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Finite Dimensional Semisimple Q-Algebras

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Abstract A Q-algebra can be represented as an operator algebra on an infinite dimensional Hilbert space. However we don't know whether a finite n-dimensional Q-algebra can be represented on a Hilbert space of dimension n except n = 1, 2. It is known that a two dimensional Q-algebra is just a two dimensional commutative operator algebra on a two dimensional Hilbert space. In this paper we study a finite n-dimensional semisimple Q-algebra on a finite n-dimensional Hilbert space. In particular we describe a three dimensional Q-algebra of the disc algebra on a three dimensional Hilbert space. Our studies are related to the Pick interpolation problem for a uniform algebra.

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1. Introduction

Let A be a uniform algebra on a compact Hausdorff space X. If I is a closed ideal of A, then the quotient algebra A/I is a commutative Banach algebra with unit. In this paper, if a Banach algebra \mathcal{B} is isometrically isomorphic to A/I, then \mathcal{B} is called a Q-algebra. (F.Bonsall and J.Duncan called \mathcal{B} an IQ-algebra.(cf. [1], p.270)) B.Cole (cf. [1], p.272) showed that any Q-algebra is an operator algebra on a Hilbert space H, that is, there exists an isometric isomorphism to an operator algebra on H. Let μ be a probability measure on X and $H^2(\mu)$ the closure of A in $L^2(\mu)$. $H^2(\mu) \cap I^{\perp}$ denotes the annihilator of I in $H^2(\mu)$. Let P be the orthogonal projection from $H^2(\mu)$ onto $H^2(\mu) \cap I^{\perp}$ and for any $f \in A$ put

$$S^{\mu}_{f}\phi=P(f\phi), \quad (\phi\in H^{2}(\mu)\cap I^{\perp}).$$

Then $S_{f+k}^{\mu} = S_f^{\mu}$ for k in I and $||S_f^{\mu}|| \le ||f+I||$. S^{μ} is the map of A/I on operators on $H^2(\mu) \cap I^{\perp}$ which sends $f+I \to S_f^{\mu}$ for each f in A. Hence S^{μ} is a contractive homomorphism from A into $B(H^2(\mu) \cap I^{\perp})$ where $B(H^2(\mu) \cap I^{\perp})$ is the set of all bounded linear operators on $H^2(\mu) \cap I^{\perp}$. The kernel of S^{μ} contains I. Then we say that S^{μ} gives a contractive representation of A/Iinto $B(H^2(\mu) \cap I^{\perp})$. If $||S_f^{\mu}|| = ||f+I||, (f \in A)$ then ker $S^{\mu} = I$ and we say that S^{μ} gives an isometric representation of A/I on $H^2(\mu) \cap I^{\perp}$.

Problem 1. Prove that any finite n-dimensional Q-algebra can be represented on a Hilbert space of finite dimension n.

If S^{μ} is isometric then we solve Problem 1. In fact, T.Nakazi and K.Takahashi (cf. [9]) solved Problem 1 for n = 2 in this way. It seems to be unknown for $n \ge 3$.

Problem 2. Describe a finite n-dimensional Q-algebra in finite n-dimensional commutative operator algebras with unit on a Hilbert space of finite dimension n.

Problem 2 is clear for n = 1 and it was proved by S.W.Drury (cf. [4]) and T.Nakazi (cf. [8]) that a 2-dimensional commutative operator algebra with unit on a Hilbert space is just a Q-algebra. J.Holbrook (cf. [6]) proved that von Neumann's inequality

$$\|p(T)\| \le \|p\|_{\infty}$$

can fail for some polynomials p in 3 variables, where $T = (T_1, T_2, T_3)$ is a triple of commuting contractions on \mathbb{C}^4 , and T_1, T_2, T_3 are simultaneously diagonalizable. Then we can construct a 4-dimensional commutative matrix algebra with unit on \mathbb{C}^4 , which is not a Q-algebra. If $n \geq 4$, then this implies that the set of all n-dimensional Q-algebra A/I is smaller than the set of all set of all n-dimensional commutative oparator algebras with unit on a n-dimensional Hilbert space. If n = 3, then Problem 2 has not been solved yet. In this paper, we concentrate on a semisimple commutative Banach algebra and we study Problem 2. In Section 2, we will prove several general results of semisimple finite dimensional Q-algebras that will be used in the latter sections. In Section 3, we will study arbitrary semisimple n dimensional Q-algebras for n = 2, 3. In Section 4, we will study the isometric representation of A/I. In Section 5, we will describe completely 3-dimensional semisimple Q-algebras of the disc algebra in 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space.

2. Semisimple and commutative matrix algebra

In this section, we study 3-dimensional commutative semisimple operator algebras on a 3-dimensional Hilbert space. In particular, we study when two such operator algebras are isometric or unitary equivalent. S.McCullough and V.Paulsen (cf. [7], Proposition 2.2) proved the similar result of Proposition 2.3. We use Lemma 2.1 to prove Proposition 2.2 and Proposition 2.3. In Example 2.6, we construct a 4-dimensional commutative matrix algebra with unit on \mathbf{C}^4 which is not a *Q*-algebra using the example of J.Holbrook (cf. [6]).

Lemma 2.1. Let $n \ge 2$ and let H be an n-dimensional Hilbert space which is spanned by $k_1, k_2, ..., k_n$. Let

$$\psi_1 = \frac{k_1}{\|k_1\|}, \quad \psi_j = \frac{k_j - \sum_{i=1}^{j-1} \langle k_j, \psi_i \rangle \psi_i}{\|k_j - \sum_{i=1}^{j-1} \langle k_j, \psi_i \rangle \psi_i\|} \ (2 \le j \le n).$$

Then $\{\psi_1, \ldots, \psi_n\}$ is an orthonormal basis for H. Let P_1, \ldots, P_n be the idempotent operators on H such that $P_i k_i = k_i$, $P_i k_j = 0$ if $i \neq j$. For $1 \leq m \leq n$, let $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$, $(1 \leq i, j \leq n)$. Then $P_m = (a_{ij}^{(m)})_{1 \leq i,j \leq n}$ is an $n \times n$ matrix such that

$$P_1 = \begin{pmatrix} B_1 \\ O \end{pmatrix}, \dots, P_m = \begin{pmatrix} O & B_m \\ O & O \end{pmatrix}, \dots, P_n = \begin{pmatrix} O & B_n \end{pmatrix},$$

where B_m is an $m \times (n - m + 1)$ matrix such that

$$B_{1} = \left(\begin{array}{ccccc} 1 & \ldots & a_{1n}^{(1)}\end{array}\right), \ \dots \ , \ B_{m} = \left(\begin{array}{ccccc} a_{1m}^{(m)} & \ldots & \ldots & a_{1n}^{(m)} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & a_{mn}^{(m)}\end{array}\right), \ \dots \ , \ B_{n} = \left(\begin{array}{cccc} a_{1n}^{(n)} \\ \vdots \\ \vdots \\ 1 \end{array}\right).$$

Then $a_{mm}^{(m)} = 1$, and for $m \ge 2$,

$$a_{im}^{(m)} = rac{\langle k_m, \psi_i
angle}{\|k_m - \sum_{h=1}^{m-1} \langle k_m, \psi_h
angle \psi_h \|},$$

and for $m+1 \leq j \leq n$,

$$a_{ij}^{(m)} = rac{-\sum_{h=m}^{j-1} \langle k_j, \psi_h
angle a_{ih}^{(m)}}{\|k_j - \sum_{h=1}^{j-1} \langle k_j, \psi_h
angle \psi_h \|}.$$

Since this lemma is proved by elementary calculations, the proof is omitted. It is well known that any *n*-dimensional commutative semisimple Banach algebra with unit I is spanned by commuting idempotents $P_1, ..., P_n$ satisfying $P_1 + ... + P_n = I$.

Proposition 2.2. In Lemma 2.1, for $1 \le m \le n$, rank $P_m = 1$, and $\mathcal{B} = \operatorname{span}\{P_1, ..., P_n\}$ is an n-dimensional semisimple commutative operator algebra with unit on H. Then $n \times n$ matrix $(a_{ij}^{(m)})$ for P_m with respect to $\{\psi_1, ..., \psi_n\}$ is $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$, and

$$P_{1} = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \vdots & \dots & 0 \end{pmatrix}, P_{2} = (a_{ij}^{(2)}) = \begin{pmatrix} 0 & a_{12}^{(2)} & a_{13}^{(2)} & \dots & a_{1n}^{(2)} \\ 0 & 1 & a_{23}^{(2)} & \dots & a_{2n}^{(2)} \\ 0 & 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 0 \end{pmatrix},$$

In Lemma 2.1, $a_{ij}^{(m)}$ is written using $k_1, ..., k_n$ and $\psi_1, ..., \psi_n$.

Proof. By the assumption of Lemma 2.1, $P_ik_i = k_i$ and $P_ik_j = 0$ if $i \neq j$. Hence rank $P_m = 1$. If $i \neq j$, then $P_iP_jk_m = \delta_{jm}P_ik_j = 0$, $(1 \leq m \leq n)$. Since $H = \text{span}\{k_1, \dots, k_n\}$, this implies that $P_iP_j = 0$ if $i \neq j$. Hence \mathcal{B} is commutative. Since $P_i^2k_m = \delta_{im}P_ik_m = P_ik_m$, $(1 \leq m \leq n)$, it follows that $P_i^2 = P_i$. Hence \mathcal{B} is semisimple and n-dimensional. Since $(P_1 + \ldots + P_n)k_m = P_mk_m = k_m$, $(1 \leq m \leq n)$, it follows that $P_1 + \ldots + P_n = I$. Hence \mathcal{B} has a unit I. This completes the proof.

Proposition 2.3. Let H be a 3-dimensional Hilbert space which is spanned by k_1, k_2, k_3 . Let $\langle \cdot, \cdot \rangle$ denote the inner product, and let $\|\cdot\|$ denote the norm of H.

$$\psi_1 = \frac{k_1}{\|k_1\|}, \quad \psi_2 = \frac{k_2 - \langle k_2, \psi_1 \rangle \psi_1}{\|k_2 - \langle k_2, \psi_1 \rangle \psi_1\|}, \quad \psi_3 = \frac{k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}$$

Then ψ_1, ψ_2, ψ_3 is an orthonormal basis in H. Let P_i be the idempotent operator on H such that $P_i k_i = k_i$, $P_i k_j = 0$ if $i \neq j$. For m = 1, 2, 3, the 3×3 matrix $(a_{ij}^{(m)})$ for P_m with respect to $\{\psi_1, \psi_2, \psi_3\}$ is $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$. Then

$$P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = (a_{ij}^{(2)}) = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = (a_{ij}^{(3)}) = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$x = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2 \rangle|^2}}, \quad y = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|},$$
$$z = \frac{-\langle k_3, \psi_2 \rangle}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}.$$

Proof. By Proposition 2.2, there exist x, y such that

$$P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 2.1,

$$x=a_{12}^{(1)}=rac{-\langle k_2,k_1
angle}{\sqrt{\|k_1\|^2\|k_2\|^2-|\langle k_1,k_2
angle|^2}},$$

and

$$y = a_{13}^{(1)} = \frac{-\sum_{h=1}^{2} \langle k_{3}, \psi_{h} \rangle a_{1h}^{(1)}}{\|k_{3} - \sum_{h=1}^{2} \langle k_{3}, \psi_{h} \rangle \psi_{h}\|} = \frac{-\langle k_{3}, \psi_{1} \rangle - \langle k_{3}, \psi_{2} \rangle x}{\|k_{3} - \langle k_{3}, \psi_{1} \rangle \psi_{1} - \langle k_{3}, \psi_{2} \rangle \psi_{2}\|}$$

By Proposition 2.2, there exist z, w such that

$$P_1 = \left(egin{array}{ccc} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}
ight), \quad P_2 = \left(egin{array}{ccc} 0 & -x & w \\ 0 & 1 & z \\ 0 & 0 & 0 \end{array}
ight), \quad P_3 = \left(egin{array}{ccc} 0 & 0 & -w - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{array}
ight),$$

because $P_1 + P_2 + P_3 = I$. By Lemma 2.1,

$$z = a_{23}^{(2)} = \frac{-\sum_{h=2}^{2} \langle k_{3}, \psi_{h} \rangle a_{2h}^{(2)}}{\|k_{3} - \sum_{h=1}^{2} \langle k_{3}, \psi_{h} \rangle \psi_{h}\|} = \frac{-\langle k_{3}, \psi_{2} \rangle}{\|k_{3} - \langle k_{3}, \psi_{1} \rangle \psi_{1} - \langle k_{3}, \psi_{2} \rangle \psi_{2}\|}$$

By Lemma 2.1,

$$a_{12}^{(1)} = rac{-\langle k_2, k_1
angle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2
angle|^2}} = -a_{12}^{(2)}.$$

Hence

$$w = a_{13}^{(2)} = \frac{-\sum_{h=2}^{2} \langle k_{3}, \psi_{h} \rangle a_{1h}^{(2)}}{\|k_{3} - \sum_{h=1}^{2} \langle k_{3}, \psi_{h} \rangle \psi_{h}\|} = \frac{-\langle k_{3}, \psi_{2} \rangle a_{12}^{(2)}}{\|k_{3} - \sum_{h=1}^{2} \langle k_{3}, \psi_{h} \rangle \psi_{h}\|} = za_{12}^{(2)} = -za_{12}^{(1)} = -xz.$$

This completes the proof.

Theorem 2.4. Let P_1, P_2, P_3 be idempotent operators defined in Proposition 2.3. Let H' be a 3-dimensional Hilbert space. Let \mathcal{B}' be a 3-dimensional semisimple commutative operator algebra on H'. Then, there are idempotent operators Q_1, Q_2, Q_3 on H', an orthonormal basis $\psi'_1, \psi'_2, \psi'_3$ in H' and complex numbers x_0, y_0, z_0 such that $\mathcal{B}' = \operatorname{span}\{Q_1, Q_2, Q_3\}$ and, as matrices relative to $\psi'_1, \psi'_2, \psi'_3$,

$$Q_1 = \begin{pmatrix} 1 & x_0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -x_0 & -x_0 z_0 \\ 0 & 1 & z_0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & x_0 z_0 - y_0 \\ 0 & 0 & -z_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let τ be the map of \mathcal{B} on \mathcal{B}' such that

$$au\left(\lambda_1P_1+\lambda_2P_2+\lambda_3P_3
ight)=\lambda_1Q_1+\lambda_2Q_2+\lambda_3Q_3, \quad (\lambda_1,\lambda_2,\lambda_3\in {f C}).$$

(1) τ is isometric if and only if

$$\begin{split} |x|^2 + |y|^2 &= |x_0|^2 + |y_0|^2, \\ (1 + |x|^2)(1 + |z|^2) &= (1 + |x_0|^2)(1 + |z_0|^2), \\ |x|^2 + xz\overline{y} &= |x_0|^2 + x_0z_0\overline{y_0}. \end{split}$$

(2) τ is induced by a unitary map from H to H' if and only if there are complex numbers u_1, u_2, u_3 such that

$$|u_1| = |u_2| = |u_3| = 1, \quad u_1 x = u_2 x_0, \quad u_1 y = u_3 y_0, \quad u_2 z = u_3 z_0.$$

 $Then \ |x|=|x_0|, \quad |y|=|y_0|, \quad |z|=|z_0|, \quad xz\overline{y}=x_0z_0\overline{y_0}.$

Proof. (1) By the theorem of B.Cole and J.Wermer (cf. [3]), τ is isometric if and only if, writing tr for trace,

$$\operatorname{tr}(P_i^*P_j) = \operatorname{tr}(Q_i^*Q_j), \quad (1 \le i, j \le 3).$$

If τ is isometric, then

$$\begin{split} 1+|x|^2+|y|^2 &= \operatorname{tr}(P_1^*P_1) = \operatorname{tr}(Q_1^*Q_1) = 1+|x_0|^2+|y_0|^2,\\ (1+|x|^2)(1+|z|^2) &= \operatorname{tr}(P_2^*P_2) = \operatorname{tr}(Q_2^*Q_2) = (1+|x_0|^2)(1+|z_0|^2),\\ |x|^2+xz\overline{y} &= \operatorname{tr}(P_1^*P_2) = \operatorname{tr}(Q_1^*Q_2) = |x_0|^2+x_0z_0\overline{y_0}. \end{split}$$

Conversely, if three equalities in (1) hold, then

$$\begin{split} \operatorname{tr}(P_1^*P_1) &= 1 + |x|^2 + |y|^2 = 1 + |x_0|^2 + |y_0|^2 = \operatorname{tr}(Q_1^*Q_1), \\ \operatorname{tr}(P_2^*P_2) &= (1 + |x|^2)(1 + |z|^2) = (1 + |x_0|^2)(1 + |z_0|^2) = \operatorname{tr}(Q_2^*Q_2), \\ \operatorname{tr}(P_1^*P_2) &= |x|^2 + xz\overline{y} = |x_0|^2 + x_0z_0\overline{y_0} = \operatorname{tr}(Q_1^*Q_2), \\ \operatorname{tr}(P_2^*P_3) &= \overline{xz}(y - xz) - |z|^2 = \overline{x_0z_0}(y_0 - x_0z_0) - |z_0|^2 = \operatorname{tr}(Q_2^*Q_3), \\ \operatorname{tr}(P_3^*P_1) &= y(\overline{xz - y}) = y_0(\overline{x_0z_0 - y_0}) = \operatorname{tr}(Q_3^*Q_1), \\ \operatorname{tr}(P_3^*P_3) &= 1 + |z|^2 + |xz - y|^2 = 1 + |z_0|^2 + |x_0z_0 - y_0|^2 = \operatorname{tr}(Q_3^*Q_3). \end{split}$$

(2) Suppose τ is induced by a unitary map $U = (u_{ij})$, $(1 \le i, j \le 3)$ from H to H'. Since $UP_1 = Q_1U$, it follows that $u_{21} = u_{31} = 0$. Since $UP_2 = Q_2U$, it follows that $u_{32} = 0$. Hence U is an upper triangular matrix. Since the columns of U are pairwise orthogonal, U is a diagonal matrix. Hence there are complex numbers u_1, u_2, u_3 such that u_1, u_2, u_3 are diagonal element of U, and $|u_1| = |u_2| = |u_3| = 1$. Since $UP_1 = Q_1U$, it follows that $u_1x = u_2x_0$, $u_1y = u_3y_0$. Since $UP_2 = Q_2U$, it follows that $u_2z = u_3z_0$. The converse is also true. This completes the proof.

Example 2.5. Let $\mathcal{B}_0 = \text{span}\{P_1, P_2, P_3\}$, where

$$P_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $\mathcal{B}_1 = \operatorname{span}\{P_1^*, P_2^*, P_3^*\}$. This is an example which is established by W.Wogen (cf. [3]). He proved that \mathcal{B}_0 and \mathcal{B}_1 are isometrically isomorphic, and not unitarily equivalent. There is another example as the following. Let $\mathcal{B}_2 = \operatorname{span}\{Q_1, Q_2, Q_3\}$, where

$$Q_1 = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -\sqrt{2} & -\sqrt{2/3} \\ 0 & 1 & 1/\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & \sqrt{2/3} \\ 0 & 0 & -1/\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then \mathcal{B}_0 and \mathcal{B}_2 are 3-dimensional commutative operator algebras with unit. By the calculation,

$$\begin{aligned} |x|^2 + |y|^2 &= |x_0|^2 + |y_0|^2 = 2, \\ (1 + |x|^2)(1 + |z|^2) &= (1 + |x_0|^2)(1 + |z_0|^2) = 4, \\ |x|^2 + xz\overline{y} &= |x_0|^2 + x_0z_0\overline{y_0} = 2. \end{aligned}$$

By (1) of Theorem 2.4, this implies that \mathcal{B}_0 and \mathcal{B}_2 are isometrically isomorphic. By (2) of Theorem 2.4, \mathcal{B}_0 and \mathcal{B}_2 are not unitarily equivalent.

3. One to one representation

In this section, we assume that A/I is *n*-dimensional and semisimple. Hence there exist $\tau_1, ..., \tau_n$ in the maximal ideal space M(A) of A such that $\tau_i \neq \tau_j$ $(i \neq j)$ and $I = \bigcap_{j=1}^n \ker \tau_j$. S^{μ} gives a contractive representation of A/I into $B(H^2(\mu) \cap I^{\perp})$ and dim $H^2(\mu) \cap I^{\perp} \leq \dim A/I = n$. We study when S^{μ} is one to one from A/I to $B(H^2(\mu) \cap I^{\perp})$. It is clear that S^{μ} is one to one if and only if dim $H^2(\mu) \cap I^{\perp} = \dim A/I$. For $1 \leq j \leq n$, there exist $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Then $f_j + I$ is idempotent in A/I and $A/I = \operatorname{span}\{f_1 + I, ..., f_n + I\}$. The following two quantities are important to study S^{μ} . For $1 \leq j \leq n$,

$$\rho_j = \sup\{|\tau_j(f)| \ ; \ f \in \cap_{l \neq j} \ker \tau_l, \ \|f\| \le 1\}$$

and

$$ho_j(\mu) = \sup\{| au_j(f)|\;;\; f\in \cap_{l
eq j} \ker au_l,\; \|f\|_\mu \leq 1\},$$

where ||f|| denotes the support of f in A and $||f||_{\mu} = \langle f, f \rangle_{\mu}^{1/2} = (\int |f|^2 d\mu)^{1/2}$. Then it is easy to see that

$$\|f_j + I\| = rac{1}{
ho_j}, \quad \|f_j + I\|_{\mu} = rac{1}{
ho_j(\mu)},$$

and

$$||f_j + I|| \ge ||S_{f_j}^{\mu}|| \ge ||f_j + I||_{\mu}.$$

If S^{μ} is one to one then τ_j has a bounded extension to $H^2(\mu)$. In fact, if S^{μ} is one to one then dim $H^2(\mu) \cap I^{\perp} = n$ and so dim $H^2(\mu) \cap (\ker \tau_j)^{\perp} = 1$ for $1 \leq j \leq n$. Then for $1 \leq j \leq n$, there exists $k_j \in H^2(\mu)$ such that

$$au_j(f)=\langle f,k_j
angle_\mu=\int_X f\overline{k_j}d\mu,\;(f\in A).$$

Proposition 3.1. There exists a one to one contractive representation S^{μ} of A/I.

Proof. Since $\tau_j \in M(A)$, there exists a positive representing measure m_j of τ_j on X. Let $\mu = \sum_{j=1}^n m_j/n$. Then

$$| au_j(f)| = |\int_X f dm_j| \le n (\int_X |f|^2 d\mu)^{1/2} = n ||f||_{\mu}, \; (f \in A).$$

Hence τ_j has a unique bounded extension $\tilde{\tau}_j$ to $H^2(\mu)$ and $\tilde{\tau}_j \neq \tilde{\tau}_i$ $(j \neq i)$. Since $H^2(\mu) \cap I^{\perp} = \bigcap_{j=1}^n \ker \tilde{\tau}_j$, dim $H^2(\mu) \cap I^{\perp} = n$. Hence $H^2(\mu) \cap I^{\perp} = \operatorname{span}\{k_1, ..., k_n\}$. Suppose $S_f^{\mu} = 0$, then $\tau_j(f)k_j = (S_f^{\mu})^*k_j = 0$. Hence $\tau_j(f) = 0$, $(1 \leq j \leq n)$. Hence $f \in \cap_j \ker \tau_j = I$. This implies that S^{μ} is one to one from A/I to $B(H^2(\mu) \cap I^{\perp})$. This completes the proof.

Theorem 3.2. Suppose that S^{μ} is a one to one contractive representation of A/I. Let k_j be a function in $H^2(\mu)$ such that $\tau_j(f) = \langle f, k_j \rangle_{\mu}$ $(f \in A)$, for $1 \leq j \leq n$. Then (1) $H^2(\mu) \cap I^{\perp} = \operatorname{span}\{k_1, ..., k_n\}$ and $H^2(\mu) \cap I_j^{\perp} = \operatorname{span}\{k_j\}$ where $I_j = \ker \tau_j$. (2) If $m_j = ||k_j||_{\mu}^{-2}|k_j|^2 d\mu$ and $m = \sum_{j=1}^n m_j/n$, then m_j is a representing measure for τ_j for each $1 \leq j \leq n$, and we may assume that μ is absolutely continuous with respect to m. (3) $||S_{f_j}^{\mu}|| = ||k_j||_{\mu} ||f_j + I||_{\mu}$ for $1 \leq j \leq n$.

Proof. (1) Since S^{μ} is one to one, τ_j has a unique bounded extension $\tilde{\tau}_j$ to $H^2(\mu)$. In fact, if S^{μ} is one to one then dim $H^2(\mu) \cap I^{\perp} = n$ and so dim $H^2(\mu) \cap (\ker \tau_j)^{\perp} = 1$ for $1 \leq j \leq n$. Then there exists $k_j \in H^2(\mu)$ such that $\tau_j(f) = \langle f, k_j \rangle_{\mu}$ $(f \in A)$. If $g \in I$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I^{\perp}$ for each j. Since $k_1, ..., k_n$ are linearly independent, $\{k_1, ..., k_n\}$ is a basis of $H^2(\mu) \cap I^{\perp}$. If $g \in I_j$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I^{\perp}$. If $g \in I_j$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I^{\perp}_j$. If $g \in I_j$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I^{\perp}_j$ for each j. Hence k_j is a basis of $H^2(\mu) \cap I^{\perp}_j$. (2) For $1 \leq j \leq n$,

$$\int_X f dm_j = \int_X f \frac{|k_j|^2}{\|k_j\|_{\mu}^2} d\mu = \frac{\langle fk_j, k_j \rangle_{\mu}}{\|k_j\|_{\mu}^2} = \frac{\tilde{\tau}_j(fk_j)}{\|k_j\|_{\mu}^2} = \frac{\tau_j(f)\tilde{\tau}_j(k_j)}{\|k_j\|_{\mu}^2} = \tau_j(f), \ (f \in A).$$

Hence m_j is a representing measure for τ_j . Let $m = \sum_{j=1}^n m_j/n$ and let $\mu = \mu^a + \mu^s$ be a Lebesgue decomposition by m. Then $H^2(\mu^a) \cap I^{\perp} = H^2(\mu) \cap I^{\perp}$ and so $H^2(\mu^s) \cap I^{\perp} = \{0\}$ where μ^a and μ^s are divided by their total masses. Hence $S_f^{\mu} = S_f^{\mu^a} \oplus S_f^{\mu^s} = S_f^{\mu^a} \oplus 0$ and so $\|S_f^{\mu}\| = \|S_f^{\mu^a}\|$ for $f \in A$.

(3) Since rank $(S_{f_j}^{\mu})^* = 1$, there exists $x_j \in H^2(\mu) \cap I^{\perp}$ such that $(S_{f_j}^{\mu})^* \phi = \langle \phi, x_j \rangle k_j = (k_j \otimes x_j) \phi$, $(\phi \in H^2(\mu) \cap I^{\perp})$. Then $\|S_{f_j}^{\mu}\| = \|(S_{f_j}^{\mu})^*\| = \|k_j \otimes x_j\| = \|k_j\|_{\mu} \|x_j\|_{\mu}$. Let P be the orthogonal projection from $H^2(\mu)$ onto $H^2(\mu) \cap I^{\perp}$. Then

$$\langle Pf_j, \phi \rangle = \langle S^{\mu}_{f_j} 1, \phi \rangle = \langle 1, (S^{\mu}_{f_j})^* \phi \rangle = \langle x_j, \phi \rangle \langle 1, k_j \rangle = \langle x_j, \phi \rangle, \quad (\phi \in H^2(\mu) \cap I^{\perp}),$$

because $\langle 1, k_j \rangle = 1$. Hence $Pf_j = x_j$. Hence

$$||f_j + I||_{\mu} = ||Pf_j||_{\mu} = ||x_j||_{\mu}.$$

Hence $||S_{f_i}^{\mu}|| = ||k_j||_{\mu} ||f_j + I||_{\mu}$. This completes the proof.

Let $G(\tau)$ denote the Gleason part of τ . If $G(\tau_i) = G(\tau_i)$, then we write $\tau_i \sim \tau_i$.

Proposition 3.3. Suppose that $\dim H^2(\mu) \cap I^{\perp} = n$, $I^1 = \bigcap_{j \in N^1} \ker \tau_j$, $I^2 = \bigcap_{j \in N^2} \ker \tau_j$, $N^1 \cap N^2 = \emptyset$ and $N^1 \cup N^2 = \{1, 2, ..., n\}$. Let $\sharp N^j$ denote the number of elements in N^j . If $\tau_j \not\sim \tau_k$ whenever $j \in N^1$ and $k \in N^2$, then $H^2(\mu) = H^2(\mu^1) \oplus H^2(\mu^2)$, $H^2(\mu) \cap I^{\perp} = (H^2(\mu^1) \cap (I^1)^{\perp}) \oplus (H^2(\mu^2) \cap (I^2)^{\perp})$, $S^{\mu}_{\phi} = S^{\mu^1}_{\phi} \oplus S^{\mu^2}_{\phi}$ and $\dim H^2(\mu^j) \cap (I^j)^{\perp} = \sharp N^j$, where $\mu = \frac{\mu^1 + \mu^2}{2}$, $\mu^1 \perp \mu^2$ and μ^j is a probability measure for j = 1, 2.

Proof. By (1) of Theorem 3.2, $H^2(\mu) \cap I^{\perp} = \operatorname{span}\{k_1, ..., k_n\}$. We may assume that $N^1 = \{1, 2, ..., l\}$ and $N^2 = \{l + 1, ..., n\}$. By (2) of Theorem 3.2, $m_j = ||k_j||_{\mu}^{-2}|k_j|^2 d\mu$ is a representing measure for τ_j for each $1 \leq j \leq n$. Put $\lambda^1 = \frac{1}{l} \sum_{j=1}^l m_j$ and $\lambda^2 = \frac{1}{n-l} \sum_{j=l+1}^n m_j$ then $\lambda^1 \perp \lambda^2$ by definitions of N^1 and N_2 . Let $\mu = \mu_0^1 + \mu_0^2$ be a Lebesgue decomposition with respect to λ^1 such that $\mu_0^1 \ll \lambda^1$ and $\mu_0^2 \perp \lambda^1$. Put $\mu^1 = \mu_0^1/||\mu_0^1||$ and $\mu^2 = \mu_0^2/||\mu_0^2||$. This completes the proof.

4. Isometric representation

In this section, we assume that A/I is *n*-dimensional and semisimple. Hence there exist $\tau_1, ..., \tau_n$ in the maximal ideal space M(A) of A such that $\tau_i \neq \tau_j$ $(i \neq j)$ and $I = \bigcap_{j=1}^n \ker \tau_j$. For $1 \leq j \leq n$, there exist $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Then $f_j + I$ is idempotent in A/I and $A/I = \operatorname{span}\{f_1+I, ..., f_n+I\}$. If S^{μ} is an isometric representation of A/I, then $||S_{f_j}^{\mu}|| = ||f_j+I||$ for $1 \leq j \leq n$. By (3) of Theorem 3.2, this implies that $||f_j + I|| = ||k_j||_{\mu} ||f_j + I||_{\mu}$. Hence, if S^{μ} is an isometric representation of A/I, then $||k_j||_{\mu} = ||f_j + I||_{\mu}$ for $1 \leq j \leq n$. Is the converse of this statement true? If n = 2, then the answer will be given in Proposition 4.4.

Theorem 4.1. Suppose that $G(\tau_i) \cap G(\tau_j) \cap G(\tau_l) = \emptyset$ if i, j and l are different from each other. Then there exists an isometric representation S^{μ} of A/I.

Proof. By Proposition 3.3, if $G(\tau_j) = \{\tau_j\}$, for all $1 \leq j \leq n$, then there exists an isometric representation S^{μ^j} of A/I_j where $I_j = \ker \tau_j$, and $\mu^i \perp \mu^j$. If $\mu = (\mu^1 + \ldots + \mu^n)/n$, then $H^2(\mu) \cap I^{\perp} = (H^2(\mu^1) \cap I^{\perp}) \oplus \ldots \oplus (H^2(\mu^n) \cap I^{\perp})$ and $S_f^{\mu} = S_f^{\mu^1} \oplus \ldots \oplus S_f^{\mu^n}$ $(f \in A)$. Therefore, the theorem is proved in the case when $G(\tau_j) = \{\tau_j\}$, for all $1 \leq j \leq n$. It is sufficient to prove the theorem when $\tau_i \sim \tau_j$ for some $i, j(i \neq j)$. Suppose $\tau_{2k-1} \sim \tau_{2k}, (1 \leq k \leq n_0)$ and $G(\tau_l) = \{\tau_l\}, (2n_0 + 1 \leq l \leq n)$ for some n_0 . Since $G(\tau_i) \cap G(\tau_j) \cap G(\tau_l) = \emptyset$, it follows that dim $A/I_{ij} = 2$ where $I_{ij} = I_i \cap I_j = \ker \tau_i \cap \ker \tau_j$. By Corollary 1 in [9], there is a probability measure μ^{ij} such that $\|S_f^{\mu^{ij}}\| = \|f + I_{ij}\|$ for all $f \in A$. By Proposition 3.3, there are probability measures $\mu^{2k-1,2k}, (1 \leq k \leq n_0)$ and $\mu^l, (2n_0 + 1 \leq l \leq n)$ such that $\mu = (\mu^{12} + \mu^{34} + \ldots + \mu^{2n_0-1,2n_0} + \mu^{2n_0+1} + \ldots + \mu^n)/(n - n_0), \ H^2(\mu) \cap I^{\perp} =$

 $(H^{2}(\mu^{12}) \cap I_{12}^{\perp}) \oplus \ldots \oplus (H^{2}(\mu^{2n_{0}-1,2n_{0}}) \cap I_{2n_{0}-1,2n_{0}}^{\perp}) \oplus (H^{2}(\mu^{2n_{0}+1}) \cap I_{2n_{0}+1}^{\perp}) \oplus \ldots \oplus (H^{2}(\mu^{n}) \cap I_{n}^{\perp}), \\ S_{f}^{\mu} = S_{f}^{\mu^{12}} \oplus \ldots \oplus S_{f}^{\mu^{2n_{0}-1,2n_{0}}} \oplus S_{f}^{\mu^{2n_{0}+1}} \oplus \ldots \oplus S_{f}^{\mu^{n}} \text{ Hence } S^{\mu} \text{ is an isometric representation of } A/I \\ \text{where } I = (\cap_{k=1}^{n_{0}} I_{2k-1,2k}) \cap (\cap_{l=2n_{0}+1}^{n} I_{l}). \text{ This completes the proof.}$

For example, we consider when n = 3 and $\tau_1 \sim \tau_2 \not\sim \tau_3$. Let $I_{12} = I_1 \cap I_2 = \ker \tau_1 \cap \ker \tau_2$. Then dim $A/I_{12} = 2$. By Corollary 1 in [9], there is a probability measure μ^{12} such that
$$\begin{split} \|S_{f}^{\mu^{12}}\| &= \|f + I_{12}\| \text{ for all } f \in A. \text{ Let } S^{\mu^{3}} \text{ be the isometric representation of } A/I_{3} \text{ where } \\ I_{3} &= \ker \tau_{3}. \text{ Let } \mu = (\mu^{12} + \mu^{3})/2. \text{ Then } \mu^{12} \perp \mu^{3}, H^{2}(\mu) \cap I^{\perp} = (H^{2}(\mu^{12}) \cap I_{12}^{\perp}) \oplus (H^{2}(\mu^{3}) \cap I_{3}^{\perp}), \\ S_{f}^{\mu} &= S_{f}^{\mu^{12}} \oplus S_{f}^{\mu^{3}}, \quad (f \in A), \ (S_{f}^{\mu^{12}})^{*}k_{j} = \overline{\tau_{j}(f)}k_{j}, \quad (j = 1, 2), \text{ and } \ (S_{f}^{\mu^{3}})^{*}k_{3} = \overline{\tau_{3}(f)}k_{3}. \text{ Hence} \end{split}$$

$$\|S_{f}^{\mu}\| = \max(\|S_{f}^{\mu^{12}}\|, \|S_{f}^{\mu^{3}}\|) = \max(\|f + I_{12}\|, |\tau_{3}(f)|) = \sup_{\nu \in (A/I)^{*}, \|\nu\| \leq 1} |\int_{X} f d\nu| = \|f + I\|.$$

Hence S^{μ} is an isometric representation of A/I where $I = I_{12} \cap I_3$. By the theorem of T.Nakazi (cf. [8]), $||f + I_{12}||$ can be written using $\rho_1 = \sup\{|\tau_1(f)|; f \in \ker \tau_2, ||f|| \le 1\}$.

Corollary 4.2. Let A be a uniform algebra and $I = \bigcap_{i=1}^{n} \ker \tau_{j}$ and $\tau_{i} \not\sim \tau_{j} (i \neq j)$. Then there exists an isometric representation S^{μ} of A/I, and $||f+I|| = \max(|\tau_1(f)|, ..., |\tau_n(f)|)$.

Proof. Since $\tau_i \not\sim \tau_j$ $(i \neq j)$, there exist probability measures $\mu^1, ..., \mu^n$ such that $\mu = (\mu^1 + ... + \mu^n)/n, \ \mu^i \perp \mu^j \ (i \neq j), \ H^2(\mu) \cap I^\perp = (H^2(\mu^1) \cap I^\perp) \oplus ... \oplus (H^2(\mu^n) \cap I^\perp),$ $S_f^{\mu} = S_f^{\mu^1} \oplus \ldots \oplus S_f^{\mu^n}$. Since $(S_f^{\mu^j})^* k_j = \overline{\tau_j(f)} k_j$, and $(S_f^{\mu^j})^*$ is a rank 1 operator on $H^2(\mu) \cap$ $(\ker \tau_j)^{\perp} = \text{span } \{k_j\}, \text{ it follows that } \|S_f^{\mu^j}\| = \|(S_f^{\mu^j})^*\| = |\tau_j(f)|.$ Then

$$||S_{f}^{\mu}|| = \max(||S_{f}^{\mu^{1}}||, ..., ||S_{f}^{\mu^{n}}||) = \max(|\tau_{1}(f)|, ..., |\tau_{n}(f)|) = \sup_{\nu \in (A/I)^{*}, ||\nu|| \leq 1} |\int_{X} f d\nu| = ||f + I||.$$

This completes the proof.

Corollary 4.3. Let A be a uniform algebra and $I = \bigcap_{i=1}^{n} \ker \tau_i$ and $\tau_i \not\sim \tau_j (i \neq j)$. Suppose that S^{μ} is an isometric representation of A/I. Then, (1) $\mu = \sum_{j=1}^{n} \mu^{j}, \ \mu^{i} \perp \mu^{j} \ (i \neq j), \ \mu^{j} \ll m^{j}$ where μ^{j} is a positive measure and m^{j} is some representing measure for τ_j . (2) $S_f^{\mu} = \sum_{j=1}^n \oplus S_f^{\mu^j}$ $(f \in A)$ where μ^j is divided by its total variation and S^{μ^j} is an isometric representation of A/I_j , where $I_j = \ker \tau_j$.

(3) S_f^{μ} is an isometric representation of a diagonal $n \times n$ matrix for any f in A.

Proof. By the proof of (2) of Theorem 3.2 and Theorem 4.1, (1), (2) and (3) holds.

If A/I is 2-dimensional and semisimple, then there exist τ_1, τ_2 in M(A) such that $\tau_1 \neq \tau_2$ and $I = \ker \tau_1 \cap \ker \tau_2$. For j = 1, 2, there exists $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Then $f_j + I$ is idempotent in A/I and $A/I = \text{span}\{f_1 + I, f_2 + I\}$. If n = 2, then

$$ho_1 = \sup\{| au_1(f)|\;;\; f\in \ker au_2,\; \|f\|\leq 1\},$$

$$\rho_1(\mu) = \sup\{|\tau_1(f)| \ ; \ f \in \ker \tau_2, \ \|f\|_{\mu} \le 1\}$$

where ||f|| denotes the supnorm of f in A and $||f||_{\mu} = \langle f, f \rangle_{\mu} = (\int |f|^2 d\mu)^{1/2}$. Then ρ_1 is a Gleason distance between τ_1 and τ_2 , and $||f_1 + I|| = 1/\rho_1$, $||f_1 + I||_{\mu} = 1/\rho_1(\mu)$. The following proposition is essentially known (cf. Lemma 3 of [9]).

Proposition 4.4. If A/I is 2-dimensional and semisimple, then the following conditions are equivalent.

(1) S^{μ} is an isometric representation of A/I.

(2) $||k_1||_{\mu} = \rho_1(\mu)/\rho_1.$

(3) $||k_1||_{\mu} = ||f_1 + I|| / ||f_1 + I||_{\mu}$.

Proof. By Theorem 3.2, (1) implies (3). By the above remark, (2) is equivalent to (3). It is sufficient to show that (3) implies (1). By Theorem 3.2, if (3) holds, then $||S_{f_1}^{\mu}|| = ||f_1 + I||$. By the above remark, this implies $||S_{f_1}^{\mu}|| = 1/\rho_1$. By the theorem of T.Nakazi (cf. [8]), if $I = \{f \in A ; \tau_1(f) = \tau_2(f) = 0\}$, then

$$\begin{split} \|f+I\| &= \sqrt{\left|\frac{\tau_1(f) - \tau_2(f)}{2}\right|^2 \left(\frac{1}{\rho_1^2} - 1\right) + \left(\frac{|\tau_1(f)| + |\tau_2(f)|}{2}\right)^2} \\ &+ \sqrt{\left|\frac{\tau_1(f) - \tau_2(f)}{2}\right|^2 \left(\frac{1}{\rho_1^2} - 1\right) + \left(\frac{|\tau_1(f)| - |\tau_2(f)|}{2}\right)^2}. \end{split}$$

Since $||S_{f_1}^{\mu}|| = 1/\rho_1$, it follows from the theorem of I.Feldman, N.Krupnik and A.Markus (cf. [5]) that

$$||f + I|| = ||\tau_1(f)S_{f_1}^{\mu} + \tau_2(f)S_{f_2}^{\mu}|| = ||S_f^{\mu}||.$$

This completes the proof.

T. Nakazi and K. Takahashi [9] proved that there exists an isometric representation of A/I in the case when dim A/I = 2. The following theorem gives a concrete matrix representation of A/I.

Theorem 4.5. Suppose A/I is 3-dimensional and semisimple. If $\tau_1 \sim \tau_2 \not\sim \tau_3$ and S^{μ} is an isometric representation of A/I, then A/I is isometric to $\{S_f^{\mu}; f \in A\} =$ span $\{S_{f_1}^{\mu}, S_{f_2}^{\mu}, S_{f_3}^{\mu}\}, S_f^{\mu} = \tau_1(f)S_{f_1}^{\mu} + \tau_2(f)S_{f_2}^{\mu} + \tau_3(f)S_{f_3}^{\mu}, and$

$$(S_{f_1}^{\mu})^* = \begin{pmatrix} 1 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_2}^{\mu})^* = \begin{pmatrix} 0 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_3}^{\mu})^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$x = rac{-\langle k_2, k_1
angle_{\mu}}{\sqrt{\|k_1\|_{\mu}^2 \|k_2\|_{\mu}^2 - |\langle k_1, k_2
angle_{\mu}|^2}}$$

Proof. This follows from Lemma 2.1 and Theorem 4.1.

If $\mathcal{B} \subset B(H)$ and dim H = 3, then

$$P_1 = \left(egin{array}{ccc} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}
ight), \quad P_2 = \left(egin{array}{ccc} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{array}
ight), \quad P_3 = \left(egin{array}{ccc} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{array}
ight),$$

It follows from a 2-dimensional case that if y = z = 0, then \mathcal{B} is a Q-algebra.

If the following condition (1) implies (2) for any distinct points $\tau_1, ..., \tau_n \in M(A)$ and complex numbers $w_1, ..., w_n$, then we say that A/I satisfies the Pick property. (1) $[(1 - w_i \overline{w_j}) k_{ji}]_{i,j=1}^n \ge 0$, where $k_{ij} = \langle k_i, k_j \rangle_{\mu}$, and $\tau_j(f) = \langle f, k_j \rangle_{\mu}$, $(f \in A)$. (2) There exists $f \in A$ such that $\tau_j(f) = w_j$, $(1 \le j \le n)$ and $||f + I|| \le 1$. The following proposition is essentially known.

Proposition 4.6. Let A/I be an n-dimensional semisimple commutative Banach algebra. Then $S^{\mu}: A/I \to B(H^2(\mu) \cap I^{\perp})$ is isometric if and only if A/I satisfies the Pick property.

Proof. Suppose S^{μ} is isometric. For any $w_1, ..., w_n \in \mathbf{C}$, there exists an $f \in A$ such that $\tau_j(f) = w_j$, $(1 \leq j \leq n)$. Suppose $[(1 - w_i \overline{w_j})k_{ji}]_{i,j=1}^n \geq 0$. For any complex numbers $\alpha_1, ..., \alpha_n$, let $k = \sum_{j=1}^n \alpha_j k_j$. Then $||k||_{\mu}^2 = \sum_{i,j=1}^n \overline{\alpha_i} \alpha_j k_{ji}$. Since $(S_f^{\mu})^* k_j = \overline{\tau_j(f)} k_j$, $(S_f^{\mu})^* k = \sum_{j=1}^n \alpha_j \overline{\tau_j(f)} k_j$. By (1),

$$||k||_{\mu}^{2} - ||(S_{f}^{\mu})^{*}k||_{\mu}^{2} = \sum_{i,j=1}^{n} \overline{\alpha_{i}} \alpha_{j} (1 - w_{i} \overline{w_{j}}) k_{ji} \ge 0.$$

Since $H^2(\mu) \cap I^{\perp}$ is spanned by $k_1, ..., k_n$, this implies that $||(S_f^{\mu})^*|| \leq 1$. Since S^{μ} is isometric, $||f + I|| = ||S_f^{\mu}|| \leq 1$. Therefore A/I satisfies the Pick property. Conversely, suppose A/I satisfies the Pick property and $||S_f^{\mu}|| = 1$. Since $(S_f^{\mu})^*k_j = \overline{\tau_j(f)}k_j$ and $||(S_f^{\mu})^*|| = 1$, it follows that

$$\sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j (1 - \tau_i(f) \overline{\tau_j(f)}) k_{ji} = \|k\|_{\mu}^2 - \|(S_f^{\mu})^* k\|_{\mu}^2 \ge 0,$$

and hence $[(1 - \tau_i(f)\overline{\tau_j(f)})k_{ji}]_{i,j=1}^n \ge 0$. By the Pick property, there exists $g \in A$ such that $||g + I|| \le 1$ and $\tau_j(g) = \tau_j(f), (1 \le j \le n)$. Therefore $||f + I|| = ||g + I|| \le 1 = ||S_f^{\mu}||$. Since the reverse inequality $||S_f^{\mu}|| \le ||f + I||$ is always holds, $||S_f^{\mu}|| = ||f + I||$. This completes the proof.

5. Q-Algebras of a Disc Algebra

In this section, we assume that A is the disc algebra and dim A/I = 3. For $f \in A$, let $||f+I|| = ||f+I||_{A/I}$. Since $M(A) = \overline{\mathbf{D}} = \{|z| \leq 1\}$, for each $1 \leq j \leq 3$, τ_j is just an evaluation functional at a point of $\overline{\mathbf{D}}$ and so we write that $\tau_1 = a$, $\tau_2 = b$ and $\tau_3 = c$, where a, b and c are in $\overline{\mathbf{D}}$. By Theorem 3.2, we may assume that a, b and c are in $\mathbf{D} = \{|z| < 1\}$. Theorem 5.2 shows that the set of all 3-dimensional semisimple Q-algebras of the disc algebra is a proper subset in the set of all 3-dimensional semisimple commutative operator algebras with unit on a Hilbert space of dimension 3. However Theorem 5.2 has not solved Problem 2 yet. We use Lemma 5.1 to prove Theorem 5.2. Let a, b, c be the distinct points in the open unit disc \mathbf{D} . Let T(a, b, c) denote the subset of \mathbf{C}^3 which consists of all $(x, y, z) \in \mathbf{C}^3$ satisfying

$$1 + |x|^{2} = \left|\frac{1 - \bar{b}a}{a - b}\right|^{2}, \quad 1 + |z|^{2} = \left|\frac{1 - \bar{c}b}{b - c}\right|^{2},$$
$$1 + |y|^{2} \left|\frac{a - b}{1 - \bar{b}a}\right|^{2} = \left|\frac{1 - \bar{a}c}{c - a}\right|^{2}.$$

This implies that $x \neq 0, y \neq 0$, and $z \neq 0$. T(a, b, c) is characterized by saying that the absolute values of x, y, z are fixed and that their argument are arbitrary. In the following, we consider some inequalities of x, y, and z. For j = 1, 2, 3, there exists $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Hence, $f_1(a) = f_2(b) = f_3(c) = 1$, and $f_1(b) = f_1(c) = f_2(a) = f_2(c) = f_3(a) = f_3(b) = 0$.

Lemma 5.1. Let a, b, c be the distinct points in **D**. Let $f \in A$. Let $I = \{g \in A ; g(a) = g(b) = g(c) = 0\}$. Let $d\mu = \frac{d\theta}{2\pi}$. (1) $S_f^{\mu} = f(a)S_{f_1}^{\mu} + f(b)S_{f_2}^{\mu} + f(c)S_{f_3}^{\mu}$, and

$$(S_{f_1}^{\mu})^* = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_2}^{\mu})^* = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_3}^{\mu})^* = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(x, y, z) \in T(a, b, c)$.

(2) $||f + I|| = ||S_f^{\mu}||$, $(f \in A)$. That is, A/I is isometrically isomorphic to the 3-dimensional semisimple commutative operator algebra on $H^2(\mu) \cap I^{\perp}$ which is spanned by

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(x, y, z) \in T(a, b, c)$.

Proof. $H^2(\mu) \cap I^{\perp}$ is a 3-dimensional Hilbert space which is spanned by

$$k_1(z) = rac{1}{1 - ar{a}z}, \quad k_2(z) = rac{1}{1 - ar{b}z}, \quad k_3(z) = rac{1}{1 - ar{c}z},$$

For orthonormal basis ψ_1, ψ_2, ψ_3 defined in Proposition 2.3,

$$\psi_1(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}, \quad \psi_2(z) = \gamma_2 \frac{z-a}{1-\bar{a}z} \frac{\sqrt{1-|b|^2}}{1-\bar{b}z}, \quad \psi_3(z) = \gamma_3 \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z} \frac{\sqrt{1-|c|^2}}{1-\bar{c}z},$$

where

$$\gamma_2 = -\left(\frac{a-b}{1-\bar{a}b}\right)^{-1} \left|\frac{a-b}{1-\bar{a}b}\right|, \quad \gamma_3 = \left(\frac{a-c}{1-\bar{a}c}\right)^{-1} \left|\frac{a-c}{1-\bar{a}c}\right| \left(\frac{b-c}{1-\bar{b}c}\right)^{-1} \left|\frac{b-c}{1-\bar{b}c}\right|.$$

Since

$$k_2-(k_2,\psi_1)\psi_1=rac{(ar b-ar a)(z-a)}{(1-ar ba)(1-ar az)(1-ar bz)},$$

it follows that

$$\|k_2-(k_2,\psi_1)\psi_1\|=\left|rac{ar{b}-ar{a}}{1-ar{b}a}
ight|rac{1}{\sqrt{1-|b|^2}}.$$

Hence

$$\psi_2 = rac{k_2 - (k_2, \psi_1)\psi_1}{\|k_2 - (k_2, \psi_1)\psi_1\|} = \gamma_2 rac{z-a}{1-ar{a}z} rac{\sqrt{1-|b|^2}}{1-ar{b}z}.$$

Since

$$k_3-\langle k_3,\psi_1
angle\psi_1-\langle k_3,\psi_2
angle\psi_2=rac{(ar{a}-ar{c})(b-ar{c})(z-a)(z-b)}{(1-ar{c}a)(1-ar{c}b)(1-ar{a}z)(1-ar{b}z)(1-ar{c}z)},$$

it follows that

$$\psi_3 = \frac{k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|} = \gamma_3 \frac{z - a}{1 - \bar{a}z} \frac{z - b}{1 - \bar{b}z} \frac{\sqrt{1 - |c|^2}}{1 - \bar{c}z}.$$

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If we calculate x, y, z using the formulas in Proposition 2.3, then it follows that $(x, y, z) \in T(a, b, c)$. Then

$$x = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2 \rangle|^2}} = \frac{\frac{-1}{1 - ba}}{\sqrt{\frac{1}{(1 - |a|^2)(1 - |b|^2)} - \frac{1}{|1 - \bar{a}b|^2}}} = \gamma_4 \frac{\sqrt{1 - |a|^2} \sqrt{1 - |b|^2}}{|a - b|},$$

where

$$\gamma_4 = -\frac{1-\bar{a}b}{|1-\bar{a}b|}.$$

Hence

$$1 + |x|^2 = \left| \frac{1 - \bar{b}a}{a - b} \right|^2.$$

Since

$$-\langle k_3,\psi_1
angle-\langle k_3,\psi_2
angle x=rac{\sqrt{1-|a|^2}}{1-ar{c}a}rac{1-ar{a}b}{ar{b}-ar{a}}rac{ar{c}-ar{b}}{1-bar{c}},$$

it follows that

$$y = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|} = \gamma_5 \frac{1 - \bar{a}b}{\bar{a} - \bar{b}} \frac{\sqrt{1 - |a|^2} \sqrt{1 - |c|^2}}{|a - c|},$$

where

$$\gamma_5 = \left(\frac{a-b}{1-a\overline{b}}\right) \left|\frac{a-b}{1-a\overline{b}}\right|^{-1} \left(\frac{b-c}{1-\overline{b}c}\right)^{-1} \left|\frac{b-c}{1-\overline{b}c}\right| \frac{|1-\overline{c}a|}{1-\overline{c}a}.$$

Since

$$\langle k_3,\psi_2
angle=\overline{\gamma_2}rac{ar c-ar a}{1-aar c}rac{\sqrt{1-|b|^2}}{1-bar c},$$

it follows that

$$z=rac{-\langle k_3,\psi_2
angle}{\|k_3-\langle k_3,\psi_1
angle\psi_1-\langle k_3,\psi_2
angle\psi_2\|}=\gamma_6rac{\sqrt{1-|b|^2}\sqrt{1-|c|^2}}{|b-c|},$$

where

$$\gamma_6 = \left(\frac{a-b}{1-\bar{a}b}\right) \left|\frac{a-b}{1-\bar{a}b}\right|^{-1} \left(\frac{c-a}{1-\bar{a}c}\right)^{-1} \left|\frac{c-a}{1-\bar{a}c}\right| \frac{|1-b\bar{c}|}{1-b\bar{c}}.$$

Since $|\gamma_2| = |\gamma_3| = |\gamma_4| = |\gamma_5| = |\gamma_6| = 1$, it follows that

$$1 + |y|^2 \left| \frac{a-b}{1-\bar{b}a} \right|^2 = \left| \frac{1-\bar{a}c}{c-a} \right|^2, \quad 1 + |z|^2 = \left| \frac{1-\bar{c}b}{b-c} \right|^2$$

Hence, (1) follows. It is sufficient to prove (2). By the theorem of D.Sarason (cf. [2], p.125, [10], Vol.1, p.231, [11]), $||f + I|| = ||S_f^{\mu}||$. Then $(S_{f_1}^{\mu})^* k_1 = k_1, (S_{f_1}^{\mu})^* k_2 = (S_{f_1}^{\mu})^* k_3 = 0, (S_{f_2}^{\mu})^* k_2 = k_2, (S_{f_2}^{\mu})^* k_3 = (S_{f_2}^{\mu})^* k_1 = 0, (S_{f_3}^{\mu})^* k_3 = k_3, \text{ and } (S_{f_3}^{\mu})^* k_1 = (S_{f_3}^{\mu})^* k_2 = 0.$ By Proposition 2.3,

$$(S_{f_1}^{\mu})^* = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_2}^{\mu})^* = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_3}^{\mu})^* = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $f - f(a)f_1 - f(b)f_2 - f(c)f_3 \in I$ and $I(H^2(\mu) \cap I^{\perp}) \subset IH^2(\mu) \subset H^2(\mu) \cap I^{\perp}$, it follows that

$$(S^{\mu}_{f}-S^{\mu}_{f(a)f_{1}+f(b)f_{2}+f(c)f_{3}})\psi=S^{\mu}_{f-f(a)f_{1}-f(b)f_{2}-f(c)f_{3}}\psi=0,\quad(\psi\in I^{\perp}_{\mu}).$$

Hence

$$S_{f}^{\mu} = S_{f(a)f_{1}+f(b)f_{2}+f(c)f_{3}}^{\mu} = f(a)S_{f_{1}}^{\mu} + f(b)S_{f_{2}}^{\mu} + f(c)S_{f_{3}}^{\mu}$$

This completes the proof.

For example, if $(a, b, c) = (0, \frac{1}{2}, \frac{1}{3})$ and $(x, y, z) = (-\sqrt{3}, 4\sqrt{2}, -2\sqrt{6})$, then the algebra span $\{P_1, P_2, P_3\}$ is isometrically isomorphic to A/I which is a Q-algebra of a disc algebra.

Theorem 5.2. Let a, b, c be the distinct points in **D**. Let $f \in A$. Let $d\mu = \frac{d\theta}{2\pi}$. Let $I = \{g \in A ; g(a) = g(b) = g(c) = 0\}$. If a 3-dimensional semisimple commutative operator algebra \mathcal{B} on $H^2(\mu) \cap I^{\perp}$ is isometrically isomorphic to A/I, then \mathcal{B} is unitarily equivalent to the 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space H spanned by P_1, P_2, P_3 such that

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

where x, y, z satisfy (1) \sim (3).

(1)
$$xyz \neq 0,$$

(2)
$$\frac{1}{\sqrt{1+|y|^2}} < \frac{1}{\sqrt{1+|x|^2}} + \frac{1}{\sqrt{1+|z|^2}},$$

(3)
$$|y| > \frac{|xz|}{\sqrt{1+|z|^2}+1},$$

Proof. By the theorem of B.Cole and J.Wermer (cf. [3]) and (2) of Theorem 2.4, we may assume that H is spanned by the orthonormal basis ψ_1, ψ_2, ψ_3 which are calculated in the proof of Lemma 5.1. By Lemma 5.1, there are complex numbers x, y, z satisfying $(x, y, z) \in T(a, b, c)$. Since

$$\begin{split} 1+|x|^2 &= \left|\frac{1-\bar{b}a}{a-b}\right|^2 > 1, \quad 1+|z|^2 = \left|\frac{1-\bar{c}b}{b-c}\right|^2 > 1, \\ 1+|y|^2 \left|\frac{a-b}{1-\bar{b}a}\right|^2 &= \left|\frac{1-\bar{a}c}{c-a}\right|^2 > 1, \end{split}$$

(1) follows. Let

$$ho(z,w) = \left|rac{z-w}{1-ar w z}
ight|.$$

Then

$$ho(a,b) = rac{1}{\sqrt{1+|x|^2}}, \quad
ho(b,c) = rac{1}{\sqrt{1+|z|^2}}, \quad
ho(c,a) = \sqrt{rac{1+|x|^2}{1+|x|^2+|y|^2}} > rac{1}{\sqrt{1+|y|^2}},$$

Since $\rho(c,a) \le \rho(a,b) + \rho(b,c),$ (2) follows. Let

$$d(z,w) = rac{1}{2}\lograc{1+
ho(z,w)}{1-
ho(z,w)}.$$

Since $d(c, a) \le d(a, b) + d(b, c)$,

$$\frac{\sqrt{1+|x|^2+|y|^2}+\sqrt{1+|x|^2}}{\sqrt{1+|x|^2+|y|^2}-\sqrt{1+|x|^2}} \le \frac{\sqrt{1+|z|^2}+1}{\sqrt{1+|z|^2}-1}\cdot \frac{\sqrt{1+|x|^2}+1}{\sqrt{1+|x|^2}-1}.$$

Hence

$$\frac{\sqrt{1+|x|^2}+1}{|y|} < \frac{\sqrt{1+|x|^2+|y|^2}+\sqrt{1+|x|^2}}{|y|} \le \frac{\sqrt{1+|z|^2}+1}{|z|} \cdot \frac{\sqrt{1+|x|^2}+1}{|x|}.$$

this implie (3). This completes the proof.

Example 5.3. In Example 2.5, \mathcal{B}_0 is isometrically isomorphic to \mathcal{B}_2 . Since $y_0 = 0$, it follows from Theorem 5.2 that \mathcal{B}_2 is not isometrically isomorphic to a 3-dimensional semisimple Q-algebra A/I where A is a disc algebra. Hence \mathcal{B}_0 is also not isometrically isomorphic to a Q-algebra A/I. Therefore \mathcal{B}_0 and \mathcal{B}_2 is the example to show that the set of all 3-dimensional semisimple Q-algebra A/I where A is a disc algebra is smaller than the set of all 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space.

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