FINANCIAL MARKETS WITH MEMORY II: INNOVATION PROCESSES AND EXPECTED UTILITY MAXIMIZATION

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Abstract. We develop a prediction theory for a class of processes with stationary increments. In particular, we prove a prediction formula for these processes from a finite segment of the past. Using the formula, we prove an explicit representation of the innovation processes associated with the stationary increments processes. We apply the representation to obtain a closed-form solution to the problem of expected logarithmic utility maximization for the financial markets with memory introduced by the first and second authors.

1. Introduction

We first recall the price process of a stock introduced in [1]. Let T be a positive constant. We consider a stock with price $S(t)$ at time $t \in [0, T]$. We suppose that $S(0)$ is a positive constant and that $S(\cdot)$ satisfies the stochastic differential equation

$$dS(t) = S(t) \left\{ m(t)dt + \sigma(t)dY(t) \right\}, \tag{1.1}$$

where the process $m(\cdot)$ of mean rate of return and the volatility process $\sigma(\cdot)$ may be random, though we are especially interested in the case in which $m(\cdot)$ is deterministic and $\sigma(\cdot)$ is a positive constant. In the standard model [12, Chapter 1], the process $Y(\cdot)$ is a one-dimensional Brownian motion. However here we assume that $Y(\cdot)$ is a continuous process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with stationary increments such that $Y(0) = 0$, and satisfies the continuous-time AR(∞)-type equation

$$\frac{dY}{dt}(t) + \int_{-\infty}^{t} a(t-s) \frac{dY}{dt}(s) ds = \frac{dW}{dt}(t) \tag{1.2}$$

(see (3.1) below for its precise formulation), where $(W(t))_{t \in \mathbf{R}}$ is a one-dimensional Brownian motion such that $W(0) = 0$, and dY/dt and dW/dt are the derivatives of $Y(\cdot)$ and $W(\cdot)$ respectively in the random distribution sense. The kernel $a(\cdot)$ is a finite, integrable, completely monotone

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function on $(0,\infty)$. In the simplest case $a(\cdot)\equiv 0$, $Y(\cdot)$ is reduced to the Brownian motion, i.e., $Y(\cdot)=W(\cdot)$.

For the filtration $\{\mathcal{F}(t)\}_{0\le t\le T}$ of the financial market, we take the augmentation of the filtration generated by $Y(\cdot)$. It follows that $Y(\cdot)$ is a (Gaussian) $\{\mathcal{F}(t)\}$-semimartingale of the form

$$Y(t)=B(t)-\int_0^t \alpha(s)ds \qquad (0\le t\le T), \tag{1.3}$$

where $\alpha(\cdot)$ is an $\{\mathcal{F}(t)\}$-adapted process and $B(\cdot)$ is an $\{\mathcal{F}(t)\}$-Brownian motion called the *innovation process* (see Section 5). Notice that $B(\cdot)$ and $W(\cdot)$ are different. The stochastic differential (1.1) may now be interpreted in the usual sense, and the solution $S(\cdot)$ is given by, for $0\le t\le T$,

$$S(t)=S(0)\exp\left[\int_0^t \sigma(s)dY(s)+\int_0^t\left\{m(s)-\frac{1}{2}\sigma(s)^2\right\}ds\right]. \tag{1.4}$$

The integral on the left-hand side of (1.2) has the effect of incorporating memory into the dynamics of $Y(\cdot)$, whence that of $S(\cdot)$. The financial market with stock price $S(\cdot)$ is complete under suitable conditions. Moreover, if $\sigma(\cdot)$ as well as the risk-free interest rate process $r(\cdot)$ are constant, then the Black–Scholes formula for option pricing holds in this market (see Section 6).

The simplest nontrivial example of $a(\cdot)$ above is $a(t)=pe^{-qt}$ for $t>0$ with $p,q\in(0,\infty)$. In this case, we have

$$Y(t)=W(t)-\int_0^t\left(\int_{-\infty}^s pe^{-(q+p)(s-u)}dW(u)\right)ds \qquad (t\in\mathbf{R}) \tag{1.5}$$

(Example 5.3). It should be noted that this (as well as (3.3) with (3.4) below) is not a semimartingale representation of $Y(\cdot)$ since $W(\cdot)$ is not $\{\mathcal{F}(t)\}$-adapted. If we further assume that $\sigma(\cdot)\equiv\sigma$ and $m(\cdot)\equiv m$ with σ and m being constants, then $S(\cdot)$ is given by, for $0\le t\le T$,

$$\begin{aligned} S(t)=S(0)\exp\Bigg\{&\sigma W(t)+\left(m-\frac{1}{2}\sigma^2\right)t \\ &-\sigma\int_0^t\left(\int_{-\infty}^s pe^{-(q+p)(s-u)}dW(u)\right)ds\Bigg\}. \end{aligned} \tag{1.6}$$

This stock price $S(\cdot)$ is worth special attention. Compared with the stock price in the Black–Scholes model, $S(\cdot)$ has two additional parameters p and q which describe the memory. As stated above, the financial market with $S(\cdot)$ is complete and the Black–Scholes formula holds in it. The difference between the market with $S(\cdot)$ and the Black–Scholes market is illustrated by the historical volatility $\mathrm{HV}(\cdot)$ defined by

$$\mathrm{HV}(t-s):=\sqrt{\frac{\mathrm{Var}\left\{\log(S(t)/S(s))\right\}}{t-s}} \qquad (t>s\ge 0),$$

where the variance is defined with respect to the physical probability measure P. While $\mathrm{HV}(\cdot)$ is constant in the Black–Scholes model, we have $\mathrm{HV}(t) = f(t)$ for $S(\cdot)$ in (1.6), where $f(t) = f(t; \sigma, p, q)$ is a decreasing function on $(0, \infty)$ defined by

$$f(t) = \sigma \sqrt{\frac{q^2}{(p+q)^2} + \frac{p(2q+p)}{(p+q)^3} \cdot \frac{(1 - e^{-(p+q)t})}{t}} \qquad (t > 0),$$

which satisfies $\lim_{t\to\infty} f(t) = q\sigma/(p+q)$ and $\lim_{t\to 0+} f(t) = \sigma$. In [2], an empirical study on the model (1.6) was carried out. There, the values $\mathrm{HV}(t)$ $(t = 1, 2, 3, \dots)$ are estimated from real market data, such as closing values of S&P 500 index. It is found that $\mathrm{HV}(t)$ is not constant unlike in the Black–Scholes model, and very often reveals features in agreement with those of $f(\cdot)$. The function $f(t)$ is fitted by nonlinear least squares, and the parameters σ, p and q are estimated in this way. It is found that the fitted $f(\cdot)$ approximates the estimated $\mathrm{HV}(\cdot)$ quite well when the market is stable.

In this paper, as a typical financial problem for the market with stock price (1.4), we consider expected logarithmic utility maximization from terminal wealth. We are especially interested in the case (1.6) explained above. In principle, we can reduce such a problem to that for the standard financial markets, as described in [12], by using the semimartingale representation (1.3) of $Y(\cdot)$. From the financial viewpoint, however, results thus obtained would not be of much value unless we have sufficient knowledge about $\alpha(\cdot)$ in (1.3). We thus need to resolve the problem of obtaining a good representation of $\alpha(\cdot)$. One of the main results of this paper is the following representation (Theorem 5.2):

$$\alpha(t) = \int_0^t k(t,s)dY(s) \qquad (0 \le t \le T), \tag{1.7}$$

where $k(t,s)$ is a deterministic function represented explicitly in terms of the $\mathrm{AR}(\infty)$-coefficient $a(\cdot)$ and the corresponding $\mathrm{MA}(\infty)$-coefficient $c(\cdot)$ (cf. Section 3). In particular, for $Y(\cdot)$ in (1.5), $k(t,s)$ has a very simple form (Example 5.3). We can regard (1.3) with (1.7) as an explicit representation of the innovation process $B(\cdot)$ in terms of $Y(\cdot)$.

To prove the representation of the form (1.7), we construct a prediction theory for $Y(\cdot)$. In particular, we prove an explicit prediction formula from a finite segment of the past for $Y(\cdot)$. We remark that, in general, it is not an easy task to obtain such an explicit finite-past prediction formula for continuous-time processes with stationary increments. In fact, known results are obtained only for special processes such as fractional Brownian motion by using their special properties (cf. [5]). In this paper, we use a general method for processes with stationary increments, as we now explain. Let $t \in (0, \infty)$. We write $M(Y)$ for the real Hilbert space spanned

by $\{Y(s) : s \in \mathbf{R}\}$ in $L^2(\Omega, \mathcal{F}, P)$ and $\|\cdot\|$ for its norm defined by $\|Z\| = E[Z^2]^{1/2}$ for $Z \in M(Y)$. Let I be a closed interval of $\mathbf{R}$ such as $[0,t]$, $(-\infty, t]$, and $[0,\infty)$. Let $M_I(Y)$ be the closed subspace of $M(Y)$ spanned by $\{Y(s) : s \in I\}$. We write P_I for the orthogonal projection from $M(Y)$ onto $M_I(Y)$, and $P_I^\perp$ for its orthogonal complement: $P_I^\perp Z = Z - P_I Z$ for $Z \in M(Y)$. Then since $Y(\cdot)$ is Gaussian, we have $E[Z|\mathcal{F}(t)] = P_{[0,t]}Z$ for $Z \in M(Y)$, where throughout the paper $E[\cdot|\cdot]$ denotes the conditional expectation with respect to the original probability measure P. In this method, we first prove the equality

$$P_{[0,t]} = \operatorname*{s-lim}_{n\to\infty} \{P_{[0,\infty)}P_{(-\infty,t]}\}^n \tag{1.8}$$

and then use it to obtain the representations of quantities related to $P_{[0,t]}$ in terms of AR(∞) and MA(∞) coefficients. It should be noticed that (1.8) is equivalent to

$$M_{[0,t]}(Y) = M_{(-\infty,t]}(Y) \cap M_{[0,\infty)}(Y) \tag{1.9}$$

(see the proof of Theorem 4.6 below). What is interesting in this method is that we consider not only the past $M_{(-\infty,t]}(Y)$ but also the future $M_{[0,\infty)}(Y)$ in the prediction from a finite segment of the past.

The above type of method was used in [8] in a simpler framework, i.e., that of discrete-time stationary processes, to obtain a representation of mean-squared prediction error. See [9] and [10] for subsequent results in the same framework. Now, unlike in these references, we develop a similar method to prove the prediction formula itself, rather than a representation of prediction error, for continuous-time processes with stationary increments. This setting is more difficult and requires new techniques. One of the key ingredients in the arguments is the proof of (1.8) or (1.9). Equalities of the type (1.9) are studied by [13], [3], and [16] for continuous-time stationary processes. A discrete-time analogue is proved in [8] by a method similar to that of [13]. In the present setting, however, we need a quite different approach.

In Section 2, we state some necessary facts about processes with stationary increments. In Section 3, we prove an infinite-past prediction formula which we need in Section 4, where we prove a finite-past prediction formula in which $P_{[0,t]}\int f(s)dY(s)$ is represented explicitly. In Section 5, we prove the representation (1.3) of $\alpha(\cdot)$ in (1.7) using the prediction formula. Finally, in Section 6, we describe the implication of (1.7) in the financial markets with stock prices (1.4) via expected logarithmic utility maximization.

2. Processes with stationary increments

In this section, we prove some facts about stationary increments processes which we need in later sections.

We denote by M the Hilbert space of $\mathbf{R}$-valued random variables, defined on a probability space $(\Omega, \mathcal{F}, P)$, with expectation zero and finite variance:

$$M := \{Z \in L^2(\Omega, \mathcal{F}, P) : E[Z] = 0\}$$

with inner product $(Z_1, Z_2) := E[Z_1 Z_2]$ and norm $\|Z_1\| := (Z_1, Z_1)^{1/2}$. By $\mathcal{D}(\mathbf{R})$, we denote the space of all $\phi \in C^\infty(\mathbf{R})$ with compact support, endowed with the usual topology. A *random distribution* (with expectation zero) is a linear continuous map from $\mathcal{D}(\mathbf{R})$ to M. We write $\mathcal{D}'(M)$ for the class of random distributions on $(\Omega, \mathcal{F}, P)$. For $X \in \mathcal{D}'(M)$, the derivative $DX \in \mathcal{D}'(M)$ is defined by $DX(\phi) := -X(d\phi/dx)$. For $X \in \mathcal{D}'(M)$ and an interval I of $\mathbf{R}$, we write $M_I(X)$ for the closed linear hull of $\{X(\phi) : \phi \in \mathcal{D}(\mathbf{R}),\ \text{supp}\ \phi \subset I\}$ in M. In particular, we write $M(X)$ for $M_I(X)$ with $I = \mathbf{R}$. A random distribution X is *stationary* if $(X(\tau_h\phi), X(\tau_h\psi)) = (X(\phi), X(\psi))$ for $\phi, \psi \in \mathcal{D}(\mathbf{R})$ and $h \in \mathbf{R}$, where τ_h is the shift operator defined by $\tau_h\phi(t) := \phi(t+h)$. We write $\mathcal{S}$ for the class of stationary random distributions on $(\Omega, \mathcal{F}, P)$. For $X \in \mathcal{S}$, we write μ_X for the spectral measure of X:

$$(X(\phi), X(\psi)) = \int_{-\infty}^{\infty} \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\mu_X(d\xi) \qquad (\phi, \psi \in \mathcal{D}(\mathbf{R})),$$

where $\hat{\phi}$ is the Fourier transform of ϕ: $\hat{\phi}(\xi) := \int_{-\infty}^{\infty} e^{-it\xi}\phi(\xi)d\xi$. See [11] for details.

In this section, we assume that $(Y(t))_{t\in\mathbf{R}}$ is a real, zero-mean, mean-square continuous process, defined on $(\Omega, \mathcal{F}, P)$, with stationary increments such that $Y(0) = 0$. Thus, for each $a \in \mathbf{R}$, the process $(\Delta Y_a(t) : t \in \mathbf{R})$ defined by $\Delta Y_a(t) := Y(a+t) - Y(t)$ is a zero-mean, weakly stationary process. As usual, we regard $Y(\cdot) \in \mathcal{D}'(M)$ by $Y(\phi) = \int_{-\infty}^{\infty} Y(t)\phi(t)dt$ for $\phi \in \mathcal{D}(\mathbf{R})$. Then it holds that $DY \in \mathcal{S}$. We now assume that

$$DY \text{ is purely nondeterministic,} \tag{2.1}$$

that is, $\bigcap_{t\in\mathbf{R}} M_{(-\infty,t]}(DY) = \{0\}$ or, equivalently, there exists a positive, even and measurable function $\Delta_{DY}(\cdot)$ on $\mathbf{R}$, called the *spectral density* of DY, satisfying $\mu_{DY}(d\xi) = \Delta_{DY}(\xi)d\xi$ and

$$\int_{-\infty}^{\infty} \frac{\Delta_{DY}(\xi)}{1+\xi^2}d\xi < \infty, \quad \int_{-\infty}^{\infty} \frac{|\log \Delta_{DY}(\xi)|}{1+\xi^2}d\xi < \infty$$

(see [15]). Let $DY(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi)Z_{DY}(d\xi)$ with $\phi \in \mathcal{D}(\mathbf{R})$ be the spectral decomposition of DY as a stationary random distribution, where Z_{DY} is the associated complex-valued random measure such that

$$E[Z_{DY}(A)\overline{Z_{DY}(B)}] = \mu_{DY}(A \cap B).$$

Since $\mu_{DY}\{0\} = 0$, we have the following spectral representation of $Y(\cdot)$ (cf. [11, Theorem 6.1]):

$$Y(t) = \int_{-\infty}^{\infty} \frac{1 - e^{-it\xi}}{i\xi} Z_{DY}(d\xi) \qquad (t \in \mathbf{R}). \tag{2.2}$$

For a closed interval I of $\mathbf{R}$, we see that $M_I(Y)$ defined above is equal to the closed real linear hull of $\{Y(t) : t \in I\}$ in $L^2(\Omega, \mathcal{F}, P)$. In particular, $M(Y)$ is equal to the closed real linear hull of $\{Y(t) : t \in \mathbf{R}\}$. Thus the definitions of $M_I(Y)$ and $M(Y)$ in this section are consistent with those in Section 1.

For $t, s \in \mathbf{R}$, we have $(Y(t), Y(s)) = (Y(-t), Y(-s))$; to see this, we use, e.g., (2.2). Hence, we may have the next definition.

Definition 2.1. We write θ for the Hilbert space isomorphism of $M(Y)$ characterized by $\theta(Y(t)) = Y(-t)$ for $t \in \mathbf{R}$.

Clearly, $\theta^{-1} = \theta$. Recall from Section 1 that, for a closed interval I of $\mathbf{R}$, P_I is the orthogonal projection of $M(Y)$ onto $M_I(Y)$. We define $-I := \{-s : s \in I\}$. Then, since $\theta P_I \theta^{-1}$ is a projection of $M(Y)$ and $\theta P_I \theta^{-1}(M(Y)) = M_{-I}(Y)$, it holds that $\theta P_I \theta^{-1} = P_{-I}$.

Since we have assumed (2.1), we have a *canonical Brownian motion* $W = (W(t))_{t \in \mathbf{R}}$ for DY; W is a Brownian motion satisfying $W(0) = 0$ and

$$M_{(-\infty,t]}(DW) = M_{(-\infty,t]}(DY) \qquad (t \in \mathbf{R}). \tag{2.3}$$

By Proposition 2.3 (4) below, it holds that $M(Y) = M(DY) = M(DW) = M(W)$. Therefore, we have the next definition.

Definition 2.2. We define the process $(W^*(t))_{t \in \mathbf{R}}$ by

$$W^*(t) := \theta(W(-t)) \qquad (t \in \mathbf{R}).$$

Since we have, for $t, s \in \mathbf{R}$,

$$\begin{aligned}(W^*(t), W^*(s)) &= (\theta(W(-t)), \theta(W(-s))) = (W(-t), W(-s)) \\ &= (W(t), W(s)),\end{aligned}$$

$W^*(\cdot)$ is also a Brownian motion such that $W^*(0) = 0$.

Proposition 2.3. *Let $t \in \mathbf{R}$ and $-t_0 \le 0 \le t_1$. Let I be a closed interval of $\mathbf{R}$. Then*

(1) $M_{[-t_0,t_1]}(Y) = M_{[-t_0,t_1]}(DY)$;
(2) $M_{(-\infty,t_1]}(Y) = M_{(-\infty,t_1]}(DY)$;
(3) $M_{[-t_0,\infty)}(Y) = M_{[-t_0,\infty)}(DY)$;
(4) $M(Y) = M(DY)$;
(5) $\theta\left(M_{(-\infty,t]}(DW)\right) = M_{[-t,\infty)}(DW^*)$;
(6) $M_{(-\infty,t_1]}(Y) = M_{(-\infty,t_1]}(DW)$;
(7) $M_{[-t_0,\infty)}(Y) = M_{[-t_0,\infty)}(DW^*)$;

(8) $\theta(\int_I f(s)dW(s)) = -\int_{-I} f(-s)dW^*(s)$ *for* $f \in L^2(I, ds)$.

Proof. For $\phi \in \mathcal{D}(\mathbf{R})$ with supp $\phi \subset [-t_0, t_1]$, we have $DY(\phi) = -Y(\phi') \in M_{[-t_0,t_1]}(Y)$, so that $M_{[-t_0,t_1]}(DY) \subset M_{[-t_0,t_1]}(Y)$. For (1), we prove the converse inclusion

$$M_{[-t_0,t_1]}(Y) \subset M_{[-t_0,t_1]}(DY). \tag{2.4}$$

Let $s \in [-t_0, 0)$. Let ρ be an element of $\mathcal{D}(\mathbf{R})$ satisfying supp $\rho \subset [-1, 1]$ and $\int_{-\infty}^{\infty} \rho(u)du = 1$. We define, for $u \in \mathbf{R}$ and large enough $n \in \mathbf{N}$, $\rho_n(u) := n\rho(nu)$ and $\phi_n(u) := \rho_n * I_{[s+(1/n),-1/n]}(u)$. Then $\phi_n \in \mathcal{D}(\mathbf{R})$ and supp $\phi_n \subset [s, 0]$. From $(1 - e^{-is\xi})/(i\xi) = -\int_s^0 e^{-iu\xi}du$ for $\xi \neq 0$, we have, for $\xi \neq 0$,

$$\left|\frac{1 - e^{-is\xi}}{i\xi} + \hat{\phi}_n(\xi)\right| \leq \int_s^0 |1 - \phi_n(u)|du \to 0 \qquad (n \to \infty).$$

On the other hand, $|(1 - e^{-is\xi}) + i\xi\hat{\phi}_n(\xi)|$ is at most

$$\begin{aligned}&|\hat{\rho}\,(\xi/n)| \cdot \left|(1 - e^{i\xi/n}) - \left(e^{-i\xi s} - e^{-i\xi(s+(1/n))}\right)\right| + \left|(1 - \hat{\rho}(\xi/n))(1 - e^{-is\xi})\right| \\ &\leq 2\,|(\xi/n)\hat{\rho}\,(\xi/n)| + 2\,|1 - \hat{\rho}(\xi/n)|\,,\end{aligned}$$

and the right-hand side is bounded and tends to 0 as $n \to \infty$. Combining, $(1 - i\xi)\{(1 - e^{-is\xi})/(i\xi) + \hat{\phi}_n(\xi)\}$ is bounded and tends to 0, as $n \to \infty$, for $\xi \neq 0$. Hence, by the dominated convergence theorem,

$$\begin{aligned}&\lim_{n\to\infty} \|Y(s) + DY(\phi_n)\|^2 \\ &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \left|(1 - i\xi)\left\{\frac{1 - e^{-is\xi}}{i\xi} + \hat{\phi}_n(\xi)\right\}\right|^2 \frac{\Delta_{DY}(\xi)}{1 + \xi^2} d\xi = 0.\end{aligned}$$

Thus $Y(s) \in M_{[-t_0,0]}(DY) \subset M_{[-t_1,t_0]}(DY)$. In the same way, we see that $Y(s) \in M_{[0,t_1]}(DY) \subset M_{[-t_0,t_1]}(DY)$ for $s \in (0, t_1]$. Moreover, $Y(0) = 0 \in M_{[-t_0,t_1]}(DY)$. Hence (2.4) holds. (2)–(4) follow immediately from (1).

Let $\phi \in \mathcal{D}(\mathbf{R})$ with supp $\phi \subset (-\infty, t]$. Then $\theta(DW(\phi))$ is equal to

$$-\int_{-\infty}^{\infty} W^*(-s)\phi'(s)ds = \int_{-\infty}^{\infty} W^*(s)\frac{d}{ds}(\phi(-s))ds = -DW^*(\phi(-\cdot)),$$

so that $\theta(DW(\phi)) \in M_{[-t,\infty)}(DW^*)$. Thus

$$\theta(M_{(-\infty,t]}(DW)) \subset M_{[-t,\infty)}(DW^*).$$

In the same way, we have $\theta(M_{[-t,\infty)}(DW^*)) \subset M_{(-\infty,t]}(DW)$, whence

$$M_{[-t,\infty)}(DW^*) = \theta^2\left(M_{[-t,\infty)}(DW^*)\right) \subset \theta(M_{(-\infty,t]}(DW)).$$

Thus (5) follows. The assertion (6) follows from (2) and (2.3), while we obtain (7) from (5), (6) and $\theta(M_{(-\infty,t_0]}(Y)) = M_{[-t_0,\infty)}(Y)$.

To prove (8), we may assume that f is of the form $f(s) = I_{(a,b]}(s)$ with $(a, b] \subset I$. Then $\theta(\int_I f(s)dW(s))$ is equal to $\theta\,(W(b) - W(a)) = W^*(-b) - W^*(-a)$ or $-\int_{-I} f(-s)dW^*(s)$. Thus (8) follows. □

3. Prediction from an infinite segment of the past

Let $(W(t))_{t\in\mathbf{R}}$ be a one-dimensional standard Brownian motion such that $W(0) = 0$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $(Y(t))_{t\in\mathbf{R}}$ be a zero-mean, mean-square continuous process with stationary increments, defined on $(\Omega, \mathcal{F}, P)$, satisfying $Y(0) = 0$ and

$$DY + a * DY = DW, \tag{3.1}$$

where DY and DW are the derivatives of the stationary increment processes $Y(\cdot)$ and $W(\cdot)$, respectively, whence *stationary* random distributions, and $a * DY$ is the convolution of a deterministic function $a(\cdot)$ and DY (see [11] and [1, Section 2]). We assume that $a(\cdot)$ is of the form

$$a(t) = I_{(0,\infty)}(t) \int_0^\infty e^{-ts}\nu(ds) \qquad (t \in \mathbf{R}), \tag{3.2}$$

where ν is a finite Borel measure on $(0, \infty)$ such that $\int_0^\infty s^{-1}\nu(ds) < \infty$. Thus $a(\cdot)$ is a bounded nonnegative function on $\mathbf{R}$, vanishing on $(-\infty, 0]$, that is completely monotone on $(0, \infty)$. Formally (3.1) can be written as (1.2).

By [1, Theorem 2.13], $Y(\cdot)$ has the following MA(∞)-type representation

$$Y(t) = W(t) - \int_0^t U(s)ds, \tag{3.3}$$

where $(U(t))_{t\in\mathbf{R}}$ is a purely nondeterministic stationary Gaussian process of the form

$$U(t) = \int_{-\infty}^t c(t-s)dW(s) \qquad (t \in \mathbf{R}) \tag{3.4}$$

with canonical representation kernel $c(\cdot)$ such that

$$c(t) = I_{(0,\infty)}(t) \int_0^\infty e^{-ts}\mu(ds) \qquad (t \in \mathbf{R}). \tag{3.5}$$

Here μ is a finite Borel measure on $(0, \infty)$ satisfying $\int_0^\infty s^{-1}\mu(ds) < 1$. Notice that $c(\cdot)$ satisfies $\int_0^\infty c(t)dt < 1$, whence

$$\int_0^\infty c(t)^2 \le c(0+) \int_0^\infty c(t)dt < \infty.$$

The kernel $c(\cdot)$ is determined from $a(\cdot)$ through the relation

$$\left\{1 + \int_0^\infty e^{izt}a(t)dt\right\}\left\{1 - \int_0^\infty e^{izt}c(t)dt\right\} = 1 \qquad (\Im z > 0). \tag{3.6}$$

Moreover, by [1, Theorem 2.13], Y satisfies (2.1).

Let I be a closed interval of $\mathbf{R}$. We define

$$\mathcal{H}_I(Y) = \left\{ f : \begin{array}{l} f \text{ is a real-valued measurable function on } I \text{ satisfying} \\ \int_I f(s)^2 ds < \infty,\ \int_{-\infty}^{\infty} \left\{ \int_I |f(u)| c(u-s) du \right\}^2 ds < \infty \end{array} \right\}.$$

This is the class of $f(\cdot)$ for which we define $\int_I f(s)dY(s)$. We write $\mathcal{H}_I^0$ for the subset of $\mathcal{H}_I(Y)$ defined by

$$\mathcal{H}_I^0 = \left\{ \sum\nolimits_{k=1}^m a_k I_{(t_{k-1},t_k]}(s) : \begin{array}{l} m \in \mathbf{N},\ -\infty < t_0 < t_1 < \cdots < t_m < \infty \\ \text{with } (t_0, t_m] \subset I,\ a_k \in \mathbf{R}\ (k = 1, \dots, m) \end{array} \right\}.$$

We call $f \in \mathcal{H}_I^0$ a *simple function* on I. For $f = \sum_{k=1}^m a_k I_{(t_{k-1},t_k]} \in \mathcal{H}_I^0$, we define the stochastic integral $\int_I f(s)dY(s)$ by

$$\int_I f(s)dY(s) := \sum_{k=1}^m a_k (Y(t_k) - Y(t_{k-1})).$$

For a real-valued function f on I, we write $f(x) = f^+(x) - f^-(x)$, where

$$f^+(x) := \max(f(x), 0), \quad f^-(x) := \max(-f(x), 0) \qquad (x \in I).$$

Definition 3.1. For $f \in \mathcal{H}_I(Y)$, we define

$$\int_I f(s)dY(s) := \lim_{n\to\infty} \int_I f_n^+(s)dY(s) - \lim_{n\to\infty} \int_I f_n^-(s)dY(s) \quad \text{in } M(Y),$$

where $\{f_n^+\}$ and $\{f_n^-\}$ are arbitrary sequences of non-negative simple functions on I such that $f_n^+ \uparrow f^+$, $f_n^- \uparrow f^-$, as $n \to \infty$, a.e.

Proposition 3.2. *For $f \in \mathcal{H}_I(Y)$, we have*

$$\begin{aligned} &\int_I f(s)dY(s) \\ &\quad = -\int_{-\infty}^{\infty} \left(\int_I f(u)c(u-s)du \right) dW(s) + \int_I f(s)dW(s). \end{aligned} \tag{3.7}$$

Proof. For $-\infty < a < b < \infty$ with $(a, b] \subset I$, we have

$$\begin{aligned} Y(b) - Y(a) &= -\int_a^b \left(\int_{-\infty}^{\infty} c(u-s)dW(s) \right) du + W(b) - W(a) \\ &= -\int_{-\infty}^{\infty} \left(\int_I I_{(a,b]}(u)c(u-s)du \right) dW(s) + \int_I I_{(a,b]}(s)dW(s), \end{aligned}$$

which implies (3.7) for $f = I_{(a,b]}$, whence for $f \in \mathcal{H}_I^0$. Let $f \in \mathcal{H}_I(Y)$ such that $f \geq 0$, and let f_n $(n = 1, 2, \dots)$ be a sequence of simple functions on I such that $0 \leq f_n \uparrow f$ a.e. Then, by the monotone convergence theorem,

we have

$$\begin{aligned}
&\left\| \int_I f_n(s)dY(s) + \int_{-\infty}^{\infty} \left(\int_I f(u)c(u-s)du \right) dW(s) - \int_I f(s)dW(s) \right\| \\
&\le \left\{ \int_I (f(s) - f_n(s))^2 ds \right\}^{1/2} \\
&\quad + \left[\int_{-\infty}^{\infty} \left\{ \int_I (f(u) - f_n(u))c(u-s)du \right\}^2 ds \right]^{1/2} \qquad \downarrow\ 0 \quad (n \to \infty).
\end{aligned}$$

Thus the proposition follows. □

Recall $M(Y)$, $\|\cdot\|$, and $M_I(Y)$ from Section 1. From the definition above, we see that $\int_I f(s)dY(s) \in M_I(Y)$ for $f \in \mathcal{H}_I(Y)$.

We define

$$b(t,s) := -a(t+s) + \int_0^s c(u)a(t+s-u)du \qquad (t,s>0). \tag{3.8}$$

We will see from (3.10) below that $b(t,s) \le 0$ for $t,s>0$.

Lemma 3.3. *We have*

$$a(t) - c(t) - \int_0^t c(u)a(t-u)du = 0 \qquad (t>0), \tag{3.9}$$

$$b(t,s) = -c(t+s) - \int_0^t a(u)c(t+s-u)du \qquad (t,s>0), \tag{3.10}$$

$$c(t+s) = -b(t,s) + \int_0^t c(t-u)b(u,s)du \qquad (t,s>0). \tag{3.11}$$

Proof. Since

$$\left(\int_0^\infty e^{izt}a(t)dt \right) \left(\int_0^\infty e^{izt}c(t)dt \right) = \int_0^\infty e^{izt} \left(\int_0^t c(u)a(t-u)du \right) dt,$$

it follows from (3.6) that $\int_0^\infty e^{izt} \left(a(t) - c(t) - \int_0^t c(u)a(t-u)du \right) dt = 0$ for $\Im z > 0$. Thus, by the uniqueness of the Laplace transform, we obtain (3.9). We obtain (3.10) from (3.9) with t replaced by $t+s$. By (3.6) and (3.10), we see that, for $s>0$, $\int_0^\infty e^{izt}c(t+s)dt$ is equal to

$$\begin{aligned}
&\left(1 - \int_0^\infty e^{izt}c(t)dt \right) \left(1 + \int_0^\infty e^{izt}a(t)dt \right) \left(\int_0^\infty e^{izt}c(t+s)dt \right) \\
&= \left(\int_0^\infty e^{izt}c(t)dt - 1 \right) \left(\int_0^\infty e^{izt}b(t,s)dt \right) \\
&= \int_0^\infty e^{izt} \left(\int_0^t c(t-u)b(u,s)du - b(t,s) \right) dt.
\end{aligned}$$

Again, by the uniqueness of the Laplace transform, we obtain (3.11). □

Here is the prediction formula for $Y(\cdot)$ from an infinite segment of the past.

Theorem 3.4. *Let $t \in [0,\infty)$ and $f \in \mathcal{H}_{[t,\infty)}(Y)$. Then $\int_0^\infty b(t-\cdot,\tau)f(t+\tau)d\tau \in \mathcal{H}_{(-\infty,t]}(Y)$ and*

$$P_{(-\infty,t]}\int_t^\infty f(s)dY(s) = \int_{-\infty}^t \left(\int_0^\infty b(t-s,\tau)f(t+\tau)d\tau\right) dY(s). \tag{3.12}$$

In particular, for $t \le T$, we have $\int_0^{T-t} b(t-\cdot,\tau)d\tau \in \mathcal{H}_{(-\infty,t]}(Y)$ and

$$P_{(-\infty,t]}Y(T) = Y(t) + \int_{-\infty}^t \left(\int_0^{T-t} b(t-s,\tau)d\tau\right) dY(s).$$

Proof. Since $f \in \mathcal{H}_{[t,\infty)}(Y)$ if and only if $|f| \in \mathcal{H}_{[t,\infty)}(Y)$, we may assume $f \ge 0$. We have $0 \le -b(t,s) = a(t+s) - \int_0^s c(u)a(t+s-u)du \le a(t+s)$ for $t,s>0$, so that

$$\begin{aligned}
&\int_{-\infty}^t \left\{\int_0^\infty b(t-s,\tau)f(t+\tau)d\tau\right\}^2 ds \\
&\le \int_0^\infty \left\{\int_0^\infty a(s+\tau)f(t+\tau)d\tau\right\}^2 ds \\
&\le \left(\int_0^\infty a(s)ds\right)^2 \left(\int_t^\infty f(s)^2 ds\right) < \infty
\end{aligned} \tag{3.13}$$

(cf. [7, Theorem 274]). It follows from (3.11) that

$$-b(t-s,\tau) = c(t-s+\tau) - \int_0^{t-s} c(t-s-u)b(u,\tau)du \quad (t>s,\ \tau>0),$$

whence, by the Fubini-Tonelli theorem, for $s<t$,

$$\begin{aligned}
&-\int_0^\infty b(t-s,\tau)f(t+\tau)d\tau \\
&= \int_0^\infty c(t-s+\tau)f(t+\tau)d\tau \\
&\qquad - \int_0^\infty d\tau f(t+\tau)\int_0^{t-s} c(t-s-u)b(u,\tau)du \\
&= \int_t^\infty c(u-s)f(u)du \\
&\qquad - \int_{-\infty}^t \left(\int_0^\infty b(t-u,\tau)f(t+\tau)d\tau\right) c(u-s)du.
\end{aligned} \tag{3.14}$$

Therefore, from (3.13) and $\int_{-\infty}^\infty \left(\int_t^\infty f(u)c(u-s)du\right)^2 ds < \infty$, we see that

$$\int_{-\infty}^\infty \left\{\int_{-\infty}^t \left(\int_0^\infty b(t-u,\tau)f(t+\tau)d\tau\right) c(u-s)du\right\}^2 ds < \infty.$$

Thus $\int_0^\infty b(t-\cdot,\tau)f(t+\tau)d\tau \in \mathcal{H}_{(-\infty,t]}(Y)$.

By Proposition 2.3 (6), we have $H_{(-\infty,t]}(Y) = H_{(-\infty,t]}(DW)$. By this as well as Proposition 3.2 and (3.14),

$$\begin{aligned}
&P_{(-\infty,t]}\int_t^\infty f(s)dY(s)\\
&\quad= -\int_{-\infty}^t \left(\int_t^\infty f(u)c(u-s)du\right)dW(s)\\
&\quad= -\int_{-\infty}^t \left\{\int_0^\infty \left(\int_0^\infty b(t-u,\tau)f(t+\tau)d\tau\right)c(u-s)du\right\}dW(s)\\
&\qquad\qquad\qquad + \int_{-\infty}^t \left(\int_0^\infty b(t-s,\tau)f(t+\tau)d\tau\right)dW(s)\\
&\quad= \int_{-\infty}^t \left(\int_0^\infty b(t-s,\tau)f(t+\tau)d\tau\right)dY(s).
\end{aligned}$$

Thus (3.12) follows. By putting $f(s) = I_{(t,T]}(s)$ in (3.12), we obtain the remaining assertions. □

For later use, we also consider the projection operator $P_{[-t,\infty)}$ with $t \geq 0$.

Proposition 3.5. *Let I be a closed interval of $\mathbf{R}$ and let $f \in \mathcal{H}_I(Y)$. Let $W^*(\cdot)$ be as in Definition 2.2. Then*
(1) *$f(-\cdot) \in \mathcal{H}_{-I}(Y)$ and $\theta\left(\int_I f(s)dY(s)\right) = -\int_{-I} f(-s)dY(s)$;*
(2) *$\int_I f(s)dY(s) = -\int_{-\infty}^\infty \left(\int_I f(u)c(s-u)du\right)dW^*(s) + \int_I f(s)dW^*(s)$.*

Proof. By simple calculation, we have

$$\int_{-\infty}^\infty \left\{\int_{-I} |f(-u)|c(u-s)du\right\}^2 ds = \int_{-\infty}^\infty \left\{\int_I |f(u)|c(u-s)du\right\}^2 ds,$$

whence $f(-\cdot) \in \mathcal{H}_{-I}(Y)$. To prove the second assertion of (1), we may assume that $f(s) = I_{(a,b]}(s)$ with $(a,b] \subset I$. However, since $I_I(-s) = I_{-I}(s)$, we have the following as desired:

$$\int_{-I} f(-s)dY(s) = Y(-a) - Y(-b) = -\theta\left(\int_I f(s)dY(s)\right).$$

By (1) and Proposition 3.2, $\int_{-I} f(-s)dY(s)$ is equal to

$$-\int_{-\infty}^\infty \left(\int_{-I} f(-u)c(u-s)du\right)dW(s) + \int_{-I} f(-s)dW(s).$$

Hence, by (1) and Proposition 2.3 (8),

$$\begin{aligned}\int_I f(s)dY(s) &= -\theta\left(\int_{-I} f(-s)dY(s)\right)\\ &= -\int_{-\infty}^{\infty}\left(\int_{-I} f(-u)c(u+s)du\right)dW^*(s) + \int_I f(s)dW^*(s)\\ &= -\int_{-\infty}^{\infty}\left(\int_I f(u)c(s-u)du\right)dW^*(s) + \int_I f(s)dW^*(s).\end{aligned}$$

Thus (2) follows. □

Theorem 3.6. *Let $t \in [0,\infty)$ and $f \in \mathcal{H}_{[t,\infty)}(Y)$. Then $\int_0^\infty b(t+\cdot,\tau)f(t+\tau)d\tau \in \mathcal{H}_{[-t,\infty)}(Y)$ and*

(3.15)
$$P_{[-t,\infty)}\int_{-\infty}^{-t} f(-s)dY(s) = \int_{-t}^{\infty}\left(\int_0^\infty b(t+s,\tau)f(t+\tau)d\tau\right)dY(s).$$

Proof. The first assertion follows from Theorem 3.4 and Proposition 3.5. Now, by Proposition 3.5, $\theta\left(P_{(-\infty,t]}\int_t^\infty f(s)dY(s)\right)$ is equal to

$$\theta P_{(-\infty,t]}\theta^{-1}\theta\left(\int_t^\infty f(s)dY(s)\right) = -P_{[-t,\infty)}\int_{-\infty}^{-t} f(-s)dY(s),$$

and $\theta(\int_{-\infty}^t(\int_0^\infty b(t-s,\tau)f(t+\tau)d\tau)dY(s))$ is equal to

$$-\int_{-t}^{\infty}\left(\int_0^\infty b(t+s,\tau)f(t+\tau)d\tau\right)dY(s).$$

Thus (3.15) follows from Theorem 3.4. □

4. Prediction from a finite segment of the past

As in Section 3, let $Y(\cdot)$ be the unique solution to (3.1) with $Y(0)=0$. This section is the technical key part of this paper. We prove a finite-past prediction formula for $Y(\cdot)$. Let $a(\cdot)$, $c(\cdot)$ and $b(t,s)$ be as in Section 3. We assume that the measure ν is nontrivial, that is, $\nu \neq 0$.

We define positive constants $K_1 \in (0,1)$ and $K_2 \in (0,\infty)$ by $K_1 := \int_0^\infty c(u)du$ and $K_2 := \int_0^\infty a(u)du$, respectively.

Proposition 4.1. (1) *We have $-\int_0^\infty b(s,\tau)d\tau = K_1-(1-K_1)\int_0^s a(u)du$ for $s>0$. In particular, $\sup_{s>0}\int_0^\infty\{-b(s,\tau)\}d\tau \le K_1$.*
(2) *We have $-\int_0^\infty b(s,\tau)ds = K_2-(1+K_2)\int_0^\tau c(u)du$ for $\tau>0$. In particular, $\sup_{\tau>0}\int_0^\infty\{-b(s,\tau)\}ds \le K_2$.*

Proof. We put $C(s) := \int_s^\infty c(u)du$ for $0<s<\infty$. Integration by parts yields $\int_0^\infty e^{-\tau y}c(\tau)d\tau = K_1 - y\int_0^\infty e^{-\tau y}C(\tau)d\tau$. Since $\int_0^\tau a(u)du \le a(0+)\tau$

for $\tau > 0$, it holds that $\lim_{\tau\to\infty} e^{-\tau y}\int_0^\tau a(u)du = 0$. Hence, by integration by parts, we get $\int_0^\infty e^{-\tau y}a(\tau)d\tau = y\int_0^\infty d\tau e^{-\tau y}\int_0^\tau a(u)du$. Therefore, using (3.6) as well as $1 = y\int_0^\infty e^{-\tau y}d\tau$ and

$$\begin{aligned}&\left(\int_0^\infty e^{-\tau y}C(\tau)d\tau\right)\left(\int_0^\infty e^{-\tau y}a(\tau)d\tau\right)\\&\qquad = \int_0^\infty e^{-\tau y}\left(\int_0^\tau a(u)C(\tau-u)du\right)d\tau,\end{aligned}$$

we find that

$$\begin{aligned}0 &= \left(1-\int_0^\infty e^{-\tau y}c(\tau)d\tau\right)\left(1+\int_0^\infty e^{-\tau y}a(\tau)d\tau\right)-1\\&= -K_1 + y\int_0^\infty e^{-\tau y}C(\tau)d\tau + (1-K_1)y\int_0^\infty e^{-\tau y}\left(\int_0^\tau a(u)du\right)d\tau\\&\qquad + y\int_0^\infty e^{-\tau y}\left(\int_0^\tau a(u)C(\tau-u)du\right)d\tau\\&= y\int_0^\infty e^{-\tau y}\left\{-K_1 + C(\tau) + (1-K_1)\int_0^\tau a(u)du\right.\\&\qquad\qquad \left. + \int_0^\tau a(u)C(\tau-u)du\right\}d\tau\end{aligned}$$

or

$$-K_1 + C(s) + (1-K_1)\int_0^s a(u)du + \int_0^s a(u)C(s-u)du = 0 \quad (s>0).$$

Thus it follows from (3.10) that, for $s > 0$, $-\int_0^\infty b(s,\tau)d\tau$ is equal to

$$C(s) + \int_0^s a(u)C(s-u)du = K_1 - (1-K_1)\int_0^s a(u)du,$$

whence (1) follows. The proof of (2) is similar to that of (1); we may exchange the roles of $a(\cdot)$ and $c(\cdot)$. □

For $s,\tau,t \in (0,\infty)$ and $n \in \mathbf{N}$, we define $b_n(s,\tau;t)$ by

$$\begin{cases} b_1(s,\tau;t) := b(s,\tau),\\ b_n(s,\tau;t) := \displaystyle\int_0^\infty b(s,u)b_{n-1}(t+u,\tau;t)du \qquad (n=2,3,\dots).\end{cases}$$

We see that $(-1)^n b_n(s,\tau;t) \geq 0$.

Proposition 4.2. *Let $t > 0$ and $n = 1,2,\dots$. Then*

(1) $\sup_{0<s<\infty}\int_0^\infty |b_n(s,\tau;t)|\,d\tau \leq (K_1)^n$;
(2) $\sup_{0<\tau<\infty}\int_0^\infty |b_n(s,\tau;t)|\,ds \leq (K_2)^n$.

Proof. We use mathematical induction on n. By Proposition 4.1 (1), (1) holds for $n = 1$. Assume that (1) holds for $n \in \mathbf{N}$. Then, by the Fubini–Tonelli theorem and Proposition 4.1 (1), we have, for $s > 0$,

$$\begin{aligned}\int_0^\infty |b_{n+1}(s,\tau;t)|\,d\tau &= \int_0^\infty du(-b(s,u)) \int_0^\infty |b_n(t+u,\tau;t)|d\tau \\ &\le \left(-\int_0^\infty b(s,u)du\right)(K_1)^n \le (K_1)^{n+1}.\end{aligned}$$

Thus (1) with n replaced by $n+1$ holds. The proof of (2) is similar; we use Proposition 4.1 (2) instead of Proposition 4.1 (1). □

We suppose that

$$-\infty < -t_0 \le 0 \le t_1 < \infty, \quad -t_0 < t_1 \tag{4.1}$$

and define a positive constant t_2 by

$$t_2 := t_0 + t_1. \tag{4.2}$$

For simplicity, we often suppress t_2 and write

$$b_n(s,\tau) := b_n(s,\tau;t_2) \qquad (s,\tau > 0,\ n = 1,2,\dots).$$

Proposition 4.3. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Then*

$$\int_0^\infty b_n(t_1 - \cdot,\tau)f(t_1+\tau)d\tau \in \mathcal{H}_{(-\infty,t_1]}(Y) \qquad (n = 1,3,5,\dots), \tag{4.3}$$

$$\int_0^\infty b_n(t_0 + \cdot,\tau)f(t_1+\tau)d\tau \in \mathcal{H}_{[-t_0,\infty)}(Y) \qquad (n = 2,4,6,\dots). \tag{4.4}$$

Proof. Without loss of generality, we may assume that $f \ge 0$. By Theorem 3.4, (4.3) holds for $n = 1$. This and Proposition 3.5 (1) imply

$$\int_0^\infty b_1(t_1 + \cdot,\tau)f(t_1+\tau)d\tau \in \mathcal{H}_{[-t_1,\infty)}(Y).$$

It follows from the definition of $\mathcal{H}_I(Y)$ that if $g \in \mathcal{H}_I(Y)$ and $J \subset I$, then the restriction of g on J is in $\mathcal{H}_J(Y)$. Hence we have

$$\int_0^\infty b_1(t_1 + \cdot,\tau)f(t_1+\tau)d\tau \in \mathcal{H}_{[t_0,\infty)}(Y).$$

However, by the Fubini-Tonelli theorem, we have, for $s > -t_0$,

$$\begin{aligned}&\int_0^\infty dub(t_0+s,u) \int_0^\infty b_1(t_1+t_0+u,\tau)f(t_1+\tau)d\tau \\ &\quad = \int_0^\infty b_2(t_0+s,\tau)f(t_1+\tau)d\tau.\end{aligned}$$

Thus, by Theorem 3.6, we get (4.4) with $n = 2$. Repeating this procedure, we obtain the proposition. □

Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. By Proposition 4.3, we may define the random variables $G_n(f)$ $(n = 1, 2, \dots)$ by

$$G_n(f) := \begin{cases} \displaystyle\int_{-t_0}^{t_1} \left(\int_0^\infty b_n(t_1 - s, \tau) f(t_1 + \tau) d\tau \right) dY(s) & (n = 1, 3, \dots), \\ \displaystyle\int_{-t_0}^{t_1} \left(\int_0^\infty b_n(t_0 + s, \tau) f(t_1 + \tau) d\tau \right) dY(s) & (n = 2, 4, \dots). \end{cases}$$

We may also define the random variables $\epsilon_n(f)$ by

$$\epsilon_n(f) := \begin{cases} \displaystyle\int_{t_1}^\infty f(s) dY(s) & (n = 0), \\ \displaystyle\int_{-\infty}^{-t_0} \left(\int_0^\infty b_n(t_1 - s, \tau) f(t_1 + \tau) d\tau \right) dY(s) & (n = 1, 3, \dots), \\ \displaystyle\int_{t_1}^\infty \left(\int_0^\infty b_n(t_0 + s, \tau) f(t_1 + \tau) d\tau \right) dY(s) & (n = 2, 4, \dots). \end{cases}$$

We set

$$P_n := \begin{cases} P_{(-\infty,t_1]} & (n = 1, 3, 5, \dots), \\ P_{[-t_0,\infty)} & (n = 2, 4, 6, \dots). \end{cases}$$

Recall that $M_{[-t_0,t_1]}(Y)$ is the closed subspace of $M(Y)$ spanned by $\{Y(s) : -t_0 \le s \le t_1\}$. We have the following inclusion:

$$M_{[-t_0,t_1]}(Y) \subset M_{(-\infty,t_1]}(Y) \cap M_{[-t_0,\infty)}(Y). \tag{4.5}$$

Proposition 4.4. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$ and $n \in \mathbf{N}$. Then*

$$P_n P_{n-1} \cdots P_1 \int_{t_1}^\infty f(s) dY(s) = \epsilon_n(f) + \sum_{k=1}^n G_k(f). \tag{4.6}$$

Proof. We use mathematical induction. By Theorem 3.4, (4.6) holds for $n = 1$. Suppose that (4.6) holds for $n = m \in \mathbf{N}$. Then, from (4.5) and

$$G_k(f) \in M_{[-t_0,t_1]}(Y) \qquad (k = 1, 2, \dots), \tag{4.7}$$

we have $P_{m+1} G_k(f) = G_k(f)$ for $k = 1, 2, \dots$. Thus

$$P_{m+1} P_m \cdots P_1 \int_{t_1}^\infty f(s) dY(s) = P_{m+1} \epsilon_m(f) + \sum\nolimits_{k=1}^m G_k(f).$$

If m is odd, then, by Theorem 3.6, $P_{m+1} \epsilon_m(f)$ is equal to

$$\begin{aligned} &\int_{-t_0}^\infty \left\{ \int_0^\infty dub(t_0 + s, u) \int_0^\infty b_m(t_2 + u, \tau) f(t_1 + \tau) d\tau \right\} dY(s) \\ &= \int_{-t_0}^\infty \left\{ \int_0^\infty b_{m+1}(t_0 + s, \tau) f(t_1 + \tau) d\tau \right\} dY(s) \\ &= \epsilon_{m+1}(f) + G_{m+1}(f), \end{aligned}$$

whence (4.6) with $n = m+1$. If m is even, then, by Theorem 3.4, we have (4.6) with $n = m+1$ in the same way. Thus the proposition follows. $\square$

Here is the key lemma.

Lemma 4.5. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Then $\|\epsilon_n(f)\| \to 0$ as $n \to \infty$.*

The proof of Lemma 4.5 is long. Therefore we first show its consequences.

Recall that $P_{[-t_0,t_1]}$ is the orthogonal projection operator from $M(Y)$ onto $M_{[-t_0,t_1]}(Y)$. We write Q for the orthogonal projection operator from $M(Y)$ onto the closed subspace $M_{(-\infty,t_1]}(Y) \cap M_{[-t_0,\infty)}(Y)$. Then we have

$$Q = \operatorname*{s-lim}_{n\to\infty} P_n P_{n-1} \cdots P_1 \tag{4.8}$$

(cf. [6, Problem 122]).

Theorem 4.6. *We have*
(1) $M_{[-t_0,t_1]}(Y) = M_{(-\infty,t_1]}(Y) \cap M_{[-t_0,\infty)}(Y)$,
(2) $P_{[-t_0,t_1]} = \text{s-lim}_{n\to\infty} P_n P_{n-1} \cdots P_1$,
(3) $\|P^{\perp}_{[-t_0,t_1]} Z\|^2 = \|P_1^{\perp} Z\|^2 + \sum_{n=1}^{\infty} \|(P_{n+1})^{\perp} P_n \cdots P_1 Z\|^2$ *for* $Z \in M(Y)$.

Proof. We claim that, for every $t \in \mathbf{R}$,

$$P_{[-t_0,t_1]} Y(t) = QY(t). \tag{4.9}$$

This claim implies that $P_{[-t_0,t_1]} Z = QZ$ for $Z \in M(Y)$. So if $Z \in M_{(-\infty,t_1]}(Y) \cap M_{[-t_0,\infty)}(Y)$, then $Z = QZ = P_{[-t_0,t_1]} Z \in M_{[-t_0,t_1]}$, which implies $M_{(-\infty,t_1]}(Y) \cap M_{[-t_0,\infty)}(Y) \subset M_{[-t_0,t_1]}(Y)$. This, together with (4.5), implies (1). The assertion (2) follows immediately from (1) and (4.8).

We derive (3) from (2). Let $Z \in M(Y)$. From the orthogonal decompositions

$$P^{\perp}_{[-t_0,t_1]} = P^{\perp}_{(-\infty,t_0]} + P^{\perp}_{[-t_0,t_1]} P_{(-\infty,t_0]} = P^{\perp}_{[-t_0,\infty)} + P^{\perp}_{[-t_0,t_1]} P_{[-t_0,\infty)}$$

of the operator $P^{\perp}_{[-t_0,t_1]}$, we obtain

$$\begin{aligned} &\left\|P^{\perp}_{[-t_0,t_1]} Z\right\|^2 = \left\|P^{\perp}_{(-\infty,t_0]} Z\right\|^2 + \left\|P^{\perp}_{[-t_0,t_1]} P_{(-\infty,t_0]} Z\right\|^2 \\ &= \left\|P^{\perp}_{(-\infty,t_0]} Z\right\|^2 + \left\|P^{\perp}_{[-t_0,\infty)} P_{(-\infty,t_1]} Z\right\|^2 + \left\|P^{\perp}_{[-t_0,t_1]} P_{[-t_0,\infty)} P_{(-\infty,t_1]} Z\right\|^2. \end{aligned}$$

Repeating this procedure, we see that, for $m \in \mathbf{N}$, $\|P^{\perp}_{[-t_0,t_1]} Z\|^2$ is equal to

$$\left\|P_1^{\perp} Z\right\|^2 + \sum_{n=1}^{m-1} \left\|(P_{n+1})^{\perp} P_n \cdots P_1 Z\right\|^2 + \left\|P^{\perp}_{[-t_0,t_1]} P_m \cdots P_1 Z\right\|^2.$$

However, by (2), $\left\|P^{\perp}_{[-t_0,t_1]} P_m \cdots P_1 Z\right\|$ tends to $\left\|P^{\perp}_{[-t_0,t_1]} P_{[-t_0,t_1]} Z\right\| = 0$ as $m \to \infty$, whence (3).

We complete the proof by proving (4.9). If $t \in [-t_0, t_1]$, then $Y(t)$ is in both $M_{[-t_0,t_1]}$ and $M_{(-\infty,t_1]}(Y) \cap M_{[-t_0,\infty)}(Y)$, and so $P_{[-t_0,t_1]}Y(t) = Y(t) = QY(t)$. Thus we may prove (4.9) for $t \in \mathbf{R} \setminus [-t_0, t_1]$. However, by symmetry, it is enough to prove (4.9) only for $t \in (t_1, \infty)$. For such t, we put $f_0(s) := I_{(t_1,t]}(s)$. Then $\int_{t_1}^{\infty} f_0(s)dY(s) = Y(t) - Y(t_1)$. Now (4.5) implies $P_{[-t_0,t_1]} = P_{[-t_0,t_1]}P_nP_{n-1}\cdots P_1$, whence $P_nP_{n-1}\cdots P_1 - P_{[-t_0,t_1]} = P^{\perp}_{[-t_0,t_1]}P_nP_{n-1}\cdots P_1$. On the other hand, it follows from (4.7) that $P^{\perp}_{[-t_0,t_1]}G_k(f) = 0$ for $k \in \mathbf{N}$ and $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Therefore, using Proposition 4.4 and Lemma 4.5, we see that

$$\begin{aligned}
&\left\|P_nP_{n-1}\cdots P_1Y(t) - P_{[-t_0,t_1]}Y(t)\right\| \\
&= \left\|P^{\perp}_{[-t_0,t_1]}P_nP_{n-1}\cdots P_1\int_{t_1}^{\infty} f_0(s)dY(s)\right\| \\
&= \left\|P^{\perp}_{[-t_0,t_1]}\left(\epsilon_n(f_0) + \sum\nolimits_1^n G_k(f_0)\right)\right\| \\
&= \left\|P^{\perp}_{[-t_0,t_1]}\epsilon_n(f_0)\right\| \le \|\epsilon_n(f_0)\| \to 0 \qquad (n \to \infty).
\end{aligned}$$

This and (4.8) imply (4.9). □

We may define, for $s \in (-t_0, t_1)$ and $\tau > 0$,

$$g(s,\tau;t_0,t_1) := \sum_{k=1}^{\infty}\{b_{2k-1}(t_1 - s,\tau;t_2) + b_{2k}(t_0 + s,\tau;t_2)\} \tag{4.10}$$

since Proposition 4.2 (1) implies $\sup_{-t_0<s<t_1}\int_0^{\infty}|g(s,\tau;t_0,t_1)|d\tau < \infty$.

Here is the finite-past prediction formula for $Y(\cdot)$.

Theorem 4.7. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Then*

$$P_{[-t_0,t_1]}\int_{t_1}^{\infty} f(s)dY(s) = \sum_{k=1}^{\infty} G_k(f), \tag{4.11}$$

the sum on the right-hand side converging in $M(Y)$. Furthermore, if $\operatorname{ess\,sup}_{t_1\le t<\infty}|f(t)| < \infty$, *then*

(4.12)

$$P_{[-t_0,t_1]}\int_{t_1}^{\infty} f(s)dY(s) = \int_{-t_0}^{t_1}\left(\int_0^{\infty} g(s,\tau;t_0,t_1)f(t_1+\tau)d\tau\right)dY(s).$$

In particular, for $t_1 \le t$, we have

$$P_{[-t_0,t_1]}Y(t) = Y(t_1) + \int_{-t_0}^{t_1}\left(\int_0^{t-t_1} g(s,\tau;t_0,t_1)d\tau\right)dY(s).$$

Proof. By Theorem 4.6 (2), we have, in $M(Y)$,

$$\lim_{n\to\infty} P_nP_{n-1}\cdots P_1\int_{t_1}^{\infty} f(s)dY(s) = P_{[-t_0,t_1]}\int_{t_1}^{\infty} f(s)dY(s).$$

Hence, letting $n \to \infty$ in (4.6) and using Lemma 4.5, we obtain (4.11). Suppose that f is essentially bounded on $[t_1, \infty)$. Then, by Proposition 4.2 (1), we have

$$\sup_{s>0} \int_0^\infty |b_n(s, \tau; t_2) f(t_1 + \tau)| d\tau \le (K_1)^n \operatorname*{ess\,sup}_{t \ge t_1} |f(t)|.$$

Since $\sum_n (K_1)^n < \infty$, (4.12) follows easily from (4.11). The last assertion follows if we put $f(s) = I_{(t_1,t]}(s)$. □

It remains to prove Lemma 4.5. For this purpose, we consider the mean-squared prediction error. For $f \in \mathcal{H}_{[t_1,\infty)}(Y)$ and $n \in \mathbf{N}$, we define $D_n(s, f) = D_n(s, f; t_0, t_1)$ by

$$D_n(s, f) := -\int_0^\infty c(u) \left(\int_0^\infty b_n(t_2 + u + s, \tau) f(t_1 + \tau) d\tau \right) du + \int_0^\infty b_n(t_2 + s, \tau) f(t_1 + \tau) d\tau \quad (s > 0).$$

We also define $D_0(s, f) = D_0(s, f; t_1)$ by

$$D_0(s, f) := -\int_0^\infty c(u) f(t_1 + s + u) du + f(t_1 + s) \qquad (s > 0).$$

From the proof of the next proposition, these integrals converge absolutely.

Proposition 4.8. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Then*

$$P_{n+1}^{\perp} \epsilon_n(f) = \begin{cases} \displaystyle\int_{t_1}^\infty D_n(s - t_1, f) dW(s) & (n = 0, 2, 4, \dots), \\ \displaystyle\int_{-\infty}^{-t_0} D_n(-t_0 - s, f) dW^*(s) & (n = 1, 3, 5, \dots). \end{cases} \tag{4.13}$$

Proof. By Propositions 3.2 and 2.3 (6), $P_1^{\perp} \epsilon_0(f)$ is equal to

$$\int_{t_1}^\infty \left(-\int_s^\infty f(u) c(u - s) du + f(s) \right) dW(s) = \int_t^\infty D_0(s - t_1, f) dW(s).$$

Thus (4.13) holds for $n = 0$. Suppose that $n = 1, 3, \dots$. Then, by Proposition 3.5 (2),

$$\begin{aligned} \epsilon_n(f) = & \int_{-\infty}^{-t_0} \left(\int_0^\infty b_n(t_1 - s, \tau) f(t_1 + \tau) d\tau \right) dW^*(s) \\ & - \int_{-\infty}^\infty \left(\int_{-\infty}^{-t_0} du c(s - u) \int_0^\infty b_n(t_1 - u, \tau) f(t_1 + \tau) d\tau \right) dW^*(s). \end{aligned} \tag{4.14}$$

Hence, by Proposition 2.3 (7), $P^{\perp}_{n+1}\epsilon_n(f) = P^{\perp}_{[-t_0,\infty)}\epsilon_n(f)$ is given by

$$\begin{aligned}
&-\int_{-\infty}^{-t_0}\left(\int_{-\infty}^{s} duc(s-u)\int_0^{\infty} b_n(t_1-u,\tau)f(t_1+\tau)d\tau\right)dW^*(s)\\
&\qquad\qquad+\int_{-\infty}^{-t_0}\left(\int_0^{\infty} b_n(t_1-s,\tau)f(t_1+\tau)d\tau\right)dW^*(s)\\
&=-\int_{-\infty}^{-t_0}\left(\int_0^{\infty} duc(u)\int_0^{\infty} b_n(t_2+u-s-t_0,\tau)f(t_1+\tau)d\tau\right)dW^*(s)\\
&\qquad\qquad+\int_{-\infty}^{-t_0}\left(\int_0^{\infty} b_n(t_2-s-t_0,\tau)f(t_1+\tau)d\tau\right)dW^*(s),
\end{aligned}$$

which is equal to $\int_{-\infty}^{-t_0} D_n(-t_0-s)dW^*(s)$. Thus we obtain (4.13) for $n = 1, 3, \dots$. The proof of (4.13) for $n = 2, 4, \dots$ is similar, and so we omit it. □

Proposition 4.9. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Then*
(1) $\|P^{\perp}_1 \int_{t_1}^{\infty} f(s)dY(s)\|^2 = \int_0^{\infty} D_0(s,f)^2 ds$;
(2) $\|P^{\perp}_{n+1}P_nP_{n-1}\cdots P_1 \int_{t_1}^{\infty} f(s)dY(s)\|^2 = \int_0^{\infty} D_n(s,f)^2 ds$ $(n = 1, 2, \dots)$;
(3) $\lim_{n\to\infty}\int_0^{\infty} D_n(s,f)^2 ds = 0$.

Proof. The assertion (1) follows immediately from (4.13) with $n = 0$. Suppose that $n \geq 1$. From (4.5) and (4.7), we have $P^{\perp}_{n+1}G_k(f) = 0$ for $k \in \mathbf{N}$. Therefore, by Propositions 4.4 and 4.8,

$$\left\|P^{\perp}_{n+1}P_nP_{n-1}\cdots P_1\int_{t_1}^{\infty} f(s)dY(s)\right\|^2 = \left\|P^{\perp}_{n+1}\epsilon_n(f)\right\|^2 = \int_0^{\infty} D_n(s,f)^2 ds,$$

whence (2).

We write Q for the orthogonal projection operator from $M(Y)$ onto $M_{(-\infty,t_1]}(Y) \cap M_{[-t_0,\infty)}(Y)$ as in the above. Then, by (2) and (4.8), we have

$$\begin{aligned}
\lim_{n\to\infty}\int_0^{\infty} D_{2n}(s,f)^2 ds &= \lim_{n\to\infty}\left\|P^{\perp}_{(-\infty,t_1]}P_{2n}P_{2n-1}\cdots P_1\int_{t_1}^{\infty} f(s)dY(s)\right\|^2\\
&= \left\|P^{\perp}_{(-\infty,t_1]}Q\int_{t_1}^{\infty} f(s)dY(s)\right\|^2 = 0.
\end{aligned}$$

Similarly, we have $\lim_n \int_0^{\infty} D_{2n+1}(s,f)^2 ds = 0$. Thus (3) follows. □

Proposition 4.10. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Then, for $t > 0$ and $n = 0, 1, \dots$, we have*

$$\int_0^{\infty} b_{n+1}(t,\tau)f(t_1+\tau)d\tau = -\int_0^{\infty} a(t+u)D_n(u,f)du. \tag{4.15}$$

Proof. We may assume that $f \geq 0$. Since $a(\cdot) \in L^1((0,\infty), ds)$, $a(\cdot) \geq 0$, and $c(\cdot) \geq 0$, using the Fubini–Tonelli theorem, we see that, for $t > 0$,

$$\begin{aligned}
&\int_0^\infty b_1(t,\tau) f(t_1+\tau) d\tau \\
&= \int_0^\infty \left(\int_0^\tau c(\tau - u) a(t+u) du - a(t+\tau) \right) f(t_1+\tau) d\tau \\
&= \int_0^\infty du a(t+u) \int_0^\infty c(\tau) f(t_1+u+\tau) d\tau - \int_0^\infty a(t+u) f(t_1+u) du,
\end{aligned}$$

which is equal to $-\int_0^\infty a(t+u) D_0(u,f) du$. Thus (4.15) holds for $n = 0$. Now we assume that $n \geq 1$. Then, by Proposition 4.2 (2) and the Fubini–Tonelli theorem, we have, for $t, \tau > 0$,

$$\begin{aligned}
b_{n+1}(t,\tau) &= \int_0^\infty b(t,s) b_n(t_2+s,\tau) ds \\
&= \int_0^\infty \left(\int_0^s c(s-u) a(t+u) du \right) b_n(t_2+s,\tau) ds \\
&\qquad\qquad - \int_0^\infty a(t+u) b_n(t_2+u,\tau) du \\
&= \int_0^\infty a(t+u) \left(\int_0^\infty c(s) b_n(t_2+s+u,\tau) ds - b_n(t_2+u,\tau) \right) du.
\end{aligned}$$

Therefore it follows from Proposition 4.3 and the Fubini–Tonelli theorem that the integral $\int_0^\infty b_{n+1}(t,\tau) f(t_1+\tau) d\tau$ is given by

$$\begin{aligned}
\int_0^\infty a(t+u) \Bigg(&\int_0^\infty ds c(s) \int_0^\infty b_n(t_2+s+u,\tau) f(t_1+\tau) d\tau \\
&- \int_0^\infty b_n(t_2+u,\tau) f(t_1+\tau) d\tau \Bigg) du,
\end{aligned}$$

which is equal to $-\int_0^\infty a(t+u) D_n(u,f) du$. Thus (4.15) holds. □

For $t, s > 0$, we define $\lambda(t,s) = \lambda(t,s;t_2)$ by

$$\lambda(t,s) := \int_0^\infty c(t+u) a(t_2+u+s) du.$$

Notice that $\lambda(t,s) < \infty$ since $0 \leq \lambda(t,s) \leq c(t) \int_{t_2+s}^\infty a(u) du < \infty$. We define an integral operator Λ on $L^2((0,\infty), ds)$ by

$$\Lambda f(t) := \int_0^\infty \lambda(t,s) f(s) ds.$$

Since $\sup_{0<t<\infty} \int_0^\infty \lambda(t,s) ds \leq M$ and $\sup_{0<s<\infty} \int_0^\infty \lambda(t,s) dt \leq M$, it follows from [7, Theorem 274] that Λ is a bounded linear operator on $L^2((0,\infty), ds)$ such that $\|\Lambda\| \leq M$, where $\|\Lambda\|$ is the operator norm of Λ and $M = \left(\int_0^\infty c(u) du\right) \left(\int_{t_2}^\infty a(u) du\right)$.

Proposition 4.11. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Then $P_{n+1}\epsilon_n(f)$ is equal to*

$$\begin{cases} \displaystyle\int_{-\infty}^{t_1} \left(\int_0^\infty \lambda(t_1 - s, u) D_{n-1}(u, f) du \right) dW(s) & (n = 0, 2, 4, \dots), \\ \displaystyle\int_{-t_0}^{\infty} \left(\int_0^\infty \lambda(t_0 + s, u) D_{n-1}(u, f) du \right) dW^*(s) & (n = 1, 3, 5, \dots). \end{cases}$$

Proof. We prove the proposition only when n is odd; the proof of the case $n = 2, 4, \dots$ is similar. By (4.14) and Propositions 2.3 (7) and 4.10, $P_{n+1}\epsilon_n(f) = P_{[-t_0,\infty)}\epsilon_n(f)$ is equal to

$$\begin{aligned} &- \int_{-t_0}^{\infty} \left(\int_{-\infty}^{-t_0} duc(s-u) \int_0^\infty b_n(t_1 - u, \tau) f(t_1 + \tau) d\tau \right) dW^*(s) \\ &= - \int_{-t_0}^{\infty} \left(\int_0^\infty dvc(t_0 + s + v) \int_0^\infty b_n(t_2 + v, \tau) f(t_1 + \tau) d\tau \right) dW^*(s) \\ &= \int_{-t_0}^{\infty} \left(\int_0^\infty dvc(t_0 + s + v) \int_0^\infty a(t_2 + v + u) D_{n-1}(u, f) du \right) dW^*(s) \end{aligned}$$

or $\int_{-t_0}^{\infty} \left(\int_0^\infty \lambda(t_0 + s, u) D_{n-1}(u, f) du \right) dW^*(s)$. Thus the proposition follows. □

We are now ready to prove Lemma 4.5.

Proof of Lemma 4.5. By Propositions 4.8, 4.11 and 4.9 (3), we have

$$\begin{aligned} \|\epsilon_n(f)\|^2 &= \int_0^\infty D_n(s, f)^2 ds + \int_0^\infty \left(\int_0^\infty \lambda(s, u) D_{n-1}(u, f) du \right)^2 ds \\ &\le \int_0^\infty D_n(s, f)^2 ds + \|\Lambda\|^2 \int_0^\infty D_{n-1}(s, f)^2 ds \to 0 \quad (n \to \infty). \end{aligned}$$

Thus the lemma follows. □

From Proposition 4.9 (1), (2) and Theorem 4.6 (3), we immediately obtain the next representation of the mean-squared prediction error.

Theorem 4.12. *Let $f \in \mathcal{H}_{[t_1,\infty)}(Y)$. Then*

$$\left\| P^{\perp}_{[-t_0,t_1]} \int_{t_1}^{\infty} f(s) dY(s) \right\|^2 = \sum_{n=0}^{\infty} \int_0^\infty D_n(s, f)^2 ds.$$

5. Representation of the innovation processes

In this section, we obtain an explicit form of the kernel $k(t, s)$ in (1.7) using Theorem 4.7. Let $Y(\cdot)$, $U(\cdot)$, $a(\cdot)$ and $c(\cdot)$ be as in Section 3. As stated in Section 1, we consider the filtration $\{\mathcal{F}(t)\}_{0 \le t \le T}$ that is the augmentation, by the null sets in $\mathcal{F}^Y(T)$, of the filtration $\{\mathcal{F}^Y(t)\}_{0 \le t \le T}$

generated by $Y(\cdot)$, i.e., $\mathcal{F}^Y(t) := \sigma(Y(u) : 0 \le u \le t)$ for $0 \le t \le T$. Let $B(\cdot)$ be the Kailath–Shiryaev innovation process defined by

$$B(t) = Y(t) + \int_0^t \alpha(s)ds \qquad (0 \le t \le T), \tag{5.1}$$

where $\alpha(\cdot)$ is a Gaussian process defined by

$$\alpha(t) = E[U(t)|\mathcal{F}(t)] \qquad (0 \le t \le T). \tag{5.2}$$

By [14, Theorem 7.16], under P, $B(\cdot)$ is a Brownian motion such that $\{\mathcal{F}(t)\}_{0\le t\le T}$ is equal to the augmentation, by the null sets in $\mathcal{F}^B(T)$, of the filtration $\{\mathcal{F}^B(t)\}_{0\le t\le T}$ generated by $B(\cdot)$. In particular, $(Y(t))_{0\le t\le T}$ is a Gaussian semimartingale with respect to $\{\mathcal{F}(t)\}_{0\le t\le T}$, and (5.1) or (1.3) gives the semimartingale representation of $Y(\cdot)$.

Proposition 5.1. *Let $t \in \mathbf{R}$. Then the function $a(t-\cdot)$ is in $\mathcal{H}_{(-\infty,t]}(Y)$. Furthermore, we have $\int_{-\infty}^t a(t-s)dY(s) = \int_{-\infty}^t c(t-s)dW(s)$.*

Proof. First we have $\int_0^\infty a^2(u)du \le a(0)\int_0^\infty a(u)du < \infty$. Next, since $a(\cdot)$ is in $L^1(\mathbf{R}, du)$ and $c(\cdot)$ is in $L^2(\mathbf{R}, du)$, $a * c$ belongs to $L^2(\mathbf{R}, du)$. Therefore $\int_{-\infty}^\infty (\int_{-\infty}^t a(t-u)c(u-s)du)^2 ds$ is equal to

$$\int_{-\infty}^\infty \left(\int_{-\infty}^{t-s} a(t-s-v)c(v)dv\right)^2 ds = \|(a*c)\|^2 < \infty,$$

where $\|\cdot\|$ is the norm of $L^2(\mathbf{R}, du)$. Thus the first assertion follows.

Notice that $a(t) = c(t) = 0$ for $t \le 0$. By Proposition 3.2 and (3.9),

$$\begin{aligned}
&\int_{-\infty}^t a(t-s)dY(s) \\
&= -\int_{-\infty}^\infty \left(\int_{-\infty}^t a(t-u)c(u-s)du\right) dW(s) + \int_{-\infty}^t a(t-s)dW(s) \\
&= \int_{-\infty}^t \left(-\int_0^{t-s} a(t-s-u)c(u)du + a(t-s)\right) dW(s) \\
&= \int_{-\infty}^t c(t-s)dW(s).
\end{aligned}$$

Thus the second assertion follows. □

Recall $g(s, \tau : t_0, t_1)$ from (4.10). For $t, \tau > 0$ and $s \in (0, t)$, we write $h(t, s, \tau)$ for $g(-s, \tau; t, 0)$, that is,

$$h(t, s, \tau) = \sum_{k=1}^\infty \{b_{2k-1}(s, \tau; t) + b_{2k}(t-s, \tau; t)\} \qquad (t, \tau > 0,\ 0 < s < t).$$

Here is the representation of $\alpha(\cdot)$.

Theorem 5.2. *We have* (1.7) *with*

$$k(t,s) = a(t-s) + \int_0^\infty h(t,s,\tau)a(t+\tau)d\tau \qquad (0<s<t<\infty). \tag{5.3}$$

Proof. Recall θ from Section 2. Let $t>0$. From Propositions 3.5 (1) and 5.1 and Theorem 4.7 with $t_0=t$, $t_1=0$, $P_{[0,t]}U(t)$ is equal to

$$\begin{aligned}
&P_{[0,t]}\left\{\int_0^t a(t-s)dY(s) + \int_{-\infty}^0 a(t-s)dY(s)\right\}\\
&= \int_0^t a(t-s)dY(s) - \theta P_{[-t,0]}\int_0^\infty a(t+s)dY(s)\\
&= \int_0^t a(t-s)dY(s) - \theta\int_{-t}^0\left(\int_0^\infty g(s,\tau;t,0)a(t+\tau)d\tau\right)dY(s)\\
&= \int_0^t a(t-s)dY(s) + \int_0^t\left(\int_0^\infty g(-s,\tau;t,0)a(t+\tau)d\tau\right)dY(s).
\end{aligned}$$

Thus the theorem follows. □

Example 5.3. Let $p,q\in(0,\infty)$. Then the kernel $a(t)=pe^{-qt}I_{(0,\infty)}(t)$ satisfies (3.2) with $\nu(ds)=p\delta_q(ds)$, and we have $c(t)=pe^{-(p+q)t}I_{(0,\infty)}(t)$ from (3.6) (cf. [1, Example 2.14]). It follows from (3.3) with (3.4) that $Y(\cdot)$ is given by (1.5). Then, for $s,\tau,t>0$,

$$b(s,\tau) = -pe^{-q(s+\tau)} + p^2e^{-q(s+\tau)}\int_0^\tau e^{-pu}du = -pe^{-qs}e^{-(q+p)\tau},$$

and so $b_2(s,\tau;t)=\int_0^\infty b(s,u)b(t+u,\tau)du = -b(s,\tau)pe^{-qt}/(2q+p)$. Repeating this procedure, we obtain $b_n(s,\tau;t)=\phi(t)^{n-1}b(s,\tau)$ for $n=1,2,\dots$, where

$$\phi(t) := -\frac{pe^{-qt}}{2q+p} \qquad (t>0).$$

Hence, for t_0, t_1, t_2 as in (4.1) and (4.2), $s\in(-t_0,t_1)$ and $\tau>0$, we have

$$\begin{aligned}
g(s,\tau;t_0,t_1) &= b(t_1-s,\tau)\sum_{k=1}^\infty \phi(t_2)^{2k-2} + b(t_0+s,\tau)\sum_{k=1}^\infty \phi(t_2)^{2k-1}\\
&= b(t_1-s,\tau)\frac{1}{1-\phi(t_2)^2} + b(t_0+s,\tau)\frac{\phi(t_2)}{1-\phi(t_2)^2}.
\end{aligned}$$

By Theorem 5.2 and elementary calculation, we see that (1.7) holds with

$$k(t,s) = p(2q+p)\frac{(2q+p)e^{qs}-pe^{-qs}}{(2q+p)^2e^{qt}-p^2e^{-qt}} \qquad (0<s<t). \tag{5.4}$$

6. Expected utility maximization for financial markets with memory

In this section, we explain the implication of Theorem 5.2, in particular, (5.4), in the financial markets with memory of [1] via expected logarithmic utility maximization.

Let T be a positive constant. Let $Y(\cdot)$ be as in Section 3 and let $\{\mathcal{F}(t)\}_{0\le t\le T}$ be as in Section 5. We consider the financial market $\mathcal{M}$ consisting of a stock with price $S(t)$ at time $t \in [0,T]$ and a share of the money market with price $S_0(t)$ at time $t \in [0,T]$. The price of the money market is governed by

$$S_0(0) = 1, \qquad dS_0(t) = r(t)S_0(t)dt \qquad (t \in [0,T]),$$

and the price of the stock satisfies (1.1) with $S(0)$ being a positive constant. We assume the following:

(1) the risk-free rate process $r(\cdot)$ is progressively measurable and satisfies $\int_0^T |r(t)|dt < \infty$ a.s.;
(2) the mean rate of return process $m(\cdot)$ is progressively measurable and satisfies $\int_0^T |m(t)|dt < \infty$ a.s.;
(3) the volatility process $\sigma(\cdot)$ is a progressively measurable process that satisfies $\int_0^T \sigma(t)^2 dt < \infty$ a.s. and $\sigma(t) > 0$ a.e. $t \in [0,T]$ a.s.;
(4) $\int_0^T \theta_0(t)^2 dt < \infty$ a.s., where

$$\theta_0(t) := \frac{m(t) - r(t)}{\sigma(t)} \qquad (0 \le t \le T);$$

(5) the following positive local martingale $(Z_0(t))_{0\le t\le T}$ is in fact a martingale:

$$\begin{aligned} Z_0(t) &= \exp\left\{ -\int_0^t [\theta_0(s) - \alpha(s)]\, dB(s) - \frac{1}{2}\int_0^t [\theta_0(s) - \alpha(s)]^2\, ds \right\} \\ &= \exp\left\{ -\int_0^t [\theta_0(s) - \alpha(s)] dY(s) - \frac{1}{2}\int_0^t [\theta_0(s)^2 - \alpha(s)^2] ds \right\}. \end{aligned}$$

A sufficient condition for (4) and (5) is that there exists a positive constant c_1 such that $P(|\theta_0(t)| \le c_1,\ \text{a.e. } t \in [0,T]) = 1$ (see [1, Proposition 3.2]). Under the above assumptions, $S_0(\cdot)$ is given by $S_0(t) = \exp\{\int_0^t r(u)du\}$ for $t \in [0,T]$ and $S(\cdot)$ by (1.4) or, for $0 \le t \le T$,

$$S(t) = S(0) \exp\left[\int_0^t \sigma(s) dB(s) + \int_0^t \left\{ m(s) - \sigma(s)\alpha(s) - \frac{1}{2}\sigma^2(s) \right\} ds \right].$$

The market $\mathcal{M}$ is complete, and if $\sigma(\cdot)$ and $r(\cdot)$ are constants, then the Black–Scholes formula for option pricing holds in it (see [12, Theorem 1.6.6 and Section 2.4]).

We define the discounted price process $\tilde{S}(\cdot)$ by $\tilde{S}(t) = S(t)/S_0(t)$ for $0 \le t \le T$. We consider the following expected logarithmic utility maximization from terminal wealth: for given $x > 0$, solve

$$V(x) = \sup_{\pi \in \mathcal{A}(x)} E[\log(X^{x,\pi}(T))],$$

where

$$\mathcal{A}(x) = \left\{ (\pi(t))_{0 \le t \le T} : \begin{array}{c} \pi(\cdot) \text{ is real-valued, progressively measurable,} \\ \int_0^T \pi^2(t)dt < \infty,\ X^{x,\pi}(t) \ge 0\ (0 \le t \le T) \text{ a.s.} \end{array} \right\},$$

and

$$X^{x,\pi}(t) = S_0(t) \left\{ x + \int_0^t \frac{\pi(u)}{S(u)} d\tilde{S}(u) \right\}.$$

The value $\pi(t)$ is the dollar amount invested in the stock at time t, whence $\pi(t)/S(t)$ is the number of units of stock held at time t. The process $X^{x,\pi}(\cdot)$ is the wealth process associated with the self-financing portfolio determined uniquely from $\pi(\cdot)$. By [12, Chapter 3, Example 7.9], there exists an optimal portfolio $\pi_0(\cdot) \in \mathcal{A}(x)$ for the problem and the optimal wealth process $X^{x,\pi_0}(\cdot)$ is given by $X^{x,\pi_0}(t) = x/H_0(t)$ with $H_0(t) := Z_0(t)/S_0(t)$ for $0 \le t \le T$. Moreover, if $S_0(\cdot)$ is deterministic, then the optimal portfolio proportion $\pi_0(t)/X^{x,\pi_0}(t)$ is given explicitly by $[\{m(t) - r(t)\}/\sigma(t)^2] - \{\alpha(t)/\sigma(t)\}$, that is,

$$\frac{\pi_0(t)}{X^{x,\pi_0}(t)} = \frac{m(t) - r(t)}{\sigma(t)^2} - \frac{1}{\sigma(t)} \int_0^t k(t,s)dY(s) \qquad (0 \le t \le T). \tag{6.1}$$

If the market does not have memory, that is, $Y(\cdot) = W(\cdot)$ or $a(\cdot) = c(\cdot) = 0$, then the integral on the right-hand side of (6.1) does not appear. On the other hand, in the presence of memory, the equality (6.1) implies that the optimal portfolio proportion at time t depends on the history of stock price from 0 through t and that the integral $\int_0^t k(t,s)dY(s)$ precisely describes the memory effect. It should be noticed that if $\sigma(\cdot) \equiv \sigma$ and $m(\cdot) \equiv m$ with σ and m being constants, then the values of $S(t)$ and $Y(t)$ are directly related with each other through

$$S(t) = S(0) \exp\left\{ \sigma Y(t) + mt - \frac{1}{2}\sigma^2 t \right\}.$$

It also should be noticed that in the simplest case $a(t) = pe^{-qt}$ which is empirically studied in [2], $k(t,s)$ is given by the elementary function (5.4) and that the parameters σ, p and q can be statistically estimated by the method of [2] which is briefly described in Section 1. Thus, if $\sigma(\cdot) \equiv \sigma$, $m(\cdot) \equiv m$ and $a(t) = pe^{-qt}$, then it is possible to calculate the right-hand side of (6.1) numerically from the data of stock prices.

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