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Canonical commutation relations, the Weierstrass Zeta function, and infinite dimensional Hilbert space representations of the quantum group $U_q(\mathfrak{sl}_2)$

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A two-dimensional quantum system of a charged particle interacting with a vector potential determined by the Weierstrass Zeta function is considered. The position and the physical momentum operators give a representation of the canonical commutation relations with two degrees of freedom. If the charge of the particle is not an integer (the case corresponding to the *Aharonov–Bohm effect*), then the representation is inequivalent to the Schrödinger representation. It is shown that the inequivalent representation induces infinite-dimensional Hilbert space representations of the quantum group $U_q(\mathfrak{sl}_2)$. Some properties of these representations of $U_q(\mathfrak{sl}_2)$ are investigated. © 1996 American Institute of Physics. [S0022-2488(96)00108-9]

I. INTRODUCTION

In a previous paper,¹ we considered a quantum system of a charged particle moving in the Euclidean plane \mathbf{R}^2 under the influence of a perpendicular magnetic field, which may be strongly singular at some fixed points $\mathbf{a}_1, \dots, \mathbf{a}_N$ in \mathbf{R}^2 . If the magnetic field is concentrated on the discrete set $\{\mathbf{a}_n\}_{n=1}^N$ in the sense of distribution, then the position and the physical momentum operators give a representation of the canonical commutation relations (CCRs) (Heisenberg relations) with two degrees of freedom.

Here we recall some technical terms in the representation theory of CCR. A set $\{\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^d\}$ consisting of a Hilbert space \mathcal{H} , a dense subspace \mathcal{D} of \mathcal{H} , and self-adjoint operators $Q_j, P_j, (j=1, \dots, d)$ is called a representation of the CCR with d degrees of freedom if $\mathcal{D} \subset \bigcap_{j,k=1}^d [D(Q_j Q_k) \cap D(Q_j P_k) \cap D(P_k Q_j) \cap D(P_j P_k)]$ [$D(T)$ denotes the domain of operator T] and the CCR,

$$[Q_j, Q_k] = 0, \neg [P_j, P_k] = 0, \neg [Q_j, P_k] = i \delta_{jk}, \neg j, k = 1, \dots, d,$$

hold on \mathcal{D} , where $[S, T] := ST - TS$. Following Putnam,² we say that a set $\{Q_j, P_j\}_{j=1}^d$ of self-adjoint operators on a Hilbert space is a *Schrödinger d system* if it is unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with d degrees of freedom. A representation $\{\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^d\}$ of CCR is called *equivalent* (resp., *inequivalent*) if $\{Q_j, P_j\}_{j=1}^d$ is (resp., not) a Schrödinger d system.

In Ref. 1 the following facts were shown: (i) the representation of CCR mentioned above is equivalent if and only if the magnetic flux is locally quantized (i.e., the magnetic flux at each point \mathbf{a}_n ($n=1, \dots, N$) is an integer multiple of $2\pi/\alpha$, where α is the charge of the particle); (ii) the inequivalent representation appearing in the case where the magnetic flux is not locally quantized may be regarded as a mathematical form of the *Aharonov–Bohm effect*.³ A geometric construction of a representation of CCR that is unitarily equivalent to the one given in Ref. 1 was considered by Kurose and Nakazato,⁴ and more detailed properties of the representation were discussed.

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To investigate differences between the equivalent and the inequivalent representations of CCR in Ref. 1, an analysis has been made on Dirac–Weyl operators defined in terms of the physical momentum operator.⁵ Moreover, the framework and results in Ref. 1 were extended to the case of a non-Abelian gauge theory.^{6,7}

Recently Wiegmann and Zabrodin⁸ considered a quantum system of a particle on a two-dimensional square lattice in a magnetic field and showed that magnetic translations on the lattice are related to finite-dimensional representations of the quantum group $U_q(\mathfrak{sl}_2)$. Inspired by their work, we investigate in this paper if there is any connection between the quantum system considered in Ref. 1 and representations of $U_q(\mathfrak{sl}_2)$. We shall show that, for a vector potential determined by the Weierstrass Zeta function, special elements of the unitary groups generated by the components of the physical momentum operator induce representations of $U_q(\mathfrak{sl}_2)$ on the Hilbert space $L^2(\mathbf{R}^2)$. This establishes a connection of a classical special function to $U_q(\mathfrak{sl}_2)$. A particular feature of the representations of $U_q(\mathfrak{sl}_2)$ given in this paper is that they have no finite-dimensional irreducible components.

The present paper is organized as follows. In Sec. II we consider a quantum system of a charged particle in \mathbf{R}^2 under the influence of a perpendicular magnetic field that may be singular at points in an *infinite lattice* [see (2.1)]. This is an extension of the framework of Ref. 1 to the case where the number of possible singular points of the magnetic field is infinite. Fundamental results in Ref. 1 continue to hold in the present case too with no significant modifications. As in the case of Ref. 1, the position operator and the physical momentum operator $\mathbf{P}=(P_1, P_2)$ of the particle give a representation of the CCR with two degrees of freedom if the magnetic field is concentrated on the infinite lattice. Spectral properties of the unitary operators generated by P_1 and P_2 are analyzed. Also, we show that, under some conditions for the vector potential of the magnetic field, P_1 and P_2 have permutation-reflection symmetries.

In Secs. III–V, we consider the special case where the vector potential $A=A_1 dx+A_2 dy$ (a 1-form) of the magnetic field is given by the Weierstrass Zeta function ζ in such a way that $\zeta=A_2+iA_1$ [see (3.2)–(3.4)]. We first show in Sec. III that, if the charge of the particle is not an integer, then special elements of the unitary groups generated by P_1 and P_2 give representations of the quantum plane. Unitary equivalences of these representations are discussed. As is shown in Ref. 8, a representation of the quantum plane with some additional properties induces representations of $U_q(\mathfrak{sl}_2)$. Applying this idea, we construct in Secs. IV and V representations of $U_q(\mathfrak{sl}_2)$ on $L^2(\mathbf{R}^2)$ and investigate some properties of them.

II. A CHARGED PARTICLE IN A MAGNETIC FIELD WITH POSSIBLE SINGULARITIES ON AN INFINITE LATTICE

A. Representation of CCR

We consider a quantum system of a charged particle with charge $\alpha \in \mathbf{R} \setminus \{0\}$ moving in the plane $\mathbf{R}^2 = \{\mathbf{r}=(x, y) | x, y \in \mathbf{R}\}$ under the influence of a perpendicular magnetic field B that may be singular at points in the infinite lattice,

$$\mathbf{Z}_{\omega_1, \omega_2}^2 := \{\Omega_{m, n} = (\omega_1 m, \omega_2 n) | m, n \in \mathbf{Z}\}, \quad (2.1)$$

where $\omega_j, j=1, 2$, are positive constants and \mathbf{Z} is the set of integers. A vector potential of the magnetic field B is given by a *continuously differentiable* real 1-form $A=A_1 dx+A_2 dy$ on the nonsimply connected domain

$$\mathbf{M} = \mathbf{R}^2 \setminus \mathbf{Z}_{\omega_1, \omega_2}^2, \quad (2.2)$$

such that

$$B = D_x A_2 - D_y A_1, \quad (2.3)$$

in the distribution sense, where D_x and D_y are the generalized partial differential operators in the variables x and y , respectively. We say that A is flat on \mathbf{M} if $B=0$ on \mathbf{M} , i.e., B is concentrated on $\mathbf{Z}_{\omega_1, \omega_2}^2$ (in the sense of distribution).

Throughout this paper we use a physical unit system such that \hbar (the Planck constant divided by 2π) = c (the light velocity) = 1. Let (Q_1, Q_2) be the position operator of the particle, i.e., Q_1 and Q_2 are the maximal multiplication operators by x and y acting in the Hilbert space $L^2(\mathbf{M}) \cong L^2(\mathbf{R}^2)$, respectively. The physical (kinetic) momentum operator $\mathbf{P}=(P_1, P_2)$ of the particle is defined by the operators

$$P_j = p_j - \alpha A_j, \quad j=1,2, \quad (2.4)$$

acting in $L^2(\mathbf{R}^2)$, where (p_1, p_2) is the momentum operator of the free particle:

$$p_1 = -iD_x, \quad p_2 = -iD_y. \quad (2.5)$$

We denote by $C_0^m(\mathbf{M})$ ($m=0,1,2,\dots$) the space of m times continuously differentiable functions on \mathbf{M} with bounded support in \mathbf{M} . In the same way as in Ref. 1, we can prove the following fact.

Lemma 2.1:

(i) Each P_j is essentially self-adjoint on $C_0^1(\mathbf{M})$. (We denote the closure of P_j by the same symbol.)

(ii) Suppose that A is flat on \mathbf{M} . Then $\{L^2(\mathbf{R}^2), C_0^2(\mathbf{M}), \{Q_j, P_j\}_{j=1}^2\}$ is a representation of the CCR with two degrees of freedom.

The analysis of the representation $\{L^2(\mathbf{R}^2), C_0^2(\mathbf{M}), \{Q_j, P_j\}_{j=1}^2\}$ in Lemma 2.1 can be done in quite the same way as in the case where $\mathbf{Z}_{\omega_1, \omega_2}^2$ is replaced by a finite discrete set in \mathbf{R}^2 (see Refs. 1, 4, and 6). Hence, as for that, we describe only results needed later. Let

$$\mathbf{S}_1 = \mathbf{R} \setminus \{m\omega_1\}_{m \in \mathbf{Z}}, \quad \mathbf{S}_2 = \mathbf{R} \setminus \{n\omega_2\}_{n \in \mathbf{Z}}.$$

For $(x, y) \in \mathbf{R} \times \mathbf{S}_2$, we can define

$$U_1(x, y) = \exp\left(-i\alpha \int_0^x A_1(x', y) dx'\right). \quad (2.6)$$

Then U_1 defines a unique unitary operator as a multiplication operator on $L^2(\mathbf{R}^2)$. Similarly, the function

$$U_2(x, y) = \exp\left(-i\alpha \int_0^y A_2(x, y') dy'\right), \quad (x, y) \in \mathbf{S}_1 \times \mathbf{R}, \quad (2.7)$$

defines a unique unitary operator as a multiplication operator on $L^2(\mathbf{R}^2)$. We then have operator equalities,

$$P_j = U_j^{-1} p_j U_j, \quad j=1,2. \quad (2.8)$$

It follows from these relations that, for all $t \in \mathbf{R}$, $\psi \in L^2(\mathbf{R}^2)$, and for almost everywhere (a.e.) (x, y) ,

$$(e^{itP_1}\psi)(x, y) = \exp\left(-i\alpha \int_x^{x+t} A_1(x', y) dx'\right) \psi(x+t, y), \quad (2.9)$$

$$(e^{itP_2}\psi)(x,y) = \exp\left(-i\alpha \int_y^{y+t} A_2(x,y')dy'\right)\psi(x,y+t). \quad (2.10)$$

Let $C_{\pm}(x,y;s,t)$ ($x,y,s,t \in \mathbf{R}$) be hook-shaped paths from (x,y) to $(x+s,y+t)$ given by

$$C_-(x,y;s,t) = \{(x+\theta s,y) | 0 \leq \theta \leq 1\} \cup \{(x+s,y+\theta t) | 0 \leq \theta \leq 1\},$$

$$C_+(x,y;s,t) = \{(x,y+\theta t) | 0 \leq \theta \leq 1\} \cup \{(x+\theta s,y+t) | 0 \leq \theta \leq 1\},$$

and set

$$C(x,y;s,t) = C_+(x,y;s,t)^{-1} \circ C_-(x,y;s,t), \quad (2.11)$$

the rectangular path: $(x,y) \rightarrow (x+s,y) \rightarrow (x+s,y+t) \rightarrow (x,y+t) \rightarrow (x,y)$. For $s,t \in \mathbf{R}$, we set

$$\mathbf{S}_1^{(s)} = \mathbf{R} \setminus \{\omega_1 m, \omega_1 m - s | m \in \mathbf{Z}\}, \quad \mathbf{S}_2^{(t)} = \mathbf{R} \setminus \{\omega_2 n, \omega_2 n - t | n \in \mathbf{Z}\}, \quad (2.12)$$

and define

$$\mathbf{M}_{s,t} = \mathbf{S}_1^{(s)} \times \mathbf{S}_2^{(t)}, \quad s,t \in \mathbf{R}. \quad (2.13)$$

For each $s,t \in \mathbf{R}$, we can define a function $\Phi_{s,t}^A$ on $\mathbf{M}_{s,t}$ by

$$\Phi_{s,t}^A(x,y) = \int_{C(x,y;s,t)} A, \quad (x,y) \in \mathbf{M}_{s,t}, \quad (2.14)$$

which physically means the magnetic flux passing through the interior domain of the closed curve $C(x,y;s,t)$. Since $\mathbf{R}^2 \setminus \mathbf{M}_{s,t}$ is a null set with respect to the two-dimensional Lebesgue measure, one can regard $\Phi_{s,t}^A$ as a real-valued function on \mathbf{R}^2 , which is a.e. finite. Hence $\Phi_{s,t}^A$ defines a unique self-adjoint multiplication operator on $L^2(\mathbf{R}^2)$. We denote this operator by the same symbol. The following theorem gives commutation relations for the one parameter unitary groups generated by Q_j and P_j , $j=1,2$.

Theorem 2.2. (cf. Theorem 2.1 in Ref. 1): *For all $s,t \in \mathbf{R}$,*

$$e^{isQ_j} e^{itP_k} = e^{-ist\delta_{jk}} e^{itP_k} e^{isQ_j}, \quad (2.15)$$

$$e^{isP_1} e^{itP_2} = e^{-i\alpha\Phi_{s,t}^A} e^{itP_2} e^{isP_1}. \quad (2.16)$$

Following Ref. 1, we say that *the magnetic flux is locally quantized if, for all $s,t \in \mathbf{R}$, $\Phi_{s,t}^A$ is a $2\pi\mathbf{Z}/\alpha$ -valued function on $\mathbf{M}_{s,t}$.*

Theorem 2.2 implies the following characterization of the representation $\{Q_j, P_j\}_{j=1}^2$ of CCR in the case where A is flat [Lemma 2.1(ii)].

Theorem 2.3: *Suppose that A is flat on \mathbf{M} . Then the representation $\{Q_j, P_j\}_{j=1}^2$ of CCR is equivalent if and only if the magnetic flux is locally quantized.*

Remark: (i) In the case where the magnetic flux is not locally quantized, formula (2.16) may be regarded as a mathematical form of the *Aharonov-Bohm effect*.⁹ Thus the inequivalent representation of $\{Q_j, P_j\}_{j=1}^2$ corresponds to the Aharonov-Bohm effect.

(ii) In operator theory, two self-adjoint operators, S and T , on a Hilbert space are said to be *strongly commuting* if their spectral measures commute. It is well known that S and T are strongly commuting if and only if $e^{iaS} e^{ibT} = e^{ibT} e^{iaS}$ for all $a,b \in \mathbf{R}$ (Theorem VIII.13 in Ref. 10). It follows from this fact and (2.16) that P_1 and P_2 are *strongly commuting if and only if the magnetic flux is locally quantized*.

Let $\omega_0 > 0$ be a number such that, for all $(m, n), (m', n') \in \mathbf{Z}^2$ with $(m, n) \neq (m', n')$, $\{\mathbf{r} \mid |\mathbf{r} - \mathbf{\Omega}_{m,n}| < \omega_0\} \cap \{\mathbf{r} \mid |\mathbf{r} - \mathbf{\Omega}_{m',n'}| < \omega_0\} = \emptyset$. The following fact is easily proven.

Lemma 2.4: Suppose that A is flat. Let $0 < \delta < \omega_0$. Then

$$\gamma_{m,n}(A) := \int_{|\mathbf{r} - \mathbf{\Omega}_{m,n}| = \delta} A^{-1} \tag{2.17}$$

is independent of δ , where the orientation of the integral on the right-hand side (rhs) is taken to be anticlockwise. Moreover, for all $s, t \in \mathbf{R}$,

$$\Phi_{s,t}^A(x, y) = \epsilon(s)\epsilon(t) \sum_{\mathbf{\Omega}_{m,n} \in D(x,y;s,t)} \gamma_{m,n}(A)^{-1} \quad (x, y) \in \mathbf{M}_{s,t},$$

where $D(x, y; s, t)$ is the interior domain of $C(x, y; s, t)$ and $\epsilon(t)$ is the sign function: $\epsilon(t) = 1$ for $t \geq 0$; $\epsilon(t) = -1$ for $t < 0$.

Theorem 2.3 and Lemma 2.4 imply the following fact.

Theorem 2.5: Suppose that A is flat. Then the representation $\{Q_j, P_j\}_{j=1}^2$ of CCR is equivalent if and only if $\gamma_{m,n}(A) \in 2\pi\mathbf{Z}/\alpha$ for all $m, n \in \mathbf{Z}$.

B. Spectral properties

For later use, we investigate spectral properties of some unitary operators. For a densely defined closed linear operator T on a Hilbert space, we denote by $\sigma(T)$ [resp., $\sigma_p(T)$] the spectrum (resp., point spectrum) of T . We set

$$\mathbb{T} = \{z \in \mathbf{C} \mid |z| = 1\}. \tag{2.18}$$

Lemma 2.6:

(i) $\sigma(P_j) = \mathbf{R}$, $\sigma_p(P_j) = \emptyset$, $j = 1, 2$.

(ii) For all $t \in \mathbf{R} \setminus \{0\}$ and $j = 1, 2$, $\sigma(e^{itP_j}) = \mathbb{T}$, $\sigma_p(e^{itP_j}) = \emptyset$.

Proof: (i) By (2.8), we have $\sigma(P_j) = \sigma(p_j) = \mathbf{R}$, $\sigma_p(P_j) = \sigma_p(p_j) = \emptyset$. (ii) This follows from part (i) and the spectral mapping theorem. ■

Lemma 2.7: For all $s, t \in \mathbf{R} \setminus \{0\}$,

$$\sigma_p(e^{isP_1}e^{itP_2}) = \emptyset, \quad \sigma_p(e^{itP_2}e^{isP_1}) = \emptyset. \tag{2.19}$$

Proof: It follows from (2.9) and (2.10) that, for all $s, t \in \mathbf{R}$,

$$e^{isP_1}e^{itP_2} = \exp\left(-i\alpha \int_{C_-(x,y;s,t)} A\right) e^{isP_1}e^{itP_2}, \tag{2.20}$$

$$e^{itP_2}e^{isP_1} = \exp\left(-i\alpha \int_{C_+(x,y;s,t)} A\right) e^{itP_2}e^{isP_1}. \tag{2.21}$$

Since $e^{isP_1}e^{itP_2}$ is unitary, we have $\sigma_p(e^{isP_1}e^{itP_2}) \subset \mathbb{T}$. Suppose that there exists a vector $\psi \in L^2(\mathbf{R}^2)$ and a constant $\lambda \in \mathbb{T}$ such that $e^{isP_1}e^{itP_2}\psi = \lambda\psi$. Then, by (2.20), we have $\exp(-i\alpha \int_{C_-(x,y;s,t)} A)\psi(x+s, y+t) = \lambda\psi(x, y)$ a.e. (x, y) . Hence $|\psi(x, y)| = |\psi(x+s, y+t)|$ a.e. (x, y) , which, together with the fact $\psi \in L^2(\mathbf{R}^2)$, implies $\psi = 0$. Thus the first formula of (2.19) follows. Similarly, using (2.21), we can prove the second one of (2.19). ■

Proposition 2.8: Consider the case where the magnetic flux is locally quantized. Then, for all $t, s \in \mathbf{R} \setminus \{0\}$,

$$\sigma(e^{isP_1}e^{itP_2}) = \sigma(e^{itP_1}e^{isP_2}) = \mathbb{T}.$$

Proof: Under the present assumption, P_1 and P_2 strongly commute [see Remark (ii) after Theorem 2.3]. Hence, by the two variable functional calculus, we see that $sP_1 + tP_2$ is essentially self-adjoint and $e^{isP_1}e^{itP_2} = e^{itP_2}e^{isP_1} = e^{i\overline{(sP_1 + tP_2)}}$, where $\overline{sP_1 + tP_2}$ is the closure of $sP_1 + tP_2$. By the two variable functional calculus and Lemma 2.6(i), we have $\sigma(\overline{sP_1 + tP_2}) = \mathbf{R}$. Thus, by the spectral mapping theorem, we obtain the desired result. \blacksquare

Remark: We have been unable to identify $\sigma(e^{isP_1}e^{itP_2})$ and $\sigma(e^{itP_2}e^{isP_1})$ in the case where the magnetic field is not locally quantized. It would be interesting to see if there appears any difference from the case where the magnetic field is locally quantized. In the case of Dirac–Weyl operators defined in terms of P_1 and P_2 , such differences exist; see Ref. 5. We leave this problem as an open problem.

C. Permutation-reflection symmetry of the physical momentum operator

Let $R_j: L^2(\mathbf{R}^2) \rightarrow L^2(\mathbf{R}^2)$, $j=1,2$, be the unitary operators defined by

$$(R_1\psi)(x,y) = \psi(-y,x), \quad (R_2\psi)(x,y) = \psi(y,-x), \quad \psi \in L^2(\mathbf{R}^2). \quad (2.22)$$

Then it is easy to see that

$$R_1R_2 = R_2R_1 = I, \quad (2.23)$$

$$R_1p_1R_1^{-1} = -p_2, \quad R_1p_2R_1^{-1} = p_1, \quad (2.24)$$

$$R_2p_1R_2^{-1} = p_2, \quad R_2p_2R_2^{-1} = -p_1. \quad (2.25)$$

As usual, we denote by $z = x + iy$ the point in the complex plane \mathbf{C} corresponding to $\mathbf{r} = (x,y) \in \mathbf{R}^2$. We set

$$\tilde{A}(z) = A_2(x,y) + iA_1(x,y). \quad (2.26)$$

Proposition 2.9:

(i) Suppose that

$$\tilde{A}(iz) = -i\tilde{A}(z). \quad (2.27)$$

Then

$$R_1P_1R_1^{-1} = -P_2, \quad R_1P_2R_1^{-1} = P_1. \quad (2.28)$$

(ii) Suppose that

$$\tilde{A}(iz) = i\tilde{A}(-z). \quad (2.29)$$

Then

$$R_2P_1R_2^{-1} = P_2, \quad R_2P_2R_2^{-1} = -P_1. \quad (2.30)$$

Proof: (i) Condition (2.27) is equivalent to that

$$A_2(-y,x) = A_1(x,y), \quad A_1(-y,x) = -A_2(x,y).$$

Hence, for all $\psi \in C_0^1(\mathbf{M})$,

$$R_1P_1R_1^{-1}\psi = -P_2\psi, \quad R_1P_2R_1^{-1}\psi = P_1\psi. \quad (2.31)$$

Since R_1 leaves $C_0^1(\mathbf{M})$ invariant bijectively and $P_j, j=1,2$, are essentially self-adjoint on $C_0^1(\mathbf{M})$ [Lemma 2.1(i)], the vector equations in (2.31) extend to operator equalities (2.28).

(ii) Similar to part (i). \square

In Sec. III, we shall consider the case of a vector potential satisfying (2.27) and (2.29).

D. Small coupling limit

Finally we consider the small coupling limit $\alpha \rightarrow 0$ of P_j .

Lemma 2.10: For all $t \in \mathbf{R}$ and $j=1,2$,

$$s\text{-}\lim_{\alpha \rightarrow 0} e^{itP_j} = e^{itp_j}, \quad (2.32)$$

where $s\text{-}\lim$ denotes strong limit.

Proof: For all $\psi \in C_0^1(\mathbf{M})$, we have $P_j \psi \rightarrow p_j \psi$ ($\alpha \rightarrow 0$). Note that $C_0^1(\mathbf{M})$ is a common core of P_j and p_j . Hence, by general convergence theorems [Theorem VIII.25 (a) and Theorem VIII.21 in Ref. 10], we obtain (2.32). \square

III. A VECTOR POTENTIAL GIVEN BY THE WEIERSTRASS ZETA FUNCTION AND REPRESENTATIONS OF THE QUANTUM PLANE

We now specialize the vector potential A . We set

$$\Omega_{m,n} = m\omega_1 + in\omega_2, \quad m, n \in \mathbf{Z}. \quad (3.1)$$

Let $\zeta(z)$ ($z \in \mathbf{C}$) be the Weierstrass Zeta function with poles at $z = \Omega_{m,n}, m, n \in \mathbf{Z}$:

$$\zeta(z) = \frac{1}{z} + \sum_{(m,n) \in \mathbf{Z}^2 \setminus \{(0,0)\}} \left(\frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2} \right). \quad (3.2)$$

In what follows, we assume that the vector potential A is given by $A = A_1 dx + A_2 dy$ with

$$A_1(\mathbf{r}) = \text{Im } \zeta(z), \quad A_2(\mathbf{r}) = \text{Re } \zeta(z), \quad (3.3)$$

so that

$$\zeta(z) = A_2(\mathbf{r}) + iA_1(\mathbf{r}). \quad (3.4)$$

Then, by the Cauchy–Riemann equation, A is flat on \mathbf{M} .

In the present case, the constant $\gamma_{m,n}(A)$ defined by (2.17) is computed as

$$\gamma_{m,n}(A) = 2\pi, \quad (3.5)$$

independently of $(m,n) \in \mathbf{Z}^2$. Hence the magnetic flux is locally quantized if and only if α is an integer. Thus the local quantization of the magnetic flux is equivalent to the ‘‘charge quantization.’’ The representation $\{Q_j, P_j\}_{j=1}^2$ of CCR in the present case is an inequivalent representation if and only if α is not an integer.

Let

$$q_\alpha = e^{2\pi i \alpha}. \quad (3.6)$$

Lemma 3.1: For all $m, n \in \mathbf{Z}$,

$$e^{in\omega_2 P_2} e^{im\omega_1 P_1} = q_\alpha^{nm} e^{im\omega_1 P_1} e^{in\omega_2 P_2}. \quad (3.7)$$

Proof: For all $(x, y) \in \mathbf{M}_{\omega_1, \omega_2}$, $D(x, y; \omega_1, \omega_2)$ contains only one point in the lattice $\mathbf{Z}_{\omega_1, \omega_2}^2$. Hence, by Lemma 2.4 and (3.5), $\Phi_{\omega_1, \omega_2}^A = 2\pi$, a.e., which, together with (2.16), implies (3.7) with $m = n = 1$. Using this relation repeatedly, we obtain (3.7) with $m, n \in \mathbf{N} \cup \{0\}$, whose adjoint gives (3.7) with $m, n < 0$. Noting that e^{itP_j} ($t \in \mathbf{R}$) is invertible with $(e^{itP_j})^{-1} = e^{-itP_j}$, we obtain (3.7) with $m \geq 0, n < 0$ or $m \leq 0, n > 0$. \blacksquare

Relation (3.7) naturally leads us to the *quantum plane*,¹¹ which is defined to be the algebra generated by two elements X, Y subject to the relation

$$qXY = YX,$$

with q a parameter.¹² We denote the quantum plane by \mathbf{C}_q^2 .

For an algebra \mathfrak{A} , a set (π, V) of a complex vector space V and an algebraic homomorphism $\pi: \mathfrak{A} \rightarrow \text{End}(V)$ is called a representation of \mathfrak{A} .

We denote by $\mathfrak{B}(L^2(\mathbf{R}^2))$ the $*$ algebra of bounded linear operators on $L^2(\mathbf{R}^2)$. The following theorem immediately follows from Lemma 3.1.

Theorem 3.2: *The following correspondence $\pi_\alpha: \{X, Y\} \rightarrow \mathfrak{B}(L^2(\mathbf{R}^2))$ defines a representation of \mathbf{C}_q^2 on $L^2(\mathbf{R}^2)$:*

$$\pi_\alpha(X) = e^{i\omega_1 P_1}, \quad \pi_\alpha(Y) = e^{i\omega_2 P_2}. \quad (3.8)$$

It should be noted that, in the representation $(\pi_\alpha, L^2(\mathbf{R}^2))$, $\pi_\alpha(X)$ and $\pi_\alpha(Y)$ are unitary operators. Also, $q_\alpha \neq 1$ if and only if $\alpha \notin \mathbf{Z}$.

We have

$$\lim_{\alpha \rightarrow 0} q_\alpha = 1, \quad (3.9)$$

which means that the small coupling limit $\alpha \rightarrow 0$ corresponds to the ‘‘classical limit’’ of deformation by the parameter q_α .

The representation $(\pi_\alpha, L^2(\mathbf{R}^2))$ has a nontrivial classical limit.

Proposition 3.3:

$$\text{s-lim}_{\alpha \rightarrow 0} \pi_\alpha(X) = e^{i\omega_1 P_1} \neq I, \quad \text{s-lim}_{\alpha \rightarrow 0} \pi_\alpha(Y) = e^{i\omega_2 P_2} \neq I,$$

where I denotes the identity operator on $L^2(\mathbf{R}^2)$.

Proof: This follows from Lemma 2.10. \blacksquare

For a subalgebra \mathfrak{M} of $\mathfrak{B}(L^2(\mathbf{R}^2))$, we denote by \mathfrak{M}' the commutant of \mathfrak{M} :

$$\mathfrak{M}' = \{T \in \mathfrak{B}(L^2(\mathbf{R}^2)) \mid TS = ST, S \in \mathfrak{M}\}. \quad (3.10)$$

Let \mathfrak{E}_α be the algebra generated by $\{e^{i\omega_1 P_1}, e^{i\omega_2 P_2}\}$:

$$\mathfrak{E}_\alpha = \pi_\alpha(\mathbf{C}_q^2 \alpha).$$

Lemma 3.4: $e^{\pm 2\pi i Q_1 / \omega_1}, e^{\pm 2\pi i Q_2 / \omega_2} \in \mathfrak{E}'_\alpha$.

Proof: This follows from (2.15). \blacksquare

By Lemma 3.1, we have

$$q_\alpha e^{-i\omega_2 P_2} e^{i\omega_1 P_1} = e^{i\omega_1 P_1} e^{-i\omega_2 P_2}, \quad q_\alpha e^{i\omega_2 P_2} e^{-i\omega_1 P_1} = e^{-i\omega_1 P_1} e^{i\omega_2 P_2}, \quad (3.11)$$

$$q_\alpha e^{-i\omega_1 P_1} e^{-i\omega_2 P_2} = e^{-i\omega_2 P_2} e^{-i\omega_1 P_1}. \quad (3.12)$$

Hence, each of $\{e^{-i\omega_2 P_2}, e^{i\omega_1 P_1}\}$, $\{e^{i\omega_2 P_2}, e^{-i\omega_1 P_1}\}$, and $\{e^{-\omega_1 P_1}, e^{-i\omega_2 P_2}\}$ gives a representation of $\mathbf{C}_{q,\alpha}^2$. We denote these representations by $(\pi_\alpha^{(j)}, L^2(\mathbf{R}^2))$, $j=1,2,3$, respectively.

Proposition 3.5: Suppose that

$$\omega_1 = \omega_2. \quad (3.13)$$

Then, each representation $(\pi_\alpha^{(j)}, L^2(\mathbf{R}^2))$ is unitarily equivalent to $(\pi_\alpha, L^2(\mathbf{R}^2))$.

Proof: It is easy to check that, if (3.13) is satisfied, then (2.27) and (2.29) hold with $\tilde{A}(z)$ replaced by $\zeta(z)$. Hence we have (2.28) and (2.30), which imply that, for all $t \in \mathbf{R}$,

$$R_1 e^{itP_1} R_1^{-1} = e^{-itP_2}, \quad R_1 e^{itP_2} R_1^{-1} = e^{itP_1}, \quad R_2 e^{itP_1} R_2^{-1} = e^{itP_2}, \quad R_2 e^{itP_2} R_2^{-1} = e^{itP_1}. \quad (3.14)$$

These relations give a unitary equivalence between $(\pi_\alpha, L^2(\mathbf{R}^2))$ and $(\pi_\alpha^{(j)}, L^2(\mathbf{R}^2))$ ($j=1,2$). Moreover, (3.14) implies that

$$R_1^2 e^{itP_1} R_1^{-2} = e^{-itP_1}, \quad R_2^2 e^{itP_2} R_2^{-2} = e^{-itP_2}.$$

It is easy to see that $R_1^2 = R_2^2$. Thus, the unitary equivalence between $(\pi_\alpha, L^2(\mathbf{R}^2))$ and $(\pi_\alpha^{(3)}, L^2(\mathbf{R}^2))$ follows. \blacksquare

Remark: (i) In the case $\omega_1 \neq \omega_2$, we have been unable to clarify whether $(\pi_\alpha, L^2(\mathbf{R}^2))$, $(\pi_\alpha^{(j)}, L^2(\mathbf{R}^2))$, $j=1,2,3$, are unitarily equivalent to each other or not.

(ii) Relation (2.15) implies that

$$q_\alpha e^{2\pi i \alpha Q_j / \omega_j} e^{i\omega_j P_j} = e^{i\omega_j P_j} e^{2\pi i \alpha Q_j / \omega_j}, \quad j=1,2.$$

Hence, for each $j=1,2$ $\{e^{2\pi i \alpha Q_j / \omega_j}, e^{i\omega_j P_j}\}$ gives a representation of $\mathbf{C}_{q,\alpha}^2$. It is a problem to clarify whether these representations and the representations $(\pi_\alpha, L^2(\mathbf{R}^2))$, $(\pi_\alpha^{(j)}, L^2(\mathbf{R}^2))$, $j=1,2,3$, are unitarily equivalent to each other or not.

In this paper we concentrate our attention on the representation $(\pi_\alpha, L^2(\mathbf{R}^2))$. The methods developed in what follows apply also to the other representations of the quantum plane.

(iii) Consider the case where α is a rational number: $\alpha = p/r$ with $p \in \mathbf{Z}$ and $r \in \mathbf{Z} \setminus \{0\}$. Then $q_\alpha^r = 1$. It follows from (3.7) that, for all $m, n \in \mathbf{Z}$ with $mn = r$, $e^{im\omega_1 P_1}$ and $e^{in\omega_2 P_2}$ commute. But, if α is irrational, then $e^{im\omega_1 P_1}$ and $e^{in\omega_2 P_2}$ do not commute for all $m, n \in \mathbf{Z} \setminus \{0\}$.

IV. REPRESENTATION OF $U_q(\mathfrak{sl}_2)$ (I)

For a complex number $q \in \mathbf{C} \setminus \{0, 1, -1\}$, the quantum group $U_q(\mathfrak{sl}_2)$ is defined to be the algebra generated by four elements E, F, K, K^{-1} subject to the following relations:¹³

$$KK^{-1} = K^{-1}K = 1,$$

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F,$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The Casimir element C of $U_q(\mathfrak{sl}_2)$ is defined by

$$C = \frac{qK - 2 + q^{-1}K^{-1}}{(q - q^{-1})^2} + FE. \quad (4.1)$$

Given a representation (π, V) of the quantum plane \mathbf{C}_q^2 such that $\pi(X)$ and $\pi(Y)$ are bijective, we can construct a representation of $U_q(\mathfrak{sl}_2)$.

Lemma 4.1 (cf. Ref. 8): Let (π, V) be as above and $a, b, a', b' \in \mathbf{C}$ be constants satisfying

$$abq = a'b'q^{-1} = -\frac{1}{(q - q^{-1})^2}. \quad (4.2)$$

Then the following correspondence $\Pi: \{E, F, K, K^{-1}\} \rightarrow \text{End}(V)$ defines a representation of $U_q(\mathfrak{sl}_2)$:

$$\Pi(E) = \pi(X)(a\pi(X) + a'\pi(X)^{-1})\pi(Y)^{-1},$$

$$\Pi(F) = \pi(Y)(b\pi(X) + b'\pi(X)^{-1})\pi(X)^{-1},$$

$$\Pi(K) = \pi(X)^2, \quad \Pi(K^{-1}) = \pi(X)^{-2}.$$

In this representation, we have

$$\Pi(C) = a'b + ab' - \frac{2}{(q - q^{-1})^2}. \quad (4.3)$$

Proof: Direct computations. \blacksquare

Remark: By (4.2), we can write $\Pi(C)$ as

$$\Pi(C) = (a' + aq)(b + b'q^{-1}).$$

In the rest of the paper, we assume that $A_j, j=1, 2$, are given by (3.3) and

$$\alpha \notin \frac{\mathbf{Z}}{2}. \quad (4.4)$$

Hence $q_\alpha^2 \neq 1$.

Let $a_\alpha, a'_\alpha, b_\alpha, b'_\alpha$ be constants satisfying

$$a_\alpha b_\alpha q_\alpha = a'_\alpha b'_\alpha q_\alpha^{-1} = -\frac{1}{(q_\alpha - q_\alpha^{-1})^2} = \frac{1}{4 \sin^2 2\pi\alpha}. \quad (4.5)$$

By Lemma 4.1 and Theorem 3.2, we have the following theorem.

Theorem 4.2: The following correspondence $\Pi_\alpha: \{E, F, K, K^{-1}\} \rightarrow \mathfrak{B}(L^2(\mathbf{R}^2))$ defines a representation of $U_{q_\alpha}(\mathfrak{sl}_2)$ on $L^2(\mathbf{R}^2)$:

$$\Pi_\alpha(E) = e^{i\omega_1 P_1} (a_\alpha e^{i\omega_1 P_1} + a'_\alpha e^{-i\omega_1 P_1}) e^{-i\omega_2 P_2},$$

$$\Pi_\alpha(F) = e^{i\omega_2 P_2} (b_\alpha e^{i\omega_1 P_1} + b'_\alpha e^{-i\omega_1 P_1}) e^{-i\omega_1 P_1},$$

$$\Pi_\alpha(K) = e^{2i\omega_1 P_1}, \quad \Pi_\alpha(K^{-1}) = e^{-2i\omega_1 P_1}.$$

We investigate basic properties of the representation $(\Pi_\alpha, L^2(\mathbf{R}^2))$.

Theorem 4.3: The representation $(\Pi_\alpha, L^2(\mathbf{R}^2))$ has no weight vectors. In particular, there exists no nonzero finite-dimensional subspace W such that (Π_α, W) gives a representation of $U_{q_\alpha}(\mathfrak{sl}_2)$.

Proof: A weight vector of weight $\lambda \in \mathbf{C}$ in the representation $(\Pi_\alpha, L^2(\mathbf{R}^2))$ is a nonzero vector ψ satisfying $\Pi_\alpha(K)\psi = q_\alpha^\lambda \psi$. Hence ψ is an eigenvector of $e^{2i\omega_1 P_1}$. But, by Lemma 2.6(ii), $e^{2i\omega_1 P_1}$ has no eigenvectors. The last assertion in Theorem 4.3 follows from the well-known fact that any nonzero finite-dimensional $U_q(\mathfrak{sl}_2)$ -module contains a highest weight vector (Proposition VI.3.3 in Ref. 11). \blacksquare

Let

$$\mathfrak{U}_\alpha = \Pi_\alpha(U_{q_\alpha}(\mathfrak{sl}_2)). \quad (4.6)$$

Theorem 4.4: *Suppose that $a'_\alpha = \bar{b}_\alpha$, $b'_\alpha = \bar{a}_\alpha$ (\bar{z} denotes the complex conjugate of the complex number z). Then*

- (i) \mathfrak{U}_α is a $*$ subalgebra of $\mathfrak{B}(L^2(\mathbf{R}^2))$.
- (ii) The representation $(\Pi_\alpha, L^2(\mathbf{R}^2))$ is completely reducible.

Proof: (i) In the present assumption, we have

$$\Pi_\alpha(E)^* = \Pi_\alpha(F),$$

where T^* denotes the adjoint of operator T . It is obvious that $\Pi_\alpha(K)^* = \Pi_\alpha(K^{-1})$. Hence it follows that \mathfrak{U}_α is self-adjoint (i.e., $T \in \mathfrak{U}_\alpha \Rightarrow T^* \in \mathfrak{U}_\alpha$). Thus part (i) follows.

(ii) As in Lemma 3.4, we have

$$\{e^{\pm 2\pi i Q_j / \omega_j}\}_{j=1}^2 \subset \mathfrak{U}'_\alpha, \quad (4.7)$$

which implies that $\mathfrak{U}'_\alpha \neq \mathbf{CI}$. As proven in part (i), \mathfrak{U}_α is self-adjoint. Hence \mathfrak{U}_α is not irreducible (Proposition 2.3.8 in Ref. 14). Let W be any closed subspace of $L^2(\mathbf{R}^2)$, which is invariant under the action of \mathfrak{U}_α . Since \mathfrak{U}_α is self-adjoint, it follows that W^\perp [the orthogonal complement of W in $L^2(\mathbf{R}^2)$] is also invariant under the action of \mathfrak{U}_α . Thus $(\Pi_\alpha, L^2(\mathbf{R}^2))$ is completely reducible. ■

For a subset \mathfrak{M} of $\mathfrak{B}(L^2(\mathbf{R}^2))$, we denote by $\bar{\mathfrak{M}}$ the closure of \mathfrak{M} with respect to the operator norm.

We denote by \mathfrak{F}_α the algebra generated by $e^{\pm 2i\omega_1 P_1}$, $e^{\pm i\omega_2 P_2}$, which is a $*$ subalgebra of $\mathfrak{B}(L^2(\mathbf{R}^2))$, so that \mathfrak{F}_α is a C^* subalgebra.

Theorem 4.5: *Suppose that $|a_\alpha| \neq |a'_\alpha|$, $|b_\alpha| \neq |b'_\alpha|$. Then*

(i)

$$\bar{\mathfrak{F}_\alpha} = \bar{\mathfrak{U}_\alpha}. \quad (4.8)$$

(ii) $\bar{\mathfrak{U}_\alpha}$ is completely reducible.

Proof: (i) For simplicity, we set $a = a_\alpha$, $a' = a'_\alpha$, $b = b_\alpha$, $b' = b'_\alpha$. We have

$$\Pi_\alpha(E) = S e^{-i\omega_2 P_2}, \quad \Pi_\alpha(F) = e^{i\omega_2 P_2} T, \quad (4.9)$$

with

$$S = a e^{2i\omega_1 P_1} + a', \quad T = b + b' e^{-2i\omega_1 P_1}.$$

Hence it follows that $\mathfrak{U}_\alpha \subset \mathfrak{F}_\alpha$, implying

$$\bar{\mathfrak{U}_\alpha} \subset \bar{\mathfrak{F}_\alpha}. \quad (4.10)$$

To prove the converse inclusion relation, we express $e^{\pm i\omega_2 P_2}$ in terms of $\Pi_\alpha(E)$, $\Pi_\alpha(F)$, $\Pi_\alpha(K)$, and $\Pi_\alpha(K^{-1})$. We first consider the case $|a|/|a'| < 1$. We can write

$$S = a \Pi_\alpha(K) + a' = a' \left(1 + \frac{a}{a'} \Pi_\alpha(K) \right).$$

We have $\|a \Pi_\alpha(K) / a'\| = |a|/|a'| < 1$, where $\|L\|$ with operator $L \in \mathfrak{B}(L^2(\mathbf{R}^2))$ denotes the operator norm of L . Hence, S is bijective with

$$S^{-1} = \frac{1}{a'} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a}{a'}\right)^n \Pi_{\alpha}(K)^n$$

in the operator norm topology. Therefore we obtain

$$e^{-i\omega_2 P_2} = S^{-1} \Pi_{\alpha}(E) \in \overline{\mathfrak{U}_{\alpha}}.$$

In the case $|a|/|a'| > 1$, we write

$$S = a \Pi_{\alpha}(K) \left(1 + \frac{a'}{a} \Pi_{\alpha}(K^{-1}) \right).$$

Then, in the same way as in the preceding case, we can show that S is bijective with

$$S^{-1} = a^{-1} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a'}{a}\right)^n \Pi_{\alpha}(K^{-1})^{n+1},$$

in the operator norm topology. Hence $e^{-i\omega_2 P_2} \in \overline{\mathfrak{U}_{\alpha}}$. Similarly, the second relation in (4.9) implies that $e^{i\omega_2 P_2} \in \overline{\mathfrak{U}_{\alpha}}$. Thus, $\overline{\mathfrak{F}_{\alpha}} \subset \overline{\mathfrak{U}_{\alpha}}$, which, together with (4.10), gives (4.8).

(ii) Relation (4.7) implies that

$$\{e^{\pm 2\pi i Q_j / \omega_j}\}_{j=1}^2 \subset \overline{\mathfrak{U}'_{\alpha}}.$$

By the preceding result, $\overline{\mathfrak{U}_{\alpha}}$ is self-adjoint (in fact, a C^* subalgebra). Thus, by the same reasoning as in the proof of Theorem 4.4(ii), we obtain the desired result. \blacksquare

Remark: In the case $a_{\alpha} = \pm a'_{\alpha}$, S is injective, but not surjective [Lemma 2.6(ii)]. Hence S^{-1} is unbounded. The same applies to T in the case $b_{\alpha} = \pm b'_{\alpha}$.

For $(m, n) \in \mathbf{Z}^2$, we define a function $F_{m,n}(x, y)$ by

$$F_{m,n}(x, y) = \begin{cases} \exp\left(-i\alpha \int_{C_{-(x,y; 2m\omega_1, n\omega_2)}} A\right); & (x, y) \in \mathbf{M}_{\omega_1, \omega_2}, \\ 0; & (x, y) \notin \mathbf{M}_{\omega_1, \omega_2}. \end{cases} \quad (4.11)$$

For $\psi \in L^2(\mathbf{R}^2)$, we set

$$\psi_{m,n}(x, y) = \psi(x + 2m\omega_1, y + n\omega_2), \quad (m, n) \in \mathbf{Z}^2. \quad (4.12)$$

Corollary 4.6: Suppose that $|a_{\alpha}| \neq |a'_{\alpha}|$, $|b_{\alpha}| \neq |b'_{\alpha}|$. Let W be any irreducible closed subspace of the representation $(\Pi_{\alpha}, L^2(\mathbf{R}^2))$. Then each nonzero vector $\psi \in W$ is cyclic and W is generated by vectors of the form $F_{m,n} \psi_{m,n}$, $(m, n) \in \mathbf{Z}^2$.

Proof: The first half is due to a general fact (Proposition 2.3.8 in Ref. 14). By Theorem 4.5(i), W is generated by vectors of the form

$$\phi_{m,n} = e^{2im\omega_1 P_1} e^{in\omega_2 P_2} \psi, \quad (m, n) \in \mathbf{Z}^2.$$

Using (2.20), we see that $\phi_{m,n} = F_{m,n} \psi_{m,n}$. \blacksquare

Corollary 4.6 clarifies the structure of any irreducible closed subspace W of the representation $(\Pi_{\alpha}, L^2(\mathbf{R}^2))$ in the case $|a_{\alpha}| \neq |a'_{\alpha}|$, $|b_{\alpha}| \neq |b'_{\alpha}|$. By Theorem 4.3, $\dim W = \infty$.

V. REPRESENTATION OF $U_q(\mathfrak{sl}_2)$ (II)

In this section, we construct representations of $U_q(\mathfrak{sl}_2)$ that are different from the one given in Sec. IV.

Lemma 5.1 (cf. Ref. 8): *Suppose that $q \in \mathbb{C} \setminus \{0, 1, -1\}$. Let (π, V) be a representation of $\mathbb{C}_{q^2}^2$ [i.e., $q^2 \pi(X)\pi(Y) = \pi(Y)\pi(X)$] with the following properties.*

- (i) $\pi(X)$ and $\pi(Y)$ are bijective on V .
- (ii) There exists a bijection $Z \in \text{End}(V)$ such that $Z^2 = \pi(Y)^{-1}\pi(X)$.

Let c_{\pm} and d_{\pm} be constants satisfying

$$c_{\pm}d_{\pm} = \pm \frac{1}{(q - q^{-1})^2}. \quad (5.1)$$

Then the following correspondences $\Pi_{\pm} : \{E, F, K, K^{-1}\} \rightarrow \text{End}(V)$ define representations of $U_q(\mathfrak{sl}_2)$:

$$\Pi_{\pm}(E) = c_{\pm}Z(\pi(X)^{-1} + \pi(Y)^{-1}),$$

$$\Pi_{\pm}(F) = d_{\pm}(\pi(X) + \pi(Y))Z^{-1},$$

$$\Pi_{\pm}(K) = \pm q^{-1}\pi(Y)^{-1}\pi(X),$$

$$\Pi_{\pm}(K^{-1}) = \pm q\pi(X)^{-1}\pi(Y).$$

In these representations, we have

$$\Pi_{\pm}(C) = \pm \frac{(1 + q^2)\pi(Y)^{-1}\pi(X) + (1 + q^{-2})\pi(X)^{-1}\pi(Y)}{(q - q^{-1})^2}. \quad (5.2)$$

Proof: Direct computations. \square

To apply Lemma 5.1 with the representation $(\pi_{\alpha}, L^2(\mathbb{R}^2))$ of $\mathbb{C}_{q_{\alpha}}^2$ given in Theorem 3.2, we need the following lemma.

Lemma 5.2: *Let U be a unitary operator on a Hilbert space \mathcal{H} . Then there exists a unitary operator T on \mathcal{H} such that*

$$T^2 = U. \quad (5.3)$$

Proof: By the spectral theorem for unitary operators, there exists a unique resolution of identity $F(\theta)$ such that $F(0) = 0$, $F(2\pi) = I$, and $U = \int_0^{2\pi} e^{i\theta} dF(\theta)$. Let $T = \int_0^{2\pi} e^{i\theta/2} dF(\theta)$. Then, by the functional calculus, T is unitary and (5.3) holds. \square

The operator $e^{-i\omega_2 P_2} e^{i\omega_1 P_1}$ is unitary on $L^2(\mathbb{R}^2)$. Hence, by Lemma 5.2, there exists a unitary operator Z_{α} on $L^2(\mathbb{R}^2)$ satisfying

$$Z_{\alpha}^2 = e^{-i\omega_2 P_2} e^{i\omega_1 P_1}. \quad (5.4)$$

Let

$$q_{\alpha}^{1/2} := e^{i\pi\alpha}. \quad (5.5)$$

Applying Lemma 5.1 with $\pi(X) = e^{i\omega_1 P_1}$, $\pi(Y) = e^{i\omega_2 P_2}$, we obtain the following theorem.

Theorem 5.3: *Let $c_{\pm}(\alpha)$ and $d_{\pm}(\alpha)$ be constants satisfying*

$$c_{\pm}(\alpha)d_{\pm}(\alpha) = \pm \frac{1}{(q_{\alpha}^{1/2} - q_{\alpha}^{-1/2})^2} = \mp \frac{1}{4 \sin^2 \pi\alpha}. \quad (5.6)$$

Then the following correspondences $\Pi_\alpha^\pm : \{E, F, K, K^{-1}\} \rightarrow \mathfrak{B}(L^2(\mathbf{R}^2))$ define representations of $U_q(\mathfrak{sl}_2)$ with $q = q_\alpha^{1/2}$:

$$\begin{aligned} \Pi_\alpha^\pm(E) &= c_\pm(\alpha) Z_\alpha(e^{-i\omega_1 P_1} + e^{-i\omega_2 P_2}), \\ \Pi_\alpha^\pm(F) &= d_\pm(\alpha)(e^{i\omega_1 P_1} + e^{i\omega_2 P_2}) Z_\alpha^{-1}, \\ \Pi_\alpha^\pm(K) &= \pm q_\alpha^{-1/2} e^{-i\omega_2 P_2} e^{i\omega_1 P_1}, \\ \Pi_\alpha^\pm(K^{-1}) &= \pm q_\alpha^{1/2} e^{-i\omega_1 P_1} e^{i\omega_2 P_2}. \end{aligned}$$

In these representations, we have

$$\Pi_\alpha^\pm(C) = \pm \frac{(1 + q_\alpha) e^{-i\omega_2 P_2} e^{i\omega_1 P_1} + (1 + q_\alpha^{-1}) e^{-i\omega_1 P_1} e^{i\omega_2 P_2}}{(q_\alpha^{1/2} - q_\alpha^{-1/2})^2}. \quad (5.7)$$

By Lemma 2.7, we obtain the following theorem.

Theorem 5.4: *The representations $(\Pi_\alpha^\pm, L^2(\mathbf{R}^2))$ have no weight vectors. In particular, there exist no nonzero finite-dimensional subspaces W_\pm , such that (Π_α^\pm, W_\pm) give representations of $U_{q_\alpha^{1/2}}(\mathfrak{sl}_2)$.*

Let

$$\mathfrak{U}_\alpha^\pm = \Pi_\alpha^\pm(U_{q_\alpha^{1/2}}(\mathfrak{sl}_2)). \quad (5.8)$$

Theorem 5.5: *Let $\overline{c_\pm(\alpha)} = d_\pm(\alpha)$. Then*

- (i) \mathfrak{U}_α^\pm are $*$ subalgebras of $\mathfrak{B}(L^2(\mathbf{R}^2))$.
- (ii) $(\Pi_\alpha^\pm, L^2(\mathbf{R}^2))$ are completely reducible.

Proof: Under the assumption, \mathfrak{U}_α^\pm are self-adjoint, since we have

$$\Pi_\alpha^\pm(E)^* = \Pi_\alpha^\pm(F), \quad \Pi_\alpha^\pm(K)^* = \Pi_\alpha^\pm(K^{-1}).$$

Note that, if $T \in \mathfrak{B}(L^2(\mathbf{R}^2))$ commutes with $e^{-i\omega_2 P_2} e^{i\omega_1 P_1}$, then T commutes with Z_α . In particular, Z_α commutes with $e^{2\pi i Q_j / \omega_j}$, $j = 1, 2$. Hence $\{e^{2\pi i Q_j / \omega_j}\}_{j=1}^2 \subset \mathfrak{U}_\alpha^\pm$. Thus, by the same reasoning as in the proof of Theorem 4.4, we obtain the desired result. ■

Note that $q_{2\alpha}^{1/2} = q_\alpha$. Hence $(\Pi_{2\alpha}^\pm, L^2(\mathbf{R}^2))$ are representations of $U_q(\mathfrak{sl}_2)$ with $q = q_\alpha$. It is natural to ask if these representations are equivalent to the representation $(\Pi_\alpha, L^2(\mathbf{R}^2))$ constructed in the preceding section.

Theorem 5.6: *The representations $(\Pi_{2\alpha}^\pm, L^2(\mathbf{R}^2))$ are not equivalent to $(\Pi_\alpha, L^2(\mathbf{R}^2))$.*

Proof: By (5.7), (2.20), and (2.21), we have, for all $\psi \in L^2(\mathbf{R}^2)$,

$$\begin{aligned} (\Pi_{2\alpha}^\pm(C)\psi)(x, y) &= \pm \frac{1 + q_\alpha^2}{(q_\alpha - q_\alpha^{-1})^2} \exp\left(-2i\alpha \int_{C_+(x, y; \omega_1, -\omega_2)} A\right) \psi(x + \omega_1, y - \omega_2) \\ &\quad \pm \frac{1 + q_\alpha^{-2}}{(q_\alpha - q_\alpha^{-1})^2} \exp\left(-2i\alpha \int_{C_-(x, y; -\omega_1, \omega_2)} A\right) \psi(x - \omega_1, y + \omega_2), \quad \text{a.e.} \end{aligned}$$

From this expression, it is seen that $\Pi_{2\alpha}^\pm(C)$ are not scalar multiples of the identity. [For example, consider a function $\psi \in L^2(\mathbf{R}^2)$, $\psi \neq 0$, with support in $S = (0, \omega_1) \times (0, \omega_2)$. Then the support of $\Pi_{2\alpha}^\pm(C)\psi$ is outside of S .] On the other hand, by (4.3), $\Pi_\alpha(C)$ is a scalar multiple of the identity. Thus, the desired result follows. ■

We can prove a more detailed fact on the inequivalence between $(\Pi_{2\alpha}^\pm, L^2(\mathbf{R}^2))$ and $(\Pi_\alpha, L^2(\mathbf{R}^2))$. For this purpose, we prepare a lemma. Let

$$h_\alpha^\pm(\lambda) = \pm \frac{(1+q_\alpha)\lambda + (1+q_\alpha^{-1})\bar{\lambda}}{(q_\alpha^{1/2} - q_\alpha^{-1/2})^2}, \quad \lambda \in \mathbf{C}. \quad (5.9)$$

Note that h_α^\pm are real valued.

Lemma 5.7:

(i)

$$\sigma(\Pi_\alpha^\pm(C)) = \{h_\alpha^\pm(\lambda) \mid \lambda \in \sigma(e^{-i\omega_2 P_2} e^{i\omega_1 P_1})\}. \quad (5.10)$$

In particular,

$$\sigma(\Pi_\alpha^\pm(C)) \subset [-R_\alpha, R_\alpha], \quad (5.11)$$

where

$$R_\alpha = \frac{2|1+q_\alpha|}{|q_\alpha^{1/2} - q_\alpha^{-1/2}|^2} = \frac{|\cos \pi\alpha|}{\sin^2 \pi\alpha}. \quad (5.12)$$

(ii)

$$\sigma_p(\Pi_\alpha^\pm(C)) = \emptyset. \quad (5.13)$$

Proof: (i) We first note that

$$U_\alpha := e^{-i\omega_2 P_2} e^{i\omega_1 P_1}$$

is unitary and $U_\alpha^{-1} = e^{-i\omega_1 P_1} e^{i\omega_2 P_2}$. We have

$$\Pi_\alpha^\pm(C) = \pm \frac{(1+q_\alpha)U_\alpha + (1+q_\alpha^{-1})U_\alpha^{-1}}{(q_\alpha^{1/2} - q_\alpha^{-1/2})^2}.$$

Relation (5.10) follows from this expression and the spectral mapping theorem for unitary operators. It is easy to see that there exists a constant $\delta_\alpha \in [0, 2\pi)$ such that

$$h_\alpha^\pm(e^{i\theta}) = \pm R_\alpha \cos(\theta + \delta_\alpha), \quad \theta \in \mathbf{R}. \quad (5.14)$$

Hence (5.11) follows.

(ii) Suppose that $\sigma_p(\Pi_\alpha^+(C)) \neq \emptyset$ and $\lambda \in \sigma_p(\Pi_\alpha^+(C))$. Then, by (5.11), $-R_\alpha \leq \lambda \leq R_\alpha$ and there exists a nonzero vector $\psi \in L^2(\mathbf{R}^2)$ such that $\Pi_\alpha^+(C)\psi = \lambda\psi$. Let E be the spectral measure of U_α :

$$U_\alpha = \int_0^{2\pi} e^{i\theta} dE(\theta).$$

Then we have

$$0 = \|\Pi_\alpha^+(C)\psi - \lambda\psi\|^2 = \int_0^{2\pi} |h_\alpha^+(e^{i\theta}) - \lambda|^2 d\|E(\theta)\psi\|^2,$$

which implies that the support of the measure $\|E(\cdot)\psi\|^2$ is included in the set $\{\theta \in [0, 2\pi] \mid h_\alpha^+(e^{i\theta}) = \lambda\}$. By (5.14), the equation $h_\alpha^+(e^{i\theta}) = \lambda$ has at most two solutions: $\theta_j \in [0, 2\pi)$, $j = 1, 2$. This implies that one of $E(\{\theta_j\})\psi$ ($j = 1, 2$) is not zero and an eigenvector of U_α with eigenvalue $e^{i\theta_j}$. But this contradicts Lemma 2.7. Thus $\sigma_p(\Pi_\alpha^+(C))$ must be empty. Similarly, we can prove $\sigma_p(\Pi_\alpha^-(C)) = \emptyset$. \blacksquare

Theorem 5.8: *Let (Π_α, W) be any irreducible component of $(\Pi_\alpha, L^2(\mathbf{R}^2))$ ($W \subset L^2(\mathbf{R}^2)$). Then (Π_α, W) is not equivalent to any irreducible component of $(\Pi_{2\alpha}^\pm, L^2(\mathbf{R}^2))$.*

Proof: By Lemma 5.7(ii), $\Pi_{2\alpha}^\pm(C)$ are not scalar multiples of the identity in any irreducible components of $(\Pi_{2\alpha}^\pm, L^2(\mathbf{R}^2))$. Thus, the desired result follows. \blacksquare

Remark: It is an open problem to clarify whether $(\Pi_\alpha^+, L^2(\mathbf{R}^2))$ is equivalent to $(\Pi_\alpha^-, L^2(\mathbf{R}^2))$ or not.

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