UNIRATIONALITY OF CERTAIN SUPERSINGULAR $K3$ SURFACES IN CHARACTERISTIC 5

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Abstract. We show that every supersingular $K3$ surface in characteristic 5 with Artin invariant ≤ 3 is unirational.

1. Introduction

We work over an algebraically closed field k.

A $K3$ surface X is called *supersingular* (in the sense of Shioda [22]) if the Picard number of X is equal to the second Betti number 22. Supersingular $K3$ surfaces exist only when the characteristic of k is positive. Artin [3] showed that, if X is a supersingular $K3$ surface in characteristic $p > 0$, then the discriminant of the Néron-Severi lattice $\mathrm{NS}(X)$ of X is written as $-p^{2\sigma(X)}$, where $\sigma(X)$ is a positive integer ≤ 10. (See also Illusie [9, Section 7.2].) This integer $\sigma(X)$ is called the *Artin invariant of* X.

A surface S is called *unirational* if the function field $k(S)$ of S is contained in a purely transcendental extension field of k, or equivalently, if there exists a dominant rational map from a projective plane $\mathbb{P}^2$ to S. Shioda [22] proved that, if a smooth projective surface S is unirational, then the Picard number of S is equal to the second Betti number of S. Artin and Shioda conjectured that the converse is true for $K3$ surfaces (see, for example, Shioda [23]):

Conjecture 1.1. Every supersingular $K3$ surface is unirational.

In this paper, we consider this conjecture for supersingular $K3$ surfaces in characteristic 5.

From now on, we assume that the characteristic of k is 5. Let $k[x]_6$ be the space of polynomials in x of degree 6, and let $\mathcal{U} \subset k[x]_6$ be the space of $f(x) \in k[x]_6$ such that the quintic equation $f'(x) = 0$ has no multiple roots. It is obvious that $\mathcal{U}$ is a Zariski open dense subset of $k[x]_6$. For $f \in \mathcal{U}$, we denote by $C_f \subset \mathbb{P}^2$ the projective plane curve of degree 6 whose affine part is defined by

$$y^5 - f(x) = 0.$$

Let $Y_f \to \mathbb{P}^2$ be the double covering of $\mathbb{P}^2$ whose branch locus is equal to C_f, and let $X_f \to Y_f$ be the minimal resolution of Y_f.

Theorem 1.2. *If f is a polynomial in $\mathcal{U}$, then X_f is a supersingular $K3$ surface with $\sigma(X_f) \leq 3$. Conversely, if X is a supersingular $K3$ surface with $\sigma(X) \leq 3$, then there exists $f \in \mathcal{U}$ such that X is isomorphic to X_f.*

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The affine part of Y_f is defined by $w^2 = y^5 - f(x)$. Hence the function field $k(X_f)$ is equal to $k(w, x, y)$, and it is contained in the purely transcendental extension field $k(w^{1/5}, x^{1/5})$ of k. Therefore we obtain the following corollary:

Corollary 1.3. *Every supersingular $K3$ surface in characteristic 5 with Artin invariant ≤ 3 is unirational.*

The unirationality of a supersingular $K3$ surface X in characteristic $p > 0$ with Artin invariant σ has been proved in the following cases: (i) $p = 2$, (ii) $p = 3$ and $\sigma \leq 6$, and (iii) p is odd and $\sigma \leq 2$. In the cases (i) and (ii), the unirationality was proved by Rudakov and Shafarevich [15], [16] by showing that there exists a structure of the quasi-elliptic fibration on X. The case (iii) follows from the result of Ogus [13],[14] that a supersingular $K3$ surface in odd characteristic with Artin invariant ≤ 2 is a Kummer surface associated with a supersingular abelian surface, and the result of Shioda [24] that such a Kummer surface is unirational. The unirationality of X in the case $(p, \sigma) = (5, 3)$ proved in this paper seems to be new.

In [19], we have shown that a supersingular $K3$ surface in characteristic 2 is birational to a normal $K3$ surface with $21A_1$-singularities, and that such a normal $K3$ surface is a purely inseparable double cover of $\mathbb{P}^2$. In [20], we have proved that a supersingular $K3$ surface in characteristic 3 with Artin invariant ≤ 6 is birational to a normal $K3$ surface with $10A_2$-singularities, and it is also birational to a purely inseparable triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$. These yield an alternative proof to the results of Rudakov and Shafarevich [15], [16] in the cases (i) and (ii) above.

In this paper, we show that a supersingular $K3$ surface in characteristic 5 with Artin invariant ≤ 3 is birational to a normal $K3$ surface with $5A_4$-singularities that is a double cover of $\mathbb{P}^2$, and then prove that such a normal $K3$ surface is isomorphic to Y_f for some $f \in \mathcal{U}$. The first step follows from the structure theorem of the Néron-Severi lattices of supersingular $K3$ surfaces due to Rudakov and Shafarevich [16]. For the second step, we investigate projective plane curves of degree 6 with $5A_4$-singularities in Section 2.

2. Projective plane curves with $5A_4$-singularities

Definition 2.1. A germ of a curve singularity in characteristic $\neq 2$ is called an A_n-singularity if it is formally isomorphic to

$$y^2 - x^{n+1} = 0,$$

(see Artin [4], and Greuel and Kröning [8].)

We assume that the base field k is of characteristic 5 until the end of the paper.

Proposition 2.2. *Let $C \subset \mathbb{P}^2$ be a reduced projective plane curve of degree 6. Then the following conditions are equivalent to each other.*

(i) *The singular locus of C consists of five A_4-singular points.*
(ii) *There exists $f \in \mathcal{U}$ such that $C = C_f$.*

For the proof, we need the following result due to Wall [26], which holds in any characteristic. Let $D \subset \mathbb{P}^2$ be an integral plane curve of degree $d > 1$, and let $I_D \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ be the closure of the locus of all $(x, l) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ such that x is a smooth point of D and l is the tangent line to D at x. Let $D^\vee \subset (\mathbb{P}^2)^\vee$ be the image of the second projection

$$\pi_D : I_D \to (\mathbb{P}^2)^\vee.$$

We equip $D^\vee$ with the reduced structure, and call it the *dual curve of* D. Note that the first projection $I_D \to D$ is birational. Therefore, by the projection π_D, we can regard the function field $k(D)$ as an extension field of the function field $k(D^\vee)$. The corresponding rational map from D to $D^\vee$ is called the *Gauss map*. We put

$$\deg \pi_D := [k(D) : k(D^\vee)].$$

We choose general homogeneous coordinates $[w_0 : w_1 : w_2]$ of $\mathbb{P}^2$, and let $F(w_0, w_1, w_2) = 0$ be the defining equation of D. We denote by $D_Q \subset \mathbb{P}^2$ the curve defined by

$$\frac{\partial F}{\partial w_2} = 0,$$

which is called the *polar curve of* D with respect to $Q = [0 : 0 : 1]$.

Proposition 2.3 (Wall [26])**.** *For a singular point s of D, we denote by $(D.D_Q)_s$ the local intersection multiplicity of D and D_Q at s. Then we have*

$$\deg \pi_D \cdot \deg D^\vee = d(d-1) - \sum_{s \in \mathrm{Sing}(D)} (D.D_Q)_s.$$

Remark 2.4. If $s \in D$ is an A_n-singular point, then the polar curve D_Q is smooth at s and the local intersection multiplicity $(D.D_Q)_s$ is $n+1$.

Proof of Proposition 2.2. Suppose that C has $5A_4$-singular points as its only singularities. Since an A_4-singular point is unibranched, C is irreducible. By Proposition 2.3 and Remark 2.4, we have

$$\deg \pi_C \cdot \deg C^\vee = 5.$$

Suppose that $(\deg \pi_C, \deg C^\vee) = (1, 5)$. Let $\nu : \widetilde{C} \to C$ be the normalization of C. Since $\deg \pi_C = 1$, we can consider $\widetilde{C}$ as a normalization of $C^\vee$. We denote by

$$\nu^\vee : \widetilde{C} \to C^\vee$$

the morphism of normalization. Let s be a singular point of C, and let $\widetilde{s} \in \widetilde{C}$ be the point of $\widetilde{C}$ that is mapped to s by ν. We can choose affine coordinates (x, y) of $\mathbb{P}^2$ with the origin s and a formal parameter t of $\widetilde{C}$ at $\widetilde{s}$ such that ν is given by

$$t \mapsto (x, y) = (\, t^2,\ t^5 + c_6\, t^6 + c_7\, t^7 + \cdots).$$

Let (u, v) be the affine coordinates of $(\mathbb{P}^2)^\vee$ such that the point $(u, v) \in (\mathbb{P}^2)^\vee$ corresponds to the line of $\mathbb{P}^2$ defined by $y = ux + v$. Then $\nu^\vee$ is given at $\widetilde{s}$ by

$$t \mapsto (u, v) = (\, 3\, c_6\, t^4 + \cdots\,,\ t^5 + \cdots).$$

(See, for example, Namba [10, p. 78].) Therefore $\nu^\vee(\widetilde{s})$ is a singular point of $C^\vee$ with multiplicity ≥ 4. We choose distinct two points $s_1, s_2 \in \mathrm{Sing}(C)$. There exists a line of $(\mathbb{P}^2)^\vee$ that passes through both of $\nu^\vee(\widetilde{s_1}) \in C^\vee$ and $\nu^\vee(\widetilde{s_2}) \in C^\vee$. This contradicts Bezout's theorem, because $\deg C^\vee = 5 < 4 + 4$. Therefore we have $(\deg \pi_C, \deg C^\vee) = (5, 1)$. Then there exists a point $P \in \mathbb{P}^2$ such that we have

$$l \in C^\vee \iff P \in l. \tag{2.1}$$

We choose homogeneous coordinates $[w_0 : w_1 : w_2]$ of $\mathbb{P}^2$ in such a way that $P = [0 : 1 : 0]$. Let L_∞ be the line $w_2 = 0$, and let (x, y) be the affine coordinates on

$\mathbb{A}^2 := \mathbb{P}^2 \setminus L_\infty$ given by $x := w_0/w_2$ and $y := w_1/w_2$. Suppose that C is defined by $h(x, y) = 0$ in $\mathbb{A}^2$. From (2.1), we have

$$h(a, b) = 0 \implies \frac{\partial h}{\partial y}(a, b) = 0. \tag{2.2}$$

Let $U_C \subset \mathbb{A}^1$ be the image of the projection $(C \setminus \mathrm{Sing}(C)) \cap \mathbb{A}^2 \to \mathbb{A}^1$ given by $(a, b) \mapsto a$. Note that U_C is Zariski dense in $\mathbb{A}^1$. Let (a_0, b_0) be a smooth point of $C \cap \mathbb{A}^2$. By (2.2), we have

$$\frac{\partial h}{\partial x}(a_0, b_0) \neq 0.$$

Hence there exists a formal power series $\gamma(\eta) \in k[[\eta]]$ such that C is defined by $x - a_0 = \gamma(y - b_0)$ locally around (a_0, b_0). By (2.2) again, $\gamma'(\eta)$ is constantly equal to 0, and hence there exists a formal power series $\beta(\eta) \in k[[\eta]]$ such that $\gamma(\eta) = \beta(\eta)^5$. Therefore the local intersection multiplicity of the line $x - a_0 = 0$ and C at (a_0, b_0) is ≥ 5. Thus we obtain the following:

(2.3) If $a \in U_C$, then the equation $h(a, y) = 0$ in y has a root of multiplicity ≥ 5.

We put

$$h(x, y) \;=\; c\, y^6 \;+\; g_1(x)\, y^5 \;+\; \cdots \;+\; g_5(x)\, y \;+\; g_6(x),$$

where c is a constant, and $g_\nu(x) \in k[x]$ is a polynomial of degree $\leq \nu$. Suppose that $c \neq 0$. We can assume $c = 1$. By (2.3), we have $g_2(a) = g_3(a) = g_4(a) = 0$ and $g_1(a)g_5(a) = g_6(a)$ for any $a \in U_C$. Since U_C is Zariski dense in $\mathbb{A}^1$, we have $g_2 = g_3 = g_4 = 0$ and $g_1 g_5 = g_6$. Then we have $h(x, y) = (y^5 + g_5(x))(y + g_1(x))$, which contradicts the irreducibility of C. Thus $c = 0$ is proved. Then, by (2.3), we have $g_1 \neq 0$ and $g_2 = g_3 = g_4 = g_5 = 0$. We put $g_1 = Ax + B$, and define a new homogeneous coordinate system $[z_0 : z_1 : z_2]$ of $\mathbb{P}^2$ by

$$\begin{cases} (z_0, z_1, z_2) := (w_0, w_1, Aw_0 + Bw_2) & \text{if } B \neq 0; \\ (z_0, z_1, z_2) := (w_2, w_1, Aw_0) & \text{if } B = 0. \end{cases}$$

Then C is defined by a homogeneous equation of the form

$$z_2 z_1^5 - F(z_0, z_2) = 0,$$

where $F(z_0, z_2)$ is a homogeneous polynomial of degree 6. We put $L'_\infty := \{z_2 = 0\}$. Defining the affine coordinates (x, y) on $\mathbb{P}^2 \setminus L'_\infty$ by $(x, y) := (z_0/z_2, z_1/z_2)$, we see that the affine part of C is defined by $y^5 - f(x)$ for some polynomial $f(x)$ of degree ≤ 6. If $\deg f < 6$, then L'_∞ would be an irreducible component of C because $\deg C = 6$. Therefore we have $\deg f = 6$. Then $C \cap L'_\infty$ consists of a single point $[0 : 1 : 0]$, and C is smooth at $[0 : 1 : 0]$. Therefore we have

$$\mathrm{Sing}(C) = \{\ (\alpha, f(\alpha)^{1/5}) \ \mid\ f'(\alpha) = 0\ \}.$$

Since C has five singular points, we have $f \in \mathcal{U}$.

Conversely, suppose that $f \in \mathcal{U}$. We show that $\mathrm{Sing}(C_f)$ consists of $5A_4$-singular points. Let $L_\infty \subset \mathbb{P}^2$ be the line at infinity. It is easy to check that $C_f \cap L_\infty$ consists of a single point $[0 : 1 : 0]$, and C_f is smooth at this point. Therefore we have $\mathrm{Sing}(C_f) = \{(\alpha, f(\alpha)^{1/5}) \,|\, f'(\alpha) = 0\}$. In particular, C_f has exactly five singular points. Let (α, β) be a singular point of C_f. Since α is a simple root of the quintic equation $f'(x) = 0$, there exists a polynomial $g(x)$ with $g(\alpha) \neq 0$ such that

$$f(x) = f(\alpha) + (x - \alpha)^2 g(x).$$

Because $\beta^5 = f(\alpha)$, the defining equation of C is written as

$$(y-\beta)^5 - (x-\alpha)^2 g(x) = 0.$$

Therefore (α, β) is an A_4-singular point of C_f. □

3. Proof of Theorem 1.2

First we show that, if $f \in \mathcal{U}$, then X_f is a supersingular $K3$ surface with Artin invariant ≤ 3. Since the sextic double plane Y_f has only rational double points as its singularities by Proposition 2.2, its minimal resolution X_f is a $K3$ surface by the results of Artin [1], [2]. Let Σ_f be the sublattice of the Néron-Severi lattice $\mathrm{NS}(X_f)$ of X_f that is generated by the classes of the (-2)-curves contracted by $X_f \to Y_f$. Then Σ_f is isomorphic to the negative-definite root lattice of type $5A_4$ by Proposition 2.2. In particular, Σ_f is of rank 20, and its discriminant is 5^5. Let $H_f \subset X_f$ be the pull-back of a line of $\mathbb{P}^2$, and put

$$h_f := [H_f] \in \mathrm{NS}(X_f).$$

Since the line at infinity $L_\infty \subset \mathbb{P}^2$ intersects C_f at a single point $[0:1:0]$ with multiplicity 6, and $[0:1:0]$ is a smooth point of C_f, the pull-back of L_∞ to X_f is a union of two smooth rational curves that intersect each other at a single point with multiplicity 3. Let L_f be one of the two rational curves, and put

$$l_f := [L_f] \in \mathrm{NS}(X_f).$$

Then h_f and l_f generate a lattice $\langle h_f, l_f \rangle$ of rank 2 in $\mathrm{NS}(X_f)$ whose intersection matrix is equal to

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

In particular, the discriminant of $\langle h_f, l_f \rangle$ is -5. Note that Σ_f and $\langle h_f, l_f \rangle$ are orthogonal in $\mathrm{NS}(X_f)$. Therefore $\mathrm{NS}(X_f)$ contains a sublattice $\Sigma_f \oplus \langle h_f, l_f \rangle$ of rank 22 and discriminant -5^6. Thus X_f is supersingular, and $\sigma(X_f) \leq 3$.

In order to prove the second assertion of Theorem 1.2, we define an even lattice S_0 of rank 22 with signature $(1, 21)$ and discriminant -5^6 by

$$S_0 := \Sigma^-_{5A_4} \oplus \langle h, l \rangle,$$

where $\Sigma^-_{5A_4}$ is the negative-definite root lattice of type $5A_4$, and $\langle h, l \rangle$ is the lattice of rank 2 generated by the vectors h and l satisfying

$$h^2 = 2, \quad l^2 = -2, \quad hl = 1.$$

Remark 3.1. This lattice $\langle h, l \rangle$ is the unique even indefinite lattice of rank 2 with discriminant -5. See Edwards [7], or Conway and Sloane [5, Table 15.2a].

Claim 3.2. For $\sigma = 1, 2, 3$, there exists an even overlattice $S^{(\sigma)}$ of S_0 with the following properties:

(i) the discriminant of $S^{(\sigma)}$ is $-5^{2\sigma}$,
(ii) the Dynkin type of the root system $\{r \in S^{(\sigma)} \mid rh = 0, r^2 = -2\}$ is $5A_4$,
(iii) the set $\{e \in S^{(\sigma)} \mid eh = 1, e^2 = 0\}$ is empty.

Here we prove that $S^{(3)} = S_0$ satisfies (ii) and (iii). Let $v = s + xh + yl$ be a vector of $S^{(3)} = S_0$, where $s \in \Sigma^-_{5A_4}$ and $x, y \in \mathbb{Z}$. If $vh = 0$ and $v^2 = -2$, then we have $2x + y = 0$ and $s^2 - 10x^2 = -2$. Since $s^2 \le 0$, we have $x = y = 0$ and hence v is a root in $\Sigma^-_{5A_4}$. Therefore $S^{(3)} = S_0$ satisfies (ii). If $vh = 1$ and $v^2 = 0$, then we have $2x + y = 1$ and $s^2 - 10x^2 + 10x - 2 = 0$. Since $s^2 \le 0$, there is not such an integer x. Hence $S^{(3)} = S_0$ satisfies (iii). Thus Claim 3.2 for $\sigma = 3$ has been proved. For the cases $\sigma = 2$ and $\sigma = 1$, see Proposition 4.1 in the next section.

Let X be a supersingular $K3$ surface with $\sigma = \sigma(X) \le 3$. By the results of Rudakov and Shafarevich [16], the isomorphism class of the lattice $\mathrm{NS}(X)$ is characterized by the following properties;

(a) even and signature $(1, 21)$, and
(b) the discriminant group is isomorphic to $\mathbb{F}_5^{\oplus 2\sigma}$.

Since the discriminant group of $S^{(\sigma)}$ is a quotient group of a subgroup of the discriminant group $\mathbb{F}_5^{\oplus 6}$ of S_0, the lattice $S^{(\sigma)}$ has also these properties. Therefore there exists an isomorphism

$$\phi \,:\, S^{(\sigma)} \xrightarrow{\sim} \mathrm{NS}(X).$$

By [16, Proposition 3 in Section 3], we can assume that $\phi(h)$ is the class $[H]$ of a nef divisor H. Note that $H^2 = h^2 = 2$. If the complete linear system $|H|$ had a fixed component, then, by Nikulin [12, Proposition 0.1], there would be an elliptic pencil $|E|$ and a (-2)-curve Γ such that $|H| = 2|E| + \Gamma$ and $E\Gamma = 1$, and the vector $e \in S^{(\sigma)}$ that is mapped to $[E]$ by ϕ would satisfy $eh = 1$ and $e^2 = 0$. Therefore the property (iii) of $S^{(\sigma)}$ implies that the linear system $|H|$ has no fixed components (see also Urabe [25, Proposition 1.7].) Then, by Saint-Donat [17, Corollary 3.2], $|H|$ is base point free. Hence we have a morphism $\Phi_{|H|} : X \to \mathbb{P}^2$ induced by $|H|$. Let

$$X \;\to\; Y_H \;\to\; \mathbb{P}^2$$

be the Stein factorization of $\Phi_{|H|}$. Then $Y_H \to \mathbb{P}^2$ is a finite double covering branched along a curve $C_H \subset \mathbb{P}^2$ of degree 6. By the property (ii) of $S^{(\sigma)}$, we see that $\mathrm{Sing}(Y_H)$ consists of $5A_4$-singular points, and hence $\mathrm{Sing}(C_H)$ also consists of $5A_4$-singular points. By Proposition 2.2, there exists an element $f \in \mathcal{U}$ such that C_H is isomorphic to C_f. Then X is isomorphic to X_f. $\square$

Remark 3.3. In [21], it is proved that a normal $K3$ surface with $5A_4$-singular points exists only in characteristic 5.

4. Classification of overlattices

Let $F \subset S_0$ be a fundamental system of roots of $\Sigma^-_{5A_4} \subset S_0$ (see Ebeling [6] for the definition and properties of a fundamental system of roots.) Then F consists of 4×5 vectors

$$e_i^{(j)} \quad (i = 1, \dots, 4,\ j = 1, \dots, 5)$$

such that

$$e_i^{(j)} e_{i'}^{(j')} = \begin{cases} 0 & \text{if } j \ne j' \text{ or } |i - i'| > 1, \\ 1 & \text{if } j = j' \text{ and } |i - i'| = 1, \\ -2 & \text{if } j = j' \text{ and } i = i', \end{cases}$$

(see Figure 4.1.) We put

FIGURE 4.1. The Dynkin diagram of type A_4

$$\mathrm{Aut}(F,h) := \{\ g \in O(S_0)\ \mid\ g(F) = F, g(h) = h\ \},$$

where $O(S_0)$ is the orthogonal group of the lattice S_0. Then $\mathrm{Aut}(F,h)$ is isomorphic to the automorphism group of the Dynkin diagram of type $5A_4$, and hence it is isomorphic to the semi-direct product $\{\pm 1\}^5 \rtimes S_5$. Note that $\mathrm{Aut}(F,h)$ acts on the dual lattice $(S_0)^\vee$ of S_0 in a natural way, and hence it acts on the set of even overlattices of S_0. We classify all even overlattices of S_0 with the properties (ii) and (iii) in Claim 3.2 up to the action of $\mathrm{Aut}(F,h)$. The main tool is Nikulin's theory of discriminant forms of even lattices [11].

The set $F \cup \{h, l\}$ of vectors form a basis of S_0. Let

$$(e_i^{(j)})^\vee \quad (i = 1,\dots,4,\ j = 1,\dots,5), \quad h^\vee \quad \text{and} \quad l^\vee$$

be the basis of $(S_0)^\vee$ dual to $F \cup \{h, l\}$. We denote by G the discriminant group $(S_0)^\vee/S_0$ of S_0, and by

$$\mathrm{pr} : (S_0)^\vee \to G$$

the natural projection. Then G is isomorphic to $\mathbb{F}_5^{\oplus 5} \oplus \mathbb{F}_5$ with basis

$$\mathrm{pr}((e_1^{(1)})^\vee),\ \dots\ ,\ \mathrm{pr}((e_1^{(5)})^\vee),\ \mathrm{pr}(h^\vee).$$

With respect to this basis, we denote the elements of G by $[x_1,\dots,x_5\,|\,y]$ with $x_1,\dots,x_5, y \in \mathbb{F}_5$. The discriminant form $q : G \to \mathbb{Q}/2\mathbb{Z}$ of S_0 is given by

$$q([x_1,\dots,x_5\,|\,y]) = -\frac{4}{5}(x_1^2 + \cdots + x_5^2) + \frac{2}{5}y^2 \mod 2\mathbb{Z}$$

The action of $\mathrm{Aut}(F,h)$ on $G = \mathbb{F}_5^{\oplus 5} \oplus \mathbb{F}_5$ is generated by the multiplications by -1 on x_i, and the permutations of $x_1,\dots,x_5$. We define subgroups $H_0,\dots,H_8$ of G by their generators as follows:

$$\begin{array}{rcl}
H_0 & := & \{0\}, \\
H_1 & := & \langle\, [0,0,2,2,2\,|\,2]\,\rangle, \\
H_2 & := & \langle\, [2,2,2,2,2\,|\,0]\,\rangle, \\
H_3 & := & \langle\, [0,1,2,2,2\,|\,1]\,\rangle, \\
H_4 & := & \langle\, [1,2,2,2,2\,|\,2]\,\rangle, \\
H_5 & := & \langle\, [0,1,1,2,2\,|\,0]\,\rangle, \\
H_6 & := & \langle\, [1,0,1,2,2\,|\,0]\,,\ [0,1,2,1,3\,|\,0]\,\rangle, \\
H_7 & := & \langle\, [1,0,0,1,1\,|\,1]\,,\ [0,1,1,1,3\,|\,3]\,\rangle, \\
H_8 & := & \langle\, [1,0,1,1,2\,|\,2]\,,\ [0,1,1,3,3\,|\,0]\,\rangle.
\end{array}$$

We then put

$$S_i := \mathrm{pr}^{-1}(H_i) \ \subset\ (S_0)^\vee.$$

the (a,b,y)-type	the roots in $h^\perp$	the set E	
$(0,0,0)$	$5A_4$	empty	$*$
$(0,2,\pm1)$	A_9+3A_4	empty	
$(0,3,\pm2)$	$5A_4$	empty	$*$
$(0,5,0)$	$5A_4$	empty	$*$
$(1,1,0)$	E_8+3A_4	empty	
$(1,3,\pm1)$	$5A_4$	empty	$*$
$(1,4,\pm2)$	$5A_4$	empty	$*$
$(2,0,\pm2)$	A_9+3A_4	empty	
$(2,2,0)$	$5A_4$	empty	$*$
$(3,0,\pm1)$	$5A_4$	empty	$*$
$(3,1,\pm2)$	$5A_4$	empty	$*$
$(4,1,\pm1)$	$5A_4$	empty	$*$
$(5,0,0)$	$5A_4$	empty	$*$

Table 4.1. The isotropic vectors in (G,q)

Proposition 4.1. *The submodules $S_0,\dots,S_8$ of $(S_0)^\vee$ are even overlattices of S_0 with the properties* (ii) *and* (iii) *in Claim 3.2. The discriminant of S_i is -5^6 for $i=0$, -5^4 for $i=1,\dots,5$, and -5^2 for $i=6,\dots,8$.*

Conversely, if S is an even overlattice of S_0 with the properties (ii) *and* (iii)*, then there exists a unique S_i among $S_0,\dots,S_8$ such that $S=g(S_i)$ holds for some $g\in\mathrm{Aut}(F,h)$.*

Proof. The mapping $S\mapsto S/S_0$ gives rise to a one-to-one correspondence between the set of even overlattices S of S_0 and the set of totally isotropic subgroups H of (G,q). The inverse mapping is given by $H\mapsto \mathrm{pr}^{-1}(H)$. If $\dim_{\mathbb{F}_5}H=d$, then the discriminant of $\mathrm{pr}^{-1}(H)$ is equal to -5^{6-2d} (see Nikulin [11].)

For $v=[x_1,\dots,x_5\,|\,y\,]\in G$, we put

$$\delta(v):=(a,b,y)\in\mathbb{Z}_{\ge0}\times\mathbb{Z}_{\ge0}\times\mathbb{F}_5,$$

where a is the number of $\pm1\in\mathbb{F}_5$ among $x_1,\dots,x_5$ and b is the number of $\pm2\in\mathbb{F}_5$ among $x_1,\dots,x_5$. Note that $\delta(v)=\delta(w)$ holds if and only if there exists $g\in\mathrm{Aut}(F,h)$ such that $g(v)=w$. A vector $v\in G$ is isotropic with respect to q if and only if $\delta(v)$ appears in the first column of Table 4.1. For each (a,b,y)-type α in Table 4.1, we choose a vector $v\in G$ such that $\delta(v)=\alpha$, and calculate the even overlattice

$$S_\alpha:=\mathrm{pr}^{-1}(\langle v\rangle)$$

of S_0. The second column of Table 4.1 presents the Dynkin type of the root system $\{r\in S_\alpha\,|\,rh=0,r^2=-2\}$, and the third column presents the set $E:=\{e\in S_\alpha\,|\,eh=1,e^2=0\}$. Hence we see that the following two conditions on a subgroup H of G are equivalent:

(I) The corresponding submodule $\mathrm{pr}^{-1}(H)$ of $(S_0)^\vee$ is an even overlattice of S_0 with the properties (ii) and (iii) in Claim 3.2.
(II) For any $v\in H$, $\delta(v)$ is an (a,b,y)–type with $*$ in Table 4.1.

Using a computer, we make the complete list of subgroups of G that satisfy the condition (II) up to the action of $\mathrm{Aut}(F,h)$. The complete set of representatives is $\{H_0,\dots,H_8\}$ above. $\square$

Remark 4.2. Since there exist no even unimodular lattices of signature $(1, 21)$ (see Serre [18, Theorem 5 in Chapter V]), all totally isotropic subgroups of (G, q) are of dimension ≤ 2 over $\mathbb{F}_5$.

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