On the degeneracy in the ground state of the $N=2$ Wess–Zumino supersymmetric quantum mechanics

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It is known that the $N=2$ Wess–Zumino supersymmetric quantum mechanical model has $p-1$ degenerate zero-energy ground states consisting of only bosonic states, where $p \geqslant 3$ is the degree of the polynomial superpotential $V(z)$ ($z \in \mathbb{C}$) of the model [Jaffe *et al.* Ann. Phys. (NY) **178**, 313 (1987)]. In this paper, the mathematical structure of the degenerate ground states is analyzed in the special case $V(z)=\lambda z^p$ ($\lambda>0$). The following facts are discovered: (i) there exists a strongly continuous one parameter unitary group acting as a symmetry group in the quantum system under consideration; (ii) the generator of the symmetry group has infinitely many eigenspaces $\mathscr{H}_m$, $m \in \mathbb{Z}$, and the bosonic part H_+ of the Hamiltonian of the model is reduced by each of them; and (iii) there exist exactly $p-1$ $\mathscr{H}_m$'s in each of which the reduced part of H_+ has a unique zero-energy ground state. It is noted also that H_+ has infinitely many generalized eigenfunctions with eigenvalue zero. Moreover, a family of operators interrelating the zero-energy ground states is constructed. The coupling constant dependence of the nonzero eigenvalues of H_+ is exactly found.

I. INTRODUCTION

The $N=2$ Wess–Zumino supersymmetric quantum mechanics (SSQM) describes the interaction between a complex bosonic degree of freedom and two fermionic degrees of freedom,[1–4] and serves as a toy model of a supersymmetric quantum field theory.[5–8] It has been shown that the model with a polynomial superpotential $V(z)$ ($z \in \mathbb{C}$) of degree $p \geqslant 2$ has exactly $p-1$ zero-energy ground state(s) consisting of only bosonic state(s) and hence, if $p \geqslant 3$, then the ground state is degenerate (see Refs. 1 and 2 for formal discussions and Ref. 3 for a mathematically rigorous analysis). However, the origin of this degeneracy has not been clarified. The present work resulted from an attempt to understand the degeneracy in the ground state.

In SSQM, supersymmetry is said to be broken if no zero-energy states exist. It is well known that, if supersymmetry is broken, the ground state is always degenerate (e.g., Refs. 1, 3, 7, and 9). In the present case, however, supersymmetry is unbroken and hence the degeneracy in the ground state is not due to supersymmetry breaking. Therefore we infer that any mechanism different from supersymmetry breaking should exist to give rise to the degeneracy in the ground state. In Ref. 1, the following conjecture was given: The potential $|\partial V(z)|^2$, which appears as a potential term of the Hamiltonian of the model, will, in general, have $p-1$ wells; this leads to the $(p-1)$-fold degeneracy in the ground state. However, as pointed out there also, this reasoning cannot be applied, e.g., to the case $V(z)=z^p$, since in this case $|\partial V(z)|^2$ has only a single well. It seems that the structure of the degeneracy is not so simple and may vary according to the "fine structure" of the superpotential.

In this paper, as a first step towards the understanding of the degeneracy in the ground state of the Wess–Zumino model, we present a mathematically rigorous analysis of the model with $V(z)=\lambda z^p$, $\lambda>0$, concentrating our attention on the mathematical structure of the degenerate ground states. We discover the following facts: (i) there exists a strongly continuous one parameter unitary group $\{R_t = e^{itL} \mid t \in \mathbb{R}\}$ acting as a symmetry group in the quantum system under consideration and the generator L has infinitely many eigenspaces $\mathscr{H}_m$ ($m \in \mathbb{Z}$) decomposing the Hilbert space of bosonic states, so that the bosonic part H_+ of the total Hamiltonian of the model is reduced by each $\mathscr{H}_m$; (ii) there exist exactly $p-1$ $\mathscr{H}_m$'s in each of which H_+ has a unique ground state (see Sec. III). Thus we see that, in the present case, the degeneracy in the ground state is connected with the existence of a symmetry group. We note also that the generator L of R_t is of a form similar to that of the usual two-dimensional rotation group with spin $\frac{1}{2}$ and differs from it only in that the spin rotation part is $p-2$ times of the usual one [see (3.5)]. This suggests that there exists a degree of freedom with spin $s=(p-2)/2$ and hence any energy level of H_+ may be degenerate with multiplicity $2s+1=p-1$, which exactly coincides with the multiplicity of the degeneracy in the ground state of H_+. We note also that H_+ has infinitely many generalized eigenfunctions with eigenvalue zero. In Sec. IV, we show that H_+ has a dilation covariance, which exactly determines the λ (coupling constant) dependence of the nonzero eigenvalues of H_+. In the last section, we construct a family of operators interrelating the degenerate ground states. These operators are related to the dilation unitary group given in Sec. IV and the rotation group in $\mathbb{C}$.

We begin with a brief review of the $N=2$ Wess–Zumino model.

II. REVIEW OF THE $N=2$ WESS–ZUMINO SSQM[1–4]

The Hilbert space of state vectors of the model is realized as

$$\mathscr{H} = L^2(\mathbb{C};\mathbb{C}^4) = L^2(\mathbb{R}^2;\mathbb{C}^4) = \mathscr{H}_+ \oplus \mathscr{H}_-, \qquad (2.1)$$

where

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$$\mathscr{H}_+ = \left\{ \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{pmatrix} \middle| f_1, f_2 \in L^2(\mathbb{C}) \right\} \cong L^2(\mathbb{C};\mathbb{C}^2)$$

and

$$\mathscr{H}_- = \left\{ \begin{pmatrix} 0 \\ 0 \\ f_1 \\ f_2 \end{pmatrix} \middle| f_1, f_2 \in L^2(\mathbb{C}) \right\} \cong L^2(\mathbb{C};\mathbb{C}^2)$$

are the subspace of "bosonic" and "fermionic" states, respectively.

The fermionic degrees of freedom in the model are given by the 4×4 matrices

$$\psi_1 = \frac{1}{2}\begin{pmatrix} 0 & I+\sigma_3 \\ I-\sigma_3 & 0 \end{pmatrix},$$

$$\psi_2 = \frac{1}{2}\begin{pmatrix} 0 & i\sigma_1+\sigma_2 \\ -i\sigma_1-\sigma_2 & 0 \end{pmatrix},$$

where σ_j, $j=1,2,3$ are the Pauli matrices and I denotes the 2×2 identity matrix. It is straightforward to check that the ψ_j's satisfy the anticommutation relations

$$\{\psi_j, \psi_k^*\} = \delta_{jk}, \quad \{\psi_j, \psi_k\} = 0, \quad j,k = 1,2.$$

The self-adjoint supercharges of the model are defined by

$$Q_1 = i(\psi_2 \bar{\partial} + \psi_2^* \partial) + i\{\psi_1(\partial V) - \psi_1^*(\partial V)^*\},$$

$$Q_2 = \psi_2 \bar{\partial} - \psi_2^* \partial + \psi_1(\partial V) + \psi_1^*(\partial V)^*,$$

where $V(z)$ $(z\in\mathbb{C})$ is an entire function on $\mathbb{C}$, denoting the superpotential of the model, and $(\partial V)^*$ denotes the complex conjugate of ∂V. The operators Q_1 and Q_2 are essentially self-abjoint on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^4)$.[3,4] We denote their closures by the same symbols.

The Hamiltonian H of the model is defined by

$$H = Q_1^2.$$

It follows that

$$Q_1^2 = Q_2^2$$

and hence $D(Q_1) = D(Q_2) = D(H^{1/2})$, where $D(A)$ denotes the domain of the operator A. Further we have

$$\{Q_1, Q_2\} f = 0, \quad f \in C_0^\infty(\mathbb{R}^2;\mathbb{C}^4).$$

On $D(\partial\bar{\partial}) \cap D(|\partial V|^2) \cap D(\partial^2 V)$, H is expressed as

$$H = -\partial\bar{\partial} - \psi_1^*\psi_2(\partial^2 V)^* - \psi_2^*\psi_1 \partial^2 V + |\partial V|^2.$$

Corresponding to the decomposition (2.1), H is decomposed as

$$H = H_+ \oplus H_-,$$

where

$$H_- = (-\partial\bar{\partial} + |\partial V|^2) I$$

and

$$H_+ = H_- + \begin{pmatrix} 0 & -i(\partial^2 V) \\ i(\partial^2 V)^* & 0 \end{pmatrix}. \tag{2.2}$$

The operators $H_\pm$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$.[3,4]

The fermion number operator or the grading operator is given by

$$N_F = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The quadruple $\{\mathscr{H}, \{Q_1, Q_2\}, H, N_F\}$ satisfies the axioms of supersymmetric quantum theory.[9,10]

The following facts have been proved.

(i) No fermionic zero-energy states exist, that is,

$$\operatorname{Ker} H_- = \{0\}$$

(see Refs. 3 and 4).

(ii) If $V(z)$ is a polynomial of degree $p\geqslant 2$, then

$$\dim \operatorname{Ker} H_+ = p-1, \tag{2.3}$$

and hence, if $p\geqslant 3$, then the bosonic zero-energy state is degenerate.[3] As for the case of nonpolynomial superpotentials, we may conjecture that $\dim \operatorname{Ker} H_+ = \infty$. It has been proved in Ref. 4 that, in the special case $V(z) = \lambda e^{\alpha z}$ $(\lambda\in\mathbb{C}\setminus\{0\}, \alpha>0)$, this is true.

III. GROUND STATES OF THE MODEL WITH A MONOMIAL SUPERPOTENTIAL

In what follows, we shall use the polar coordinate representation

$$z = re^{i\theta}, \quad r\in\mathbb{R}_+ = (0,\infty), \quad \theta\in[0,2\pi],$$

and the canonical identification

$$L^2(\mathbb{C}) = L^2(\mathbb{R}_+, d\mu) \otimes L^2(0,2\pi), \tag{3.1}$$

where $d\mu(r)$ is the measure on $\mathbb{R}_+$ given by

$$d\mu(r) = r\,dr. \tag{3.2}$$

In this section we consider the case of the monomial superpotential

$$V(z) = \lambda z^p, \quad \lambda>0, \quad p\geqslant 3, \tag{3.3}$$

so that H_+ given by (2.2) takes the form

$$H_+ = -\frac{1}{4r}\left(\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial\theta^2}\right) + \lambda^2 p^2 r^{2(p-1)} + \lambda i p(p-1) r^{p-2} \begin{pmatrix} 0 & -e^{i(p-2)\theta} \\ e^{-i(p-2)\theta} & 0 \end{pmatrix}, \tag{3.4}$$

on a suitable dense domain.

By (2.3), the zero-energy level of H_+ is degenerate with multiplicity $p-1\geqslant 2$. Our aim is to investigate the mathematical structure of the $p-1$ degenerate zero-energy ground states.

We first show that there exists a symmetry group acting in the quantum system governed by the bosonic Hamiltonian H_+. Let $\partial/\partial\theta$ be the generalized derivative with the periodic boundary condition in $[0,2\pi]$. Then, the operator

$$L = i\frac{\partial}{\partial\theta} + (p-2)\cdot\frac{\sigma_3}{2} \tag{3.5}$$

is self-adjoint with $D(L) = D(\partial/\partial\theta) \subset \mathscr{H}_+$ and hence

$$R_t = e^{itL}, \quad t\in\mathbb{R}, \tag{3.6}$$

generate a strongly continuous one parameter unitary group on $\mathscr{H}_+$.

Lemma 3.1: For all $t,s\in\mathbb{R}$,

$$R_t e^{isH_+} = e^{isH_+} R_t. \tag{3.7}$$

Proof: Direct computations show that, for all $t \in \mathbb{R}$, R_t takes $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$ into $D(H_+)$ and

$$H_+ R_t \Psi = R_t H_+ \Psi, \quad \Psi \in C_0^\infty(\mathbb{R}^2; \mathbb{C}^2). \tag{3.8}$$

Since $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$ is a core of H_+ (see Sec. II), (3.8) extends to all $\Psi \in D(H_+)$ showing at the same time that R_t takes $D(H_+)$ into $D(H_+)$. Then it follows that the resolvents, and hence the spectral projections of H_+, commute with R_t. Therefore we obtain (3.7). □

Lemma 3.1 is equivalent to saying that H_+ and L commute in the proper sense (e.g., Ref. 11, §VIII. 5) and shows that the unitary group $\{R_t\}_{t \in \mathbb{R}}$ is a symmetry group of the quantum system under consideration. We note that the generator L is only different in the coefficient of the spin rotation part $\sigma_3/2$ from that of the usual two-dimensional rotation group with spin $\frac{1}{2}$ (i.e., the case $p = 3$). Therefore, L may be regarded as the generator of a rotation group "distorted" with respect to spin degree of freedom. The form of L suggests that there exists a degree of freedom with spin $s = (p-2)/2$ and hence that any energy level of H_+ may be degenerate with multiplicity $2s + 1 = p - 1$, which exactly coincides with the multiplicity of the degeneracy in the ground state of H_+.

The spectrum $\sigma(L)$ of L is purely discrete and given as

$$\sigma(L) = \{m - (p/2) \mid m \in \mathbb{Z}\}. \tag{3.9}$$

The eigenspace of L with eigenvalue $m - (p/2)$ is given by

$$\mathscr{H}_m = \left\{ \begin{pmatrix} f \otimes e_{p-1-m} \\ g \otimes e_{1-m} \end{pmatrix} \mid f, g \in L^2(\mathbb{R}_+, d\mu) \right\}, \tag{3.10}$$

where

$$e_n(\theta) = e^{in\theta}/\sqrt{2\pi}, \quad n \in \mathbb{Z}. \tag{3.11}$$

The Hilbert space $\mathscr{H}_+$ is decomposed as

$$\mathscr{H}_+ = \bigoplus_{m=-\infty}^{\infty} \mathscr{H}_m. \tag{3.12}$$

Lemma 3.1 implies that H_+ is reduced by each $\mathscr{H}_m$. We denote by $H_{+,m}$ the reduced part of H_+ to $\mathscr{H}_m$:

$$H_{+,m} = H_+ \restriction \mathscr{H}_m. \tag{3.13}$$

We set

$$F_\alpha(r) = \lambda r^{p-1} K_{\alpha/p}(2\lambda r^p), \quad r > 0, \quad \alpha \in \mathbb{R}, \tag{3.14}$$

where $K_\nu(z)$ is the modified Bessel function of the third kind (e.g., Ref. 12, §7.2.2). Let

$$\Phi_m = F_m \otimes e_{p-1-m}, \quad m \in \mathbb{Z}. \tag{3.15}$$

Proposition 3.2: For $m = 1, \ldots, p-1$, $H_{+,m}$ has a unique zero-energy ground state (up to constant multiples), which is given by

$$\Omega_m = \begin{pmatrix} \Phi_m \\ -i\Phi_{p-m}^* \end{pmatrix}. \tag{3.16}$$

Proof: It follows from direct computations employing the formulas

$$zK_\nu'(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$$

and

$$K_\nu(z) = K_{-\nu}(z) \tag{3.17}$$

(e.g., Ref. 12) that, for $m = 1, \ldots, p-1$, $\Omega_m \in D(H_{+,m})$ and

$$H_{+,m} \Omega_m = 0. \tag{3.18}$$

Combining this fact with (2.3), we conclude that Ω_m is a unique zero-energy state (up to constant multiples) of $H_{+,m}$. □

Remarks: (1) Each component of the zero-energy state Ω_m $(m = 1, \ldots, p-1)$ is analytic in z and z^*. In fact, we have

$$\Phi_m(r,\theta) = \frac{\pi\lambda^{(p-m)/p} z^{p-1-m}}{2\sin(m\pi/p)} \sum_{n=0}^{\infty} \frac{\lambda^{2n} z^{np} z^{*np}}{n!} \times \left\{ \frac{1}{\Gamma(-(m/p) + n + 1)} - \frac{\lambda^{2m/p} z^m z^{*m}}{\Gamma((m/p) + n + 1)} \right\},$$

where $\Gamma(z)$ is the gamma function. This follows from the expansion of $K_\nu(x)$ in x (e.g., Ref. 12, §7.2.2).

(2) For $m \neq 1, \ldots, p-1$, Ω_m is not in $\mathscr{H}_+$ because of the singularity at $r = 0$, but Eq. (3.18) still holds; that is, Ω_m is a generalized eigenfunction of H_+ with eigenvalue zero. Hence H_+ has infinitely many generalized eigenfunctions with eigenvalue zero. This also is a remarkable phenomenon.

IV. DILATION COVARIANCE AND THE COUPLING CONSTANT DEPENDENCE OF EIGENVALUES

Before analyzing interrelations between the zero-energy states Ω_m, we show that H_+ is dilation covariant. To express the dependence of H_+ on the coupling constant λ, we write H_+ as

$$H_+ = H_+(\lambda). \tag{4.1}$$

Let

$$(u_t f)(r) = tf(tr), \quad f \in L^2(\mathbb{R}_+, d\mu), \quad t > 0. \tag{4.2}$$

Then, it is easy to see that, for all $t > 0$, u_t is unitary on $L^2(\mathbb{R}_+, d\mu)$, strongly continuous in t, and satisfies

$$u_{t=1} = I, \quad u_t u_s = u_s u_t = u_{ts}, \quad t, s > 0, \tag{4.3}$$

where I denotes identity. Hence $\{u_t\}_{t>0}$ forms a strongly continuous unitary representation of the Abelian group (the "dilation group") $\mathbb{R}_+$. The operator u_t naturally extends to $\mathscr{H}_+$ as

$$U_t = (u_t \otimes I) \oplus (u_t \otimes I). \tag{4.4}$$

Obviously $\{U_t\}_{t>0}$ has the same properties as those of $\{u_t\}_{t>0}$ stated above.

Lemma 4.1: For all $\lambda > 0$ and $t > 0$,

$$U_t H_+(\lambda) U_t^{-1} = t^{-2} H_+(\lambda t^p). \tag{4.5}$$

Proof: This follows from the transformation properties

$$U_t \frac{\partial}{\partial r} U_t^{-1} = \frac{1}{t} \frac{\partial}{\partial r}, \quad U_t r U_t^{-1} = tr. \quad \square$$

It is known that the spectrum $H_+(\lambda)$ is purely discrete.[3] Let

$$0 < E_1(\lambda) < E_2(\lambda) < \cdots < E_n(\lambda) < E_{n+1}(\lambda) < \cdots \tag{4.6}$$

be the nonzero eigenvalues of $H_+(\lambda)$.

Proposition 4.2: For each $n \geqslant 1$, there exists a constant $\alpha_n > 0$ independent of λ such that

$$E_n(\lambda) = \alpha_n \lambda^{2/p}, \quad \lambda > 0. \tag{4.7}$$

Proof: By (4.5), we have

$$\sigma(H_+(\lambda)) = t^{-2}\sigma(H_+(\lambda t^p)),$$

and hence, for all $n \geqslant 1$,

$$E_n(\lambda t^p) = t^2 E_n(\lambda),$$

which gives

$$E_n(\lambda)/\lambda^{2/p} = E_n(\mu)/\mu^{2/p},$$

for all μ, $\lambda > 0$. Therefore $\alpha_n \equiv E_n(\lambda)/\lambda^{2/p}$ is a constant independent of λ. Hence we obtain (4.7). □

Remark: The zero-energy states $\Omega_m \equiv \Omega_m(\lambda)$ are also dilation covariant:

$$U_t \Omega_m(\lambda) = \Omega_m(\lambda t^p), \quad \lambda > 0, \quad t > 0.$$

V. INTERRELATION BETWEEN THE ZERO-ENERGY GROUND STATES

In this section, we construct a family of operators interrelating the zero-energy states Ω_m.

For $\nu > \mu > (\nu - 1)/2$, the function

$$a_{\mu\nu}(t) = t^{p(1-\nu)-1}/(t^{2p} - 1)^{1+\mu-\nu}, \quad t > 1, \tag{5.1}$$

is integrable on $(1, \infty)$ with respect to dt. Hence

$$A_{\mu\nu} = \int_1^\infty a_{\mu\nu}(t) u_t \, dt \tag{5.2}$$

defines a bounded linear operator on $L^2(\mathbb{R}_+, d\mu)$, where the integral on the right-hand side of (5.2) is taken in the operator norm topology.

Let

$$W_\beta = \left\{ f : \mathbb{R}_+ \to \mathbb{C} \text{ measurable} \,\middle|\, \|f\|_\beta^2 \equiv \int_0^\infty (1 + r^2)^\beta |f(r)|^2 \, d\mu(r) < \infty \right\} \tag{5.3}$$

and

$$W_\infty = \bigcap_{\beta > 0} W_\beta \tag{5.4}$$

with the Fréchet topology generated by the norms $\|\cdot\|_\beta$. We shall denote by $\mathscr{L}(W_\beta, W_\gamma)$ the family of all bounded linear operators from W_β to W_γ. Let M be the multiplication operator on $L^2(\mathbb{R}_+, d\mu)$ given by

$$(Mf)(r) = r f(r). \tag{5.5}$$

Lemma 5.1: Let $\alpha \geqslant 0$ and

$$\nu > \mu > (\nu - 1)/2 - \alpha/2p. \tag{5.6}$$

Then, for all $\beta \geqslant 0$, $M^\alpha A_{\mu\nu} \in \mathscr{L}(W_{\alpha+\beta}, W_\beta)$. In particular, $M^\alpha A_{\mu\nu}$ takes continuously W_∞ into W_∞.

Proof: It is easy to see that, for all $t > 0$, $M^\alpha u_t \in \mathscr{L}(W_{\alpha+\beta}, W_\beta)$ with

$$\|M^\alpha u_t f\|_\beta \leqslant \frac{2^{\beta/2}(1 + t^{2\beta})^{1/2}}{t^{\alpha+\beta}} \|f\|_{\alpha+\beta}, \quad f \in W_{\alpha+\beta}.$$

Under condition (5.6), we have

$$\int_1^\infty a_{\mu\nu}(t) \frac{(1 + t^{2\beta})^{1/2}}{t^{\alpha+\beta}} dt < \infty.$$

Therefore, $M^\alpha A_{\mu\nu} \in \mathscr{L}(W_{\alpha+\beta}, W_\beta)$. □

We introduce the operator

$$B_{\mu\nu} = \frac{2p\lambda^{\nu-\mu}}{\Gamma(\nu - \mu)} M^{p(\nu-\mu)} A_{\mu\nu}, \tag{5.7}$$

which obeys the following composition law.

Lemma 5.2: For all $\mu < \nu < \rho$,

$$B_{\mu\nu} B_{\nu\rho} = B_{\mu\rho}. \tag{5.8}$$

Proof: The operator $B_{\mu\nu}$ is written as

$$(B_{\mu\nu} f)(r) = \int_0^\infty b_{\mu\nu}(r,s) f(s) d\mu(s),$$

with

$$b_{\mu\nu}(r,s) = \frac{2p\lambda^{\nu-\mu}}{\Gamma(\nu - \mu)} a_{\mu\nu}\left(\frac{s}{r}\right) r^{p(\nu-\mu)-2} \chi_{(r,\infty)}(s),$$

where $\chi_{(r,\infty)}$ is the characteristic function of the interval (r, ∞). Then, by direct computations, we can show that

$$\int_0^\infty b_{\mu\nu}(r,s) b_{\nu\rho}(s,t) d\mu(s) = b_{\mu\rho}(r,t)$$

holds. Therefore, (5.8) follows. □

We note also that $B_{\mu\nu}$ is dilation covariant:

$$u_t B_{\mu\nu} u_t^{-1} = t^{p(\nu-\mu)} B_{\mu\nu}, \quad t > 0. \tag{5.9}$$

Lemma 5.3: Let F_α be given by (3.14). Then, $F_\alpha \in W_\infty$, $0 \leqslant \alpha < p$, and

$$F_\alpha = B_{\alpha/p, \beta/p} F_\beta, \quad 0 \leqslant \alpha < \beta < p. \tag{5.10}$$

Proof: The fact $F_\alpha \in W_\infty$ follows from the asymptotic properties of the Bessel function

$$K_\nu(x) \sim (\pi/2x)^{1/2} e^{-x}, \quad \text{as } x \to \infty,$$

$$K_\nu(x) \sim \text{const}\, x^{-\nu}, \quad \text{as } x \downarrow 0 \quad (\nu > 0),$$

$$K_0(x) \sim \text{const} \log x, \quad \text{as } x \downarrow 0$$

(see, e.g., Ref. 12). To prove (5.10), we note that the following formula holds [Ref. 12, 7.14.2, formula (50)]:

$$\frac{2^{\nu-\mu-1}\Gamma(\nu - \mu)}{a^{\nu-\mu} y^\mu} K_\mu(ay) = \int_0^\infty \frac{x^{2(\nu-\mu)-1}}{(x^2 + y^2)^{\nu/2}} K_\nu(a\sqrt{x^2 + y^2}) dx, \tag{5.11}$$

for $\operatorname{Re} \nu > \operatorname{Re} \mu$, $a > 0$, and $y > 0$. By direct computations we see that (5.11) with $a = 2\lambda$, $y = r^p$, $\mu = \alpha/p$, and $\nu = \beta/p$ is equivalent to (5.10). □

Let u be the operator on $L^2(0, 2\pi)$ defined by

$$(uf)(\theta) = e^{-i\theta} f(\theta), \quad f \in L^2(0, 2\pi), \tag{5.12}$$

and $T_{\mu\nu}^{\pm}$ be the operators on $\mathscr{H}_+$ given by

$$T_{\mu\nu}^{+} = (B_{-\mu/p, -\nu/p} \otimes u^{\mu-\nu}) \oplus (B_{(p-\mu)/p, (p-\nu)/p} \otimes u^{\mu-\nu}), \quad \mu > \nu, \tag{5.13}$$

$$T_{\mu\nu}^{-} = (B_{\mu/p, \nu/p} \otimes u^{\mu-\nu}) \oplus (B_{(\mu-p)/p, (\nu-p)/p} \otimes u^{\mu-\nu}), \quad \mu < \nu. \tag{5.14}$$

Proposition 5.4: For all $m, n = 1, \ldots, p - 1$, $m < n$,

$$\Omega_n = T_{nm}^{+} \Omega_m, \quad \Omega_m = T_{mn}^{-} \Omega_n. \tag{5.15}$$

Proof: Property (3.17) implies that

$$F_\alpha = F_{-\alpha}.$$

Obviously we have

$$ue_m = e_{m-1}, \quad m \in \mathbb{Z}. \tag{5.16}$$

Combining these facts with (3.15) and (5.10), we obtain (5.15). □

Equation (5.15) shows that $T^{\pm}_{mn}$ are operators interrelating the zero-energy states Ω_m. It follows from (5.15) also that Ker H_+ has the following structures:

$$\begin{aligned}\operatorname{Ker} H_+ &= \left\{\left(\beta_1 + \sum_{j=2}^{p-1} \beta_j T^+_{j1}\right)\Omega_1 \,\middle|\, \beta_j \in \mathbb{C}, \right. \\ &\qquad \left. j = 1,\ldots,p-1\right\} \\ &= \left\{\left(\sum_{j=1}^{p-2} \beta_j T^-_{j,p-1} \right.\right. \\ &\qquad \left.\left. + \beta_{p-1}\right)\Omega_{p-1} \,\middle|\, \beta_j \in \mathbb{C},\ j = 1,\ldots,p-1\right\}.\end{aligned}$$

Finally we briefly discuss properties of $T^{\pm}_{\mu\nu}$. By (5.16), we see that, if $\mu - \nu \in \mathbb{Z}$, then $T^{\pm}_{\mu\nu}: \mathscr{H}_m \to \mathscr{H}_{m+\mu-\nu}$, $m \in \mathbb{Z}$. Further, $T^{\pm}_{\mu\nu}$ are dilation covariant:

$$U_t T^{\pm}_{\mu\nu} U_t^{-1} = t^{p(\nu-\mu)} T^{\pm}_{\mu\nu}, \quad t > 0.$$

This follows from (5.9).

Remark: We have not been able to clarify the commutation relations of $T^{\pm}_{\mu\nu}$ with themselves and H_+. It seems that they are not so simple.

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