $v^{-1}_{\lambda,l} = \xi_\lambda(x^{-1}_{\lambda,l})(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l})$.
(3) $u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l} = \eta_\lambda(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l})(x^{-1}_{\lambda,l})$.

Proof. (1) By the definition of x and y,

$$\begin{aligned}(((\lambda(x^{-1}_{\lambda,l}))(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}))y)x &= \lambda \\ &= (((\lambda(v^{-1}_{\lambda,l}))(u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l}))u)v \\ &= (((\lambda(v^{-1}_{\lambda,l}))(u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l}))y)x.\end{aligned}$$

The cancellation law (3.4) of $(L,\cdot)$ and (3.5) imply the desired result.
(2) From Lemma 4.12 (1), it suffices to prove that $y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l} = \xi_{\lambda(x^{-1}_{\lambda,l})}(x)(v^{-1}_{\lambda,l})$. It follows from (1), the definition of x, (3.11), and (3.13) that $\eta_{\lambda(x^{-1}_{\lambda,l})}(v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r})(x) = u$. By means of (3.12), we deduce

$$m(\lambda(x^{-1}_{\lambda,l}))(x, v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r}) = m(\lambda(x^{-1}_{\lambda,l}))(\xi_{\lambda(x^{-1}_{\lambda,l})}(x)(v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r}), u).$$

This equation and (1) induce

$$\begin{aligned}&m((\lambda(x^{-1}_{\lambda,l}))(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}))(m((\lambda(x^{-1}_{\lambda,l}))(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}))(y, \xi_{\lambda(x^{-1}_{\lambda,l})}(x)(v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r})), u) \\ =\ & m((\lambda(x^{-1}_{\lambda,l}))(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}))(y, m(\lambda(x^{-1}_{\lambda,l}))(x, v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r})) \\ =\ & m((\lambda(v^{-1}_{\lambda,l}))(u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l}))(m((\lambda(v^{-1}_{\lambda,l}))(u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l}))(y,x), v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r}) \\ =\ & m((\lambda(v^{-1}_{\lambda,l}))(u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l}))(m((\lambda(v^{-1}_{\lambda,l}))(u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l}))(u,v), v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r}) \\ =\ & u.\end{aligned}$$

As a consequence of the cancellation law (3.4) of $(L, m((\lambda(x^{-1}_{\lambda,l})(y^{-1}_{\lambda(x^{-1}_{\lambda,l})})))$ and the above equation,

$$m((\lambda(x^{-1}_{\lambda,l}))(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}))(y, \xi_{\lambda(x^{-1}_{\lambda,l})}(x)(v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r})) = e_L.$$

By taking Lemma 3.6 into account, this equation implies

$$\xi_{\lambda(x^{-1}_{\lambda,l})}(x)(v^{-1}_{\lambda,l}) = \xi_{\lambda(x^{-1}_{\lambda,l})}(x)(v^{-1}_{\lambda(v^{-1}_{\lambda,l}),r}) = y^{-1}_{(\lambda(x^{-1}_{\lambda,l}))(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}),r} = y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}.$$

(3) By means of (1) and (3.12),

$$\begin{aligned}m(\lambda)(\xi_\lambda(x^{-1}_{\lambda,l})(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}), \eta_\lambda(y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l})(x^{-1}_{\lambda,l})) &= m(\lambda)(x^{-1}_{\lambda,l}, y^{-1}_{\lambda(x^{-1}_{\lambda,l}),l}) \\ &= m(\lambda)(v^{-1}_{\lambda,l}, u^{-1}_{\lambda(v^{-1}_{\lambda,l}),l}).\end{aligned}$$

From (2) and the cancellation law (3.3) of $(L, m(\lambda))$, we get the desired result. □

The dynamical YB map $R(\lambda)$ associated with H, X, and ϕ is said to satisfy the unitary condition if $R(\lambda)PR(\lambda) = P$ for all $\lambda \in H$.

Theorem 5.5. *Let $V = [(L, G, \pi)]$ be an object of the category $\mathcal{A}$. The following conditions are equivalent*:
(1) $R^V(\lambda)$ *satisfies the unitary condition*;

(2) $\sigma^V(\lambda)^2 = \mathrm{id}_{L\times L}$ *for all* $\lambda \in L$;

(3) $\xi_\lambda(u)(v) = (\eta_{(\lambda u)v}(u^{-1}_{\lambda,r})(v^{-1}_{\lambda u,r}))^{-1}_{\lambda,l}$ *for all* $\lambda,\ u,\ v \in L$;

(4) *all the groups* $(L, \star_\lambda, e_L)$ *are abelian.*

Remark 4.14, Lemma 4.17, and Theorem 5.5 imply the following corollary.

Corollary 5.6. *Let* $V = [(L, G, \pi)]$ *be an element of* $Ob(\mathcal{A})$. $R^V(\lambda)$ *satisfies the unitary condition if and only if the group* G *is abelian.*

The definition of $\sigma^V(\lambda)$ immediately shows that the conditions (1) and (2) in Theorem 5.5 are equivalent.

Proof ((2) $\Leftrightarrow$ (3)). Let λ, u, and v be elements of L. Define the elements u_i and v_i $(i = 1, 2)$ of L by $(u_1, v_1) = \sigma^V(\lambda)^{-1}(u, v)$ and $(u_2, v_2) = \sigma^V(\lambda)(u, v)$. From (3.12),

$$m(\lambda)(u_1, v_1) = m(\lambda)\sigma^V(\lambda)(u_1, v_1) = m(\lambda)(u, v) = m(\lambda)(u_2, v_2).$$

By virtue of Lemma 5.2, the condition (3) means that $u_1 = u_2$, and the cancellation law (3.3) of $(L, m(\lambda))$ implies that $v_1 = v_2$ as a result. We have thus proved (2) from (3). The rest of the proof is immediate. □

The following lemma is useful to prove that the conditions (3) and (4) are equivalent.

Lemma 5.7. $u \star_\lambda v = m(\lambda)(v, (\eta_{\lambda u}(v)(u^{-1}_{\lambda,r}))^{-1}_{\lambda v,l})$ $(\forall \lambda, u, v \in L)$.

Proof. Because $v = m(\lambda)(u, m(\lambda u)(\xi_{\lambda u}(u^{-1}_{\lambda,r})(v), \eta_{\lambda u}(v)(u^{-1}_{\lambda,r})))$, we obtain

$$m(\lambda)(v, \big(\eta_{\lambda u}(v)(u^{-1}_{\lambda,r})\big)^{-1}_{(\lambda u)\xi_{\lambda u}(u^{-1}_{\lambda,r})(v),r}) = u \star_\lambda v.$$

On account of Lemma 3.6, $\big(\eta_{\lambda u}(v)(u^{-1}_{\lambda,r})\big)^{-1}_{(\lambda u)\xi_{\lambda u}(u^{-1}_{\lambda,r})(v),r} = (\eta_{\lambda u}(v)(u^{-1}_{\lambda,r}))^{-1}_{\lambda v,l}$, and we conclude the lemma. □

Proof ((3) $\Leftrightarrow$ (4)). Lemma 3.6 induces that $u^{-1}_{\lambda u,l} \star_{\lambda u} v = m(\lambda u)(u^{-1}_{\lambda u,l}, \xi_\lambda(u)(v))$. With the aid of Lemma 5.7,

$$v \star_{\lambda u} (u^{-1}_{\lambda u,l}) = m(\lambda u)(u^{-1}_{\lambda u,l}, \big(\eta_{(\lambda u)v}(u^{-1}_{\lambda u,l})(v^{-1}_{\lambda u,r})\big)^{-1}_{(\lambda u)u^{-1}_{\lambda u,l},\ l}),$$

and the conditions (5.2) and (5.3) are equivalent as a result:

$$\text{the condition (3) of Theorem 5.5;} \tag{5.2}$$

$$u^{-1}_{\lambda u,l} \star_{\lambda u} v = v \star_{\lambda u} (u^{-1}_{\lambda u,l}) \text{ for all } \lambda, u, v \in L. \tag{5.3}$$

Because Lemma 3.6 implies that $u \star_\lambda v = (u^{-1}_{\lambda u,l})^{-1}_{(\lambda u)(u^{-1}_{\lambda u,l}),l} \star_{(\lambda u)(u^{-1}_{\lambda u,l})} v$, (5.3) is equivalent to (4). □

6. Examples

The present section describes several examples of the dynamical YB maps constructed in Theorem 3.7 and Proposition 3.8.

Let $\mathbb{K}$ denote the field $\mathbb{R}$, or $\mathbb{C}$, and let V be a vector space over $\mathbb{K}$, together with a mapping $N : V \to \mathbb{K}$ satisfying the following properties:

$$\begin{aligned} &(1) \quad N(u) = 0 \text{ if and only if } u = 0. \\ &(2) \quad N(\alpha u) = |\alpha|^2 N(u) \quad (\alpha \in \mathbb{K}, u \in V). \end{aligned}$$

V is an abelian group (and a loop, of course) with respect to the addition $+$. Define the mapping $\pi : V \to V$ by

$$\pi(u) = \begin{cases} N(u)^{-1}u, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

This mapping π is bijective: the inverse of π is as follows.

$$\pi^{-1}(u) = \begin{cases} \overline{N(u)}^{-1}u, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

Theorem 3.7 and Corollary 5.6 give a dynamical YB map $R^{[(V,V,\pi)]}(\lambda)$ satisfying the unitary condition.

In place of $\theta^{(V,V,\pi)}(u)$, we will use $\theta(u)$. For a non-zero element u of V, $\theta(2u) \neq \theta(u)\theta(u)$ because

$$\begin{aligned} &\theta(u)(-N(u)^{-1}u) = -N(u)^{-1}u, \\ &\theta(2u)(-N(u)^{-1}u) = (1/2)N(u)^{-1}u. \end{aligned}$$

From Proposition 4.10, $R^{[(V,V,\pi)]}(\lambda)$ is dependent on the dynamical parameter λ.

If $V = \mathbb{K}$, and $N(u) = |u|^2$, then the mapping $\xi_\lambda(u) = \xi_\lambda^{(V,V,\pi)}(u)$ $(\lambda, u \in V)$ defined by (4.6) is as follows:

$$\xi_\lambda(u)(v) = \begin{cases} \frac{\lambda^2 v}{(\lambda+u)^2+uv}, & \text{if } \lambda v(\lambda+u)(\lambda+u+v)\{(\lambda+u)^2+uv\} \neq 0, \\ -\frac{(\lambda+u)(\lambda+u+v)}{v}, & \text{if } v(\lambda+u)(\lambda+u+v) \neq 0 \\ & \text{and } \lambda\{(\lambda+u)^2+uv\} = 0, \\ -\frac{\lambda^2}{\lambda+v}, & \text{if } (\lambda+u)(\lambda+u+v) = 0 \text{ and } \lambda v(\lambda+v) \neq 0, \\ v, & \text{if } v(\lambda+u)(\lambda+u+v) = 0 \text{ and } \lambda v(\lambda+v) = 0. \end{cases}$$

Let W be a vector space over $\mathbb{K}$. Suppose that there exist mappings $\psi : W \to V$ and $\rho : V \to W$ satisfying $\psi\rho = \mathrm{id}_V$. By using Proposition 3.8 (1), $R^{[(V,V,\pi)]}(\lambda)$ gives birth to a dynamical YB map associated with W, V, and ϕ. Here $\phi(\lambda, u) = \rho(\psi(\lambda) + u)$ $(\lambda \in W, u \in V)$. Proposition 3.8 (2) similarly produces dynamical YB maps from $R^{[(V,V,\pi)]}(\lambda)$.

Remark 6.1. Let h be a finite-dimensional abelian Lie algebra over $\mathbb{C}$, and $\{e_i\}$ a basis of h. Define the mapping N from h to $\mathbb{C}$ by $N(u) = \sum_i |u_i|^2$ $(u = \sum_i u_i e_i, u_i \in \mathbb{C})$. Let ρ denote an isomorphism from h to h^*, and let us define the mapping ϕ from $h^* \times h \to h^*$ by $\phi(\lambda, u) = \lambda + \rho(u)$ $(\lambda \in h^*, u \in h)$.

Proposition 3.8 (1) implies that $R(\lambda) = R^{[(h,h,\pi)]}(\rho^{-1}(\lambda))$ $(\lambda \in h^*)$ is a dynamical YB map associated with h^*, h, and ϕ.

Let V denote the $\mathbb{C}$-vector space generated by the set h. For $\lambda \in h^*$ and $u, v \in h$, define $(u', v') = R(\lambda)(u, v)$. The condition (3.12) induces that $u' + v' = u + v$, and $\rho(u') + \rho(v') = \rho(u) + \rho(v)$ as a result. Because $R(\lambda)$ is bijective and $wt = -\rho$, the above $R(\lambda)$ produces a dynamical R-matrix $\widetilde{R}(\lambda) : V \otimes V \to V \otimes V$ (See Section 2.).

Let us next introduce an example for the case that L_V $(V \in Ob(\mathcal{A}))$ is finite. The straightforward computation shows that $R^V(\lambda) = \mathrm{id}_{L_V \times L_V}$, if the order of L_V is less than or equal to three.

Let $L_6 = \{1, 2, \ldots, 6\}$ be a loop, together with a binary operation $(\cdot)$ presented in Table 1 [5]. Here $4 \cdot 6 = 1$.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	6	5	1	4	2
4	4	5	6	2	3	1
5	5	3	1	6	2	4
6	6	4	2	5	1	3

Table 1. Multiplication table of L_6

Remark 6.2. The binary operation $(\cdot)$ on the loop L_6 is not associative because $(5 \cdot 4) \cdot 6 \neq 5 \cdot (4 \cdot 6)$. By the definition of $m(5)$, $m(5)(4, 6) \neq 4 \cdot 6$.

Let g denote a generator of C_6, the cyclic group of order 6. Define the bijection π from L_6 to C_6 by $\pi(k) = g^{k-1}$. Theorem 3.7 and Corollary 5.6 imply that $R^{[(L_6,C_6,\pi)]}(\lambda)$ is a dynamical YB map satisfying the unitary condition, and we deduce, owing to Proposition 4.10, that $R^{[(L_6,C_6,\pi)]}(\lambda)$ is dependent on the dynamical parameter λ. The Tables 2 and 3 are the mappings $R^{[(L_6,C_6,\pi)]}(2)$ and $R^{[(L_6,C_6,\pi)]}(3)$. Here $R^{[(L_6,C_6,\pi)]}(2)(2, 4) = (6, 6)$.

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	1	2	3	4	5	6
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
2	$(2,1)$	$(6,4)$	$(5,3)$	$(6,6)$	$(4,5)$	$(6,2)$
3	$(3,1)$	$(3,4)$	$(3,3)$	$(3,5)$	$(3,2)$	$(3,6)$
4	$(4,1)$	$(4,6)$	$(2,3)$	$(5,5)$	$(5,4)$	$(2,2)$
5	$(5,1)$	$(2,5)$	$(4,3)$	$(5,2)$	$(4,4)$	$(2,6)$
6	$(6,1)$	$(6,5)$	$(6,3)$	$(2,4)$	$(5,6)$	$(4,2)$

TABLE 2. $R^{[(L_6,C_6,\pi)]}(2)(u,v)$

	1	2	3	4	5	6
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
2	$(2,1)$	$(4,4)$	$(5,3)$	$(6,6)$	$(6,5)$	$(6,2)$
3	$(3,1)$	$(3,4)$	$(3,3)$	$(3,5)$	$(3,2)$	$(3,6)$
4	$(4,1)$	$(4,5)$	$(2,3)$	$(2,2)$	$(5,4)$	$(5,6)$
5	$(5,1)$	$(2,5)$	$(4,3)$	$(2,4)$	$(4,6)$	$(5,2)$
6	$(6,1)$	$(2,6)$	$(6,3)$	$(5,5)$	$(6,4)$	$(4,2)$

TABLE 3. $R^{[(L_6,C_6,\pi)]}(3)(u,v)$

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