On the Existence of a Stationary Equilibrium in a Stochastic Growth Model with Many Consumers*

Jun-ichi Itaya
Associate Professor
Otaru University of Commerce

1. INTRODUCTION

In the recent monetary literature [6,7,15,16,22]etc., the intertemporal optimization models under uncertainty play essential roles in investigation of properties of a monetary economy. These intertemporal optimization models, irrespective of the degree of uncertainty (perfect foresight, rational expectation, etc.) assume identical agents who have the same preferences as well as endowments. In other words, all these models have a single agent in an economy. Though it is true that this assumption helps to keep such intertemporal optimization models operational and tractable, it is sometimes restrictive and gives limited results; Specifically, in the competitive equilibrium which they define, there are no transactions or exchanges among consumers. Since the consumers are assumed to be homogeneous, prices (or price functions) are so determined as to balance their consumptions and storages between a current and a future date. This result seems to be crucially dependent on the assumption of identical agents.

On the other hand, the standard general equilibrium models of Arrow, Debrue, and McKenzie [1,19] (A.D.M. model) deal with many agents who have diverse preferences and endowments. It has been shown that transactions among agemts can achieve Pareto optimal allocation at a

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competitive equilibrium in their models. If we introduce the concept of contingent markets into A.D.M. model, we can formally extend this static model to the dynamic case. But there is no explicit capital accumulation in their models with the contingent markets.

In this paper we investigate a dynamic general equilibrium model with many consumers under uncertainty. Since we assume the specific class of utility functions, our model can be thought of as a special case of A.D.M. model with contingent claims. Our motivations are twofold; First, our model explicitly includes capital accumulation in the A.D.M. model under uncertainty. Secondary, in order to present the model which analyzes a dynamic monetary model, it is necessary to incorporate many consumers into intertemporal optimization models.

In this paper, it shall be shown that the existence of a stationary equilibrium and stationary equilibrium path will be proven and shall also apply some results which are developed in optimal stochastic growth literatures [8,9,10,13,18,23,28,29]. Especially, we shall use the Ky Fan's Fixed Point Theorem in the infinite dimensional space. Our existence proof can be applicable to the existence problem in the static general equilibrium of the infinite dimension. However, the more dynamic aspects of our model like turnpike properties--the convergence to a stationary equilibrium path--are left as our future works.

In Section 3 of this paper, we shall explain the model in detail. In Section 4 we shall summarize several definitions and theorems which are proved in this paper. In Section 5 we shall discuss the relationship between my model and other models. Section 6 is a key part of the present paper giving the rigorous proofs for the existence and optimality of a stationary equilibrium.

In the next section, some mathematical notations are presented and defined.

2. VECTOR SPACE NOTATION

R^L is a L-dimensional Euclidean space. R^{Lc} denotes the subspace of R^L, i.e., $R^{Lc} = \{x \varepsilon R^L | x_k = 0 \text{ if } k \notin L_c\}$. R^{Lo} and R^{Lp} are defined similarly. R^L_+ denotes the nonnegative orthant of R^L. The norm of any $x \varepsilon R^L$ is denoted by $||x||$, which implies

$$||x|| = \max\{|x_k|; j=1, \ldots L\} .$$

Let $(S, \mathscr{S})$ be a measurable space consisting of a set S and a sigma-algebra $\mathscr{S}$ of subsets of S. $L_{\infty,L}(S, \mathscr{S}, \sigma)$ is the set of functions from S to R^L which are measurable with respect to σ and are essentially bounded. Throughout this paper, measure theoretic words like 'measurable' and 'integrable' are used in the sense of Lebesgue. $L_{\infty,L_o}(,)$, $L_{\infty,L_p}(,)$, and $L_{\infty,L_c}(,)$ are the subspaces of $L_{\infty,L}(,)$. $L^+_{\infty,L}(S, \mathscr{S}, \sigma)$ is defined by

$$\{x \varepsilon L_{\infty,L}(S, \mathscr{S}, \sigma) \; x \geqq 0 \quad \text{almost surely}\} .$$

For simplicity, we sometimes abbreviate 'almost surely' to 'a.s.'. $L^+_{\infty,L_o}(,)$, $L^+_{\infty,L_p}(,)$, and $L^+_{\infty,L_c}(,)$ are defined similarly. The norm of L_∞ space is given by

$$||x||_\infty = \sup\{\alpha \varepsilon R_+ | \sigma(\{t \varepsilon E, x(t) > \alpha\}) = 0\} .$$

$L_{1,L}(S, \mathscr{S}, \sigma)$ denotes the set of functions from S to R^L which are $\mathscr{S}$-measurable and integrable with respect to the measure σ. The norm of L_1 space is given by

$$||x||_1 = \int |x| d\sigma .$$

The weak-star topology is defined as follows: Let $\Psi \varepsilon L_{1,L}(S, \mathscr{S}, \sigma)$ be a linear functional on $L_{\infty,L}(S, \mathscr{S}, \sigma)$. The linear functional is defined by $\Psi \cdot x$ for $x \varepsilon L_{\infty,L}(S, \mathscr{S}, \sigma)$. The weak-star topology on $L_{\infty,L}(S, \mathscr{S}, \sigma)$ is the weakest topology such that each functional $\Psi \varepsilon L_{1,L}(S, \mathscr{S}, \sigma)$ is continuous.

3. THE MODEL

Commodities

There are L types of commodities in our economy which are physically distinguishable and are finite number at each date.

Commodities are classified into three groups as follows;

L_o= a set of primary goods like labor, $L_o \subset L$.

L_c= a set of consumption goods, $L_c \subset L$.

L_p = a set of producible goods like capital equipments, $L_p \subset L$.

Throughout this paper, we assume

(L1) $L_i \neq \phi$ (i=0, c, p), $L_o \cup L_p = L$, $L_p \cap L_o = \phi$, and
$L_c \cap L_i \neq \phi$ (i=0, p) .

Environment

In a real world, firms and the consumers are faced with various types of uncertainty, such as climate variations, uncertainty about resources available, technological inventions in future, etc. In this paper we assume that there is uncertainty about the production possibilities of firms and the supply of endowment of consumers. This uncertainty in our model is represented by the occurrence of some states of the nature, s_t , at period t, independent of the firm's and consumer's decision. These uncertainties happen similarly in successive future periods, too. It seems natural to represent such a sequence of the states of the world by a stochastic process defined on some probability space, where the probability distribution on the set of sequences of states is perfectly known to all agents.

Following Dana [10] and Zilcah [28, 29], instead of specifying the particular stochastic process, we will consider the more general case that the probability distribution σ on the sequence of states is given and it is not affected by the history.

The environment is, therefore, represented by a following probability space $(S, \mathcal{S}, \sigma)$ where S is assumed to be set of all doubly infinite sequence $s=(s_t)$, $-\infty<t<\infty$, i.e., $S= \times_{-\infty}^{\infty} R_t$ where R is the set of all possible states of the environment and complete separable metric space independent of time. S is endowed with the product topology. $\mathcal{S}$ is the sigma field generated by all the cylinder sets in S, i.e. $\mathcal{S}$ is the Borel sigma field. $\{\mathcal{S}_t\}_{t=1}^{\infty}$ is a monotonic increasing sequence of sigma field. σ is a probability measure on $(S, \mathcal{S})$.

Define the shift operator, $T:S \to S$ and $(Ts)_t = s_{t+1}$ for $-\infty < t < \infty$. To keep our analysis tractable, we assume that the stochastic process is stationary. Mathematically,

(S1) T and T^{-1} are measure preserving, where T^{-1} is an inverse operator of T.

This assumption implies that the stochastic process $\{s_t\}_{t=1}^{\infty}$ is stationary, i.e., $\sigma(TA)= \sigma(A)= \sigma(T^{-1}A)$ for every $A \in S$.

Firms

Following McKenzie [19] and Yano [25], we shall aggregate all firms' technologies into one social production set. Under the uncertain environment the aggregate production possibility set is expressed by

$$Y(s)= \{(\alpha,\beta)\in R_+^L \times R_+^{Lp} \mid \beta\in\phi(s,\alpha)\},$$

where $\alpha\in R_+^L$ are inputs in one period, $\beta\in R_+^{Lp}$ are outputs in the succeeding period, and the correspondence $\phi(s, \alpha)$ is the set of possible output β is the state of the world is s and inputs α.

For each state s, the technology set Y(s) satisfies the following neoclassical assumptions,

(P1) Y(s) is a convex cone with vertex 0 for all s.

(P2) Free-disposal; If $(\alpha, \beta)\in Y(s)$, $\alpha'\geqq\alpha$ and $\beta\leqq\beta'$, then $(\alpha', \beta')\in Y(s)$ for all s.

(P3) Necessity of primary goods to production; For all s if $(\alpha,\beta)\varepsilon Y(s)$ and $\alpha^{L}=0$, then $\beta=0$.

(P4) Primary inputs are not producible; If $(\alpha, \beta)\varepsilon Y(s)$, then all coordinates of β corresponding to primary goods are zero.

(P5) Existence of a stock expansible by the factor ρ^{-1}; There exists a positive number r and $(\bar{\alpha}, \bar{\beta})\varepsilon Y(s)$ such that $\rho T^{-1}\bar{\beta}_k(s)-\bar{\alpha}_k(s)\geq r$, for all $k\varepsilon L_p$.

(P6) The correspondence $\phi:S\to\phi(s,\alpha)$ is $\mathcal{S}_1$ -measurable for a fixed α.

(P7) For each s, $\phi(s, \alpha)$ is upper-semi continuous.

(P1) implies that the technology exhibits constant returns to scale. (P2)∿(P5) are standard assumptions in the capital theory [20]. (P6) is justified by that the production possibility during the period 0 depends on the history of the environment up to the period 1. (P7) implies that the convex cone Y(s) is closed.

The firm are owned by consumers. Since the production possibility set is a cone, there is no profit which can be distributed to the consumers. However, in order to begin production at the period 0, the firm needs some initial endowments of producible goods, α^{o}, like capital equipments. We assime that each consumer also has an initial endowment of producible goods at the period 0 and rents it out to the firm. Therefore, the firm must pay a rent to shareholders at each period.

Especially, in a stationary equilibrium the firm pays a fixed rent to the shareholders at each period. Given the price sequence, the firm chooses an intertemporal production plan so as to maximize its total discounted expected profits over the infinite horizon. That is,

$$(3.1)\ \max \sum_{t=1}^{\infty} \rho^{t}[\int \Psi^{t+1}(s)\beta^{t+1}(s)d\sigma - \int \Psi^{t}(s)\alpha^{t}(s)d\sigma]$$

for $(\alpha^t, \beta^{t+1}) \varepsilon T^t G$

$\Psi^t \varepsilon L^+_{1,L}(S, \mathcal{S}_t, \sigma)$ for t=0, 1, 2, ...

$G = \{(\alpha, \beta) | \alpha, T^{-1}\beta \varepsilon L$ and $\beta(s) \varepsilon \phi(s, \alpha(s))$ a.s.$\}$ is the set of all technologically feasible stationary programmes, where L is the set of all $\mathcal{S}_o$-measurable functions from S to R^L_+ [see Lemma 1, p.16].

When we consider about a stationary equilibrium, the firm's problem becomes more simple,

$$(3.2) \quad \max \int \Psi(\rho T^{-1}\beta - \alpha) d\sigma \quad \text{for all} \quad (\alpha, \beta) \varepsilon G$$

at each period, given a stationary price sequence, $\{\Psi^t\}^\infty_{t=1} = \{\rho^t \bar{\Psi}\}^\infty_{t=1}$ where $\bar{\Psi}(s) \varepsilon L^+_{1,L}(S, \mathcal{S}_o, \sigma)$.

Consumers

There are I consumers, where I is a positive finite integer. This number of I is constant over time. Each consumer lives over the infinite horizon and has a different preference and a different endowment.

The endowment of consumer is given by $w_i \varepsilon L^+_{\infty, L_o}(S, \mathcal{S}_o, \sigma)$ for all i and all t. This implies that the supply of resources for the factor market is bounded.

The preference of consumer i is represented by the utility function, $u_i = R^{Lc}_+ \rightarrow R$ for each i. Further, the utility function is time-additively separable and satisfies the expected utility hypothesis. That implies that the following utility function uniquely exists,

$$(3.3) \quad \int_A u(x(s)) d\sigma, \quad \text{for any } A \varepsilon \mathcal{S}.$$

In our model the consumer maximizes the total discounted expected utility of consumption, (3.3), over the infinite periods under the given price sequence and the following budget set. That is,

$$(3.4) \quad \max \sum_{t=1}^{\infty} \rho^t \int u_i(x^t_i(s)) d\sigma,$$

sub. to

$\{x_i^t\}_{t=1}^{\infty} \varepsilon \beta_i(\Psi)$,

where ρ is a discount factor,

Ψ is a price system, $\Psi = \{\Psi^t\}_{t=1}^{\infty} \varepsilon \mathop{\times}\limits_{t=1}^{\infty} L_{1,L}^{+}(s, \mathcal{S}_t, \sigma)$,

$$(3.5)\quad \beta_i(\Psi) = \{\{x_i^t\}_{t=1}^{\infty} \varepsilon \mathop{\times}\limits_{t=1}^{\infty} L_{\infty,L}^{+}(s, \mathcal{S}_t, \sigma) \mid \sum_{t=1}^{\infty} \int \Psi^t x_i^t d\sigma \leq \sum_{t=1}^{\infty} \int \Psi^t w_i^t d\sigma + d_i \Psi^o \alpha^o \text{ and } w_i^t = w_i \text{ for all } t\} .$$

where Ψ^o is an initial price and α^o is an initial stock of producible goods which are not random variables. d_i is a share of initial producible goods held by consumer i. Although the consumers are faced with uncertainty in future, we can say that every consumer is in full information in the sense that he knows all distribution functions of future price sequence $\{\Psi^t\}_{t=1}^{\infty}$ and w_i.The same thing holds to firms. In the consumer's budget, (3.5) his income consists of total expected value endowments and a rent revenue from firms.

In a stationary equilibrium, given a stationary price sequence, the consumer's maximization problem (3.5) is also reduced to

$$(3.6)\quad \max \int u_i(x_i(s))d\sigma ,$$

sub. to $x_i \varepsilon \tilde{\beta}_i(\Psi)$

where $\tilde{\beta}_i(\Psi) = \{\int \Psi x_i d\sigma \leq \int \Psi w_i d\sigma + (\rho^{-1} - 1) d_i \int \Psi \alpha d\sigma, \Psi \varepsilon L_{1,L}^{+}(S, \mathcal{S}_o, \sigma)\}$

We assume;

(C1) $w_i \varepsilon L_{\infty,L_o}^{+}(s, \mathcal{S}_o, \sigma)$ for all i and all t.

(C2) u_i is continuous and concave.

(C3) All consumers have a common discount factor $0<\rho<1$.

(C4) Strong monotonicity; for all i $u_i(,): R_+^{Lc} \to R$ is strongly monotone.

(C5) $w_i \gg 0$ a.s. for all i.

(C6) There exists $w \varepsilon L^+_{\infty,L_o}(S, \mathscr{S}_o, \sigma)$ and $(\alpha, \beta) \varepsilon G$ for all t such that $T^{-1}\beta(s) - \alpha(s) + w \geqq 0$ a.s., where $w = \sum_{i=1}^{I} w_i$.

(C1) implies that the endowment of primary goods for consumer i is the same and bounded at every period. (C3) may look restrictive, but we can relax it to a homogeneous discount case without difficulty. (C5) implies that every consumer has a positive amount of all primary goods. (C6) implies that the resources and technologies of society together permit the supply of all goods, i.e., primary goods and producible goods. These two assumptions guarantee that every consumer has positive income in an equilibrium.

4. DEFINITIONS AND THEOREMS RELATED TO A STATIONARY EQUILIBRIUM

Denote the economy as follows;

$$E = \{(u_i, \rho, w_i), Y, d_i : i=1,2,\ldots I\}.$$

We shall give the following definitions;

<u>feasible allocation and feasible allocation path</u>

A feasible stationary allocation is a stationary allocation $\{(\alpha,\beta), (x_i^I)_{i=1}\} \varepsilon L^+_{\infty,L}(S, \mathscr{S}_o, \sigma) \times L^+_{\infty,L_p}(S, \mathscr{S}_1, \sigma) \times (\mathop{\times}_{i=1}^{I} L_{\infty,L_c}(S, \mathscr{S}_o, \sigma))$ such that

(i) $(\alpha, \beta) \varepsilon G$,

(ii) $\Sigma x_i = \alpha - T^{-1}\beta + \Sigma w_i$.

A feasible allocation path from the initial stocks $\alpha^o \varepsilon R^{Lp}_+$ is a stochastic process $\{(\alpha^t, \beta^{t+1}), (x_i^t)_{i=1}^I\}_{t=0}^{\infty} \varepsilon \mathop{\times}_{t=1}^{\infty} \{L^+_{\infty,L}(s, \mathscr{S}_t, \sigma) \times L^+_{\infty,L_p}(S, \mathscr{S}_{t+1}, \sigma)) \times (\mathop{\times}_{t=1}^{I} L^+_{\infty,L_c}(S, \mathscr{S}_t, \sigma))\}$ such that

(i) $(\alpha^t, \beta^{t+1}) \varepsilon T^t G$ for $t=0,1,\ldots$,

(ii) $\Sigma x_i^t = \alpha^t - T^{-1}\beta^t + \Sigma w_i$ for $t=0,1,\ldots$,

This is a non-stationary path.

A feasible stationary allocation path from α^o is infinitely repeated by a feasible stationary allocation.

<u>stationary equilibrium</u>

A stationary equilibrium of E is $((\alpha^*(s),\ \beta^*(s))$ $(x_i^* S(s))_{i=-1}^I$, $\Psi^{*t}(s))$, satisfying that

(i) Stationary prices;

$$\Psi^{*t}(s) = \rho^t \bar{\Psi}^*(s) \text{ where } \bar{\Psi}^*(s) \varepsilon L_{1,L}^+(s, \mathcal{S}_o, \sigma) \text{ and } \Psi^{*t}(s) \neq 0 \text{ a.s.}$$

(ii) Utility maximization;

$$x_i^*(s) \text{ maximizes } \int u_i(x_i(s))d\sigma \ ,$$

sub. to

$$\int \Psi^* x_i \, d\sigma \leqq \int \Psi^* w_i \, d\sigma + (\rho^{-1} - 1)\int \Psi \alpha^* d\sigma$$

where $w_i \varepsilon L_{\infty o L}^+(S, \mathcal{S}_o, \sigma)$.

(iii) Profit maximization;

$$(\alpha^*, \beta^*) \text{ maximizes } \int \Psi^*(\rho T^{-1}\beta - \alpha)d\sigma \text{ for all } (\alpha, \beta)\varepsilon G.$$

(iv) Market clearing condition;

$$\alpha^*(s) - T^{-1}\beta^*(s) + \sum_{i=1}^{I} w_i(s) = \sum_{i=1}^{I} x_i^*(s) \quad \text{a.s.}$$

<u>stationary equilibrium path</u>

A stationary equilibrium path of E is a $\{(\alpha^{*t}(s),\ \beta^{*t+1}(s)),$

$$(x_i^{*t}(s))_{i=1}^I,\ \Psi^{*t}(s)\}_{t=1}^\infty , \quad \text{satisfying that}$$

(i) Stationary price path;

$$\{\Psi^{*t}\}_{t=1}^\infty = \{\rho^t \Psi^*(\rho)\}_{t=1}^\infty \ \varepsilon \ \overset{\infty}{\underset{t=1}{\times}} (L_{1,L}^+(s, \mathcal{S}_o, \sigma)^t$$

and $\Psi^t(s) \neq 0$ a.s. for all t.

(ii) $x_i^{*t} = x_i^* \varepsilon L_{\infty,L_c}^+(s, \mathcal{S}_o, \sigma)$ for all t, which also satisfies (ii) of a stationary equilibrium at each period.

(iii) $(\alpha^{*t}, \beta^{*t+1}) = (\alpha^*, \beta^*)$ for all t, which also satisfies (iii) of a stationary equilibrium at each period.

(iv) Market clearing condition;

$$\alpha^*(s) - T^{-1}\beta^*(s) + \sum_{i=1}^{I} w_i(s) = \sum_{i=1}^{I} x_i^*(s) \quad \text{a.s. for all t.}$$

<u>Pareto optimality</u>

A feasible allocation path $\{((\alpha^t, \beta^{t+1}), (x_i^t)_{i=1}^I)\}_{t=1}^{\infty}$ is Pareto Optimal if there exists no feasible allocation path $\{((\bar{\alpha}^t, \bar{\beta}^{t+1}), (\bar{x}_i^t)_{i=1}^I\}$ such that

$$\sum_{t=1}^{\infty} \rho^t \int u_i(\bar{x}_i^t)\, d\sigma \geq \sum_{t=1}^{\infty} \rho^t \int u_i(x_i^t)\, d\sigma \quad \text{for all } i$$

with strict inequality for at least one i.

Under the assumptions (L1), (S1), (P1)$\sim$(P7) and (C1)$\sim$(C6), we shall prove the following theorems;

Theorem 1

For an economy $E = \{(u_i, \rho, w_i), Y, d_i \;;\; i=1,2,\ldots I\}$ there is a stationary equilibrium.

Theorem 2

For an econbomy $E = \{(u_i, \rho, w_i), Y, d_i;\; i=1,2,\ldots I\}$ at every period, there is a stationary equilibrium path.

Theorem 3

The allocation of any stationary equilibrium path is Pareto optimal.

At the stationary equilibrium path with a discount factor ρ, the same current stationary prices, Ψ, are established at every period and the same allocation is repeated. The consumer receives the same rent, $(\rho^{-1}-1)d_i$, at every period. He solves the same optimization problem (3.6) to get the stationary consumption plan at every period. The producer maximizes his profit, $\int \Psi(\rho T^{-1}\beta - \alpha)\, d\sigma$ at every period, too. It should be noted that at a stationary equilibrium path with a transfer payment $(\rho^{-1}-1)d_i$ the marginal utility of income of consumer i which is denoted by γ_i be constant, which is the multiplier associated with the budget constraint in (3.6). This property of γ_i plays an important role in proving the existence proof of a stationary equilibrium.

5. RELATION TO THE LITERATURE

We have already mentioned the relationship to monetary literatures.

The model in this paper is derived from the capital theory (See the excellent survey of this field, Mckenzie 20). In most of the discussion of this field the object function is a single utility function. Depending on the context, we can interpret this single utility function as a social welfare function or an individual utility function over time.

An exception to the above is seen in T. Bewley's model [3]. Bewley has applied the capital theoretical model to the general equilibrium model. His model, however, is a deterministic model and also assumed the differentiability of utility functions and production functions.

Yano [25] has generalized T. Bewley's model to the Mckenzie economy having an aggregate convex cone production set. My model is also based on the Mckenzie economy, but differs from Yono's model which is a deterministic one.

My model has several similar points to T. Bewley's model [4] which includes the uncertainty and each firm has an individual convex production set. An important different point between mine and T. Bewley's model is the way of proving the existence of a stationary equilibrium; According to Bewley's [4], he has used his existence results of his earlier paper [2] in the infinite dimensional commodity space. His procedure to prove the existence of a competitive equilibrium is as follows; First, he proves the existence of a competitive equilibrium in a sub-economy which has a finite dimension, using the ordinal Kakutani's fixed point theorem. Then, using the Hahn-Banach theorem, this equilibrium price is extended to a positive functional in the infinite dimensional space. Finally, the limit of a competitive equilibrium of sub-economy is proven to be an equilibrium in the infinite dimension space.

Compared to the above-mentioned indirect method, I have used a more direct method, which applies the generalized Kakutani fixed point theorem to the locally convex linear topological space [11]. Therefore, in the infinite dimensional space we directly construct the Negishi mapping [1,21] which satisfies the continuity in the appropriate topology from the compact convex set to itself in the appropriate topology.

To the best of my knowledge, Majumdar and Zilcha's paper [18] is the first one to use the Ky Fan's fixed point theorem. I think that this direct method is easier and more powerful than Bewley's method in proving the existence of a competitive equilibrium in an infinite dimensional commodity space.

If we assume that one consumer and one firm exist in an economy, our proving way becomes similar to Majumdar and Zilcha's [18] way of proving the existence of Modified Golden-Rule under uncertainty. Since my model has many consumers, we have to construct a different mapping (like Negishi mapping) from theirs.

Lucas and Stocky [17] also constructed the optimal growth model with many consumers. But their model was constructed under perfect certainty, and the preference and the technology are recursive. Instead of proving the existence of a competitive equilibrium, they proved the existence of (dynamic) Pareto optimum allocation. In addition, their proof depends crucially on the contraction mapping theorem under the Lipshitz condition.

6. PROOFS

In this section, I first prove several lemmas about the properties of the set $\tilde{G}^{\rho}$ and boundedness of an allocation. Then using these lemmas, I shall prove main theorems.

Lemma 1

There are measurable selections $(\alpha, \beta)\varepsilon Y(s)$ where α is $\mathcal{S}_0$-measurable and β is $\mathcal{S}_1$-measurable.

Proof

By the assumption (P6), the graph of Y(s) is Borel measurable. Hence, we can apply the "Measurable Choice Theorem".
(See lemma 1, p.55 in Hildenbrand [12])

Q.E.D.

This lemma implies that $G\neq\phi$.

Define a set of feasible production programs denoted by $\tilde{G}^{\rho}$ as follows:

$$\tilde{G}^{\rho}=\{(\alpha,\ \beta),(\alpha,\ \beta)\varepsilon G:\ T^{-1}\beta(s)+\sum_{i=1}^{I}w_i(s)\geqq\alpha(s)\quad \text{a.s.}\}\ .$$

Lemma 2 (Properties of $\tilde{G}^{\rho}$)

(i) $\tilde{G}^{\rho}$ is non-empty.

(ii) $\tilde{G}^{\rho}$ is convex.

(iii) $\tilde{G}^{\rho}$ is weak-star closed.

Proof

By $G\neq\phi$ and the assumption (C6), $\tilde{G}^{\rho}\neq\phi$. Hence, (i) is proved. (ii) is clear.

In order to prove (iii), it suffices to show that the intersection of $\tilde{G}^{\rho}$ with any bounded and closed subset is weak-star closed. Since the weak-star topology for any norm-bounded subset of L_∞ is metrizable [24], it suffices to prove that it is sequentially closed.

Suppose that sequence (α^n, β^n) in $\tilde{G}^\rho$ converges to (α, β) in the weak-star topology, then for every measurable set A in $\mathcal{S}_0$,

(4.1) $\int_A \alpha^n d\sigma \to \int_A \alpha d\sigma$ (i.e. weak-star convergence).

For every measurable set A' in $\mathcal{S}_1$,

(4.2) $\int_{A'} \beta^n d\sigma \to \int_{A'} \beta d\sigma$

and by definition, for all A in $\mathcal{S}_0$,

$$\int_A T^{-1}\beta^n d\sigma + \int_A \sum_{i=1}^{I} w_i d\sigma \geq \int_A \alpha^n d\sigma .$$

In the limit, by the fact (4.1) and (4.2),

$$\int_A T^{-1}\beta d\sigma + \int_A \sum_{i=1}^{I} w_i d\sigma \geq \int_A \alpha d\sigma .$$

Therefore,

$$T^{-1}\beta(s) + \sum_{i=1}^{I} w_i(s) \geq \alpha(s) \quad \text{a.s.}$$

Thus, $(\alpha, \beta) \in \tilde{G}^\rho$.

Q.E.D.

Lemma 3

Given $w_i \in L^+_{\infty,L}(S, \mathcal{S}_0, \sigma)$, there is a constant integer k such that $||\alpha||_\infty \leq k$ and $||\beta||_\infty \leq k$ for all (α, β) in $\tilde{G}^\rho$. Moreover, $||x_i||_\infty \leq k$ for all $i \in I$.

Proof

By the definition of $\tilde{G}^\rho$,

(4.3) $\alpha(s) \leq T^{-1}\beta(s) + \sum_{i=1}^{I} w_i(s)$ a.s.,

(4.4) $\alpha_k(s) \leq \sum_{i=1}^{I} w_i(s)$ a.s. for $k \in L_0$.

So there exists a $k>0$ such that $||\alpha_{L_0}(S)||_\infty \leq k$ by assumption (C1).

Let (α^n, β^n), $n \in N$ be a net in $\tilde{G}^\rho$. where N is a directed set. Now, suppose that there is a net $(\alpha^n, \beta^n) \in \tilde{G}^\rho$ in the weak-star topology such that

$$\lim_{n\to\infty} ||\beta^n||_\infty \to \infty .$$

Let $(\bar{\alpha}^n, \bar{\beta}^n) = ||\beta^n||_\infty^{-1}(\alpha^n, \beta^n)$.

Since this net is norm-bounded, $(\bar{\alpha}^n, \bar{\beta}^n)$ has a convergent subnet in the weak-star topology, which I index n again. Let $(\bar{\alpha}, \bar{\beta})$ be the limit of this net. Since $\tilde{G}^\rho$ is closed in the weak-star topology, $(\bar{\alpha}, \bar{\beta}) \varepsilon \tilde{G}^\rho$. Then $\bar{\alpha}_{L_o}=0$ a.s. But $\bar{\beta}\geqq 0$ a.s. and $||\bar{\beta}||_\infty=0$. Hence, $\bar{\beta}>0$ a.s. This contradicts the necessity of primary goods. Therefore, there exists a k such that for all $(\alpha, \beta) \varepsilon \tilde{G}^\rho$, $||\beta(s)||_\infty \leqq k$. Since $||\beta(s)||_\infty = ||T^{-1}\beta(s)||_\infty$ by assumption (S1) and $||\alpha_{L_p}(s)||_\infty \leqq ||T^{-1}\beta(s)||_\infty$, we have $||\alpha_{L_p}(s)||_\infty \leqq k$. That is, $||\alpha(s)||_\infty = ||(\alpha_{L_o}(s), \alpha_{L_p}(s))||_\infty \leqq k$.

Finally, by the feasible condition,

$$\sum_{i=1}^{I} x_i(s) \leqq T^{-1}\beta(s) - \alpha(s) + \sum_{i=1}^{I} w_i(s) \text{ a.s.,}$$

we have $||\sum_{i=1}^{I} x_i(s)||_\infty \leqq k$.

Thus, $||x_i(s)||_\infty \leqq k$ for all i.

This completes the lemma.

Q.E.D.

Lemma 4

$\tilde{G}^\rho$ is compact in the weak-star topology and metrizable.

Proof

By lemma 2(iii), lemma 3, and the Banach-Alaoglu theorem [24], it follows compactness. Since L_1 is separable, by the corollary of Banach-Alaoglu theorem [Theorem 3.16, 24] $\tilde{G}^\rho$ is metrizable.

Q.E.D.

The Proof of Theorem 1

The proof of Theorem 1 will be carried out through several steps, Step 1-Step 6.

<u>Step 1</u>

In order that we deal with the only essentially-bounded price function Ψ, we define the following subset of L_∞,

$$S^L=\{\Psi\varepsilon I_{\infty,L}(\mathscr{S}_o)\,|\,||x||_1=1 \quad \text{and } \Psi(s)\varepsilon k \quad \text{a.s.}\},$$

where $k=\{a\ \varepsilon\ R^L_+ \mid a_i \leqq N \quad i=1,\ldots L$ and N is a large integer$\}$.

We want to show that S^L is a compact subset of L_∞ in the weak-star topology. Since $(S, \mathscr{S}, \sigma)$ is seprable, this topology on S^L is metrizable (See Theorem 3.16 in [24]).

Since S^L is clearly norm-bounded, it is enough to show that S^L is closed in the weak-star topology (See Theorem 3.15 in [24]).

Since the following argument to prove it is standard, we shall omit it (See [18]).

Step 2

Define a correspondence F from S^L into $\tilde{G}^\rho$

$$F(\Psi)=\{(\tilde{\alpha},\ \tilde{\beta})\varepsilon\tilde{G}^\rho \,|\, \int\Psi(\rho T^{-1}\tilde{\beta}-\tilde{\alpha})d\sigma$$

$$\geqq\int\Psi(\rho T^{-1}\beta-\alpha)d\sigma \quad \text{for all } (\alpha,\ \beta)\varepsilon\tilde{G}^\rho \quad .$$

We will show that $F(\Psi)\neq\phi$ for all $\Psi\varepsilon S^L$.

Since $\tilde{G}^\rho$ is metrizable, we can use a sequence.

Suppose that $(\alpha^n,\ \beta^n)\ \varepsilon\ \tilde{G}^\rho$ converges to $(\tilde{\alpha},\ \tilde{\beta})$ in the weak-star topology. Then $(\tilde{\alpha},\ \tilde{\beta})\ \varepsilon\ \tilde{G}^\rho$, because $\tilde{G}^\rho$ is weak-star closed. It is also clear that by the above assumption,

$$\int\Psi(\rho T^{-1}\beta^n-\alpha^n)d\sigma \rightarrow \int\Psi(\rho T^{-1}\tilde{\beta}-\tilde{\alpha})d\sigma \quad \text{as } n\rightarrow\infty .$$

That is, the linear functional $\int\Psi(\rho T^{-1}\beta-\alpha)d\sigma$ is weak-star continuous on $\tilde{G}^\rho$. This implies that the maximum on $\tilde{G}^\rho$ is attained, since $\tilde{G}^\rho$ is weak-star compact. Therefore, $F(\Psi)\neq\phi$. By definition, we can easily know that $F(\Psi)$ is weak-star closed. And $F(\Psi)\subset\tilde{G}^\rho$, so $F(\Psi)$ is norm-bounded. Therefore, $F(\Psi)$ is weak-star compact.

Also, we can easily demonstrate that $F(\Psi)$ is convex-valued as follows; Let $(\tilde{\alpha}^1,\ \tilde{\beta}^1)$ and $(\tilde{\alpha}^2,\ \tilde{\beta}^2)\ \varepsilon\ F(\Psi)$.

Then, for any $t\ \varepsilon\ (0,\ 1)$,

$$\int\Psi[\rho T^{-1}(t\tilde{\beta}^1+(1-t)\tilde{\beta}^2)-(t\tilde{\alpha}^1+(1-t)\tilde{\alpha}^2)]d\sigma$$

$$=t\int\Psi[\rho T^{-1}\tilde{\beta}^1-\tilde{\alpha}^1]\,d\sigma+(1-t)\int\Psi[\rho T^{-1}\tilde{\beta}^2-\tilde{\alpha}^2]\,d\sigma$$

$$\geqq t\int\Psi[\rho T^{-1}\beta^1-\alpha^1]\,d\sigma+(1-t)\int\Psi[\rho T^{-1}\beta^2-\alpha^2]\,d\sigma$$

$$=\int\Psi[\rho T^{-1}\{t\beta^1+(1-t)\beta^2\}-\{t\alpha^1+(1-t)\alpha^2\}]\,d\sigma\ .$$

Therefore, $F(\Psi)$ is convex-valued.

Finally, we must show that $F(\Psi)$ is upper-semi continuous in the weak-star topology.

Suppose $\Psi^n\to\Psi$ in the weak-star topology. And let $(\alpha^n, \beta^n)\varepsilon F(\Psi_n)$ for all n and (α^n, β^n) converges to (α, β) in the weak-star topology.

By the above assumptions and profit maximization, we get, for each n,

$$0=\int\Psi^n(\rho T^{-1}\beta^n-\alpha^n)\,d\sigma \geqq \int\Psi^n(\rho T^{-1}\tilde{\beta}-\tilde{\alpha})\,d\sigma\ .$$

As $n\to\infty$,

$$0=\int\Psi(\rho T^{-1}\beta-\alpha)\,d\sigma \geqq \int\Psi(\rho T^{-1}\tilde{\beta}-\tilde{\alpha})\,d\sigma\ .$$

Therefore, $(\alpha, \beta)\ \varepsilon\ F(\Psi)$.

That is, $F(\Psi)$ is upper-semi continuous in the weak-star topology.

Step 3

Let (α, β) be a feasible stationary allocation, i.e. $(\alpha, \beta)\varepsilon\tilde{G}^\rho$, so $(x_i)_{i=1}^I$ satisfies

$$\sum_{i=1}^I x_i= T^{-1}\beta-\alpha+\sum_{i=1}^I w_i\geqq 0 \quad \text{a.s.}$$

Define the set of weights of individuals such that

$$U^I=\{\gamma\varepsilon R_+^I \mid \sum_{i=1}^I\gamma_i=1\}\ .$$

Define

$$X(x,\gamma)=\{(\tilde{x}_i)_{i=1}^I\ \varepsilon\ \mathop{\times}_{i=1}^I L_\infty^+,L_c(\mathcal{F}_o)\mid \sum_{i=1}^I\gamma_i Eu_i(\tilde{x}_i)\geqq\sum_{i=1}^I\gamma_i Eu_i(x_i)\ \text{for}$$

any $\gamma\varepsilon U^I$ and all $(\alpha,\beta)\ \varepsilon\ \tilde{G}^\rho\}$, where $E\{\cdot\}$ is an expectations operator. By applying the Aumann's measurable selection theorem [12] to the graph $G(x)=\{(s,x_1,\ldots,x_I)\ \varepsilon S\times(R_+^{Lc}\times\ldots\times R_+^{Lc})\mid\Sigma\gamma_i Eu_i(\tilde{x}_i(s))\geqq\Sigma\gamma_i Eu_i(x_i(s))$ for any $\gamma\varepsilon U^I$

and all $(\alpha, \beta) \varepsilon \tilde{G}^{\rho}\}$, we get $\tilde{x}_i \varepsilon L^{+}_{\infty,L_c}(S, \mathscr{S}_t, \sigma)$ for all i such that $\Sigma\gamma_i u_i(\tilde{x}_i(s)) \geqq \Sigma\gamma_i u_i(x_i(s))$ a.s.
Thus, $X(x, \gamma) \neq \phi$.

Since u_i is continuous and concave, $X(x, \gamma)$ is convex and closed in the weak-star topology.

Let us define a correspondence $V((\alpha, \beta), \gamma)$ on $\tilde{G}^{\rho} \times U^I$ to S^L .
For all $(\alpha, \beta) \varepsilon \tilde{G}^{\rho}$, i.e., $\Sigma x_i = T^{-1}\beta - \alpha + \Sigma w_i \geqq 0$ a.s., we define the aggregate expenditure function in a society as follows,

$$V((\alpha,\beta);\gamma)=\{\Psi\varepsilon S^L | \int \Psi\Sigma x_i \leqq \int \Psi\Sigma\tilde{x}_i \quad \text{for all } (\tilde{x}_i)_{i=1}^I \varepsilon\, X(x,\gamma)\}.$$

This mapping is

$$V((\alpha,\beta);\gamma):\ \tilde{G}^{\rho} \times U^L \rightarrow S^L \ .$$

Note that $\tilde{G}^{\rho}$ and S^L are weak-star compact and U^I is a compact convex set in the I-dimensional Euclidean space.
Clearly, $V((\alpha, \beta), \gamma) \neq \phi$.

Lemma 5
$V((\alpha, \beta);\gamma)$ is closed in the weak-star topology and norm-bounded, i.e., it is weak-star compact.

Proof

By the previous argument, $X(x,\gamma)$ is a closed convex set in the weak-star topology. Also, $(x_i)_{i=1}^I \varepsilon \mathring{X}(x, \gamma)$ in the strong topology of $L^{+}_{\infty,L_c}(\mathscr{S}_0)$. By the separation theorem ([12] and [24]), there is a linear functional Λ in $ba^{+}_L(S, \mathscr{S}_t, \sigma)$, the set of linear functionals on $L_{\infty,L}(S, \mathscr{S}_t, \sigma)$, which are continuous with respect to the sup-remum norm, $\Lambda \neq 0$, such that

(5.1) $\Lambda x \leqq \Lambda\tilde{x}$ for all $\tilde{x} \varepsilon X(x,\gamma)$.

Since u_i is increasing by the assumption (C4) for all i, $\Lambda > 0$.

By the standard argument of Radner [23], we use the following facts;

For Λ in $L_\infty^*(\mathcal{S}_t)$, there is a bounded finitely additive vector-valued measure ν such that $\Lambda x = \int x d\nu$ for all $x \varepsilon L_{\infty,L_c}^+(\mathcal{S}_o)$.

By the Yoshida-Hewitt theorem [26],

$$\nu = \nu_c + \nu_\rho ,$$

where ν_c is a non-negative countably additive measure on $(S, \mathcal{S}_o)$. ν_p is a non-negative purely additive measure on $(S, \mathcal{S}_o)$.

Applying the Radon-Nikodym theorem [27], there exists a unique Ψ in $L_{1,L}^+$ such that

$$\int x d\nu = \int \Psi x d\sigma$$

Thus, we have

$$\Lambda x = \int \Psi x d\sigma + \int x d\nu_p .$$

Since our production set satisfies the "Exclusion Assumption" that Bewley [2, 4] has showed, there is a sequence $\{A_n^t\}_{n=1}^\infty$ in $\mathcal{S}_t$, $A_n \subset A_{n+1}$, $\sigma(A_n) \uparrow 1$ as $n \to \infty$, and $\lim_{n\to\infty} \nu_p(A_n) = 0$.

Therefore, from (5.1) there exists $\Psi \varepsilon L_{1,L}(\mathcal{S}_t)$ such that

$$\int \Psi x d\sigma \leqq \int \Psi \tilde{x} d\sigma$$

for all $(\tilde{x})_{i=1}^I \varepsilon X(x, \gamma)$.

From the monotonicity assumption (C4), $\Psi > 0$ a.s. Thus, $\Psi \varepsilon L_{1,L}^+(\mathcal{S}_t)$.

Finally, in order that $V((\alpha, \beta), \gamma)$ is closed in the weak-star topology, we assume that $\Psi^n \to \Psi$ as $n \to \infty$ in the weak-star topology, since S^L is also metrizable (See Step 1).

$\int_B \Psi^n x d\sigma \leqq \int_B \Psi^n \tilde{x} d\sigma$ and $\Psi \varepsilon S^L$ as $n \to \infty$ on both sides of the inequality for any measurable set, we get

$$\int_B \Psi x d\sigma \leqq \int_B \Psi \tilde{x} d\sigma \text{ and } \Psi \varepsilon S .$$

That is, $\Psi \varepsilon V((\alpha,\beta),\gamma)$.
Thus, $V((\alpha,\beta), \gamma)$ is closed in the weak-star topology.
Since $V((\alpha, \beta), \gamma) \subset S^L$, where S^L is norm-bounded, $V((\alpha, \beta), \sigma)$ is weak-star compact.

Lemma 6

$V((\alpha, \beta), \gamma)$ is upper-semi continuous in the weak-star topology.

Proof

Since $\tilde{G}^\rho$ is metrizable, to prove the lemma, we will prove that for any sequence (α^n, β^n) in $\tilde{G}^\rho$ which converges to (α, β) in the weak-star topology, $\gamma^n \varepsilon U^I$ converges to γ in the strong topology, and for each n, $\Psi^n \varepsilon V((\alpha^n, \beta^n), \gamma^n)$, then $\Psi \varepsilon V((\alpha, \beta), \gamma)$.

Suppose $\Psi^n \varepsilon V((\alpha^n, \beta^n), \gamma^n)$.

By assumption, $(\alpha^n, \beta^n) \varepsilon \tilde{G}^\rho$, that is,

$$\sum_{i=1}^{I} x_i^n = T^{-1}\beta^n - \alpha^n + \Sigma w_i \geqq 0 \text{ a.s.},$$

where the sequence x_i^n corresponds to the sequence $(\alpha^n, \beta^n) \varepsilon \tilde{G}^\rho$. Since $\tilde{G}^\rho$ is closed in the weak-star topology, $(\alpha, \beta) \varepsilon \tilde{G}^\rho$, that is, as $n \to \infty$,

$$\sum_{i=1}^{I} x_i = T^{-1}\beta - \alpha + \Sigma w_i \geqq 0 \text{ a.s.}$$

Since U^I is closed in the strong topology, the limit γ is in U^I. By the definition of $X(x, \gamma)$, for each n,

$$X(x^n,\gamma^n) = \{(\tilde{x}_i)_{i=1}^I \ \varepsilon \ \mathop{\times}_{i=1}^{I} L^+_{\infty,L_c}(\mathscr{S}_o) \mid \sum_{i=1}^{I} \gamma_i^n E u_i(\tilde{x}_i) \geqq \sum_{i=1}^{I} \gamma_i^n E u_i(x_i^n) \text{ for } \gamma^n \varepsilon U^I \text{ and } (\alpha^n,\beta^n) \varepsilon \tilde{G}^\rho\} .$$

Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_I)$ be in $X(x, \gamma)$.
For any ε/I, define $\tilde{x}_i(\varepsilon) = \tilde{x}_i + (\varepsilon/I)\cdot e$ for all i where $e = (1,1,\ldots,1) \varepsilon R^{Lc}$.

By the result of Dana [10], we know that when u_i is concave and continuous, $\int u_i(x_i)d\sigma$ is upper-semi continuous in the weak-star topology. This implies that by definition of u.s.c., if x_i^n converges to x_i in the weak-star topology, then

$$\lim_{n\to\infty} \sup \int u_i(x_i^n)d\sigma \leqq \int u_i(x_i)d\sigma \text{ for all i.}$$

Therefore, for large n enough and a given $\varepsilon/I>0$, $\int u_i(\tilde{x}_i(\varepsilon))d\sigma > \int u_i(x_i^n)d\sigma$ for all i. Multiplying γ_i^n and summing up for all i, we have

$$\sum_{i=1}^{I} \gamma_i^n Eu_i(\tilde{x}_i(\varepsilon)) > \sum_{i=1}^{I} \gamma_i^n Eu_i(x_i^n) \ .$$

Thus, $\tilde{x}(\varepsilon)\varepsilon X(x^n, \gamma^n)$.

Furthermore, $\int \Psi^n \Sigma\tilde{x}_i(\varepsilon)d\sigma = \int \Sigma\Psi^n\tilde{x}_i(\varepsilon)d\sigma = \int \Sigma\Psi^n\tilde{x}_i d\sigma+\varepsilon \geqq \int \Sigma\Psi^n\tilde{x}_i d\sigma$
As $n\to\infty$,

$$\int \Psi\tilde{x}d\sigma \geqq \int \Psi x d\sigma, \text{ i.e. } \Psi\varepsilon V(\alpha,\beta),\gamma),$$

where $\tilde{x}=\sum_{i=1}^{I}\tilde{x}_i$ and $x=\sum_{i=1}^{I}x_i$.

This completes the lemma.

Step 4

We define the adjustment function of the weights which represents the contribution of individual utility level to a social utility function.

Since $(x_i)_{i=1}^{I}$ and (α, β) are norm-bounded by lemma 3 and $\Psi\varepsilon S^L$, Ψx_i and Ψw_i are uniformly integrable. Therefore, there exists a positive real number A such that for any feasible allocation $((x_i)_{i=1}^{I}, (\alpha, \beta))$ and $\{\Psi\}$.

$$\sum_{i=1}^{I} \left| \int \Psi w_i d\sigma + (\rho^{-1}-1)d_i \int \Psi\alpha d\sigma - \int \Psi x_i d\sigma \right| < A \ .$$

Simplifying it yields,

$$\sum_{i=1}^{I} \left| M_i - \int \Psi x_i d\sigma \right| < A \ .$$

where $M_i \equiv \int \Psi w_i d\sigma + (\rho^{-1}-1)d_i \int \Psi\alpha d\sigma$, i.e.,

M_i is income of consumer i at a stationary equilibrium.

Then, for any $\gamma \varepsilon U^i$ and $((\alpha, \beta), \Psi) \varepsilon \tilde{G}^\rho \times S^L$, we define the following function,

$$\beta_i = \max [0, \gamma_i + \frac{1}{A} (M_i - \int \Psi x_i d\sigma)] \quad ,$$

$$\beta_i' = \frac{\beta_i}{\sum_i \beta_i} \qquad \text{for all } i \; .$$

Note that all M_i are exogenously determined.

Denote the function $f_i((\alpha, \beta), \Psi, \gamma)$ as follows;

$$f_i((\alpha,\beta),\Psi,\gamma) = \{\beta_i' = \frac{\beta_i}{\Sigma\beta_i} | \beta_i = \max [0, \gamma_i + \frac{1}{A}(M_i - \int\Psi x_i d\sigma)]$$

where $\Sigma x_i = T^{-1}\beta - \alpha + \Sigma w_i$ a.s.$\}$ for all i .

That is,

$$f_i : U^I \times S^L \times \tilde{G}^\rho \rightarrow U^I \; .$$

This is a point to point mapping, that is, it is unique and continuous function with respect to (α, β), Ψ, and γ in the weak-star topology, because $f_i(,)$ is continuous with respect to γ in the strong topology and $f_i(,)$ is continuous with respect to (α, β) and Ψ in the weak-star topology.

By the definition of U^I, U^I is clearly compact in the strong topology and also convex.

Step 5

Let us define the following correspondence from $\tilde{G}^\rho \times S^L \times U^I$ to $\tilde{G}^\rho \times S^L \times U^I$ as follows;

$$H((\alpha,\beta),\Psi,\gamma) = V((\alpha,\beta),\gamma) \times F(\Psi) \times f((\alpha,\beta),\Psi,\gamma)$$

where $f(,) = (f_1(,), f_2(), \ldots, f_I(,)) \varepsilon R_+^I$.

The mapping H is convex-valued and upper-semi-continuous in the weak-star topology, since each component, $V(,)$, $F(,)$, and $f(,)$ has the above properties by the previous discussion through Step 2 to Step 4.

Furthermore, $V((\alpha, \beta), \gamma)$, $F(\Psi)$, and $f((\alpha, \beta), \Psi, \gamma)$ have the weak-star compact-value for any $(\alpha,\beta) \varepsilon \tilde{G}^\rho$, $\Psi \varepsilon S^L$, and $\gamma \varepsilon U^i$, so the Cartesian product of these mappings,

$H(\alpha, \beta), \Psi, \gamma)$, have the weak-star compact value.

Also it is clear that the Cartesian product $\tilde{G}^{\rho} \times S^{L} \times U^{I}$ (i.e., the domain of $H(,)$) is weak-star compact by the previous arguments.

Therefore, all conditions of the Ky Fan's Fixed Point Theorem for the topological vector space which has the locally convex topology are satisfied by the mapping $H(,)$ (See Theorem 1 in [11]). Thus, there exists a fixed point $((\alpha^*, \beta^*), \Psi^*, \gamma^*)$ such that $(\alpha^*, \beta^*) \varepsilon F(\Psi^*)$, $\Psi^* \varepsilon V((\alpha^*, \beta^*), \gamma^*)$, and $\Psi^* \varepsilon f((\alpha^*, \beta^*), \Psi^*, \gamma^*)$.

Step 6

Finally, we will show that this fixed point satisfies the conditions of a stationary equilibrium. By definition of $F(\Psi)$,

$$\int \Psi^*(\rho T^{-1}\beta^* - \alpha^*)d\sigma \geqq \int \Psi^*(\rho T^{-1}\beta - \alpha)d\sigma \quad \text{for all } (\alpha, \beta)\varepsilon\tilde{G}^{\rho}$$

which implies profit maximization of the integrate firm in a society. That is, the condition (iii) of a stationary equilibrium is satisfied.

Next, it will be verified that the market clearing condition (iv) is satisfied. Since $(\alpha^*, \beta^*)\varepsilon\tilde{G}^{\rho}$,

$$x^*(=\sum_{i=1}^{I} x_i^*) = T^{-1}\beta^* - \alpha^* + \Sigma w_i \geqq 0 \quad \text{a.s.}$$

But we must exclude the following situation,

$$x^* = T^{-1}\beta^* - \alpha^* + \Sigma w_i = 0 \quad \text{a.s.}$$

Suppose that.

By assumption (P5), there exists $(\bar{\alpha}, \bar{\beta})\varepsilon\tilde{G}^{\rho}$ such that $\rho T^{-1}\bar{\beta}_k - \bar{\alpha}_k > 0$ a.s. for all $k \varepsilon L_p$. Then

$$0 < \int \Psi^*(\rho T^{-1}\bar{\beta} - \bar{\alpha})d\sigma \leqq \int \Psi^*(\rho T^{-1}\beta^* - \alpha^* + \Sigma w_i)d\sigma$$

$$\leqq \int \Psi^*(T^{-1}\beta^* - \alpha^* + \Sigma w_i)d\sigma = 0$$

which is contradiction. Thus, $\int \Psi^* x_i^* d\sigma > 0$ and the market clearing condition is satisfied.

By the market clearing condition, multiplying Ψ^* on both sides, we get

$$\begin{aligned}\Sigma_i \int \Psi^* x_i^* &= \int \Psi^* T^{-1} \beta^* d\sigma - \int \Psi^* \alpha^* d\sigma + \sum_i \int \Psi^* w_i d\sigma \\ &= \int \Psi^* [T^{-1}\beta^* - \alpha^*] d\sigma + \sum_i \int \Psi^* w_i d\sigma \\ &= \int \rho^{-1} \Psi^* [\rho T^{-1}\beta^* - \rho\alpha^*] d\sigma + \sum_i \int \Psi^* w_i d\sigma \\ &= \int \rho^{-1} \Psi^* [\rho T^{-1}\beta^* - \alpha^*] d\sigma + \int \Psi^* (\rho^{-1} - 1)\alpha^* d\sigma + \sum_i \Psi^* w_i d\sigma\end{aligned}$$

by profit maximization (=0),

$$\begin{aligned}&= \int \Psi^* (\rho^{-1} - 1)\alpha^* d\sigma + \sum_i \int \Psi^* w_i d\sigma \\ &= \sum_i d_i \int \Psi^* (\rho^{-1} - 1)\alpha^* d\sigma + \sum_i \int \Psi^* w_i d\sigma \\ &= \sum_i M_i \ .\end{aligned}$$

The last two equalities follow from the facts that $\Sigma d_i = 1$ and the definition of M_i.

We seek to show that each individual's budget constraint holds;
that is, $\int \Psi^* x_i^* d\sigma = M_i$ for all i.
By the fact

$$\gamma^* = f((\alpha^*, \beta^*), \Psi^*, \gamma^*)$$

and construction of f(,),
each $M_i - \int \Psi^* x_i^* d\sigma$ has the same sign.

Using it and the fact, $\sum_i M_i - \sum_i \int \Psi^* x_i^* d\sigma = 0$,
we get

$$M_i - \int \Psi^* x_i^* d\sigma = 0 \quad ,$$

or

$$M_i = \int \Psi^* x_i^* d\sigma \quad \text{for all i.}$$

Now, we can show that the last condition holds, i.e., the utility maximization of every consumer under the above budget constraint holds. Since $\Psi^* \varepsilon V((\alpha^*, \beta^*), \gamma^*)$, we

know that
whenever

$$\sum_i^I \gamma_i^* Eu_i(x_i) \geq \sum_i^I \gamma_i^* Eu_i(x_i^*),$$

then $\int \Psi^* x d\sigma \geq \int \Psi^* x^* d\sigma$

where $x = \sum_i^I x_i$.

Let $x_i = x_i^*$ for $i \neq i'$. Then if we cancel all terms on both sides except i', we have

whenever $Eu_{i'}(x_{i'}) \geq Eu_{i'}(x_{i'}^*)$,

then $\int \Psi^* x_i' d\sigma \geq \int \Psi^* x_{i'}^* d\sigma$.

Repeating the above procedure over all i, we get the same result for all i. Thus, for each i x_i^* minimizes $\Psi^* x_i$ over $x_i \underset{\sim}{<} x_i$, given the price function Ψ^*. Since $\int \Psi^* x_i^* d\sigma = M_i$ for all i, we can rewrite the above result as follows,

whenever $Eu_i(x_i) \geq Eu_i(x_i^*)$,

then $\int \Psi^* x_i d\sigma \geq M_i$ $(= \int \Psi^* x_i^* d\sigma)$ for all i.

This implies the utility maximization of each consumer (=condition (ii) of a stationary equilibrium). This completes the proof of Theorem 1.

Q.E.D.

Proof of Theorem 2

This is the direct result of Theorem 5.1. Since the economy $E = \{u_i, \rho, w_i), Y, d_i : i=1, \ldots I\}$ repeats every period, the same allocation, $\{(\alpha^*, \beta^*), (x_i^*)_{i=1}^I\}$ is realized at every period. Therefore, the stationary prices $\rho^t \Psi$ are also realized, given a discount factor ρ.

Proof of Theorem 3

I don't give a detailed proof for this theorem, since the proof is completely routine. Please see the paper of Bewley's (Theorem 4.3 in 4).

Q.E.D.

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