A theorem on essential self-adjointness with application to Hamiltonians in nonrelativistic quantum field theory

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(Received 9 October 1990; accepted for publication 26 February 1991)

An abstract theorem is given on essential self-adjointness of operators in infinite direct sum of Hilbert spaces and is applied to a class of Hamiltonians in nonrelativistic quantum field theory to prove their essential self-adjointness.

I. INTRODUCTION

In this paper we present an abstract theorem on essential self-adjointness of operators in infinite direct sum of Hilbert spaces and apply it to a class of Hamiltonians in nonrelativistic quantum field theory (QFT) to prove their essential self-adjointness.

About 20 years ago, D. Masson and W. K. McClary[1] gave an interesting proof of the essential self-adjointness of the Hamiltonian of $(\phi^4)_2$ theory with a space cutoff. Their proof makes use of some specific properties of the interaction Hamiltonian acting in the boson Fock space over $L^2(\mathbb{R})$. We have found that their method can be formulated in an abstract way to give a criterion for essential self-adjointness of operators in infinite direct sum of Hilbert spaces. This is a background of the present work.

The outline of the present paper is as follows. In Sec. II we first state the abstract theorem mentioned above and then prove it. The proof of the theorem is quite similar to that of Masson and McClary in Ref. 1, but, we give it for completeness. Section III is devoted to the application of the theorem to a class of models in nonrelativistic QFT. Each model in the class describes a quantum system of a finite number of nonrelativistic particles interacting with some quantum scalar fields. In Sec. IV we discuss some examples: the linear polaron model,[2-5] the RWA oscillator,[6-8] a model of a bounded electron interacting with a quantized radiation field,[9-11] their generalizations, and scalar quantum electrodynamics with cutoffs.[12]

II. THE ABSTRACT THEOREM

Let $\mathscr{H}_n$, $n = 0,1,2,\ldots$, be Hilbert spaces and

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n \tag{2.1}$$

be the infinite direct sum of $\mathscr{H}_n$, $n\geqslant 0$. Every vector $f\in\mathscr{H}$ is a sequence $f = \{f^{(n)}\}_{n=0}^{\infty}$ of vectors $f^{(n)}\in\mathscr{H}$ with

$$\|f\|^2 = \sum_{n=0}^{\infty} \|f^{(n)}\|^2 < \infty.$$

We identify $f^{(n)}$ with the vector $\{0,0,\ldots,0,f^{(n)},0,\ldots\}\in\mathscr{H}$ [the $(n+1)$th component is $f^{(n)}$ and all the other components are zero]. We introduce the subspace

$$\mathscr{D}_0 = \{f\in\mathscr{H} | f^{(n)} = 0 \text{ for all but finitely many } n\}, \tag{2.2}$$

which is dense in $\mathscr{H}$, and the degree operator ("number operator") $\hat{N}$ by

$$(\hat{N}f)^{(n)} = nf^{(n)}, \quad n\geqslant 0, \tag{2.3}$$

with domain

$$D(\hat{N}) = \left\{ f\in\mathscr{H} \middle| \sum_{n=0}^{\infty} n^2\|f^{(n)}\|^2 < \infty \right\}.$$

The operator $\hat{N}$ is self-adjoint and non-negative.

Let A be a self-adjoint operator in $\mathscr{H}$ which is reduced by each $\mathscr{H}_n$, so that for all $n\geqslant 0$,

$$A: D(A)\cap\mathscr{H}_n \to \mathscr{H}_n. \tag{2.4}$$

is self-adjoint. It is easy to see that A is essentially self-adjoint on the dense subspace

$$\mathscr{D} = D(A)\cap\mathscr{D}_0. \tag{2.5}$$

Let B be a symmetric operator in $\mathscr{H}$ that satisfies the following conditions (B1) and (B2).

(B1) $\mathscr{D}_0 \subset D(B)$ and there exist a constant $c>0$ and a linear operator L in $\mathscr{H}$ such that $D(L) \supset D([(A+B)\upharpoonright\mathscr{D}]^*)$,

$$L: D(L)\cap\mathscr{H}_n \to \mathscr{H}_n,$$

for all $n\geqslant 0$, and

$$|(f,Bg)| \leqslant c\|Lf\|\|(\hat{N}+1)^2 g\|, \quad f,g\in\mathscr{D}.$$

(B2) There exists an integer $p\geqslant 0$ such that for all $f\in\mathscr{D}_0$,

$$(f^{(m)}, Bf^{(n)}) = 0 \quad \text{unless } |m-n| = 0,1,\ldots,p.$$

The following theorem gives a criterion for the essential self-adjointness of the operator $A+B$.

Theorem 2.1: Let A and B be as above. Suppose that $A+B$ is bounded from below. Then $A+B$ is essentially self-adjoint on $\mathscr{D}$.

To prove this theorem, we prepare a lemma, which may be interesting in its own right.

Lemma 2.2: Let S and T be symmetric operators in $\mathscr{H}$ and

$$R \equiv S + T$$

is strictly positive on a subspace $D\subset D(S)\cap D(T)$, i.e., for a constant $\lambda>0$,

$$R\geqslant\lambda \text{ on } D. \tag{2.6}$$

Let $\{f_n\}_{n=1}^{\infty}\subset D$ be a sequence satisfying the following conditions (i)–(iv):

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(i) $f_1 \neq 0$;

(ii) $(f_m, f_n) = 0$, unless $m = n$;

(iii) $(f_m, Sf_n) = 0$, unless $m = n$;

(iv) $(f_m, Tf_n) = 0$, for $|m - n| \geqslant 2$;

(v) $(f_{n-1}, Tf_n) + (f_n, Rf_n) + (f_{n+1}, Tf_n) = 0$, $n \geqslant 1$, (2.7)

where we set $f_0 = 0$. Then, for all $n \geqslant 1$, $f_n \neq 0$ and

$$(f_{n+1}, Tf_n) < 0. \tag{2.8}$$

Moreover,

$$\sum_{n=1}^{\infty} \frac{1}{|(f_{n+1}, Tf_n)|} < \infty. \tag{2.9}$$

Proof: Conditions (i) and (ii) imply that $\Sigma_{j=1}^n f_j \neq 0$ for all $n \geqslant 1$. Therefore (2.6) gives

$$\left(\sum_{j=1}^n f_j, R \sum_{j=1}^n f_j \right) > 0. \tag{2.10}$$

Using conditions (iii)–(v), we can show that the left-hand side of (2.10) equals $-(f_{n+1}, Tf_n)$. Hence, $f_{n+1} \neq 0$ and (2.8) follows.

To prove (2.9), we define a set of numbers $\{a_n\}_{n=1}^{\infty}$ by the following recursion relations:

$$a_1 = 1, \tag{2.11}$$

$$a_{n-1}(f_{n-1}, Tf_n) + a_n(f_n, Rf_n) + a_{n+1}(f_{n+1}, Tf_n) = \mu a_n \|f_n\|^2, \quad n \geqslant 1, \tag{2.12}$$

where μ is a constant with $0 < \mu < \lambda$ and we set $a_0 = 0$. It is easy to see that for all $n \geqslant 2$, a_n is real. Let

$$g_n = a_n f_n$$

and

$$\hat{R} = R - \mu,$$

which is strictly positive. Multiplying (2.12) by a_n, we have

$$(g_{n-1}, Tg_n) + (g_n, \hat{R} g_n) + (g_{n+1}, Tg_n) = 0.$$

Hence, we can apply the preceding result with f_n, R, and S replaced by g_n, $\hat{R}$, and $S - \mu$, respectively, to obtain

$$0 > (g_{n+1}, Tg_n) = -a_{n+1} a_n |(f_{n+1}, Tf_n)|.$$

Since $a_1 = 1 > 0$, this inequality implies that for all $n \geqslant 2$, $a_n > 0$. Multiplying (2.7) [resp. (2.12)] by a_n^2 (resp. a_n) and making the subtraction to eliminate the term (f_n, Rf_n), we obtain

$$(a_n - a_{n+1})|(f_{n+1}, Tf_n)| = \mu a_n \|f_n\|^2 + (a_{n-1} - a_n)|(f_{n-1}, Tf_n)|, \quad n \geqslant 1. \tag{2.13}$$

Taking $n = 1$ in this equation, we see that $a_1 > a_2$. It then turns out that for all $n \geqslant 1$, $a_n > a_{n+1}$. Combining this result with (2.13), we obtain

$$\frac{1}{|(f_{n+1}, Tf_n)|} \leqslant \frac{a_n - a_{n+1}}{\mu \|f_1\|^2},$$

which implies that

$$\sum_{n=1}^{N} \frac{1}{|(f_{n+1}, Tf_n)|} \leqslant \frac{a_1}{\mu \|f_1\|^2}.$$

Thus (2.9) follows. ■

Proof of Theorem 2.1: Without loss of generality, we can assume that for a constant $\gamma > 0$,

$$A + B \geqslant \gamma$$

on $\mathscr{D}$. Throughout the proof, we set

$$C = A + B.$$

It is sufficient to prove that Ker $(C \restriction \mathscr{D})^* = \{0\}$ (e.g., Theorem X.26 in Ref. 13). Let $g \in \mathrm{Ker}(C \restriction \mathscr{D})^*$. Then

$$(g, Cf) = 0, \tag{2.14}$$

for all $f \in \mathscr{D}$. By (2.4), we have

$$(g^{(n)}, Af) = (g, Af^{(n)}) = (g, (C - B) f^{(n)}) = -(g, Bf^{(n)}).$$

Using (B1) and (B2), we obtain

$$|(g^{(n)}, Af)| \leqslant c(n + p + 1)^2 \|Lg\| \|f\|.$$

Hence, the map: $f \to (g^{(n)}, Af)$ defines uniquely a continuous linear functional on $\mathscr{H}$. Hence, by the Riesz lemma, there exists a vector $\psi_n \in \mathscr{H}$ such that

$$(g^{(n)}, Af) = (\psi_n, f), \quad f \in \mathscr{D}.$$

Since $\mathscr{D}$ is a core of A, this equation extends to all $f \in D(A)$, which implies that $g^{(n)} \in D(A^*) = D(A)$. Hence, $g^{(n)} \in \mathscr{D}$.

If $p = 0$, then $B: \mathscr{H}_n \to \mathscr{H}_n$ for all $n \geqslant 0$ and hence

$$C: \mathscr{D} \cap \mathscr{H}_n \to \mathscr{H}_n,$$

for all $n \geqslant 0$. Therefore, putting $f = g^{(n)} \in \mathscr{D}$ into (2.14), we have

$$0 = (g, Cg^{(n)}) = (g^{(n)}, Cg^{(n)}) \geqslant \gamma \|g^{(n)}\|^2.$$

Hence, $g^{(n)} = 0$ for all $n \geqslant 0$, i.e., $g = 0$.

Let $p \geqslant 1$ and define

$$h_n = \sum_{j=0}^{p-1} g^{(pn+j)} \in \mathscr{D}.$$

Putting $f = h_n$ into (2.14), we have

$$(g, Ch_n) = 0,$$

which implies that

$$(h_{n-1}, Bh_n) + (h_n, Ch_n) + (h_{n+1}, Bh_n) = 0, \quad n \geqslant 0,$$

where we set $h_{-1} = 0$. It is easy to see that

$$(h_m, h_n) = 0 \quad (h_m, Ah_n) = 0,$$

unless $m = n$ and

$$(h_m, Bh_n) = 0,$$

for $|m - n| \geqslant 2$. Suppose that $g \neq 0$. Then, for some n, $h_n \neq 0$. Hence, we can define

$$N = \min\{n \geqslant 0 | \ \|h_n\| \neq 0\}.$$

Then we can apply Lemma 2.2 with $f_n = h_{N+n-1}$ $(n \geqslant 1)$, $S = A, T = B$, to obtain

$$K = \sum_{n=N}^{\infty} \frac{1}{|(h_{n+1}, Bh_n)|} < \infty.$$

Using the Schwarz inequality, we have

$$\sum_{n=N}^{\infty}\left\{\frac{\|Lh_{n+1}\|\|h_n\|}{|(h_{n+1},Bh_n)|}\right\}^{1/2}$$
$$\leqslant K^{1/2}\left\{\sum_{n=N}^{\infty}\|Lh_{n+1}\|^2\right\}^{1/4}\left\{\sum_{n=N}^{\infty}\|h_n\|^2\right\}^{1/4}$$
$$\leqslant\sqrt{Kp\|Lg\|\|g\|}. \tag{2.15}$$

On the other hand, we have by (B1)

$$\frac{\|Lh_{n+1}\|\|h_n\|}{|(h_{n+1},Bh_n)|}\geqslant\frac{1}{cp^2(n+1)^2},$$

which implies that the left-hand side of (2.15) diverges. This is a contradiction. Thus g must be zero. ■

III. ESSENTIAL SELF-ADJOINTNESS OF A CLASS OF HAMILTONIANS IN NONRELATIVISTIC QUANTUM FIELD THEORY

In this section we apply Theorem 2.1 to prove the essential self-adjointness of a class of Hamiltonians in nonrelativistic quantum field theory. The Hamiltonians we consider correspond to models of a finite number of nonrelativistic particles interacting with some quantum scalar fields.

For a mathematical generality, we assume that the scalar fields under consideration are over $\mathbb{R}^d$ with $d\geqslant1$. The Hilbert space for state vectors of the particle system is taken to be $L^2(\mathbb{R}^N)$. We denote by $q=(q_1,...,q_N)\in\mathbb{R}^N$ the coordinate variable of $\mathbb{R}^N$ and define the "momentum operator" p by

$$p=(p_1,...,p_N),$$

with

$$p_j=-i\frac{\partial}{\partial q_j}, \tag{3.1}$$

where $i=\sqrt{-1}$ and the partial derivatives are taken in the generalized sense.

The mathematical framework for the quantum scalar fields is given as follows: Let $\mathscr{K}$ be the M direct sum of $L^2(\mathbb{R}^d)$ $(M\geqslant1)$:

$$\mathscr{K}=\underbrace{L^2(\mathbb{R}^d)\oplus\cdots\oplus L^2(\mathbb{R}^d)}_{M\text{ times}} \tag{3.2}$$

and $S^n(\mathscr{K})$ $(n\geqslant1)$ be the n-fold symmetric tensor product of $\mathscr{K}$:

$$S^n(\mathscr{K})=\otimes_s^n\mathscr{K} \tag{3.3}$$

[we set $S^0(\mathscr{K})=\mathbb{C}$]. The Hilbert space for the scalar fields is taken to be the symmetric Fock space over $\mathscr{K}$:

$$\mathscr{F}_s(\mathscr{K})=\bigoplus_{n=0}^{\infty}S^n(\mathscr{K}). \tag{3.4}$$

We denote by $a(F)$ $(F\in\mathscr{K})$ the annihilation operator in $\mathscr{F}_s(\mathscr{K})$ (antilinear in F) and by N_b the number operator. The mapping: $L^2(\mathbb{R}^d)\ni f\to f_r=(0,...,0,f,0,...,0)\in\mathscr{K}$ (the rth component is f and the other components are zero) defines an embedding of $L^2(\mathbb{R}^d)$ into $\mathscr{K}$. Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}^d$. Then the mapping: $\mathscr{S}(\mathbb{R}^d)\ni f\to a(f_r)$ defines an operator-valued distribution; we denote its kernel by $a_r(k)$:

$$a(f_r)=\int a_r(k)f(k)dk.$$

The operator-valued distributions $\{a_r(k)\}_{r=1}^M$ satisfy the canonical commutation relations:

$$[a_r(k),a_s(p)^*]=\delta_{rs}\delta(k-p),$$
$$[a_r(k),a_s(p)]=0, r,s=1,...,M.$$

Let $\mathscr{F}_0(\mathscr{K})$ be the finite particle vector space of $\mathscr{F}_s(\mathscr{K})$:

$$\mathscr{F}_0(\mathscr{K})=\{\Psi=\{\Psi^{(n)}\}_{n=0}^{\infty}\in\mathscr{F}_s(\mathscr{K})\,|\,\Psi^{(n)}=0\text{ for all but finitely many }n\}. \tag{3.5}$$

For each $W_{m,n}\in L^2(\mathbb{R}^{d(m+n)})$ $(m+n\geqslant1, m,n\geqslant0)$ and $r_i,s_j=1,...,M$, we can define a unique closed linear operator $W_{m,n}(a_{r_1}^*,...,a_{r_m}^*;a_{s_1},...,a_{s_n})$ in $\mathscr{F}_s(\mathscr{K})$ with

$$D(W_{m,n}(a_{r_1}^*,...,a_{r_m}^*;a_{s_1},...,a_{s_n}))\supset\mathscr{F}_0(\mathscr{K}) \tag{3.6}$$

such that $\mathscr{F}_0(\mathscr{K})$ is a core for $W_{m,n}(a_{r_1}^*,...,a_{r_m}^*;a_{s_1},...,a_{s_n})$ and

$$W_{m,n}(a_{r_1}^*,...,a_{r_m}^*;a_{s_1},...,a_{s_n})=\int_{\mathbb{R}^{d(m+n)}}W_{m,n}(k_1,...,k_m,\xi_1,...,\xi_n)\times\left(\prod_{i=1}^{m}a_{r_i}(k_i)^*\right)\left(\prod_{j=1}^{n}a_{s_j}(\xi_j)\right)dk\,d\xi \tag{3.7}$$

as a quadratic form on $\mathscr{F}_0(\mathscr{K})\times\mathscr{F}_0(\mathscr{K})$ (see Theorem X.44 in Ref. 13). Some fundamental properties of the operator $W_{m,n}(a_{r_1}^*,...,a_{r_m}^*;a_{s_1},...,a_{s_n})$ are summarized in the following lemma.

Lemma 3.1: (i) If k and l are non-negative integers such that $k+l=m+n$, then

$$(1+N_b)^{-k/2}W_{m,n}(a_{r_1}^*,...,a_{r_m}^*;a_{s_1},...,a_{s_n})(1+N_b)^{-l/2}$$

is a bounded operator with

$$\|(1+N_b)^{-k/2}W_{m,n}(a_{r_1}^*,...,a_{r_m}^*;a_{s_1},...,a_{s_n})(1+N_b)^{-l/2}\|\leqslant C(k,l)\|W\|_{L^2},$$

where $C(k,l)>0$ is a constant.

(ii) Let

$$\widetilde{W}_{m,n}(k_1,...,k_m,\xi_1,...,\xi_n)=W_{n,m}(\xi_1,...,\xi_n,k_1,...,k_m)^*.$$

Then

$$D(\widetilde{W}_{m,n}(a_{s_1}^*,...,a_{s_m}^*;a_{r_1},...,a_{r_n}))\subset D(W_{n,m}(a_{r_1}^*,...,a_{r_n}^*;a_{s_1},...,a_{s_m})^*)$$

and

$$\widetilde{W}_{m,n}(a_{s_1}^*,...,a_{s_m}^*;a_{r_1},...,a_{r_n})=W_{n,m}(a_{r_1}^*,...,a_{r_n}^*;a_{s_1},...,a_{s_m})^*$$

on $D(\widetilde{W}_{m,n}(a_{s_1}^*,...,a_{s_m}^*;a_{r_1},...,a_{r_n}))$.

(iii) $W_{m,n}(a_{r_1}^*,...,a_{r_m}^*;a_{s_1},...,a_{s_n})$ maps $S^k(\mathscr{K})$ into $S^{k+m-n}(\mathscr{K})$ (resp. $\{0\}$) for $k\geqslant n$ (resp. $k<n$).

For proof of this lemma, see Theorem X.44 in Ref. 13.

Let $\omega_r(k), r=1,...,M,$ be non-negative measurable functions on $L^2(\mathbb{R}^d)$ with $\omega_r\in L^2_{\rm loc}(\mathbb{R}^d)$ and

$$\hat{\omega}=\oplus_{r=1}^{M}\omega_r$$

be the direct sum of ω_r as multiplication operators. We define

$$H_F=d\Gamma(\hat{\omega}) \tag{3.8}$$

to be the second quantization of the operator $\hat{\omega}$. We have

$$H_F = \sum_{r=1}^{M} \int \omega_r(k) a_r(k)^* a_r(k) dk$$

as a quadratic form on

$$[\mathscr{F}_0(\mathscr{K}) \cap D(H_F)] \times [\mathscr{F}_0(\mathscr{K}) \cap D(H_F)].$$

The Hilbert space of the coupled system of the particles and the scalar fields is defined by

$$\mathscr{F} = L^2(\mathbb{R}^N) \otimes \mathscr{F}_s(\mathscr{K}). \tag{3.9}$$

Every closed operator A (resp. B) in $L^2(\mathbb{R}^N)$ [resp. $\mathscr{F}_s(\mathscr{K})$] extends to $\mathscr{F}$ as $A \otimes I$ (resp. $I \otimes B$), where I denotes identity. In what follows, however, we shall denote them by the same symbols, provided that there is no danger of confusion.

We now consider the following Hamiltonian:

$$\begin{aligned} H = {} & \sum_{j=1}^{N} \frac{p_j^2}{2m_j} + H_F + \sum_{1 \leqslant k \leqslant 4} \{\lambda_{j_1 \cdots j_k} q_{j_1} \cdots q_{j_k} + \mu_{j_1 \cdots j_k} p_{j_1} \cdots p_{j_k}\} \\ & + \sum_{1 \leqslant k \leqslant 4} \sum_{1 \leqslant l \leqslant k} \{\nu_{j_1 \cdots j_k} p_{j_1} \cdots p_{j_l} q_{j_{l+1}} \cdots q_{j_k} + \nu^*_{j_1 \cdots j_k} q_{j_{l+1}} \cdots q_{j_k} p_{j_1} \cdots p_{j_l}\} + \sum_{1 \leqslant n+m \leqslant 4} \{W_{m,n}^{(r_1,\dots,r_m,s_1,\dots,s_n)}(a^*_{r_1},\dots,a^*_{r_m};a_{s_1},\dots,a_{s_n}) \\ & + W_{m,n}^{(r_1,\dots,r_m,s_1,\dots,s_n)}(a^*_{r_1},\dots,a^*_{r_m};a_{s_1},\dots,a_{s_n})^*\} + \sum_{1 \leqslant k+m+n \leqslant 4} \sum_{1 \leqslant l \leqslant k} \{\gamma_{j_1 \cdots j_k} p_{j_1} \cdots p_{j_l} q_{j_{l+1}} \cdots q_{j_k} \\ & \times V_{m,n}^{(r_1,\dots,r_m,s_1,\dots,s_n)}(a^*_{r_1},\dots,a^*_{r_m};a_{s_1},\dots,a_{s_n}) + \gamma^*_{j_1 \cdots j_k} q_{j_{l+1}} \cdots q_{j_k} p_{j_1} \cdots p_{j_l} V_{m,n}^{(r_1,\dots,r_m,s_1,\dots,s_n)}(a^*_{r_1},\dots,a^*_{r_m};a_{s_1},\dots,a_{s_n})^*\}, \end{aligned} \tag{3.10}$$

where $m_j > 0$, $\lambda_{j_1 \cdots j_k}$, $\mu_{j_1 \cdots j_k} \in \mathbb{R}$, $\nu_{j_1 \cdots j_k}$, $\gamma_{j_1 \cdots j_k} \in \mathbb{C}$, are constants,

$$W_{m,n}^{(r_1,\dots,s_n)}, \quad V_{m,n}^{(r_1,\dots,s_n)} \in L^2(\mathbb{R}^{d(m+n)}),$$

and summations with respect to the repeated indices j_l, r_i, s_i, are understood. The operator H is a sum of operators of polynomial type with degree less than or equal to 4 in $q_j, p_j, a_r^\#, j = 1,\dots,N, r = 1,\dots,M$. By Lemma 3.1(i) and the fact that p_j and q_j leave $\mathscr{S}(\mathbb{R}^N)$ invariant, H is a symmetric operator with

$$D(H) \supset \mathscr{S}(\mathbb{R}^N) \hat{\otimes} (\mathscr{F}_0(\mathscr{K}) \cap D(H_F)) \equiv \mathscr{D}_H, \tag{3.11}$$

where $\hat{\otimes}$ denotes algebraic tensor product. We prove the following theorem.

Theorem 3.2: Suppose that H is bounded from below on $\mathscr{D}_H$. Then H is essentially self-adjoint on $\mathscr{D}_H$.

To prove Theorem 3.2, we need a preliminary. To apply Theorem 2.1 to the present case, we must rewrite $\mathscr{F}$ as an infinite direct sum. To this end, we make use of the Fock–Hermite–Wiener decomposition of $L^2(\mathbb{R}^N)$. We first recall this decomposition. Let $\nu_j > 0, j = 1,\dots,N$, be constants and introduce the annihilation and creation operators for the particles as follows:

$$b_j = (1/\sqrt{2\nu_j})(ip_j + \nu_j q_j), \tag{3.12}$$

$$b_j^\dagger = (1/\sqrt{2\nu_j})(-ip_j + \nu_j q_j), \quad j = 1,\dots,N, \tag{3.13}$$

which leave $\mathscr{S}(\mathbb{R}^N)$ invariant and satisfy the commutation relations

$$[b_j, b_k^\dagger] = \delta_{jk}, \quad [b_j, b_k] = 0, j,k = 1,\dots,N, \tag{3.14}$$

on $\mathscr{S}(\mathbb{R}^N)$. We have from (3.12) and (3.13)

$$q_j = (1/\sqrt{2\nu_j})(b_j + b_j^\dagger), \tag{3.15}$$

$$p_j = i\sqrt{\nu_j/2}(b_j^\dagger - b_j). \tag{3.16}$$

Let

$$\psi_0 = \prod_{j=1}^{N} \left\{ \left(\frac{\nu_j}{\pi}\right)^{1/4} e^{-\nu_j q_j^2/2} \right\}. \tag{3.17}$$

Then we have $\|\psi_0\| = 1$ and

$$b_j \psi_0 = 0, \quad j = 1,\dots,N. \tag{3.18}$$

For $z = (z_1,\dots,z_N) \in \mathbb{C}^N$, we define the operators $b(z)$ and $b(z)^\dagger$ by

$$b(z) = \sum_{j=1}^{N} z_j^* b_j, \quad b(z)^\dagger = \sum_{j=1}^{N} z_j b_j^\dagger. \tag{3.19}$$

Then (3.14) is equivalent to the following commutation relations:

$$[b(z), b(w)^\dagger] = (z,w)_{\mathbb{C}^N}, \quad [b(z), b(w)] = 0, z,w \in \mathbb{C}^N. \tag{3.20}$$

Let

$$\mathscr{M}_0 = \{\alpha\psi_0 \mid \alpha \in \mathbb{C}\},$$

$$\mathscr{M}_n = \left\{ \sum_{j_1,\dots,j_n=1}^{J_1,\dots,J_n} \alpha_{j_1 \cdots j_n} b(z_{j_1})^\dagger \cdots b(z_{j_n})^\dagger \psi_0 \,\middle|\, \alpha_{j_1 \cdots j_n} \in \mathbb{C}, z_{j_k} \in \mathbb{C}^N, J_k \geqslant 1 \right\}, \quad n \geqslant 1. \tag{3.21}$$

Then one can easily see that

$$\mathscr{M}_n = \{P_n(q)\psi_0 \mid P_n\text{:polynomials of order } n\} \subset \mathscr{S}(\mathbb{R}^N). \tag{3.22}$$

Hence, for all $n \geqslant 0$, $\mathscr{M}_n$ is finite dimensional. Moreover, we can show that $\mathscr{M}_n \perp \mathscr{M}_m$ for $m \neq n$ and

$$L^2(\mathbb{R}^N) = \bigoplus_{n=0}^{\infty} \mathscr{M}_n \cong \mathscr{F}_s(\mathbb{C}^N). \tag{3.23}$$

This is the desired decomposition of $L^2(\mathbb{R}^N)$. In the decomposition (3.23), the degree operator is given by

$$N_p = \sum_{j=1}^{N} b_j^\dagger b_j. \tag{3.24}$$

Using (3.23), the entire Hilbert space $\mathscr{F}$ is decomposed as

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}_n, \tag{3.25}$$

where

$$\mathscr{F}_n = \bigoplus_{l+m=n} \mathscr{M}_l \otimes S^m(\mathscr{K}). \quad (3.26)$$

One can easily show that

$$\|b_j^{\#}\psi\| \leqslant \|(N_p+1)^{1/2}\psi\|, \quad \psi \in D(N_p^{1/2}), j=1,...,N. \quad (3.27)$$

Using (3.15), (3.16), and (3.27), we can prove the following estimate:

$$\|p_{j_1}\cdots p_{j_m} q_{i_1}\cdots q_{i_n}\psi\| \leqslant C_{m,n}\|(N_p+1)^{(m+n)/2}\psi\|,$$
$$\psi \in D(N_p^{(m+n)/2}), \quad (3.28)$$

where $C_{m,n} > 0$ is a constant.

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2: Write H as

$$H = H_0 + H_I,$$

with

$$H_0 = \sum_{j=1}^N \frac{1}{2m_j}(p_j^2 + \nu_j^2 q_j^2) + H_F$$

and

$$H_I = -\sum_{j=1}^N \frac{\nu_j^2}{2m_j} q_j^2 + \sum_{1\leqslant k\leqslant 4}\{\lambda_{j_1\cdots j_k} q_{j_1}\cdots q_{j_k} + \mu_{j_1\cdots j_k} p_{j_1}\cdots p_{j_k}\}$$
$$+ \sum_{1\leqslant k\leqslant 4}\sum_{1\leqslant l\leqslant k}\{\nu_{j_1\cdots j_k} p_{j_1}\cdots p_{j_l} q_{j_{l+1}}\cdots q_{j_k} + \nu^*_{j_1\cdots j_k} q_{j_{l+1}}\cdots q_{j_k} p_{j_1}\cdots p_{j_l}\} + \sum_{1\leqslant n+m\leqslant 4}\{W_{m,n}^{(r_1,...,r_m,s_1,...,s_n)}(a^*_{r_1},...,a^*_{r_m};a_{s_1},...,a_{s_n})$$
$$+ W_{m,n}^{(r_1,...,r_m,s_1,...,s_n)}(a^*_{r_1},...,a^*_{r_m};a_{s_1},...,a_{s_n})^*\} + \sum_{1\leqslant k+n+m\leqslant 4}\sum_{1\leqslant l\leqslant k}\{\gamma_{j_1\cdots j_k} p_{j_1}\cdots p_{j_l} q_{j_{l+1}}\cdots q_{j_k}$$
$$\times V_{m,n}^{(r_1,...,r_m,s_1,...,s_n)}(a^*_{r_1},...,a^*_{r_m};a_{s_1},...,a_{s_n}) + \gamma^*_{j_1\cdots j_k} q_{j_{l+1}}\cdots q_{j_k} p_{j_1}\cdots p_{j_l} V_{m,n}^{(r_1,...,r_m,s_1,...,s_n)}(a^*_{r_1},...,a^*_{r_m};a_{s_1},...,a_{s_n})^*\},$$

The operator H_0 is self-adjoint and positive with

$$D(H_0) = \cap_{j=1}^N \{D(p_j^2)\cap D(q_j^2)\}\cap D(H_F).$$

Since H_0 is written as

$$H_0 = \sum_{j=1}^N \left\{\frac{\nu_j}{m_j} b_j^\dagger b_j + \frac{\nu_j}{2m_j}\right\} + H_F,$$

it follows that H_0 is reduced by each $\mathscr{F}_n$. Let

$$\mathscr{D}_0 = \{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathscr{F} | \Psi^{(n)} \in \mathscr{F}_n,$$
$$\Psi^{(n)} = 0 \text{ for all but finitely many } n\}$$

and

$$\mathscr{D} = \mathscr{D}_0 \cap D(H_0).$$

By (3.22), we can show that

$$\mathscr{D} = \{P(q)\psi_0 | P\text{:polynomials}\}$$
$$\hat{\otimes}[D(H_F)\cap \mathscr{F}_0(\mathscr{K})] \subset \mathscr{D}_H. \quad (3.29)$$

The degree operator in $\mathscr{F}$ represented as (3.25) is given by

$$\hat{N} = N_p + N_b.$$

By Lemma 3.1 (iii) and the fact that $b_j^\dagger$ (resp. b_j) maps $\mathscr{M}_k$ into $\mathscr{M}_{k+1}$ (resp. $\mathscr{M}_{k-1}$), we see that for all $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty, \Phi = \{\Phi^{(n)}\}_{n=0}^\infty \in \mathscr{D}_0$,

$$(\Psi^{(n)}, H_I \Phi^{(m)}) = 0, \quad |m-n| > 4.$$

Moreover, by Lemma 3.1(i) and (3.28), we can show that

$$\|H_I\Psi\| \leqslant \sum_{k+l\leqslant 2} C_{k,l}\|(N_p+1)^k(N_b+1)^l\Psi\|, \quad \Psi\in\mathscr{D}_0,$$

where $C_{k,l} > 0$ is a constant. It is easy to see that

$$\|(N_p+1)^k(N_b+1)^l\Psi\| \leqslant \|(\hat{N}+2)^{k+l}\Psi\|$$
$$\leqslant 2^{k+l}\|(\hat{N}+1)^{k+l}\Psi\|.$$

Hence, we get

$$\|H_I\Psi\| \leqslant C\|(\hat{N}+1)^2\Psi\|,$$

with $C > 0$ being a constant. Thus we can apply Theorem 2.1 with $\mathscr{H} = \mathscr{F}, A = H_0, B = H_I$ to conclude that H is essentially self-adjoint on $\mathscr{D}$, which, together with (3.29), gives Theorem 3.2. ■

IV. EXAMPLES

In this section we discuss some concrete examples of the Hamiltonian H given by (3.10). We follow the notations in Sec. III unless otherwise stated.

Example 1: Let us take $\mathscr{K} = L^2(\mathbb{R}^d)$ (i.e., the case $M=1$), so that

$$\mathscr{F} = L^2(\mathbb{R}^N)\otimes \mathscr{F}_s(L^2(\mathbb{R}^d)).$$

Let $\omega_1 = \omega$ and consider the Hamiltonian

$$H_1 = p^2/2m + V(q) + d\Gamma(\omega)$$
$$+ \sum_{j=1}^N q_j \int (\lambda_j(k)a(k)^* + \lambda_j(k)^* a(k))dk$$
$$+ \int \frac{|\Sigma_{j=1}^N \lambda_j(k) q_j|^2}{\omega(k)} dk. \quad (4.1)$$

Here, $m > 0$ is a constant, $V(q)$ is a polynomial of the form

$$V(q) = \sum_{1\leqslant k\leqslant 4} g_{j_1\cdots j_k} q_{j_1}\cdots q_{j_k}, \quad (4.2)$$

with $g_{j_1\cdots j_k}\in\mathbb{R}$ and λ_j is a measurable function on $\mathbb{R}^d$ with $\lambda_j, \lambda_j/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. We assume that V is bounded from below:

$$\inf_{q\in\mathbb{R}^d} V(q) > -\infty. \quad (4.3)$$

In the case where $N=1$ and $V(q) = Kq^2/2$ with a constant $K>0$, H_1 gives the Hamiltonian of a model of laser,[2] which was discussed rigorously in Refs. 4 and 5, in connection with the problem of Lamb shift and spontaneous emission of light in quantum electrodynamics. The case where $N = d = 3$ and

$V(q) = Kq^2/2$ is the linear polaron model.[3] The Hamiltonian H_1 with $N=1$ and with a nonquadratic V was proposed by Caldeira and Leggett[11] to discuss quantum tunneling and coherence with dissipation. For quantum coherence, V is taken to be a double well potential, e.g., $V(q) = g(1-q^2)^2$ with a constant $g>0$. The model given by (4.1) is a generalization of these models.

Let

$$\mathscr{D}_{H_1} = \mathscr{S}(\mathbb{R}^N)\,\hat{\otimes}\,[\mathscr{F}_0(L^2(\mathbb{R}^d))\cap D(d\Gamma(\omega))]. \quad (4.4)$$

Theorem 4.1: The operator H_1 is bounded from below and essentially self-adjoint on $\mathscr{D}_{H_1}$.

Proof: Since H_1 is of the form of the operator H defined by (3.10), we need only to show that it is bounded from below. Then Theorem 3.2 gives the essential self-adjointness of H_1 on $\mathscr{D}_{H_1}$. Introducing the operators

$$H_b = d\Gamma(\omega) + \sum_{j=1}^{N} q_j \int (\lambda_j(k)a(k)^* + \lambda_j(k)^* a(k))dk + \int \frac{|\Sigma_{j=1}^N \lambda_j(k)q_j|^2}{\omega(k)}dk$$

and

$$H_p = p^2/2m + V(q), \quad (4.5)$$

we can write H_1 as

$$H_1 = H_p + H_b.$$

By (4.3), H_p is bounded from below. We note that for all $\Psi\in\mathscr{D}_{H_1}$,

$$(\Psi,H_b\Psi) = \int \left(\left(\sqrt{\omega(k)}a(k) + \sum_{j=1}^N \frac{\lambda_j(k)q_j}{\sqrt{\omega(k)}}\right)\Psi, \left(\sqrt{\omega(k)}a(k) + \sum_{j=1}^N \frac{\lambda_j(k)q_j}{\sqrt{\omega(k)}}\right)\Psi\right)dk.$$

The right-hand side is non-negative and hence $(\Psi,H_b\Psi)\geqslant 0$. Thus it follows that H_1 is bounded from below on $\mathscr{D}_{H_1}$. ■

Example 2: Let $b_j^\#$ be given by (3.12) and (3.13) and V be as in Example 1. Let

$$H_2 = p^2/2m + V(q) + d\Gamma(\omega) + \sum_{j=1}^N \int (\lambda_j(k)b_j a(k)^* + \lambda_j(k)^* b_j^\dagger a(k))dk + \sum_{i,j=1}^N \int dk \frac{\lambda_i(k)^*\lambda_j(k)}{\omega(k)} b_i^\dagger b_j, \quad (4.6)$$

This model is a variant of Example 1. The case where $N=1$ and

$$H_p = (\nu_1/m)b_1^\dagger b_1$$

is called the RWA oscillator (e.g., Refs. 6–8).

Theorem 4.2: The operator H_2 is bounded from below and essentially self-adjoint on $\mathscr{D}_{H_1}$.

Proof: The operator H_2 is of the form of H given by (3.10). Hence, we need only to prove the boundedness from below of H_2 on $\mathscr{D}_{H_1}$. This can be done in the same way as in the proof of Theorem 4.1; in fact, we can show that $H_2 - H_p \geqslant 0$ on $\mathscr{D}_{H_1}$. ■

Example 3: We consider the following case:

$$N=d, \quad m_j = m, \quad j=1,\dots,d,$$
$$\mathscr{K} = \oplus^{d-1}L^2(\mathbb{R}^d) \quad (\text{i.e., } M=d-1),$$
$$\omega_r(k) = \omega(k), \quad r=1,\dots,d-1,$$

so that

$$\mathscr{F} = L^2(\mathbb{R}^d)\otimes\mathscr{F}_s(\oplus^{d-1}L^2(\mathbb{R}^d)).$$

This choice gives a framework to discuss models of a d-dimensional electron coupled to a quantized radiation field.[9,10,14,15] Let $\rho(x)$ be a real distribution on $\mathbb{R}^d$ such that its Fourier transform $\hat{\rho}(k)$ is a measurable function with

$$\hat{\rho}/\sqrt{\omega}, \quad |k|\hat{\rho}/\sqrt{\omega}\in L^2(\mathbb{R}^d) \quad (4.7)$$

and $\{e^{(r)}(k)\}_{r=1}^{d-1}$ be a set of vectors in $\mathbb{R}^d$ ("polarization vectors" of a photon with momentum k) such that

$$e^{(r)}(k)\cdot e^{(s)}(k) = \delta_{rs}, \quad k\cdot e^{(r)}(k) = 0, r,s=1,\dots,d-1.$$

The time zero radiation field with cutoff $\hat{\rho}$ is defined by

$$A_j(x;\rho) = \int \frac{1}{\sqrt{2\omega(k)}} e_j^{(r)}(k)\{\hat{\rho}(k)a_r(k)^* \times e^{-ikx} + \hat{\rho}(k)^* a_r(k)e^{ikx}\}dk, \quad j=1,\dots,d.$$

We consider the following Hamiltonian:

$$H_3 = \frac{1}{2m}\left(p - eA(0;\rho) - \lambda e\sum_{j=1}^d q_j(\partial_j A)(0;\rho)\right)^2 + V(q) + d\Gamma(\oplus^{d-1}\omega), \quad (4.8)$$

where $e\in\mathbb{R}$ is a parameter denoting the elementary charge, $\lambda\in\mathbb{R}$, and $V(q)$ is a polynomial given by (4.2) with (4.3) $(N=d-1)$. The case $\lambda=0$ corresponds to the usual dipole approximation. The case where $V(q) = \epsilon q^2/2$ $(\epsilon>0)$ and $\lambda=0$ has been discussed in Ref. 10 (cf. also Ref. 9). Let

$$\mathscr{D}_{H_3} = \mathscr{S}(\mathbb{R}^d)\,\hat{\otimes}\,[D(d\Gamma(\oplus^{d-1}\omega)) \cap\mathscr{F}_0(\oplus^{d-1}L^2(\mathbb{R}^d))]. \quad (4.9)$$

Theorem 4.3: The operator H_3 is bounded from below and essentially self-adjoint on $\mathscr{D}_{H_3}$.

Proof: The operator H_3 is also of the form of H given by (3.10). It is obvious that H_3 is bounded from below on $\mathscr{D}_{H_3}$. Thus, applying Theorem 3.1, we get the desired result. ■

Remarks: In the case $\lambda=0$, the second condition for $\hat{\rho}$ in (4.7) can be dropped. The above theorem slightly improves the result in Ref. 10 on the essential self-adjointness of H_3 with $V(q) = \epsilon q^2/2$ and with $\lambda=0$ in the sense that concerning the condition for ρ, we need to assume only the first condition in (4.7), while in Ref. 10, we assume, in addition to the first condition in (4.7), $\sqrt{\omega}\rho\in L^2(\mathbb{R}^d)$.

Example 4: Scalar quantum electrodynamics with cutoffs. We consider a quantum system of a charged scalar field interacting with a radiation field. The Fock space to describe such a system is given by

$$\mathscr{F} = \mathscr{F}_s(\mathscr{K})$$

(i.e., the case "$N=0$" in the framework given in Sec. III), where $\mathscr{K}$ is given by (3.2) with $M = 2 + (d-1)$. We set

$$a_1(k) = b(k), \quad a_2(k) = c(k), \quad \omega_1(k) = \omega_2(k) = \mu(k),$$
$$\omega_3(k) = \cdots = \omega_{d+1}(k) = \omega(k),$$

and rename $a_r(k), r=3,\dots,2+(d-1)$ as $a_r(k), r$

$= 1,\ldots,d-1$, respectively. Let η be a measurable function on $\mathbb{R}^d$ such that

$$\hat{\eta}/\sqrt{\mu}, \quad |k|\hat{\eta}/\sqrt{\mu}, \quad \sqrt{\mu}\hat{\eta}\in L^2(\mathbb{R}^d).$$

Then the time-zero charged scalar field and its conjugate with the ultraviolet cutoff $\hat{\eta}$ are defined by

$$\phi(x;\eta) = \int \frac{1}{\sqrt{2\mu(p)}} \{\hat{\eta}(p)c(p)^* e^{-ipx} + \hat{\eta}(p)^* b(p) e^{ikp}\} dp,$$

$$\pi(x;\eta) = i\int \sqrt{\frac{\mu(p)}{2}} \{\hat{\eta}(p)b(p)^* e^{-ipx} - \hat{\eta}(p)^* c(p) e^{ikp}\} dp.$$

We consider the following Hamiltonian:

$$H_4 = \int g(x)\{\pi(x;\eta)^*\pi(x;\eta) + (\nabla + ieA(x;\rho))\phi(x;\eta)^*(\nabla - ieA(x;\rho))\phi(x;\eta) + m^2\phi(x;\eta)^*\phi(x;\eta)\}dx + H_{EM}, \tag{4.10}$$

where

$$H_{EM} = \sum_{r=1}^{d-1} \int \omega(k) a_r(k)^* a_r(k) dk,$$

and $g\in\mathscr{S}(\mathbb{R}^d)$ is positive.

Theorem 4.4: The operator H_4 is non-negative and essentially self-adjoint on $D(H_{EM})\cap\mathscr{F}_0(\mathscr{K})$.

Proof: One can easily check that H_4 is of the form of H given by (3.10) and is non-negative on $D(H_{EM})\cap\mathscr{F}_0(\mathscr{K})$. Thus we can apply Theorem 3.1 to obtain the desired result.

For formal aspects of the model H_4 without cutoffs, see, e.g., Ref. 12.

ACKNOWLEDGMENT

This work was supported by the Grant-In-Aid 02740094 for science research from the Ministry of Education, Japan.

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