Ekedahl-Oort Strata and the First Newton Slope Strata

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Abstract

We investigate stratifications on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g in characteristic $p > 0$. In this paper we give an easy algorithm determining the first Newton slope of any generic point of each Ekedahl-Oort stratum.

1 Introduction

We study p-divisible groups and Barsotti-Tate truncated level one groups (BT_1) in characteristic $p > 0$ and conclude some results about stratifications on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties over fields of characteristic p.

For a p-divisible group, we can define its Newton polygon. The isogeny classes of p-divisible groups are classified by Newton polygons. From a Newton polygon, we have a finite set of rational numbers λ_i $(i = 1, \cdots, t)$ satisfying $0 \leq \lambda_1 \leq \cdots \leq \lambda_t \leq 1$ (cf. §2.2). We call λ_1 *the first Newton slope*. An abelian variety X defines its p-divisible group

$$X[p^\infty] := \mathrm{ind.lim.}\, X[p^i]$$

whose Newton polygon is symmetric, i.e., it satisfies $\lambda_i + \lambda_{t+1-i} = 1$ for $i = 1, \cdots, t$. The symmetric Newton polygon with $\lambda_1 = 1/2$ (hence all $\lambda_i = 1/2$) is called supersingular and is denoted by σ. For a symmetric Newton polygon ξ ending at $(2g, g)$, we can define its NP-stratum W_ξ in $\mathcal{A}_g$ (cf. §2.2).

On the other hand, Ekedahl and Oort defined another new stratification which is now called the EO-stratification. This stratification is defined by isomorphism classes of p-kernels of principally polarized abelian varieties (we shall give a brief review in §2.3). We will denote by S_φ an EO-stratum.

Thus we have two stratifications on $\mathcal{A}_g$. Our basic problem is to find an easy criterion for $S_\varphi \subset W_\xi$. This is still an open problem in general. For the supersingular case $\xi = \sigma$, Oort gave an answer (cf. (4.0.3)), which played an important role in determining whether S_φ is irreducible or not. Our chief aim is to generalize his result. More precisely speaking, let λ be a rational number and Z_λ the locus where the first Newton slope is not less than λ. Then we have a necessary and sufficient condition for $S_\varphi \subset Z_\lambda$ (Cor. 4.2). This is a corollary to our main theorem (Th. 4.1), in which we determine the first Newton slope of any generic point of S_φ.

This theorem can be regarded as a variant of the result of Goren and Oort ([3], Th. 5.4.11) on reductions modulo inert primes of Hilbert modular varieties. For these modular varieties, they computed the first Newton slope of any generic point of the generalized a-number locus which is an analogue of EO-stratum. Note that in their cases the Newton polygon of every point is determined only by its first Newton slope, as shown by themselves, and therefore the computation of the first Newton slopes gave the complete answer.

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Notations. We fix once for all a rational prime p. All base fields and all base schemes will be in characteristic p. For non-negative integers m, n we denote by $\gcd(m, n)$ the greatest common divisor where for convenience we set $\gcd(m, 0) = \gcd(0, m) = m$ for $\forall m \in \mathbb{Z}_{\geq 0}$.

2 Stratifications

Let us review the definitions of the stratifications we will deal with. We recall some known facts.

2.1 Dieudonné theory

Let K be a perfect field of characteristic p and $W(K)$ the ring of infinite Witt vectors with coordinates in K. Let A_K be the p-adic completion of the associative ring

$$W(K)[\mathcal{F}, \mathcal{V}]/(\mathcal{F}x - x^\rho \mathcal{F}, \mathcal{V}x^\rho - x\mathcal{V}, \mathcal{F}\mathcal{V} - p, \mathcal{V}\mathcal{F} - p, \forall x \in W(K))$$

with the Frobenius automorphism ρ on $W(K)$. Note A_K is not commutative unless $K = \mathbb{F}_p$. *A Dieudonné module over* $W(K)$ is a left A_K-module which is finitely generated as a $W(K)$-module.

We use the covariant Dieudonné theory, which says that there is a canonical categorical equivalence $\mathbb{D}$ from the category of p-torsion finite commutative group schemes (resp. p-divisible groups) over K to the category of Dieudonné modules over $W(K)$ which are of finite length (resp. free as $W(K)$-modules). We write F and V for "Frobenius" and "Verschiebung" on commutative group schemes. Note the covariant Dieudonné functor $\mathbb{D}$ satisfies $\mathbb{D}(F) = \mathcal{V}$ and $\mathbb{D}(V) = \mathcal{F}$. For a p-torsion finite commutative group scheme G, we have $\mathrm{length}(G) = \mathrm{length}(\mathbb{D}(G))$.

2.2 The NP-stratification

For $m, n \in \mathbb{Z}_{\geq 0}$ with $\gcd(m, n) = 1$, we define a p-divisible group $G_{m,n}$ over $\mathbb{F}_p$ by

$$\mathbb{D}(G_{m,n}) = A_{\mathbb{F}_p}/A_{\mathbb{F}_p}(\mathcal{F}^m - \mathcal{V}^n). \tag{2.2.1}$$

Let K be a field of characteristic p and k an algebraically closed field containing K. Let $\mathcal{G}$ be a p-divisible group over K. By the Dieudonné-Manin classification, see [5] and [1], $\mathcal{G}$ is isogeneous over k to

$$\bigoplus_{i=1}^{t} G_{m_i,n_i} \tag{2.2.2}$$

for some finite set of pairs (m_i, n_i) with $\gcd(m_i, n_i) = 1$. Set $\lambda_i = n_i/(m_i + n_i)$. We can suppose

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_t \tag{2.2.3}$$

without losing generality. We call λ_i *the i-th Newton slope.* The height of $\mathcal{G}$ is equal to $\sum_{i=1}^{t}(m_i+n_i)$ and the dimension of $\mathcal{G}$ is equal to $\sum_{i=1}^{t} m_i$. The Newton polygon $\mathrm{NP}(\mathcal{G})$ of $\mathcal{G}$ is the line graph which starts at $(0,0)$ and ends at $(\sum_{i=1}^{t}(m_i+n_i), \sum_{i=1}^{t} n_i)$ and whose break-points are of the form $(\sum_{i=1}^{j}(m_i+n_i), \sum_{i=1}^{j} n_i)$ for some $1 \leq j \leq t-1$. We write this Newton polygon as

$$[m_1, n_1] + \cdots + [m_t, n_t]. \tag{2.2.4}$$

(This is usually written as $(m_1, n_1) + \cdots + (m_t, n_t)$. But in this paper, this would be confusing, because we use a similar symbol for another notion.) In general, a Newton polygon is a line graph obtained in this way from some finite set of pairs (m_i, n_i) of non-negative integers with $\gcd(m_i, n_i) = 1$. For an abelian variety X, we write $\mathrm{NP}(X) = \mathrm{NP}(X[p^\infty])$. We say, for two Newton polygons ξ, ξ' with the same end point, that $\xi' \prec \xi$ if every point of ξ' is not below ξ.

For a symmetric Newton polygon ξ ending at $(2g, g)$, we define its *NP-stratum* by

$$W_\xi = \{(X,\eta) \in \mathcal{A}_g \mid \mathrm{NP}(X) \prec \xi\},$$

which has a natural structure of closed subscheme of $\mathcal{A}_g$ by Grothendieck and Katz ([4], Th. 2.3.1 on p. 143). We also define the open NP-stratum by

$$W_\xi^0 = \{(X,\eta) \in \mathcal{A}_g \mid \mathrm{NP}(X) = \xi\},$$

which is a locally closed subscheme of $\mathcal{A}_g$.

For a rational number $\lambda = n/(m+n)$ with $m \geq n \geq 0$ and $\gcd(m,n) = 1$ and for a natural number e with $e(m+n) \leq g$, let $\xi_{\lambda,e}$ be the lowest Newton polygon with Newton slopes $\lambda_i = \lambda$ for all $1 \leq i \leq e$. We set $Z_{\lambda,e} = W_{\xi_{\lambda,e}}$ and write $Z_\lambda = Z_{\lambda,1}$. We denote by $Z_{\lambda,e}^0$ the locally closed subscheme of $\mathcal{A}_g$ which consists of principally polarized abelian varieties with the Newton slopes $\lambda_i = \lambda$ for all $1 \leq i \leq e$. Note that $W_{\xi_{\lambda,e}}^0 \subset Z_{\lambda,e}^0$ and in many cases $Z_{\lambda,e}^0 \neq W_{\xi_{\lambda,e}}^0$.

2.3 The EO-stratification

Let K be a field of characteristic p.

Definition 2.1. (1) A finite commutative group scheme G over K is said to be *a Barsotti-Tate truncated level one group scheme (denoted by* BT_1*) over* K if

$$\mathrm{Im}(V : G^{(p)} \to G) = \mathrm{Ker}(F : G \to G^{(p)}),$$
$$\mathrm{Im}(F : G \to G^{(p)}) = \mathrm{Ker}(V : G^{(p)} \to G).$$

(2) A symmetric BT_1 over K is a pair (G, ι) of a BT_1 G over K and a group isomorphism ι over K from G to its Cartier dual G^D.

Definition 2.2. (1) *An elementary sequence of length* g is a map

$$\varphi : \{0, 1, \cdots, g\} \longrightarrow \{0, 1, \cdots, g\}$$

satisfying $\varphi(0) = 0$ and $\varphi(i-1) \leq \varphi(i) \leq \varphi(i-1) + 1$ for $1 \leq i \leq g$. We shall frequently write φ as $(\varphi(1), \varphi(2), \cdots, \varphi(g))$.

(2) *A final sequence of length* $2g$ is a map

$$\psi : \{0, 1, \cdots, 2g\} \longrightarrow \{0, 1, \cdots, g\}$$

satisfying $\psi(i-1) \le \psi(i) \le \psi(i-1)+1$ for $1 \le i \le 2g$ with $\psi(0) = 0$ and $\psi(2g-i) = g-i+\psi(i)$ for $0 \le i \le 2g$. We shall frequently write ψ as $(\psi(1), \cdots, \psi(g); \psi(g+1), \cdots, \psi(2g))$.

(3) Let φ be an elementary sequence. The map

$$\psi : \{0, 1, \cdots, 2g\} \longrightarrow \{0, 1, \cdots, g\}$$

defined by $\psi(i) = \varphi(i)$ and $\psi(2g-i) = g-i+\varphi(i)$ for $0 \le i \le g$ is called *the final sequence stretched from* φ.

Let G be a symmetric BT_1 over K. For any subgroup scheme H of G over $\overline{K}$ and for any word w of V, F^{-1}, we define $w \cdot H$ inductively by

$$V \cdot H := VH^{(p)} \quad \text{and} \quad F^{-1} \cdot H := F^{-1}(H^{(p)} \cap FG). \tag{2.3.1}$$

Then there exists a unique elementary sequence φ of a certain length g such that for any word w of V, F^{-1} we have

$$\psi(\mathrm{length}(w \cdot G)) = \mathrm{length}(Vw \cdot G), \tag{2.3.2}$$

where ψ is the final sequence stretched from φ, see [7], (2.3) and (5.6). Moreover in [7], Prop. 9.6, it was proved that there exists a filtration over $\overline{K}$

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{2g} = G \tag{2.3.3}$$

with $\mathrm{length}(G_i) = i$ such that

$$V \cdot G_i = G_{\psi(i)} \quad \text{and} \quad F^{-1} \cdot G_i = G_{g+i-\psi(i)} \quad \text{for} \quad 0 \le i \le 2g. \tag{2.3.4}$$

A filtration as in (2.3.3) satisfying (2.3.4) is called *a final filtration of* G.

Remark 2.3. Although φ is uniquely determined by G, the final filtration is not unique, see [7], Rem. 9.6.

Thus we have a canonical map

$$\mathrm{ES} : \{\text{symmetric } \mathrm{BT}_1 \text{ of length } 2g \text{ over } K\}/K\text{-isom.} \longrightarrow \{\text{elementary sequence of length } g\}.$$

The following deep result is due to Oort, [7], (9.4):

Theorem 2.4. *If* K *is algebraically closed, the map* ES *is bijective.*

For a principally polarized abelian variety (X, μ), we have a BT_1 $X[p]$ and a symmetry

$$\iota_\mu : X[p] \simeq X^t[p] \simeq X[p]^D$$

where the second isomorphism is a canonical one ([6], III, Cor. 19.2). Thus for each (X, μ) we have an elementary sequence $\mathrm{ES}(X[p], \iota_\mu)$, which will be simply denoted by $\mathrm{ES}(X)$.

For a principally polarized abelian variety (X, μ), its p-rank $f(X)$ is defined by $X[p](\overline{K}) = (\mathbb{Z}/p\mathbb{Z})^{f(X)}$ and its a-number $a(X)$ is defined to be $\dim_{\overline{K}} \mathrm{Hom}_{\overline{K}}(\alpha_p, X)$. These invariants depend only on $(X[p], \iota_\mu)$. In fact, if we put $\varphi = \mathrm{ES}(X)$, then $f(X) = \max\{i \,|\, \varphi(i) = i\}$ and $a(X) = g - \varphi(g)$.

For each elementary sequence φ, the EO-stratum S_φ is defined to be the subset of $\mathcal{A}_g$ consisting of points $y \in \mathcal{A}_g$ where y comes over some field from a principally polarized abelian variety X_y such that $\mathrm{ES}(X_y) = \varphi$, see [7], (5.11). As shown in [7], (3.2), S_φ has a natural structure of a locally closed reduced subscheme of $\mathcal{A}_g$.

There are two partial orderings on the set of elementary sequences of length g.

Definition 2.5. Let φ and φ' be elementary sequences of length g.

(Bruhat ordering) We say $\varphi' \leq \varphi$ if $\varphi'(i) \leq \varphi(i)$ for all $i = 1, \cdots, g$.

(Geometric ordering) We say $\varphi' \subset \varphi$ if $S_{\varphi'}$ is contained in the Zariski closure $\overline{S}_\varphi$ of S_φ.

We shall use the fundamental results of [7] on the EO-stratification:

Theorem 2.6. (1) *S_φ is quasi-affine. Furthermore $\overline{S}_\varphi$ is connected unless $\varphi(g) = 0$.*

(2) *Any irreducible component of S_φ has dimension $\sum_{i=1}^{g} \varphi(i)$. In particular S_φ is not empty for every φ.*

(3) *$\varphi' \leq \varphi$ implies $\varphi' \subset \varphi$.*

(4) *$\varphi' \subset \varphi$ is equivalent to $S_{\varphi'} \cap \overline{S}_\varphi \neq \emptyset$.*

3 Combinatorics arising from elementary sequences

In order to describe our main theorem, we introduce some new notions: Ψ-sets, Φ-sets, and invariants: λ_φ, e_φ. We derive some basic properties of them.

3.1 The slope λ_φ associated with φ

Let φ be an elementary sequence and ψ the final sequence stretched from φ. Let G be a symmetric BT_1 over an algebraically closed field k with $\mathrm{ES}(G) = \varphi$. Choose one of the final filtrations

$$G_* : \quad 0 = G_0 \subset \cdots \subset G_g \subset \cdots \subset G_{2g} = G.$$

We define a map

$$\tilde{\Psi} : \quad \{G_1, \cdots, G_{2g}\} \longrightarrow \{G_1, \cdots, G_{2g}\}$$

by sending G_i to

$$\begin{cases} V \cdot G_i = G_{\psi(i)} & \text{if } \psi(i) \neq 0, \\ F^{-1} \cdot G_i = G_{g+i-\psi(i)} = G_{g+i} & \text{if } \psi(i) = 0. \end{cases} \tag{3.1.1}$$

We get a non-empty subset

$$\mathcal{D} := \bigcap_{j=1}^{\infty} \mathrm{Im}\, \tilde{\Psi}^j$$

of the set $\{G_1, \cdots, G_{2g}\}$. Then $\tilde{\Psi}$ induces an automorphism

$$\Psi: \quad \mathcal{D} \longrightarrow \mathcal{D}.$$

Set $\mathcal{C} := \mathcal{D} \cap \{G_{g+1}, G_{g+2}, \cdots, G_{2g}\}$.

Definition 3.1. *The slope associated with* φ is the rational number

$$\lambda_\varphi = \sharp\mathcal{C}/\sharp\mathcal{D}. \tag{3.1.2}$$

Remark 3.2. One can regard $(\mathcal{D}, \Psi)$ as a pair of a subset of $\{1, \cdots, 2g\}$ and an automorphism on it, and the pair depends only on φ. Hence all combinatorial objects defined only from $(\mathcal{D}, \Psi)$ are independent of the choice of a final filtration.

Lemma 3.3. *$\mathcal{C} = \emptyset$ if and only if $\varphi(1) = 1$.*

Proof. If $\varphi(1) = 1$, then $\psi(i) \neq 0$ for all i. By definition this means $\Psi(H) = V \cdot H$ for any $H \in \mathcal{D}$. Thus $\mathcal{D} = V \cdot \mathcal{D} \subset \{G_1, \cdots, G_g\}$. Hence $\mathcal{C} = \emptyset$.

If $\varphi(1) = 0$, then $\psi(i) < i$ for $i = 1, \cdots, 2g$; hence some power of V kills G. Thus $\mathcal{D}$ contains an element of the form $F^{-1} \cdot G_i = G_{g+i}$ for some $i \geq 1$, i.e., $\mathcal{C} \neq \emptyset$. □

Suppose $\varphi(1) = 0$. For any subgroup scheme H of G, let $l(H)$ denote the least integer l such that $V^{l+1} \cdot H = 0$. Obviously for any two subgroup schemes H, H' with $H \subset H'$, we have $l(H) \leq l(H')$. We introduce an automorphism

$$\Phi: \quad \mathcal{C} \longrightarrow \mathcal{C} \tag{3.1.3}$$

which is defined by sending G_i to $F^{-1}V^{l(G_i)} \cdot G_i$. Note

$$\mathcal{D} = \{V^i \cdot H \mid H \in \mathcal{C},\ 0 \leq i \leq l(H)\}. \tag{3.1.4}$$

In particular we have $\sharp\mathcal{D} = \sum_{H \in \mathcal{C}} (l(H) + 1)$.

Lemma 3.4. *Assume $\varphi(1) = 0$. For $i \geq g + 1$, we have $l(G_i) \geq 1$.*

Proof. Since $\psi(g + 1) = g - (g - 1) + \varphi(g - 1) \geq 1$, we have $V \cdot G_{g+1} \neq 0$, i.e., $l(G_{g+1}) \geq 1$. By $G_{g+1} \subset G_i$, we have $l(G_i) \geq l(G_{g+1}) \geq 1$. □

Lemma 3.5. (1) $0 \leq \lambda_\varphi \leq 1/2$.

(2) $\lambda_\varphi = 0$ *if and only if* $\varphi(1) = 1$.

(3) $\lambda_\varphi = 1/2$ *if and only if* $\varphi([(g + 1)/2]) = 0$.

Proof. (2) Note $\lambda_\varphi = 0$ is equivalent to $\mathcal{C} = \emptyset$. Then this follows immediately from Lem. 3.3.

(1) It suffices to show that $\lambda_\varphi \leq 1/2$ for $\varphi(1) = 0$. Suppose $\varphi(1) = 0$. By Lem. 3.4, we have $l(H) \geq 1$ for every $H \in \mathcal{C}$. Then $\lambda_\varphi = \sharp\mathcal{C}/\sum_{H \in \mathcal{C}}(l(H) + 1)$ is at most 1/2.

(3) Suppose $\lambda_\varphi = 1/2$. Then we have $l(H) = 1$ for all $H \in \mathcal{C}$. Let G_i be the biggest element of $\mathcal{C}$. Note $F^{-1}V \cdot G_i = G_i$. Then we have $\psi(i) = i - g$. If $i \geq g + [(g + 1)/2]$, then we have $i - g \geq [(g + 1)/2]$ and $\varphi(i - g) = \text{length}\, V^2 \cdot G_i = 0$ by $l(G_i) = 1$. Thus $\varphi([(g + 1)/2]) = 0$. If $i < g + [(g + 1)/2]$, then $\varphi(2g - i) = \psi(i) + g - i = 0$ with $2g - i \geq [(g + 1)/2]$. Hence $\varphi([(g + 1)/2]) = 0$.

Conversely assume $\varphi([(g+1)/2]) = 0$. It suffices to show that $l(H) = 1$ for all $H \in \mathcal{C}$. Let H be an element of $\mathcal{C}$. Then there exists $H' \in \mathcal{D}$ such that $V \cdot H' = 0$ and $H = F^{-1} \cdot H'$. Set $j = \text{length}\, H'$, i.e., $H' = G_j$. Then $V \cdot H' = 0$ implies $\psi(j) = 0$. We also have $H = G_{g+j}$. Since $\varphi([(g+1)/2]) = 0$ implies $\psi(v) \le \max\{0, v - [(g+1)/2]\}$, we have

$$\begin{aligned} \psi(g+j) &= j + \psi(g-j) \\ &\le \begin{cases} j & \text{if } j \ge [g/2], \\ j + (g - j - [(g+1)/2]) = [g/2] & \text{if } j < [g/2]. \end{cases} \end{aligned}$$

By $\psi(j) = 0$ and $\psi([g/2]) = 0$, we have $\psi^2(g+j) = 0$ and therefore $V^2 \cdot H = G_{\psi^2(g+j)} = 0$. This means $l(H) = 1$. □

3.2 Ψ-sets and Φ-sets

Definition 3.6. (1) *A Ψ-set in $\mathcal{D}$ (or simply in φ)* is a subset of $\mathcal{D}$ which is stable under the action of the group generated by Ψ. We call $\mathcal{D}$ the *full* Ψ-set.

(2) *A Ψ-cycle in $\mathcal{D}$* is an orbit in $\mathcal{D}$ under the action of the group generated by Ψ.

Definition 3.7. (1) *A Φ-set in $\mathcal{C}$ (or simply in φ)* is a subset of $\mathcal{C}$ which is stable under the action of the group generated by Φ. We call $\mathcal{C}$ the *full* Φ-set.

(2) *A Φ-cycle in $\mathcal{C}$* is an orbit in $\mathcal{C}$ under the action of the group generated by Φ.

For a Φ-set $\mathcal{P}$ in $\mathcal{C}$, we get a Ψ-set

$$\mathcal{Q}_{\mathcal{P}} := \{V^i \cdot H \mid H \in \mathcal{P},\ 0 \le i \le l(H)\},$$

which is called *the Ψ-set associated with* $\mathcal{P}$. Conversely for a Ψ-set $\mathcal{Q}$ in $\mathcal{D}$, we get a Φ-set

$$\mathcal{P}_{\mathcal{Q}} := \mathcal{Q} \cap \mathcal{C} = \mathcal{Q} \cap \{G_{g+1}, \cdots, G_{2g}\},$$

which is called *the Φ-set associated with* $\mathcal{Q}$. Thus there is a canonical bijection from the set of Ψ-sets to the set of Φ-sets.

Let $\mathcal{Q}$ be a Ψ-set. We denote by $e(\mathcal{Q})$ the cardinal number of the set of Ψ-cycles in $\mathcal{Q}$ and set

$$e_\varphi = e(\mathcal{D}). \tag{3.2.1}$$

Since $\mathcal{D} \neq \emptyset$, we have $e_\varphi \ge 1$. Let $\mathcal{P}$ be the Φ-set associated with $\mathcal{Q}$. Set $c(\mathcal{Q}) := \sharp\mathcal{P}$ and $d(\mathcal{Q}) := \sharp\mathcal{Q}$ and put $c = c(\mathcal{D})$ and $d = d(\mathcal{D})$. Note $\lambda_\varphi = c/d$. Since $0 \le \lambda_\varphi \le 1/2$, there are non-negative integers m_φ and n_φ such that

$$\lambda_\varphi = n_\varphi/(m_\varphi + n_\varphi) \tag{3.2.2}$$

with $\gcd(m_\varphi, n_\varphi) = 1$ and $m_\varphi \ge n_\varphi$.

We write $\mathcal{D}$ as $\{I_1, \cdots, I_d\}$ with $I_1 \subset \cdots \subset I_d$. Note $I_i \in \mathcal{C} \Leftrightarrow d - c < i \le d$. Since $V(I_i) = 0 \Leftrightarrow 1 \le i \le c$, we have $\Psi(I_i) \subset \Psi(I_j)$ if $c < i \le j$ or $i \le j \le c$ and $\Psi(I_i) \supset \Psi(I_j)$ if $i \le c < j$. Thus

$$\Psi(I_i) = \begin{cases} I_{i+d-c} & \text{if } \ i \le c, \\ I_{i-c} & \text{if } \ i > c. \end{cases}$$

We define a bijective map

$$\tau : \quad \mathcal{D} \longrightarrow \mathbb{Z}/d\mathbb{Z}$$

by sending I_i to the class of $i-1$. Then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\tau} & \mathbb{Z}/d\mathbb{Z} \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle -c} \\ \mathcal{D} & \xrightarrow[\tau]{} & \mathbb{Z}/d\mathbb{Z}. \end{array}$$

Then clearly we obtain

Proposition 3.8. (1) *$e_\varphi = \gcd(c,d)$ and therefore $c = e_\varphi n_\varphi$ and $d = e_\varphi(m_\varphi + n_\varphi)$;*

(2) *for any Ψ-cycle Q in $\mathcal{D}$, there is an integer i with $0 \le i < e_\varphi$ such that $\tau(Q) = (i + e_\varphi\mathbb{Z})/d\mathbb{Z}$.*

Corollary 3.9. *For any Ψ-set $\mathcal{Q}$ in $\mathcal{D}$, we have $c(\mathcal{Q}) = e(\mathcal{Q})n_\varphi$ and $d(\mathcal{Q}) = e(\mathcal{Q})(m_\varphi + n_\varphi)$. In particular $e(\mathcal{Q}) = \gcd(c(\mathcal{Q}), d(\mathcal{Q}))$.*

Proof. It suffices to show that for any Ψ-cycle Q in $\mathcal{Q}$ we have $\sharp Q = m_\varphi + n_\varphi$ and $\sharp P = n_\varphi$, where P is the Φ-cycle associated with Q. By Prop. 3.8 (2), we have $\tau(Q) = (i + e_\varphi\mathbb{Z})/d\mathbb{Z}$ for some $0 \le i < e_\varphi$. Hence $\sharp Q = d/e_\varphi = m_\varphi + n_\varphi$ and $\sharp P = \sharp\{j \mid d - c \le i + e_\varphi j < d\} = c/e_\varphi = n_\varphi$. □

Corollary 3.10. $e_\varphi \le [g/m_\varphi]$.

Proof. Since $\mathcal{D} \setminus \mathcal{C} \subset \{G_1, \cdots, G_g\}$, we have an inequality $d - c \le g$. By Prop. 3.8 (1) we get $e_\varphi m_\varphi \le g$. □

Definition 3.11. Let $\mathcal{Q}$ be a Ψ-set in $\mathcal{D}$.

(1) *The absolute shape of* $\mathcal{Q}$ is the subset of $\mathbb{Z}^2$ defined by

$$\mathrm{AS}(\mathcal{Q}) = \{(u,v) \mid \Psi(G_u) = G_v,\ G_u \in \mathcal{Q}\}.$$

(2) Let us write $\mathcal{Q}$ as $\{J_1, \cdots, J_{d(\mathcal{Q})}\}$ with $J_1 \subset \cdots \subset J_{d(\mathcal{Q})}$. *The relative shape of* $\mathcal{Q}$ is the subset of $\mathbb{Z}^2$ defined by

$$\mathrm{RS}(\mathcal{Q}) = \{(i,j) \mid \Psi(J_i) = J_j\}.$$

Proposition 3.12. *Let φ and φ' be two elementary sequences (possibly of different lengths). Let $\mathcal{Q}$ and $\mathcal{Q}'$ be Ψ-sets in φ and φ' respectively. We have $\lambda_\varphi = \lambda_{\varphi'}$ and $e(\mathcal{Q}) = e(\mathcal{Q}')$ if and only if $\mathrm{RS}(\mathcal{Q}) = \mathrm{RS}(\mathcal{Q}')$ as subsets of $\mathbb{Z}^2$.*

Proof. First note that $\lambda_\varphi = \lambda_{\varphi'}$ and $e(\mathcal{Q}) = e(\mathcal{Q}')$ is equivalent to $c(\mathcal{Q}) = c(\mathcal{Q}')$ and $d(\mathcal{Q}) = d(\mathcal{Q}')$ by Cor. 3.9.

If $\mathrm{RS}(\mathcal{Q}) = \mathrm{RS}(\mathcal{Q}')$, then we have $c(\mathcal{Q}) = c(\mathcal{Q}')$ and $d(\mathcal{Q}) = d(\mathcal{Q}')$. In fact $c(\mathcal{Q}) = \sharp\{(i,j) \in \mathrm{RS}(\mathcal{Q}) \mid i < j\}$ and $d(\mathcal{Q}) = \sharp\,\mathrm{RS}(\mathcal{Q})$.

Conversely assume $c(\mathcal{Q}) = c(\mathcal{Q}')$ and $d(\mathcal{Q}) = d(\mathcal{Q}')$. Let $J_1 \subset \cdots \subset J_{d(\mathcal{Q})}$ be as in Def. 3.11 (2). Consider the bijection

$$\tau_\mathcal{Q} : \quad \mathcal{Q} \longrightarrow \mathbb{Z}/d(\mathcal{Q})\mathbb{Z}$$

sending J_j to the class of $j-1$. By the definition of Ψ, we have $\Psi(J_j) = J_{j+d(\mathcal{Q})-c(\mathcal{Q})}$ for $j \le c(\mathcal{Q})$ and $\Psi(J_j) = J_{j-c(\mathcal{Q})}$ for $j > c(\mathcal{Q})$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\tau_{\mathcal{Q}}} & \mathbb{Z}/d(\mathcal{Q})\mathbb{Z} \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle -c(\mathcal{Q})} \\ \mathcal{Q} & \xrightarrow[\tau_{\mathcal{Q}}]{} & \mathbb{Z}/d(\mathcal{Q})\mathbb{Z}. \end{array}$$

Thus $\mathrm{RS}(\mathcal{Q})$ depends only on $c(\mathcal{Q})$ and $d(\mathcal{Q})$, and therefore $\mathrm{RS}(\mathcal{Q}) = \mathrm{RS}(\mathcal{Q}')$. □

3.3 Explicit description of Φ-cycles

In this subsection we assume $\varphi(1) = 0$. Let $m = m_\varphi$ and $n = n_\varphi$. By the assumption, we have $\lambda_\varphi = n/(m+n) > 0$ and therefore $n > 0$.

Let P be a Φ-cycle. By Cor. 3.9 the cardinal number of P is equal to n and $\sum_{H\in P}(l(H)+1) = m+n$. We write P as $\{L_1, \cdots, L_n\}$ with $L_1 \subset L_2 \subset \cdots \subset L_n$ and define a bijective map

$$\varpi : \quad P \longrightarrow \mathbb{Z}/n\mathbb{Z}$$

by sending L_i to the class of $i-1$. Our aim is to describe Φ as an automorphism of $\mathbb{Z}/n\mathbb{Z}$.

Lemma 3.13. *We have $|l(G_i) - l(G_j)| \le 1$ for any $i, j \ge g$.*

Proof. By the definition of l, we have $l(G_g) = l(V \cdot G_{2g}) = l(G_{2g}) - 1$. Then this lemma follows immediately from $l(G_g) \le l(G_i) \le l(G_{2g})$ for all $g \le i \le 2g$. □

For $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the smallest integer $\ge x$.

Lemma 3.14. *For any $H \in P$, we have $l(H) \in \{\lceil m/n \rceil - 1, \lceil m/n \rceil\}$. At least one $H \in P$ has $l(H) = \lceil m/n \rceil$.*

Proof. First note $|l(H) - l(H')| \le 1$ for any $H, H' \in P$ by Lem. 3.13. Let $y = \max\{l(H) | H \in P\}$. Then we have

$$m = \sum_{H\in P} l(H) \le ny < \sum_{H \in P}(l(H)+1) = m+n.$$

Hence $y = \lceil m/n \rceil$. □

Let $P_1 = \{H \in P \,|\, l(H) = \lceil m/n \rceil\}$ and $P_2 = P \setminus P_1$ and set $n_1 = \sharp P_1$ and $n_2 = \sharp P_2$. Then we have

$$P_1 = \{L_{n_2+1}, L_{n_2+1}, \cdots, L_n\} \quad \text{and} \quad P_2 = \{L_1, L_1, \cdots, L_{n_2}\}.$$

Lemma 3.15. (1) *$n_1 = m + n - n\lceil m/n \rceil$ and $n_2 = n\lceil m/n \rceil - m$.*

(2) *$l(L_i) = \lceil (m-n+i)/n \rceil$.*

Proof. (1) Note $m + n = \sum_{H\in P}(l(H)+1) = n_1(\lceil m/n \rceil + 1) + n_2(\lceil m/n \rceil)$. By $n = n_1 + n_2$, we have $m + n = n_1 + n\lceil m/n \rceil$. Hence $n_1 = m + n - n\lceil m/n \rceil$. Then $n_2 = n - n_1 = n\lceil m/n \rceil - m$.

(2) Set $l'(i) := \lceil (m-n+i)/n \rceil$. Then for $0 \le j < n_2$ we have

$$l'(n_2 - j) = \lceil (m - n + n_2 - j)/n \rceil = \lceil \lceil m/n \rceil - 1 - (j/n) \rceil = \lceil m/n \rceil - 1,$$

and for $1 \le j \le n_1$ we have

$$l'(n_2 + j) = \lceil (m - n + n_2 + j)/n \rceil = \lceil \lceil m/n \rceil - 1 + (j/n) \rceil = \lceil m/n \rceil.$$

These mean $l(L_i) = l'(i)$ for all $1 \le i \le n$. □

Proposition 3.16. *We have the commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{\varpi} & \mathbb{Z}/n\mathbb{Z} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle +m} \\ P & \xrightarrow[\varpi]{} & \mathbb{Z}/n\mathbb{Z}. \end{array}$$

Proof. First for $L, L' \in P_i$ $(i = 1, 2)$ with $L \subset L'$, we have $\Phi(L) \subset \Phi(L')$. Secondly let $L \in P_1$ and $L' \in P_2$. Obviously $L \supset L'$. We claim $\Phi(L) \subset \Phi(L')$. Indeed $V \cdot L \subset V \cdot G = G_g \subset L'$ and therefore $\Phi(L) = F^{-1}V^{\lceil m/n \rceil} \cdot L = F^{-1}V^{\lceil m/n \rceil - 1} \cdot (V \cdot L)$ is contained in $F^{-1}V^{\lceil m/n \rceil - 1} \cdot L' = \Phi(L')$. Hence we have

$$\Phi(L_j) = \begin{cases} L_{j+n_1} & \text{if } j \le n_2, \\ L_{j-n_2} & \text{if } j > n_2. \end{cases}$$

Since $n_1 \equiv -n_2 \equiv m \pmod{n}$ by Lem. 3.15 (1), we have the required commutative diagram. □

We denote by $H_0 = H_0(P)$ the biggest element of P. Set

$$H_i = \Phi^i(H_0) \quad \text{and} \quad l_i = l(H_i) \quad \text{for} \quad i \in \mathbb{Z}_{\ge 0}. \tag{3.3.1}$$

Note $P = \{H_0, \cdots, H_{n-1}\}$ and $H_i = H_{i+n}$ for all $i \ge 0$. We define natural numbers r_i $(i \in \mathbb{Z})$ by

$$r_i \equiv im \pmod{n} \qquad \text{and} \qquad 0 < r_i \le n. \tag{3.3.2}$$

By using these we can describe some basic properties on Φ-cycles:

Corollary 3.17. *Let P, L_i, H_i and l_i be as above. We have*

(1) $H_i = L_{r_i}$ *for* $i \in \mathbb{Z}_{\ge 0}$*;*

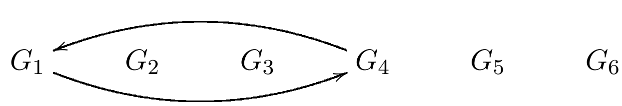
(2) $l_i = \lceil (m - n + r_i)/n \rceil$ *for* $i \in \mathbb{Z}_{\ge 0}$.

Proof. Since $\varpi_j(H_0)$ is the class of $n - 1$, the class of $\varpi_j(H_i)$ is given by $r_i - 1$ for $i \in \mathbb{Z}_{\ge 0}$ by Prop. 3.16, i.e., $H_i = L_{r_i}$. Then from Lem. 3.15 (2), we get $l_i = l(L_{r_i}) = \lceil (m - n + r_i)/n \rceil$. □

Remark 3.18. In order to compute $\lambda_\varphi = n/(m + n)$ for a given φ, it suffices to find a Φ-cycle P and calculate $\sharp P$ and l_i $(i = 0, \cdots, n - 1)$. In fact $n = \sharp P$ and $m = \sum_{i=0}^{n-1} l_i$.

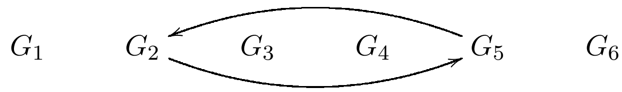
Here we give some examples.

Example 3.19. (1) Let $g = 3$. Consider $\varphi = (0, 0, 1)$. Then $\psi = (0, 0, 1; 1, 2, 3)$.



We have a Φ-cycle with $H_0 = G_4$. Indeed $V \cdot G_4 = G_{\psi(4)} = G_1$ and $V^2 \cdot G_4 = V \cdot G_1 = G_{\psi(1)} = 0$. Also $F^{-1}V \cdot G_4 = F^{-1} \cdot G_1 = G_4$.

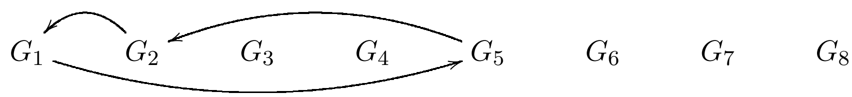
There is another Φ-cycle:



Indeed put $H_0 = G_5$. Then $V \cdot G_5 = G_{\psi(5)} = G_2$ and $V^2 \cdot G_g = V \cdot G_2 = G_{\psi(2)} = 0$. Also $F^{-1}V \cdot G_5 = F^{-1} \cdot G_2 = G_5$.

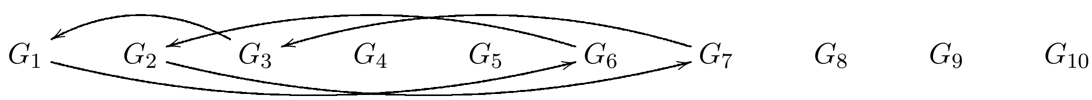
Note these two give all Φ-cycles in G_*, i.e., $e_\varphi = 2$. We have $n = 1$ and $l_0 = 1$, Thus $\lambda_\varphi = 1/2$.

(2) Let $\varphi = (0, 1, 1, 2)$. Then $\psi = (0, 1, 1, 2; 2, 3, 3, 4)$.



We have a unique Φ-cycle with $H_0 = G_5$, i.e., $e_\varphi = 1$. In this case, $l_0 = 2$; hence we have $n = 1$, $m = 2$; thus $\lambda_\varphi = 1/3$.

(3) Let $\varphi = (0, 0, 1, 1, 2)$. Then $\psi = (0, 0, 1, 1, 2; 2, 3, 3, 4, 5)$.



We have a unique Φ-cycle: $(H_0, H_1) = (G_7, G_6)$, i.e., $e_\varphi = 1$. In this case, $l_0 = 2$ and $l_1 = 1$. Thus we have $n = 2$, $m = 3$ and $\lambda_\varphi = 2/5$.

(4) Let $\varphi = (0, 0, 1, 2, 2, 3)$. Then $\psi = (0, 0, 1, 2, 2, 3; 3, 4, 4, 4, 5, 6)$. There are two Φ-cycles, i.e., $e_\varphi = 2$. Those are given by $H_0 = G_7$ and $H_0 = G_8$. In the both cases, we have $l_0 = 2$; hence $\lambda_\varphi = 1/3$.

(5) We can check that there are two Φ-cycles for $\varphi = (0, 0, 0, 0, 1, 2, 2, 2, 2, 3)$. Those are given by $(H_0, H_1) = (G_{13}, G_{11})$ and $(H_0, H_1) = (G_{14}, G_{12})$. In the both cases, we have $l_0 = 2$ and $l_1 = 1$. Thus $\lambda_\varphi = 2/5$ and $e_\varphi = 2$.

4 Main results

Let φ be an elementary sequence. The main purpose of this paper is to show

Theorem 4.1. *Any generic point of S_φ has the Newton slopes $\lambda_i = \lambda_\varphi$ for all $1 \leq i \leq e_\varphi$. (All the other Newton slopes are $\geq \lambda_\varphi$, see (2.2.3).)*

As an obvious conclusion of this theorem, we have

Corollary 4.2. *$S_\varphi \subset Z_\lambda$ if and only if $\lambda_\varphi \geq \lambda$.*

Proof. Suppose $S_\varphi \subset Z_\lambda$. By the definition of Z_λ, Th. 4.1 implies $\lambda_\varphi \geq \lambda$. Conversely suppose $\lambda_\varphi \geq \lambda$. Clearly $Z_{\lambda_\varphi} \subset Z_\lambda$. Since Z_{λ_φ} is closed by Grothendieck and Katz [4], Th. 2.3.1, we have $S_\varphi \subset Z_{\lambda_\varphi}$ by Th. 4.1. □

By Lem. 3.5 (3) and the fact $W_\sigma = Z_{1/2}$, this corollary can be viewed as a generalization of Oort's result:

$$S_\varphi \subset W_\sigma \iff \varphi([(g+1)/2]) = 0. \tag{4.0.3}$$

5 Decompositions of symmetric BT_1's

The first aim of this section is to give a criterion for the existence of decompositions of symmetric BT_1's in terms of self-dual (V,F^{-1})-subsets. After that, we shall apply the criterion to symmetric BT_1's which have a Ψ-set making a (V,F^{-1})-subset. Then we will obtain a key result (Cor. 5.26), where we shall see that these symmetric BT_1's have direct summands which come from minimal p-divisible groups.

5.1 (V,F^{-1})-cycles and self-dual (V,F^{-1})-subsets

Let us recall the notion of (V,F^{-1})-cycles (cf. [7], (2.5)), which is different from that of Ψ-cycles, and introduce the notion of self-dual (V,F^{-1})-subsets.

Let φ be an elementary sequence of length g and ψ its stretched final sequence. Choose a symmetric BT_1 G with $\mathrm{ES}(G)=\varphi$ and a final filtration G_* as in (2.3.3). Set

$$\Gamma=\{G_i/G_{i-1}\,|\,i=1,2,\cdots,2g\}. \tag{5.1.1}$$

There exists a bijection

$$\pi:\quad \Gamma\longrightarrow\Gamma \tag{5.1.2}$$

defined by sending G_i/G_{i-1} to

$$\begin{cases} V\cdot G_i/V\cdot G_{i-1}=G_{\psi(i)}/G_{\psi(i-1)} & \text{if } \psi(i-1)<\psi(i),\\ F^{-1}\cdot G_i/F^{-1}\cdot G_{i-1}=G_{g+i-\psi(i)}/G_{g+i-1-\psi(i-1)} & \text{if } \psi(i-1)=\psi(i). \end{cases} \tag{5.1.3}$$

Definition 5.1. (1) *A (V,F^{-1})-subset* is a subset of Γ which is stable under the action of the group generated by π. We call Γ the *full* (V,F^{-1})-subset.

(2) *A (V,F^{-1})-cycle* is an orbit in Γ under the action of the group generated by π.

In [7], (5.1), an operation $\perp$ on the set of subgroup schemes of G was introduced. Recall it satisfies $G_i^\perp=G_{2g-i}$ and therefore it defines an operation on Γ, denoted by the same symbol $\perp$, by sending $\gamma=G_i/G_{i-1}$ to $\gamma^\perp=G_{i-1}^\perp/G_i^\perp=G_{2g+1-i}/G_{2g-i}$. Since $\perp$ on Γ satisfies

$$\pi\cdot\perp=\perp\cdot\pi, \tag{5.1.4}$$

the operation $\perp$ respects the disjoint union into (V,F^{-1})-cycles.

Definition 5.2. A (V,F^{-1})-subset Λ is said to be *self-dual* if $\Lambda=\Lambda^\perp$.

From now on, let Λ be a self-dual (V,F^{-1})-subset. Note $\sharp\Lambda$ is even because $\gamma^\perp\neq\gamma$ for all $\gamma\in\Gamma$. Set $g_\Lambda=\sharp\Lambda/2$. Let us write $\{u\in\{1,\cdots,2g\}\,|\,G_u/G_{u-1}\in\Lambda\}$ as $\{u_1,u_2,\cdots,u_{2g_\Lambda}\}$ with $u_1<\cdots<u_{2g_\Lambda}$. By the self-duality of Λ, we have

$$u_i+u_{2g_\Lambda+1-i}=2g+1. \tag{5.1.5}$$

Definition 5.3. We associate to Λ an elementary sequence φ_Λ of length g_Λ defined by

$$\varphi_\Lambda(i)=\sharp\{u\in\{u_1,\cdots,u_i\}\,|\,\varphi(u-1)<\varphi(u)\} \tag{5.1.6}$$

for $i=1,\cdots,g_\Lambda$. We call φ_Λ *the type of* Λ.

Note the final sequence ψ_Λ stretched from φ_Λ satisfies

$$\psi_\Lambda(i) = \sharp\{u \in \{u_1, \cdots, u_i\} \,|\, \psi(u-1) < \psi(u)\} \tag{5.1.7}$$

for all $i = 1, \cdots, 2g_\Lambda$. For convenience we set $u_0 = 0$. Then one can check

$$V \cdot G_{u_i} = G_{u_{\psi_\Lambda(i)}} \quad \text{and} \quad F^{-1} \cdot G_{u_i} = G_{u_{g_\Lambda + i - \psi_\Lambda(i)}} \quad \text{for} \quad 0 \le i \le 2g_\Lambda. \tag{5.1.8}$$

Definition 5.4. *The relative shape of* Λ is the subset of $\mathbb{Z}^2$ defined by

$$\mathrm{rs}(\Lambda) = \{(i,j) \,|\, \pi(G_{u_i}/G_{u_i - 1}) = G_{u_j}/G_{u_j - 1}\}.$$

Proposition 5.5. *We have*

$$\varphi_\Lambda(i') = \sharp\{(i,j) \in \mathrm{rs}(\Lambda) \,|\, i \le i' \text{ and } i \ge j\} \quad \text{for} \quad 1 \le i' \le g_\Lambda. \tag{5.1.9}$$

In particular the type φ_Λ of Λ is determined by $\mathrm{rs}(\Lambda)$.

Proof. Comparing (5.1.6) and (5.1.9), it suffices to show
Claim: for $(i,j) \in \mathrm{rs}(\Lambda)$ with $i \le g_\Lambda$, the condition $i \ge j$ is equivalent to $\varphi(u_i - 1) < \varphi(u_i)$.

First note that by (5.1.5) the condition $i \le g_\Lambda$ is equivalent to $u_i \le g$, and by definition the condition $(i,j) \in \mathrm{rs}(\Lambda)$ is equivalent to

$$\begin{cases} u_j = \varphi(u_i) & \text{if } \varphi(u_i - 1) < \varphi(u_i), \\ u_j = g + u_i - \varphi(u_i) & \text{if } \varphi(u_i - 1) = \varphi(u_i). \end{cases}$$

Proof of Claim: If $\varphi(u_i - 1) < \varphi(u_i)$, then we have $u_j = \varphi(u_i)$. In particular we get $u_i \ge u_j$ and therefore $i \ge j$. Conversely assume $i \ge j$. If $u_j = g + u_i - \varphi(u_i)$ holds, then we have $u_j \ge u_i$ and therefore $i \le j$. Hence if $i > j$, then $\varphi(u_i - 1) < \varphi(u_i)$ has to hold. The remaining case is the case of $i = j$. Let us suppose $i = j$ and show $\varphi(u_i - 1) < \varphi(u_i)$. If $\varphi(u_i - 1) = \varphi(u_i)$, then we have $u_i = g + u_i - \varphi(u_i)$, i.e., $\varphi(u_i) = g$. Hence we get $g = \varphi(u_i) = \varphi(u_i - 1) \le u_i - 1$, i.e., $u_i \ge g + 1$, which contradicts $i \le g_\Lambda$. □

5.2 Construction of decompositions

Let k be an algebraically closed field in characteristic p. All BT_1's considered in this subsection will be over k.

Let φ be an elementary sequence of length g and ψ the final sequence stretched from φ. Let G be a symmetric BT_1 with elementary sequence φ. For two symmetric BT_1's G' and G'', the direct sum $G' \oplus G''$ becomes a symmetric BT_1 canonically. In this subsection we give a sufficient condition for $G \simeq G' \oplus G''$ as symmetric BT_1's.

We begin by recalling the construction ([7], (9.1)) of the quasi-polarized Dieudonné module A_φ such that $\mathbb{D}(G) \simeq A_\varphi$. For this we consider the sets

$$\begin{aligned} \{1 \le i \le 2g \,|\, \psi(i-1) < \psi(i)\} &= \{\mathfrak{m}_1 < \cdots < \mathfrak{m}_g\}, \\ \{1 \le i \le 2g \,|\, \psi(i-1) = \psi(i)\} &= \{\mathfrak{n}_g < \cdots < \mathfrak{n}_1\}. \end{aligned}$$

Note $\mathfrak{m}_i + \mathfrak{n}_i = 2g+1$ for $1 \le i \le g$. First A_φ is the k-vector space of dimension $2g$ generated by $Z_1, \cdots, Z_{2g}$. We put $X_i = Z_{\mathfrak{m}_i}$ and $Y_i = Z_{\mathfrak{n}_i}$ for $1 \le i \le g$. The operation $\mathcal{F}$ on A_φ is defined by

$$\mathcal{F}(X_i) = Z_i \quad \text{and} \quad \mathcal{F}(Y_i) = 0 \quad \text{for} \quad 1 \le i \le g$$

and the quasi-polarization on A_φ is the alternating pairing defined by

$$\langle X_i, Y_j\rangle = \delta_{ij}, \quad \langle X_i, X_j\rangle = 0, \quad \langle Y_i, Y_j\rangle = 0 \quad \text{for} \quad 1 \le i,j \le g.$$

Note these determine the operation $\mathcal{V}$ on A_φ, in fact

$$\mathcal{V}(Z_i) = 0 \quad \text{and} \quad \mathcal{V}(Z_{2g-i+1}) = \pm Y_i \quad \text{for} \quad 1 \le i \le g$$

where $\mathcal{V}(Z_{2g-i+1}) = +Y_i$ if $Z_{2g-i+1} \in \{Y_1, \cdots, Y_g\}$ and $\mathcal{V}(Z_{2g-i+1}) = -Y_i$ if $Z_{2g-i+1} \in \{X_1, \cdots, X_g\}$.

Note

$$Z_z \in \{X_1, \cdots, X_g\} \quad \Longrightarrow \quad Z_z = X_{\psi(z)} \quad \text{for} \quad 1 \le z \le 2g, \tag{5.2.1}$$

since $\psi(\mathfrak{m}_i) = i$ for $1 \le i \le g$.

We put $A_{\varphi,i} := k\langle Z_1, \cdots, Z_i\rangle$ for $0 \le i \le 2g$. Then $A_{\varphi,i}$ is a $k[\mathcal{F}, \mathcal{V}]$-submodule of A_φ. Thus we have a filtration $0 = A_{\varphi,0} \subset A_{\varphi,1} \subset \cdots \subset A_{\varphi,2g} = A_\varphi$. Note this yields a final filtration $0 \subset G_1 \subset \cdots \subset G_{2g} = G$. Let $\Gamma = \{G_i/G_{i-1} \,|\, 1 \le i \le 2g\}$.

Proposition 5.6. *Assume Γ is decomposed as $\Gamma = \Lambda' \sqcup \Lambda''$ for some self-dual (V, F^{-1})-subsets Λ' and Λ''. Set $\varphi' := \varphi_{\Lambda'}$ and $\varphi'' := \varphi_{\Lambda''}$. Then there are two symmetric* BT_1*'s G' and G'' such that $G \simeq G' \oplus G''$ with* $\mathrm{ES}(G') = \varphi'$ *and* $\mathrm{ES}(G'') = \varphi''$*.*

Proof. It suffices to construct an isomorphism $A_\varphi \simeq A_{\varphi'} \oplus A_{\varphi''}$ as quasi-polarized Dieudonné modules. Let $\{Z_1, \cdots, Z_{2g}\} = \{X_1, \cdots, X_g\} \sqcup \{Y_1, \cdots, Y_g\}$ be the basis of A_φ as above.

Let ψ' and ψ'' be the final sequences stretched from φ' and φ'' respectively and set $g' = g_{\Lambda'}$ and $g'' = g_{\Lambda''}$. Let us write Λ' as $\{G_{u'_i}/G_{u'_i-1} \,|\, i = 1, \cdots, 2g'\}$ and Λ'' as $\{G_{u''_i}/G_{u''_i-1} \,|\, i = 1, \cdots, 2g''\}$. Set $Z'_i = Z_{u'_i}$ for all $1 \le i \le 2g'$ and define $\mathfrak{m}'_i$ and $\mathfrak{n}'_i$ by

$$\begin{aligned} \{1 \le i \le 2g' \,|\, \psi(u'_i - 1) < \psi(u'_i)\} &= \{\mathfrak{m}'_1 < \cdots < \mathfrak{m}'_{g'}\}, \\ \{1 \le i \le 2g' \,|\, \psi(u'_i - 1) = \psi(u'_i)\} &= \{\mathfrak{n}'_{g'} < \cdots < \mathfrak{n}'_1\}. \end{aligned}$$

Write $X'_i = Z'_{\mathfrak{m}'_i}$ and $Y'_i = Z'_{\mathfrak{n}'_i}$. Similarly we define Z''_i $(1 \le i \le 2g'')$, $\mathfrak{m}''_i, \mathfrak{n}''_i$ $(1 \le i \le g'')$ and X''_i, Y''_i $(1 \le i \le g'')$. From the assumption $\Gamma = \Lambda' \sqcup \Lambda''$, we get $\{X_1, \cdots, X_g\} = \{X'_1, \cdots, X'_{g'}\} \sqcup \{X''_1, \cdots, X''_{g''}\}$ and $\{Y_1, \cdots, Y_g\} = \{Y'_1, \cdots, Y'_{g'}\} \sqcup \{Y''_1, \cdots, Y''_{g''}\}$. Since $\psi'(\mathfrak{m}'_i) = i$ for $1 \le i \le g'$ by the definition of ψ', we have

$$Z'_u \in \{X'_1, \cdots, X'_{g'}\} \quad \Longrightarrow \quad Z'_u = X'_{\psi'(u)} \quad \text{for} \quad 1 \le u \le 2g'. \tag{5.2.2}$$

Put $A' := k\langle Z'_1, \cdots, Z'_{2g'}\rangle$ and $A'' := k\langle Z''_1, \cdots, Z''_{2g''}\rangle$. First we claim

$$\langle Z'_i, Z''_j\rangle = 0 \quad \text{for} \quad 1 \le i \le g' \text{ and } 1 \le j \le g'', \tag{5.2.3}$$

$$\langle X'_i, Y'_j\rangle = \delta_{ij}, \quad \langle X'_i, X'_j\rangle = 0, \quad \langle Y'_i, Y'_j\rangle = 0 \quad \text{for} \quad 1 \le i,j \le g', \tag{5.2.4}$$

$$\langle X''_i, Y''_j\rangle = \delta_{ij}, \quad \langle X''_i, X''_j\rangle = 0, \quad \langle Y''_i, Y''_j\rangle = 0 \quad \text{for} \quad 1 \le i,j \le g''. \tag{5.2.5}$$

Indeed since Λ' is self-dual, we have $u'_{\mathfrak{m}'_i} + u'_{\mathfrak{n}'_i} = 2g + 1$; hence $\langle X'_i, Y'_j\rangle = \langle Z'_{\mathfrak{m}'_i}, Z'_{\mathfrak{n}'_j}\rangle = \delta_{ij}$. Similarly $\langle X''_i, Y''_j\rangle = \delta_{ij}$. Then the others must be 0, because for each i $(1 \le i \le 2g)$ there is only one j $(1 \le j \le 2g)$ such that $\langle Z_i, Z_j\rangle \neq 0$.

Next we show that $\mathcal{F}A' \subset A'$ and $\mathcal{F}A'' \subset A''$. (Then we also have $\mathcal{V}A' \subset A'$ and $\mathcal{V}A'' \subset A''$ by using the quasi-polarizations.) If $\psi(u'_i) > \psi(u'_i - 1)$, then $Z_{u'_i} = X_{\psi(u'_i)}$ and $G_{\psi(u'_i)}/G_{\psi(u'_i)-1} \in \Lambda'$. Hence

$$\mathcal{F}(Z_{u'_i}) = Z_{\psi(u'_i)} \in A'. \tag{5.2.6}$$

If $\psi(u'_i) = \psi(u'_i - 1)$, then $Z_{u'_i} = Y_j$ for some j and therefore

$$\mathcal{F}(Z_{u'_i}) = 0 \in A'. \tag{5.2.7}$$

Thus we have $\mathcal{F}A' \subset A'$. Similarly we get $\mathcal{F}A'' \subset A''$.

The equations (5.2.6) and (5.2.7) are paraphrased as

$$\mathcal{F}(X'_j) = Z'_j \quad \text{and} \quad \mathcal{F}(Y'_j) = 0 \quad \text{for} \quad 1 \le j \le g'. \tag{5.2.8}$$

By definition, (5.2.4) and (5.2.8) say that A' is isomorphic to $A_{\varphi'}$. Similarly we have $A'' \simeq A_{\varphi''}$. Then by (5.2.3) we get an isomorphism $A_\varphi \simeq A_{\varphi'} \oplus A_{\varphi''}$ as required. □

For symmetric BT_1's G' and G'', let φ' and φ'' be the elementary sequences of G' and G'' respectively. We denote by $\varphi' \oplus \varphi''$ the elementary sequence of $G' \oplus G''$. Also for any elementary sequences $\varphi_1, \cdots, \varphi_e$, we define $\varphi_1 \oplus \cdots \oplus \varphi_e$ similarly.

Corollary 5.7. *Suppose Γ is decomposed as $\Gamma = \Lambda_1 \sqcup \cdots \sqcup \Lambda_e$ for some self-dual (V, F^{-1})-subsets Λ_i of Γ. Then we have $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_e$ with $\varphi_i = \varphi_{\Lambda_i}$.*

Remark 5.8. By Cor. 5.7, it is easy to get a decomposition $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_e$ as above for given φ. However unexpectedly it is so complicated in general to determine the explicit form of $\varphi_1 \oplus \cdots \oplus \varphi_e$ for given φ_i $(1 \le i \le e)$.

5.3 Minimal p-divisible groups

In this subsection we shall review the theory of minimal p-divisible groups, developed in [10].

Definition 5.9. For non-negative integers m, n with $\gcd(m, n) = 1$, we define a p-divisible group $H_{m,n}$ over $\mathbb{F}_p$ by

$$\mathbb{D}(H_{m,n}) = \bigoplus_{i=0}^{m+n-1} \mathbb{Z}_p x_i$$

with $\mathcal{F}, \mathcal{V}$ operations:

$$\mathcal{F}x_i = x_{i+n} \quad \text{and} \quad \mathcal{V}x_i = x_{i+m} \quad \text{for} \quad \forall i \in \mathbb{Z}_{\ge 0} \tag{5.3.1}$$

where x_i $(i \in \mathbb{Z}_{\ge m+n})$ are defined as satisfying $x_{i+m+n} = px_i$ for $i \in \mathbb{Z}_{\ge 0}$.

For an arbitrary perfect field K, the Dieudonné module $\mathbb{D}(H_{m,n} \otimes K)$ has a $W(K)$-basis $\{x_0, \cdots, x_{m+n-1}\}$ satisfying the equations (5.3.1), which is called *a minimal basis of* $\mathbb{D}(H_{m,n} \otimes K)$.

For a Newton polygon $\xi = \sum_{i=1}^t [m_i, n_i]$, we denote by $H(\xi)$ the p-divisible group

$$\bigoplus_{i=1}^{t} H_{m_i, n_i}. \tag{5.3.2}$$

Note the Newton polygon of $H(\xi)$ is equal to ξ.

Definition 5.10. A p-divisible group $\mathcal{G}$ is called *minimal* if there exist a Newton polygon ξ and an isomorphism from $\mathcal{G}$ to $H(\xi)$ over an algebraically closed field.

In the proof of the main theorem, we shall use

Theorem 5.11 (Oort, [10]). *Let $\mathcal{G}$ be a p-divisible group over an algebraically closed field k. If $\mathcal{G}[p] \simeq H(\xi)[p] \otimes k$, then $\mathcal{G} \simeq H(\xi) \otimes k$ over k.*

Let ξ be a symmetric Newton polygon. By [8], Prop. 3.7, there exists a principal quasi-polarization ζ on $H(\xi)$, which is unique up to isomorphism of $H(\xi)$. We set

$$\varphi_\xi := \mathrm{ES}(H(\xi)[p], \zeta[p]).$$

Lemma 5.12. *Let ξ be the Newton polygon $[m,n]+[n,m]$ with $\gcd(m,n)=1$ and $m \geq n \geq 0$. Put $g = m+n$. Let G be the symmetric BT_1 $(H(\xi)[p], \zeta[p])$. Then*

(1) $\varphi_\xi = (\underbrace{0,\cdots,0}_{n}, \underbrace{1,2,\cdots,m-n}_{m-n}, \underbrace{m-n,\cdots,m-n}_{n})$*;*

(2) *G has a unique final filtration $0 = G_0 \subset G_1 \subset \cdots \subset G_{2g} = G$;*

(3) *$Q_{m,n} := \{G_1, \cdots, G_m, G_{g+1}, \cdots, G_{g+n}\}$ is the unique Ψ-cycle;*

(4) $\lambda_{\varphi_\xi} = n/(m+n)$.

Proof. Set $N = \mathbb{D}(G)$. Let $\{x_0, \cdots, x_{g-1}\}$ and $\{y_0, \cdots, y_{g-1}\}$ be minimal bases of $\mathbb{D}(H_{m,n})$ and $\mathbb{D}(H_{n,m})$ respectively. We define a basis $\{z_1, \cdots, z_{2g}\}$ of N over $\mathbb{F}_p$ by

$$(z_1, \cdots, z_{2g}) = (x_{g-1}, \cdots, x_n, y_{g-1}, \cdots, y_m, x_{n-1}, \cdots, x_0, y_{m-1}, \cdots, y_0).$$

Set $N_i = \mathbb{F}_p\langle z_1, \cdots, z_i\rangle$ and let G_i be the subgroup scheme of G such that $\mathbb{D}(G_i) = N_i$. By using $\mathcal{F}x_i = x_{i+n}$, $\mathcal{V}x_i = x_{i+m}$, $\mathcal{F}y_i = y_{i+m}$ and $\mathcal{V}y_i = y_{i+n}$, one can check that

$$\mathcal{F}N_i = \begin{cases} 0 & \text{if } \ i \leq n, \\ N_{i-n} & \text{if } \ n < i \leq m, \\ N_{m-n} & \text{if } \ m < i \leq g, \\ N_{i-g+m-n} & \text{if } \ g < i \leq g+n, \\ N_m & \text{if } \ g+n < i \leq g+m, \\ N_{i-g} & \text{if } \ g+m < i \leq 2g, \end{cases} \qquad \mathcal{V}^{-1}N_i = \begin{cases} N_{g+i} & \text{if } \ i \leq n, \\ N_{g+n} & \text{if } \ n < i \leq m, \\ N_{g+i-m+n} & \text{if } \ m < i \leq g, \\ N_{2g-m+n} & \text{if } \ g < i \leq g+n, \\ N_{g+i-m} & \text{if } \ g+n < i \leq g+m, \\ N_{2g} & \text{if } \ g+m < i \leq 2g. \end{cases}$$

In particular this implies that $0 \subset G_1 \subset \cdots \subset G_{2g} = G$ is a final filtration and also (1) holds. (We will show later that G_* is the unique final filtration.)

Then since we have $\Psi(G_i) = G_{i+g}$ for $1 \leq i \leq n$, $\Psi(G_i) = G_{i-n}$ for $n < i \leq m$, $\Psi(G_i) = G_{i-g+m-n}$ for $g < i \leq g+n$ and $\Psi(G_i) = G_m$ for $g+n < i \leq g+m$, we get $\Psi(Q_{m,n}) = Q_{m,n}$, i.e., $Q_{m,n}$ is a Ψ-set. We denote by $P_{m,n}$ the Φ-set associated with $Q_{m,n}$. Then $P_{m,n} = \{G_{g+1}, \cdots, G_{g+n}\}$. Since $\sharp P_{m,n} = n$ and $\sharp Q_{m,n} = m+n$, we have $\lambda_{\varphi_\xi} = n/(m+n)$. Also since $e(Q_{m,n}) = \gcd(m,n) = 1$, the Ψ-set $Q_{m,n}$ is a Ψ-cycle. By $[g/m] = 1$, Cor. 3.10 implies that there is only one Ψ-cycle in φ_ξ.

Finally let us show that G_* is the unique final filtration. We claim that any element of $Q_{m,n}$ is written as $w \cdot G$ for some word w of V, F^{-1}. Indeed $V^2 \cdot G = V \cdot G_g = G_{m-n} \in Q_{m,n}$; then since $Q_{m,n}$ is a Ψ-cycle, every element of $Q_{m,n}$ is of the form $w \cdot G$ for some word w of V, F^{-1}. Put $U = \{u \,|\, G_u \in Q_{m,n}\}$. Since $U \cup \{2g - u \,|\, u \in U\} \cup \{g, 2g\} = \{i \in \mathbb{Z} \,|\, 1 \leq i \leq 2g\}$, any G_i $(1 \leq i \leq 2g)$ is either of the form $w \cdot G$ or of the form $(w \cdot G)^\perp$ for some word w of V, F^{-1}. $\square$

5.4 Dualized Ψ-sets

Let $\mathcal{Q}$ be a Ψ-set in $\mathcal{D}$. Let H_0 be the biggest element of $\mathcal{Q}$ and v_0 the integer satisfying $H_0 = G_{v_0}$. Set $H_i = \Phi^i(H_0)$ and $l_i = l(H_i)$ for any $i \in \mathbb{Z}_{\geq 0}$.

Lemma 5.13. *If $\lambda_\varphi < 1/2$, then we have $\psi(v_0) \leq 2g - v_0$.*

Proof. Since $H_0 = F^{-1}V^{l_{n-1}} \cdot H_{n-1}$, we have $\text{length}(V^{l_{n-1}} \cdot H_{n-1}) = v_0 - g$. In particular $\psi(v_0 - g) = 0$. By the assumption $\lambda_\varphi < 1/2$, Lem. 3.5 (3) implies $v_0 - g \leq [g/2]$ and therefore $v_0 \leq 3g - v_0$. Hence $\psi(v_0)$ is less than or equal to

$$\psi(3g - v_0) = \psi(2g - (v_0 - g)) = g - (v_0 - g) + \psi(v_0 - g) = 2g - v_0 + \psi(v_0 - g).$$

By $\psi(v_0 - g) = 0$, we have $\psi(v_0) \leq 2g - v_0$. □

We define a bijection

$$\wedge : \quad \{G_1, \cdots, G_{2g}\} \longrightarrow \{G_1, \cdots, G_{2g}\}$$

by sending G_i to $G_i^\wedge = G_{2g+1-i}$.

Lemma 5.14. (1) *$\mathcal{Q} \cap \mathcal{Q}^\wedge \neq \emptyset$ implies $\lambda_\varphi = 1/2$.*

(2) *For any $H \in \mathcal{Q} \cap \mathcal{Q}^\wedge$ we have $\Psi(H) = \Psi(H^\wedge)^\wedge$.*

Proof. (1) Assume $\mathcal{Q} \cap \mathcal{Q}^\wedge \neq \emptyset$ and $\lambda_\varphi < 1/2$. Let G_i be an element of $\mathcal{Q} \cap \mathcal{Q}^\wedge$, i.e., $G_i, G_{2g+1-i} \in \mathcal{Q}$. Let $j = \min\{i, 2g+1-i\}$. Note $j \leq g$ and $G_j, G_{2g+1-j} \in \mathcal{Q}$. Since $G_j \in \mathcal{Q} \setminus \mathcal{P}$, there is an element I of $\mathcal{Q}$ such that $G_j = V \cdot I$. By $I \subset G_{v_0}$, we have $G_j \subset V \cdot G_{v_0} = G_{\psi(v_0)}$. Thus $j \leq \psi(v_0)$. Hence $2g + 1 - j \geq 2g + 1 - \psi(v_0) \geq v_0 + 1$ by Lem. 5.13. By the definition of v_0, we have $G_{2g+1-j} \notin \mathcal{Q}$. This is a contradiction.

(2) If $\lambda_\varphi = 1/2$, then for $G_j \in \mathcal{Q}$, we have $\Psi(G_j) = G_{j-g}$ if $j > g$ and $\Psi(G_j) = G_{j+g}$ if $j \leq g$. Let $H = G_i \in \mathcal{Q} \cap \mathcal{Q}^\wedge$. If $i > g$, then $G_i^\wedge = G_{2g+1-i}$ with $2g + 1 - i \leq g$; hence $\Psi(G_i^\wedge) = G_{3g+1-i}$; thus $\Psi(G_i^\wedge)^\wedge = G_{2g+1-(3g+1-i)} = G_{i-g} = \Psi(G_i)$. If $i \leq g$, then $2g+1-i > g$; hence $\Psi(G_i^\wedge) = G_{g+1-i}$; thus $\Psi(G_i^\wedge)^\wedge = G_{2g+1-(g+1-i)} = G_{i+g} = \Psi(G_i)$. □

Thus we can define a map

$$\overline{\Psi} : \quad \mathcal{Q} \cup \mathcal{Q}^\wedge \longrightarrow \mathcal{Q} \cup \mathcal{Q}^\wedge$$

by sending $G_i \in \mathcal{Q}$ to $\Phi(G_i) \in \mathcal{Q}$ and $G_i \in \mathcal{Q}^\wedge$ to $\Phi(G_i^\wedge)^\wedge \in \mathcal{Q}^\wedge$.

Definition 5.15. The set $\overline{\mathcal{Q}} := \mathcal{Q} \cup \mathcal{Q}^\wedge$ is called *the dualized Ψ-set.*

Definition 5.16. Let $\mathcal{Q}$ be a Ψ-set in $\mathcal{D}$. Writing $\overline{\mathcal{Q}}$ as $\{B_1, \cdots, B_{\sharp\overline{\mathcal{Q}}}\}$ with $B_1 \subset \cdots \subset B_{\sharp\overline{\mathcal{Q}}}$. *The dualized relative shape of* $\mathcal{Q}$ is the subset of $\mathbb{Z}^2$ defined by

$$\overline{\mathrm{RS}}(\mathcal{Q}) = \{(i, j) \,|\, \overline{\Psi}(B_i) = B_j\}.$$

Proposition 5.17. *Assume $\lambda_\varphi < 1/2$. Then $\overline{\mathrm{RS}}(\mathcal{Q})$ is determined by* $\mathrm{RS}(\mathcal{Q})$.

Proof. Let $\mathcal{Q} = \{I_1, \cdots, I_{d(\mathcal{Q})}\}$ with $I_1 \subset \cdots \subset I_{d(\mathcal{Q})}$ and $\mathcal{Q}^\wedge = \{J_1, \cdots, J_{d(\mathcal{Q})}\}$ with $J_1 \subset \cdots \subset J_{d(\mathcal{Q})}$. Note $I_{d(\mathcal{Q})} = G_{v_0}$ and $J_1 = G^\wedge_{v_0}$. By Lem. 5.14 (1), we have $\mathcal{Q} \cap \mathcal{Q}^\wedge = \emptyset$. Thus $\sharp\overline{\mathcal{Q}} = 2d(\mathcal{Q})$. Let $B_1, \cdots, B_{2d(\mathcal{Q})}$ be as in Def. 5.16. We claim $(B_1, \cdots, B_{2d(\mathcal{Q})})$ is equal to

$$(I_1, \cdots, I_{d(\mathcal{Q})-c(\mathcal{Q})}, J_1, \cdots, J_{c(\mathcal{Q})}, I_{d(\mathcal{Q})-c(\mathcal{Q})+1}, \cdots, I_{d(\mathcal{Q})}, J_{c(\mathcal{Q})+1}, \cdots, J_{d(\mathcal{Q})}).$$

By definition, this proposition follows immediately from this claim. Let us show this claim. Since $I_i^\wedge = J_{d(\mathcal{Q})+1-i}$ for all $1 \le i \le d(\mathcal{Q})$, it suffices to show (1) $I_i \subset J_j$ for $1 \le i \le d(\mathcal{Q}) - c(\mathcal{Q})$ and $1 \le j \le d(\mathcal{Q})$ and (2) $J_j \subset I_i$ for $d(\mathcal{Q}) - c(\mathcal{Q}) < i \le d(\mathcal{Q})$ and $1 \le j \le c(\mathcal{Q})$.

(1) If $1 \le i \le d(\mathcal{Q}) - c(\mathcal{Q})$, then there is an element H of $\mathcal{Q}$ such that $I_i = V \cdot H$. For any $1 \le j \le d(\mathcal{Q})$, we have $V \cdot H \subset V \cdot G_{v_0} = G_{\psi(v_0)} \subset G_{2g-v_0} = G^\wedge_{v_0+1} \subset G^\wedge_{v_0} \subset J_j$.

(2) For $d(\mathcal{Q}) - c(\mathcal{Q}) < i \le d(\mathcal{Q})$, we have $G_{g+1} \subset I_i$. Hence if $1 \le j \le c(\mathcal{Q})$, we have $J_j = I^\wedge_{d(\mathcal{Q})+1-j} \subset G^\wedge_{g+1} = G_g \subset I_i$. □

5.5 Ψ-sets making (V, F^{-1})-subsets

Let φ be an elementary sequence and G a symmetric BT_1 with $\mathrm{ES}(G) = \varphi$. Choose a final filtration G_* of G. Let $\mathcal{C}$ and $\mathcal{D}$ be as in §3.1. Let $\mathcal{Q}$ be a Ψ-set in $\mathcal{D}$.

Definition 5.18. We say that $\mathcal{Q}$ *makes a* (V, F^{-1})*-subset* if $\pi(G_i/G_{i-1}) = G_j/G_{j-1}$ is equivalent to $\Psi(G_i) = G_j$ for every $G_i \in \mathcal{Q}$.

Note if $\mathcal{Q}$ makes a (V, F^{-1})-subset, the subset $\Gamma_\mathcal{Q} := \{G_i/G_{i-1} \,|\, G_i \in \mathcal{Q}\}$ of Γ is a (V, F^{-1})-subset.

Lemma 5.19. *A* Ψ*-set* $\mathcal{Q}$ *makes a* (V, F^{-1})*-subset if and only if* $\psi(i) > \psi(i-1)$ *for every* i *satisfying* $G_i \in \mathcal{Q}$ *and* $\psi(i) > 0$.

Proof. Compare the definitions of Ψ and π, see (3.1.1) and (5.1.3). □

Lemma 5.20. *Assume a* Ψ*-set* $\mathcal{Q}$ *makes a* (V, F^{-1})*-subset. Let* $\mathcal{Q} = Q_1 \sqcup \cdots \sqcup Q_{e(\mathcal{Q})}$ *be the decomposition into* Ψ*-cycles of* $\mathcal{Q}$*. Then* Q_j *makes a* (V, F^{-1})*-subset and we have* $\Gamma_\mathcal{Q} = \Gamma_{Q_1} \sqcup \cdots \sqcup \Gamma_{Q_{e(\mathcal{Q})}}$.

Proof. By Lem. 5.19, any Ψ-cycle in $\mathcal{Q}$ makes a (V, F^{-1})-subset, if $\mathcal{Q}$ makes a (V, F^{-1})-subset. The decomposition of $\Gamma_\mathcal{Q}$ is obvious. □

Note

$$(G_i/G_{i-1})^\perp = G^\perp_{i-1}/G^\perp_i = G_{2g+1-i}/G_{2g-i} = G^\wedge_i/G^\wedge_{i+1} \tag{5.5.1}$$

for $1 \le i \le 2g$, where we set $G^\wedge_{2g+1} = \{0\}$ for convenience. Hence we have

$$\Gamma^\perp_\mathcal{Q} = \{G_i/G_{i-1} \,|\, G_i \in \mathcal{Q}^\wedge\}. \tag{5.5.2}$$

Let $\Lambda_\mathcal{Q}$ denote the self-dual (V, F^{-1})-subset $\Gamma_\mathcal{Q} \cup \Gamma^\perp_\mathcal{Q} = \{G_i/G_{i-1} \,|\, G_i \in \overline{\mathcal{Q}}\}$ with $\overline{\mathcal{Q}} = \mathcal{Q} \cup \mathcal{Q}^\wedge$.

Lemma 5.21. *Let* $\mathcal{Q}$ *be a* Ψ*-set making a* (V, F^{-1})*-subset. Then* $\mathrm{rs}(\Lambda_\mathcal{Q}) = \overline{\mathrm{RS}}(\mathcal{Q})$.

Proof. It suffices to show

$$\pi(G_i/G_{i-1}) = G_j/G_{j-1} \iff \overline{\Psi}(G_i) = G_j$$

for any $G_i \in \overline{\mathcal{Q}}$. Since $\mathcal{Q}$ makes a (V, F^{-1})-subset, this equivalence for $G_i \in \mathcal{Q}$ is obvious by definition. Let us consider the case of $G_i \in \mathcal{Q}^\wedge$. Note $\pi(G_i/G_{i-1}) = G_j/G_{j-1}$ is equivalent to $\pi(G_i^\wedge/G_{i+1}^\wedge) = G_j^\wedge/G_{j+1}^\wedge$ by (5.1.4) and (5.5.1). Since $G_i^\wedge \in \mathcal{Q}$, this is equivalent to $\Psi(G_i^\wedge) = G_j^\wedge$. Clearly $\Psi(G_i^\wedge) = G_j^\wedge$ is equivalent to $\overline{\Psi}(G_i) := \Psi(G_i^\wedge)^\wedge = G_j$. □

Lemma 5.22. *Suppose $\lambda_\varphi < 1/2$. Let $\mathcal{Q}$ be a Ψ-set making a (V, F^{-1})-subset. Then $\Gamma_\mathcal{Q} \cap \Gamma_\mathcal{Q}^\perp = \emptyset$.*

Proof. Lem. 5.14 (1) says $\mathcal{Q} \cap \mathcal{Q}^\wedge = \emptyset$. Hence $\Gamma_\mathcal{Q} \cap \Gamma_\mathcal{Q}^\perp = \emptyset$ by (5.5.2). □

Lemma 5.23. *Suppose $\lambda_\varphi < 1/2$. Let $\mathcal{Q}$ be a Ψ-set making a (V, F^{-1})-subset. Then the type of $\Lambda_\mathcal{Q}$ is determined by $\mathrm{RS}(\mathcal{Q})$. (See Def. 5.3 for the definition of types.)*

Proof. By Prop. 5.5 the type of $\Lambda_\mathcal{Q}$ is determined by $\mathrm{rs}(\Lambda_\mathcal{Q})$. Lem. 5.21 says that $\mathrm{rs}(\Lambda_\mathcal{Q}) = \overline{\mathrm{RS}}(\mathcal{Q})$. In Prop. 5.17 we showed that $\overline{\mathrm{RS}}(\mathcal{Q})$ is determined by $\mathrm{RS}(\mathcal{Q})$. □

First we investigate the following special case:

Lemma 5.24. *Let m, n and ξ be as in Lem. 5.12. Consider the case of $\varphi = \varphi_\xi$. Let $Q_{m,n}$ be the Ψ-cycle in φ_ξ obtained in Lem. 5.12 (3). Then*

(1) *$Q_{m,n}$ makes a (V, F^{-1})-cycle;*

(2) *$\Gamma = \Lambda_{Q_{m,n}}$. In particular the type of $\Lambda_{Q_{m,n}}$ is φ_ξ.*

Proof. Let ψ_ξ be the final sequence stretched from φ_ξ. By Lem. 5.12 (1), ψ_ξ equals

$$(\underbrace{0, \cdots, 0}_{n}, \underbrace{1, \cdots, m-n}_{m-n}, \underbrace{\cdots, m-n}_{n}; \underbrace{m-n+1, \cdots, m}_{n}, \underbrace{\cdots, m}_{m-n}, \underbrace{m+1, \cdots, g}_{n}). \tag{5.5.3}$$

Recall $Q_{m,n} = \{G_1, \cdots, G_m, G_{g+1}, \cdots, G_{g+n}\}$. Note $G_i \in Q_{m,n}$ and $\psi_\xi(i) > 0$ if and only if $n+1 \le i \le m$ or $g+1 \le i \le g+n$; in this case we have $\psi_\xi(i-1) < \psi_\xi(i)$ by (5.5.3). Hence by Lem. 5.19, we obtain (1). Obviously we have $Q_{m,n} \sqcup Q_{m,n}^\wedge = \{G_1, \cdots, G_{2(m+n)}\}$ and therefore (2) holds. □

Let us return to the general situation.

Proposition 5.25. *Let φ be an elementary sequence with $\lambda_\varphi < 1/2$. Set $m = m_\varphi$ and $n = n_\varphi$. Let $\mathcal{Q}$ be a Ψ-set in φ. If $\mathcal{Q}$ makes a (V, F^{-1})-subset, then $\Lambda_\mathcal{Q}$ is of type $\varphi_\xi^{\oplus e(\mathcal{Q})}$ with $\xi = [m, n] + [n, m]$.*

Proof. Let $\mathcal{Q} = Q_1 \sqcup \cdots \sqcup Q_{e(\mathcal{Q})}$ be the decomposition into Ψ-cycles of $\mathcal{Q}$. By Lem. 5.20 and Lem. 5.22, we have $\Lambda_\mathcal{Q} = \Lambda_{Q_1} \sqcup \cdots \sqcup \Lambda_{Q_{e(\mathcal{Q})}}$ with $\Lambda_{Q_i} = \Gamma_{Q_i} \sqcup \Gamma_{Q_i}^\perp$. By Cor. 5.7, the type of $\Lambda_\mathcal{Q}$ is the direct sum of the types of $\Lambda_{Q_1}, \cdots, \Lambda_{Q_{e(\mathcal{Q})}}$.

It suffices to show that the type of Λ_{Q_i} $(i = 1, \cdots, e(\mathcal{Q}))$ is equal to φ_ξ. Let $Q_{m,n}$ be the Ψ-cycle in φ_ξ. By Lem. 5.24, $Q_{m,n}$ makes a Ψ-cycle and the type of $\Lambda_{Q_{m,n}}$ is φ_ξ. By Prop. 3.12, we have $\mathrm{RS}(Q_i) = \mathrm{RS}(Q_{m,n})$, since $\lambda_\varphi = n/(m+n) = \lambda_{\varphi_\xi}$ and $e(Q_i) = 1 = e(Q_{m,n})$. Hence by Lem. 5.23, the type of Λ_{Q_i} is equal to the type φ_ξ of $\Lambda_{Q_{m,n}}$. □

Corollary 5.26. *Let φ, m, n and $\mathcal{Q}$ be as in Prop. 5.25. If $\mathcal{Q}$ makes a (V, F^{-1})-subset, then there exists an elementary sequence φ' of length $g - e(\mathcal{Q})(m+n)$ such that*

$$\varphi = \varphi' \oplus \varphi_\xi^{\oplus e(\mathcal{Q})} \tag{5.5.4}$$

with $\xi = [m,n] + [n,m]$.

Proof. By Prop. 5.25 the type of $\Lambda_\mathcal{Q}$ is $\varphi_\xi^{\oplus e(\mathcal{Q})}$ with $\xi = [m,n] + [n,m]$. Set $\Lambda' := \Gamma \setminus \Lambda_\mathcal{Q}$. Obviously Λ' is a self-dual (V, F^{-1})-subset. Let φ' be the type of Λ'. Then Cor. 5.7 shows $\varphi = \varphi' \oplus \varphi_\xi^{\oplus e(\mathcal{Q})}$. □

6 Proof of the main theorem

In [2], Th. 11.5, Ekedahl and van der Geer proved that S_φ is irreducible if $\lambda_\varphi < 1/2$. Hence in order to prove Th. 4.1 for $\lambda_\varphi < 1/2$, it suffices to show

$$(\mathbf{A})\ S_\varphi \subset Z_{\lambda_\varphi} \quad \text{and} \quad (\mathbf{B})\ S_\varphi \cap Z^0_{\lambda_\varphi, e_\varphi} \neq \emptyset.$$

For $\lambda_\varphi = 1/2$, it suffices to show (A). Indeed $Z_{1/2}$ is the supersingular locus W_σ; hence (A) $S_\varphi \subset Z_{1/2}$ implies that any Newton slope of any generic point of S_φ is 1/2.

In this section, we shall always mean by a p-divisible group (resp. a BT_1) a p-divisible group (resp. a BT_1) over an algebraically closed field k.

6.1 Proof of (A)

Let φ be an elementary sequence and G a symmetric BT_1 with $\mathrm{ES}(G) = \varphi$. Choose a final filtration G_* of G. We set $m = m_\varphi$ and $n = n_\varphi$ (see (3.2.2) for the definition of m_φ and n_φ). Let P be a Φ-cycle and let H_0 be the biggest element of P. Set $H_i = \Phi^i(H_0)$ and $l_i = l(H_i)$ ($i \in \mathbb{Z}$).

The next proposition is the key for the proof of (A).

Proposition 6.1. *Assume $\varphi(1) = 0$. Then for any p-divisible group $\mathcal{G}$ over k with $G \simeq \mathcal{G}[p]$, we have*

$$V^{1+\sum_{j=j_0+1}^{j_1}(l_j+1)} \cdot \mathcal{G} \subset p^{j_1-j_0}\mathcal{G} \tag{6.1.1}$$

for all integers $j_1 \geq j_0 \geq 0$. In particular, for any $\alpha \in \mathbb{N}$ we have

$$V^{1+\alpha(m+n)} \cdot \mathcal{G} \subset p^{\alpha n}\mathcal{G}. \tag{6.1.2}$$

We prepare a general lemma in order to prove Prop. 6.1. For a $k[\mathcal{F}, \mathcal{V}]$-submodule S of $N = M/pM$, we define an A_k-submodule of M by $\langle\langle S \rangle\rangle := \{x \in M \mid (x \bmod p) \in S\}$.

Lemma 6.2. *We have*

(1) $\langle\langle \mathcal{V}^{-1}S \rangle\rangle = p^{-1}(\mathcal{F}\langle\langle S \rangle\rangle \cap pM)$. *In particular, if $\mathcal{F}S = 0$ then $\langle\langle \mathcal{V}^{-1}S \rangle\rangle = p^{-1}\mathcal{F}\langle\langle S \rangle\rangle$.*

(2) $\langle\langle \mathcal{F}S \rangle\rangle = \mathcal{F}\langle\langle S \rangle\rangle + pM$.

Proof. (1) follows from the direct calculation:

$$\begin{aligned} \langle\!\langle \mathcal{V}^{-1}S \rangle\!\rangle &= \{x \in M | (x \bmod p) \in \mathcal{V}^{-1}S\} = \{x \in M | (\mathcal{V}x \bmod p) \in S\} \\ &= \mathcal{V}^{-1}\{\mathcal{V}x \in \mathcal{V}M | (\mathcal{V}x \bmod p) \in S\} = \mathcal{V}^{-1}(\langle\!\langle S \rangle\!\rangle \cap \mathcal{V}M) = p^{-1}(\mathcal{F}\langle\!\langle S \rangle\!\rangle \cap pM). \end{aligned}$$

(2) By definition $\mathcal{F}\langle\!\langle S \rangle\!\rangle + pM$ is contained in $\langle\!\langle \mathcal{F}S \rangle\!\rangle$. Any element x of $\langle\!\langle \mathcal{F}S \rangle\!\rangle = \{x \in M | x \bmod p \in \mathcal{F}S\}$ is of the form $x = py + \mathcal{F}z$ for some $y \in M$ and for some $z \in \langle\!\langle S \rangle\!\rangle$. This means $\langle\!\langle \mathcal{F}S \rangle\!\rangle \subset \mathcal{F}\langle\!\langle S \rangle\!\rangle + pM$. □

Proof of Prop. 6.1. Set $M = \mathbb{D}(\mathcal{G})$ and $N = M/pM$. Put $E_j = \mathbb{D}(H_j)$, which is a $k[\mathcal{F}, \mathcal{V}]$-submodule of N. Note $\mathcal{F}^{l_j+1}E_j = 0$ and $E_{j+1} = \mathcal{V}^{-1}\mathcal{F}^{l_j}E_j$.

By Lem. 6.2, for any j, we can compute $\langle\!\langle E_{j+1} \rangle\!\rangle = \langle\!\langle \mathcal{V}^{-1}\mathcal{F}^{l_j}E_j \rangle\!\rangle$ as

$$p^{-1}\mathcal{F}(\mathcal{F}^{l_j}\langle\!\langle E_j \rangle\!\rangle + pM) = p^{-1}\mathcal{F}^{l_j+1}\langle\!\langle E_j \rangle\!\rangle + \mathcal{F}M.$$

Inductively we can show that $\langle\!\langle E_{j_1+1} \rangle\!\rangle = \langle\!\langle \mathcal{V}^{-1}\mathcal{F}^{l_{j_1}} \cdots \mathcal{V}^{-1}\mathcal{F}^{l_{j_0+1}}\mathcal{V}^{-1}\mathcal{F}^{l_{j_0}}E_{j_0} \rangle\!\rangle$ is equal to

$$p^{-j_1+j_0-1}\mathcal{F}^{\sum_{j=j_0}^{j_1}(l_j+1)}\langle\!\langle E_{j_0} \rangle\!\rangle + \sum_{r=j_0}^{j_1} p^{-j_1+r}\mathcal{F}^{1+\sum_{j=r+1}^{j_1}(l_j+1)}M. \tag{6.1.3}$$

Since $\langle\!\langle E_{j_1+1} \rangle\!\rangle$ is an A_k-submodule of M, the term of $r = j_0$ of (6.1.3) is contained in M. Hence $\mathcal{F}^{1+\sum_{j=j_0+1}^{j_1}(l_j+1)}M \subset p^{j_1-j_0}M$. Thus we obtain the inclusion (6.1.1).

For any $\alpha \in \mathbb{Z}_{\geq 0}$, considering the case of $j_1 - j_0 = \alpha n$ in (6.1.1), we have the inclusion (6.1.2). Here we used $l_{i+n} = l_i$ for all $i \geq 0$ and $\sum_{i=0}^{n-1} l_i = m$. □

Proposition 6.3. *For any p-divisible group $\mathcal{G}$ with $\mathcal{G}[p] \simeq G$, the first Newton slope of $\mathcal{G}$ is greater than or equal to λ_φ.*

Proof. If $\varphi(1) = 1$, then $\lambda_\varphi = 0$ and therefore there is nothing to prove.

Assume $\varphi(1) = 0$. Let ω be the slope-function of M (see [1], IV. 5), which is a continuous real-valued function on $\mathbb{R}$. It suffices to show $\omega(\lambda_\varphi) = 2g\lambda_\varphi$, because this implies that the first slope of M is not less than λ_φ by the definition of the slope function ([1], p. 86). By the inclusion (6.1.2), for any $\varepsilon \in \mathbb{Q}_{>0}$ and for any $\beta \in \mathbb{N}$ with $\beta\varepsilon \in \mathbb{N}$, we get $\mathcal{F}^{\beta(m+n+\varepsilon)}M \subset p^{\beta n}M$. Hence by [1], Cor. on p. 88,

$$\begin{aligned} \omega\left(\frac{n}{m+n+\varepsilon}\right) &= \lim_{\beta\to\infty} \frac{1}{\beta(m+n+\varepsilon)} \operatorname{length}(M/(\mathcal{F}^{\beta(m+n+\varepsilon)}M + p^{\beta n}M)) \\ &= \lim_{\beta\to\infty} \frac{1}{\beta(m+n+\varepsilon)} \operatorname{length}(M/p^{\beta n}M) \\ &= 2g\frac{n}{m+n+\varepsilon}. \end{aligned}$$

The continuity of ω implies $\omega(\lambda_\varphi) = 2g\lambda_\varphi$. □

Corollary 6.4. $S_\varphi \subset Z_{\lambda_\varphi}$.

Proof. Let (X, η) be a principally polarized abelian variety with $\mathrm{ES}(X[p], \iota_\eta) = \varphi$. Then Prop. 6.3 says that the p-divisible group $X[p^\infty]$ has the first Newton slope $\geq \lambda_\varphi$. This means $(X, \eta) \in Z_{\lambda_\varphi}$. □

6.2 Proof of (B)

Proposition 6.5. *Let φ be an elementary sequence with $\lambda_\varphi < 1/2$ and $\mathcal{Q}$ a Ψ-set in φ. Assume that φ is minimal in the Bruhat ordering among elementary sequences in which there is a Ψ-set $\mathcal{Q}'$ satisfying $\mathrm{AS}(\mathcal{Q}') = \mathrm{AS}(\mathcal{Q})$ (see Def. 3.11 (1) for the definition of $\mathrm{AS}(\mathcal{Q})$). Then $\mathcal{Q}$ makes a (V, F^{-1})-subset.*

Proof. By Lem. 5.19, it suffices to prove

$$\psi(i) > \psi(i-1) \quad \text{for all } G_i \in \mathcal{Q} \text{ satisfying } \psi(i) > 0. \tag{6.2.1}$$

First show (6.2.1) for $i \leq g$. Suppose $\psi(i-1)$ was equal to $\psi(i)$. We define an elementary sequence φ_1 by

$$\varphi_1(j) = \begin{cases} \varphi(j) - 1 & \text{if } j < i \text{ and } \varphi(j) = \varphi(i), \\ \varphi(j) & \text{otherwise.} \end{cases}$$

Then clearly $\varphi_1 < \varphi$. Let ψ_1 be the final sequence stretched from φ_1. We claim that

$$\psi_1(i') = \psi(i') \quad \text{for all } G_{i'} \in \mathcal{Q}. \tag{6.2.2}$$

Indeed, for $i' \leq g$, since $\mathcal{Q}$ is a Ψ-set, we have $\varphi(i') \neq \varphi(i)$ for any $G_{i'} \in \mathcal{Q}$ with $i' \leq g$ and $i' \neq i$; then by the definition of φ_1, (6.2.2) holds for $i' \leq g$. Let us show (6.2.2) for $i' > g$. Note $\psi_1(j) = \psi(j)$ for $g < j \leq 2g - i$ by the definition of φ_1. Let H_0 be the biggest element of $\mathcal{Q}$, and let v_0 be the integer such that $H_0 = G_{v_0}$. Recall $\psi(v_0) \leq 2g - v_0$ (Lem. 5.13). For every $G_{i'} \in \mathcal{Q}$ with $i' > g$, we have $i' \leq v_0 \leq 2g - \psi(v_0) \leq 2g - i$. Thus (6.2.2) holds also for $i' > g$. Clearly (6.2.2) implies that φ_1 has a Ψ-set with the same absolute shape as that of $\mathcal{Q}$. This contradicts the minimality of φ.

Similarly we can show (6.2.1) for $i > g$. Suppose $\psi(i) = \psi(i-1)$. We define an elementary sequence φ_2 by

$$\varphi_2(j) = \begin{cases} \varphi(j) - 1 & \text{if } j > 2g - i \text{ and } \psi(2g - j) = \psi(i) \\ \varphi(j) & \text{otherwise.} \end{cases}$$

Since $\varphi_2(2g - i + 1) = \varphi(2g - i + 1) - 1$, we have $\varphi_2 < \varphi$. Let ψ_2 be the final sequence stretched from φ_2. We claim that

$$\psi_2(i') = \psi(i') \quad \text{for all } G_{i'} \in \mathcal{Q}. \tag{6.2.3}$$

Indeed for $i' > g$, since $\mathcal{Q}$ is a Ψ-set, we have $\psi(i') \neq \psi(i)$ for any $G_{i'} \in \mathcal{Q}$ with $i' > g$ and $i' \neq i$; then by the definition of φ_2, (6.2.3) holds for $i' > g$. Since for all i' with $G_{i'} \in \mathcal{Q}$ and $i' \leq g$ we have $i' \leq \psi(v_0) \leq 2g - v_0 \leq 2g - i$, we get (6.2.3) also for $i' \leq g$ by the definition of φ_2. It follows from (6.2.3) that φ_2 has a Ψ-set with the same absolute shape as that of $\mathcal{Q}$. This contradicts the minimality of φ. □

Corollary 6.6. *Let φ be an elementary sequence of length g with $\lambda_\varphi < 1/2$. Set $m = m_\varphi$ and $n = n_\varphi$. Then there exists an elementary sequence φ' of length $g - e_\varphi(m+n)$ such that*

$$\varphi' \oplus \varphi_\xi^{\oplus e_\varphi} \leq \varphi \tag{6.2.4}$$

with $\xi = [m, n] + [n, m]$.

Proof. Let $\mathcal{D}$ be the full Ψ-set in φ. Choose a minimal elementary sequence φ'' in the Bruhat ordering such that $\varphi'' \le \varphi$ and there is a Ψ-set $\mathcal{Q}''$ in φ'' satisfying $\mathrm{AS}(\mathcal{Q}'') = \mathrm{AS}(\mathcal{D})$. By Prop. 3.12, $\mathrm{AS}(\mathcal{Q}'') = \mathrm{AS}(\mathcal{D})$ implies $\lambda_\varphi = \lambda_{\varphi''}$ and $e_\varphi = e(\mathcal{Q}'')$. In particular we have $\lambda_{\varphi''} < 1/2$; then we can apply Prop. 6.5 to the pair (φ'', Q''); hence $\mathcal{Q}''$ makes a (V, F^{-1})-subset. By Cor. 5.26 we have $\varphi'' = \varphi' \oplus \varphi_\xi^{\oplus e_\varphi}$. ☐

Corollary 6.7. *$S_\varphi \cap Z^0_{\lambda_\varphi, e_\varphi} \neq \emptyset$.*

Proof. If $\lambda_\varphi = 1/2$, then this follows from (A): $S_\varphi \subset Z_{1/2} = Z^0_{1/2, e_\varphi}$ and Th. 2.6 (2): $S_\varphi \neq \emptyset$.

Assume $\lambda_\varphi < 1/2$. Let φ' be the elementary sequence obtained in Cor. 6.6. Then $\varphi' \oplus \varphi_\xi^{\oplus e_\varphi} \le \varphi$. By Th. 2.6 (3), we have $S_{\varphi' \oplus \varphi_\xi^{\oplus e_\varphi}} \subset \overline{S}_\varphi$. There exist principally polarized abelian varieties X, Y such that $\mathrm{ES}(X) = \varphi'$ and $\mathrm{ES}(Y) = \varphi_\xi^{\oplus e_\varphi}$ by Th. 2.6 (2). Then $X \times Y$ gives a point of $S_{\varphi' \oplus \varphi_\xi^{\oplus e_\varphi}}$. Since $X \times Y \in \overline{S}_\varphi \subset Z_{\lambda_\varphi}$ by (A), the first Newton slope of $X \times Y$ is $\ge \lambda_\varphi$ and therefore the first Newton slope of X is $\ge \lambda_\varphi$. Th. 5.11 says that $\mathrm{NP}(Y)$ is the Newton polygon $e_\varphi([m,n]+[n,m])$. By Grothendieck and Katz ([4], Th. 2.3.1), there exists a point x of S_φ whose Newton polygon $\succ \mathrm{NP}(X \times Y)$. Then it follows from (A) $S_\varphi \subset Z_{\lambda_\varphi}$ that the point x has the Newton slopes $\lambda_1 = \cdots = \lambda_{e_\varphi} = \lambda_\varphi$. ☐

Corollary 6.8. (1) *The elementary sequences φ and φ' in Cor. 5.26 satisfy $\lambda_{\varphi'} \ge \lambda_\varphi$.*

(2) *In Cor. 6.6 we can choose φ' satisfying $\lambda_{\varphi'} \ge \lambda_\varphi$ in addition.*

Proof. (1) Let $\mathcal{Q}$ and ξ be as in Cor. 5.26. Applying Cor. 6.7 to φ', there exists a principally polarized abelian variety X' such that $\mathrm{ES}(X') = \varphi'$ and the first Newton slope of X' is equal to $\lambda_{\varphi'}$. We also take a principally polarized abelian variety Y' with $\mathrm{ES}(Y') = \varphi_\xi^{\oplus e(\mathcal{Q})}$. Note $X' \times Y' \in S_\varphi$. If $\lambda_{\varphi'} < \lambda_\varphi$, then the first Newton slope of $X' \times Y'$ is $< \lambda_\varphi$. However this contradicts (A) $S_\varphi \subset Z_{\lambda_\varphi}$.

(2) Applying (1) to φ'' in the proof of Cor. 6.6, we have $\lambda_{\varphi'} \ge \lambda_{\varphi''}$. Hence (2) follows from $\lambda_{\varphi''} = \lambda_\varphi$. ☐

7 Examples

For given elementary sequence φ, it is easy to compute λ_φ (see Rem. 3.18). However it is difficult in general to enumerate φ satisfying $\lambda_\varphi = \lambda$ for given rational number λ with $0 \le \lambda \le 1/2$. In this section we give some examples for which we can do that.

7.1 Elementary sequences φ with big $m_\varphi + n_\varphi$

Proposition 7.1. *Let g be a positive integer and m, n positive integers with $\gcd(m,n) = 1$ and $m > n$. Assume that $m + n = g$ or $g - 1$. Then every elementary sequence φ of length g with $\lambda_\varphi = n/(m+n)$ is either of the form*

$$\varphi_{min} := (\underbrace{0, \cdots, 0}_{n}, \underbrace{1, 2, \cdots, m-n}_{m-n}, \underbrace{m-n, \cdots, m-n}_{g-m})$$

or of the form

$$\varphi^+_{min} := (\underbrace{0,\cdots,0}_{n},\underbrace{1,2,\cdots,m-n}_{m-n},\underbrace{m-n,\cdots,m-n}_{g-m-1},m-n+1).$$

Proof. First we claim

$$\varphi_{\min} = \begin{cases} \varphi_\xi & \text{if} \quad m+n=g, \\ \varphi_\xi \oplus (0)\ (= \varphi_{\xi+[1,1]}) & \text{if} \quad m+n=g-1 \end{cases}$$

with $\xi = [m,n]+[n,m]$. In the case of $m+n=g$, this is nothing but Lem. 5.12 (1). Let us show this claim for $m+n=g-1$. The full (V,F^{-1})-set $\Gamma = \{G_i/G_{i-1} \,|\, 1 \le i \le 2g\}$ of $\varphi_{\min}$ is decomposed into two self-dual (V,F^{-1})-subsets $\Lambda' = \{G_i/G_{i-1} \,|\, i = m+1, g+n+1\}$ and $\Lambda = \Gamma \setminus \Lambda'$. One can easily check that $\varphi_\xi = \varphi_\Lambda$ and $\varphi_{\Lambda'} = (0)$. Then Cor. 5.7 shows this claim.

The elementary sequence $\varphi_{\min}$ has the unique Ψ-cycle whose absolute shape is

$$\{(i,g+i)|i=1,\cdots,n\} \sqcup \{(n+i,i)|i=1,\cdots,m-n\} \sqcup \{(g+i,m-n+i)|i=1,\cdots,n\}. \quad (7.1.1)$$

In particular $\lambda_{\varphi_{\min}} = n/(m+n)$.

Let φ be an elementary sequence with $\lambda_\varphi = n/(m+n)$. By Cor. 6.6 and its proof, there is an elementary sequence φ' of length ≤ 1 such that $\varphi_\xi \oplus \varphi' \le \varphi$ and $\varphi_\xi \oplus \varphi'$ has a Ψ-cycle with the same absolute shape as that of the Ψ-cycle in φ. Then φ' must be () if $m+n=g$ and (0) if $m+n=g-1$. (Here () is the elementary sequence of length 0 and (0) is the elementary sequence of length 1 sending 1 to 0.) This means $\varphi_{\min} = \varphi_\xi \oplus \varphi'$; hence $\varphi_{\min} \le \varphi$. Then by (7.1.1) the final sequence ψ stretched from φ has to be of the form:

$$(\underbrace{0,\cdots,0}_{n},\underbrace{1,\cdots,m-n}_{m-n},\underbrace{a_1,\cdots,a_{g-m}}_{g-m};\underbrace{m-n+1,\cdots,m}_{n},*,\cdots,*)$$

for some integers $a_1,\cdots,a_{g-m}$ satisfying $m-n \le a_1 \le \cdots \le a_{g-m} \le m-n+1$. If $a_{g-m} = m-n$, then $a_i = m-n$ for all i, i.e., $\varphi = \varphi_{\min}$. Otherwise we have $a_{g-m} = m-n+1$; then $a_{g-m-1} = \varphi(g-1) = \psi(g+1)-1 = m-n$; hence $a_i = m-n$ for all $i < g-m$; thus we have $\varphi = \varphi^+_{\min}$. □

7.2 EO-strata contained in the almost supersingular locus

The almost supersingular locus is the second smallest NP-stratum. More precisely speaking, this is defined to be W_ν with

$$\nu = \begin{cases} \left[\frac{g+1}{2},\frac{g-1}{2}\right] + \left[\frac{g-1}{2},\frac{g+1}{2}\right] & \text{if } g \text{ is odd,} \\ \left[\frac{g}{2},\frac{g-2}{2}\right] + [1,1] + \left[\frac{g-2}{2},\frac{g}{2}\right] & \text{if } g \text{ is even.} \end{cases} \quad (7.2.1)$$

If $g \ge 3$, the condition $\xi \prec \nu$ is equivalent to that the first Newton slope of ξ is greater than or equal to $(g-1)/2g$ for odd g and $(g-2)/2(g-1)$ for even g.

Assume $g \ge 3$. Let φ be an elementary sequence of length g with $S_\varphi \not\subset W_\sigma$. Then $S_\varphi \subset W_\nu$ if and only if φ equals either

$$\varphi_\nu = (\overbrace{0,\cdots,0}^{[(g-1)/2]},1,\cdots,1,1) \quad \text{or} \quad \varphi^+_\nu := (\overbrace{0,\cdots,0}^{[(g-1)/2]},1,\cdots,1,2) \quad (7.2.2)$$

by Prop. 7.1. One may ask whether each of S_{φ_ν} and $S_{\varphi_\nu^+}$ intersects with the supersingular locus or not. Since φ_ν comes from the minimal p-divisible group $H(\nu)$, we conclude that $S_{\varphi_\nu} \subset W_\nu^0$ by Th. 5.11; hence $S_{\varphi_\nu} \cap W_\sigma = \emptyset$. On the other hand, we expect $W_\sigma \cap S_{\varphi_\nu^+} \neq \emptyset$ for all $g \geq 3$. We intend to prove this and to enumerate the irreducible components of $W_\sigma \cap S_{\varphi_\nu^+}$ in a future paper.

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