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On a Non-Local Interaction in the Quantum Electrodynamics

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On replacing the local interaction terms appearing in the equations of motion in the ordinary quantum electrodynamics with non-local interaction ones, it is shown, from the calculation of the second order radiative correction to the current operator, that the self-energy of the electron can be made finite with an appropriate choice of non-local four-dimensional form factors. The observed value of the anomalous magnetic moment may play a role in the determination of the behavior of the four-demensional form factors employed for the non-localization of the interaction between the electromagnetic and electron fields.

§ 1. Introduction

Thus far various types of non-local field theory have been proposed by many authors, which may be divided into two main classes: one treats free fields themselves principally as non-local and the other attempts to non-localize the interaction part alone. The proposition made here, belonging to the latter class, is performed, however simple it may be, with an intention to develop it into more comprehensive formulation in the future.

Recently RAYSKI[1] has discussed field theories with non-localized interaction through quantizing the McMANUS fieid[2] in accordance with the method of YANG-FELDMAN.[3] His field theories are of course Lorentz-invariant; the McMANUS field, though classical, has an extended charge distribution in a Lorentz-invariant manner by introducing a four dimensional form function. Following McMANUS or RAYSKI we shall introduce here four-dimensional form factors which are delta-like functions in order to retain a correspondence to the usual field theory, but otherwise unspecified for the time being. Employing these weight functions we average four-dimensionally each field operator appearing in the interaction terms in the field

equations and replace the ordinary interaction terms with the ones thus modified. After such replacements we can follow the computational method of KÄLLÉN[4] and SCHWINGER[5] to evaluate the second order radiative correction to the current operator, the investigation of whose one-electron part reveals us that the self-mass of the electron can be made finite to the corresponding order provided an appropriate choice is made on the functional form of the form factors.

In addition the observed value of the anomalous magnetic moment of the electron may play a role to control the behavior of the form functions. Our theory is obviously Lorentz-invariant. The evasion of the Lagrangian formalism, however, will remain unsatisfactory, unless further examination is performed as to whether or not this directly replacing method may be applicable to more instances in the field theory.

§ 2. Calculation of the second order non-local radiative correction to the current operator and the concomitant discussion

The equation of motion of the interacting electromagnetic and electron fields are*

$$\left(\gamma\frac{\partial}{\partial x}+m\right)\psi(x)=\frac{ie}{2}\left\{A_\nu(x),\gamma_\nu\psi(x)\right\}+\delta m\psi(x), \tag{1}$$

and

$$\Box A_\mu(x)=-\frac{ie}{2}\left[\bar{\psi}(x),\gamma_\mu\psi(x)\right], \tag{2}$$

where the constant m is the observed mass of the electron and δm is its electromagnetic mass. In order to replace the local interaction terms with non-local counterparts we shall introduce two four-dimensional form functions, $F(x)$ and $G(x)$ that are both delta-like functions in correspondence to the usual field theory, accordingly even functions of the coordinates, but may well be different from each other; $F(x)$ weights the electromagnetic field and $G(x)$ does the electron field as follows:

$$\tilde{A}_\mu(x)=\int dx'F(x-x')A_\mu(x'), \tag{3}$$

* Throughout this paper we shall used the natural untis, $\hbar=c=1$.

and

$$\tilde{\psi}(x) = \int dx' G(x-x')\, \psi(x')\,, \tag{4}$$

The tilde means that the quantity is four-dimensionally averaged with one of these form factors. We shall make no further specification on the form factors except some later references and the determination of their definitive form is beyond our discussion.

Using the averaged quantities we can rewrite the equations of motion (1) and (2):

$$\left(\gamma\frac{\partial}{\partial x}+m\right)\psi(x) = \frac{ie}{2}\{\tilde{A}_\nu(x), \gamma_\nu\tilde{\psi}(x)\} + \delta m\psi(x)\,, \tag{5}$$

and

$$\Box A_\mu(x) = -\frac{ie}{2}[\tilde{\psi}(x), \gamma_\mu\tilde{\psi}(x)]\,. \tag{6}$$

To proceed with the calculation we shall expand field operators, current operator and the electromagnetic mass δm in a power series in the coupling constant e

$$A_\mu(x) = \sum_{n=0}^{\infty} e^n A_\mu^{(n)}(x), \qquad \tilde{A}_\mu(x) = \sum_{n=0}^{\infty} e^n \tilde{A}_\mu^{(n)}(x)\,, \tag{7a, b}$$

$$\psi(x) = \sum_{n=0}^{\infty} e^n \psi^{(n)}(x), \qquad \tilde{\psi}(x) = \sum_{n=0}^{\infty} e^n \tilde{\psi}^{(n)}(x)\,, \tag{8a, b}$$

$$j_\mu(x) = \sum_{n=0}^{\infty} e^{n+1} j_\mu^{(n)}(x)\,, \tag{9}$$

and

$$\delta m = \sum_{n=1}^{\infty} e^n \delta m^{(n)}\,. \tag{10}$$

From Eqs. (6), (7), and (9) we get

$$j_\mu^{(n)}(x) = \frac{i}{2}\sum_{m=0}^{n}[\tilde{\psi}^{(m)}(x), \gamma_\mu\tilde{\psi}^{(n-m)}(x)]\,. \tag{11}$$

From these expansions and the field equations (5) and (6) we obtain

$$\left(\gamma\frac{\partial}{\partial x}+m\right)\psi^{(n+1)}(x) = \frac{i}{2}\sum_{m=0}^{n}\{\tilde{A}_\mu^{(m)}(x), \gamma_\mu\tilde{\psi}^{(n-m)}(x)\} + \sum_{m=1}^{n+1}\delta m^{(m)}\psi^{(n+1-m)}(x)\,, \tag{12}$$

and

$$\Box A_\mu^{(n+1)}(x) = -\frac{i}{2}\sum_{m=0}^{n}[\tilde{\psi}^{(m)}(x), \gamma_\mu\tilde{\psi}^{(n-m)}(x)]\,. \tag{13}$$

If we use as boundary conditions for Eqs. (12) and (13) ($n \neq 0$)

$$\psi^{(n)}(x) \to 0\,, \quad A^{(n)}(x) \to 0 \qquad \text{for} \quad x_0 \to \infty \tag{14}$$

and

$$\psi^{(n)}(x)\Psi_0 \to 0\,, \quad A^{(n)}(x)\Psi_0 \to 0 \qquad \text{for} \quad x_0 \to -\infty\,, \tag{15}$$

where Ψ_0 denotes the constant state vector in the Heisenberg representation referred to throughout this paper, then Eqs. (12) and 13 are formally integrated as follows:

$$\begin{aligned}\psi^{(n+1)}(x) = &-\frac{i}{2}\sum_{m=0}^{n}\int \bar{S}(x-x')\{\tilde{A}_\mu^{(m)}(x')\,\gamma_\mu\bar{\psi}^{(n-m)}(x')\}\,dx' \\ &-\sum_{m=1}^{n+1}\delta m^{(m)}\int \bar{S}(x-x')\psi^{(n+1-m)}(x')\,d\iota'\,,\end{aligned} \tag{16}$$

and

$$A_\mu^{(n+1)}(x) = \frac{i}{2}\sum_{m=0}^{n}\int \bar{D}(x-x')[\bar{\psi}^{(m)}(x'),\,\gamma_\mu\bar{\psi}^{(n-m)}(x')]\,dx' \tag{17}$$

with the help of the Green functions defined by

$$\left(\gamma\frac{\partial}{\partial x}+m\right)\bar{S}(x) = -\delta(x) \tag{18}$$

and

$$\Box\bar{D}(x) = -\delta(x)\,, \tag{19}$$

whose employment suffices to get the correct expectation values.

In the first approximation we obtain

$$\psi^{(1)}(x) = -i\int\bar{S}(x-x')\gamma_\mu\tilde{A}_\mu^{(0)}(x')\bar{\psi}^{(0)}(x')dx' - \delta m^{(1)}\int\bar{S}(x-x')\psi^{(0)}(x')dx' \tag{20}$$

and

$$A_\mu^{(1)}(x) = \frac{i}{2}\int\bar{D}(x-x')[\bar{\psi}^{(0)}(x'),\,\gamma_\mu\bar{\psi}^{(0)}(x')]\,dx'\,. \tag{21}$$

From Eqs. (11) and (20) we obtain for the first approximation of the current operator

$$\begin{aligned}j_\mu^{(1)}(x) = &\frac{1}{2}\int[\bar{\psi}^{(0)}(x),\,\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\nu\bar{\psi}^{(0)}(x')]\,\tilde{A}_\nu^{(0)}(x')\,dx' \\ &-\frac{i}{2}\delta m^{(1)}\int[\bar{\psi}^{(0)}(x),\,\gamma_\mu\bar{\bar{S}}(x-x')\psi^{(0)}(x')]\,dx' \\ &+\frac{1}{2}\int[\bar{\psi}^{(0)}(x')\gamma_\nu\bar{\bar{S}}(x'-x),\,\gamma_\mu\bar{\psi}^{(0)}(x)]\,\tilde{A}_\nu^{(0)}(x')\,dx' \\ &-\frac{i}{2}\delta m^{(1)}\int[\bar{\psi}^{(0)}(x')\bar{\bar{S}}(x'-x),\,\gamma_\mu\bar{\psi}^{(0)}(x)]\,dx'\,.\end{aligned} \tag{22}$$

On computing the one-electron part of Eq. (22) we can readily see that

$$\delta m^{(1)} = 0 \tag{23}$$

because the first and third terms vanish due to the operator $\widetilde{A}_{\nu}^{(0)}(x')$.

It is necessary to proceed to the next approximation for the calculation of $j_{\mu}^{(2)}(x)$. As such we obtain

$$\begin{aligned}\psi^{(2)}(x) = &-\frac{1}{2}\iint \bar{S}(x-x')\gamma_{\mu}\bar{\bar{S}}(x'-x'')\gamma_{\nu}\bar{\psi}^{(0)}(x'')\{\widetilde{A}_{\mu}^{(0)}(x'), \widetilde{A}_{\nu}^{(0)}(x'')\}dx'dx'' \\ &+\frac{1}{4}\iint \bar{S}(x-x')\gamma_{\mu}\{\bar{\psi}^{(0)}(x'),[\bar{\psi}^{(0)}(x''),\gamma_{\nu}\bar{\psi}^{(0)}(x'')]\}\delta_{\mu\nu}\bar{\bar{D}}(x'-x'')dx'dx'' \\ &-\delta m^{(2)}\int \bar{S}(x-x')\psi^{(0)}(x')dx' ,\end{aligned} \tag{24}$$

From Eqs. (20) and (24) we can calculate the second order radiative correction to the current operator in the one-electron case as follows:

$$\begin{aligned}\langle j_{\mu}^{(2)}(x)\rangle_{1,0} = &\frac{i}{2}[\bar{\psi}^{(0)}(x), \gamma_{\mu}\bar{\psi}^{(2)}(x)]_{1,0} + \frac{i}{2}[\bar{\psi}^{(1)}(x), \gamma_{\mu}\bar{\psi}^{(1)}(x)]_{1,0} \\ &+\frac{i}{2}[\bar{\psi}^{(2)}(x), \gamma_{\mu}\bar{\psi}^{(0)}(x)]_{1,0} .\end{aligned} \tag{25}$$

The first term is

$$\begin{aligned}&\frac{i}{2}[\bar{\psi}^{(0)}(x),\ \gamma_{\mu}\bar{\psi}^{(2)}(x)]_{1,0} \\ &= -\frac{i}{4}\iint[\bar{\psi}^{(0)}(x),\gamma_{\mu}\bar{\bar{S}}(x-x')\gamma_{\lambda}\bar{\bar{S}}(x'-x'')\gamma_{\nu}\bar{\psi}^{(0)}(x'')]_{1}\delta_{\lambda\nu}\bar{\bar{D}}(x'-x'')dx'dx'' \\ &\quad+\frac{i}{8}\iint[\bar{\psi}^{(0)}(x),\gamma_{\mu}\bar{\bar{S}}(x-x')\gamma_{\lambda}\{\bar{\psi}^{(0)}(x'),[\bar{\psi}^{(0)}(x''),\gamma_{\nu}\bar{\psi}^{(0)}(x'')]\}]_{1} \\ &\quad\times\delta_{\lambda\nu}\bar{\bar{D}}(x'-x'')dx'dx'' \\ &\quad-\frac{i}{2}\delta m^{(2)}\int[\bar{\psi}^{(0)}(x),\gamma_{\mu}\bar{\bar{S}}(x-x')\psi^{(0)}(x')]_{1}dx' ,\end{aligned} \tag{26}$$

where the last factor in the first term stems from the vacuum expectation value of the anticommutator of electromagnetic field operators, viz.,

$$\begin{aligned}\langle\{\widetilde{A}_{\lambda}^{(0)}(x'),\ \widetilde{A}_{\nu}^{(0)}(x'')\}\rangle_{0} &= \iint dx_1dx_2F(x'-x_1)F(x''-x_2)\delta_{\lambda\nu}D^{(1)}(x_1-x_2) \\ &= \delta_{\lambda\nu}\bar{\bar{D}}^{(1)}(x'-x'') ,\end{aligned} \tag{27}$$

since

$$\langle\{A_\lambda^{(0)}(x'),A_\nu^{(0)}(x'')\}\rangle_0=\delta_{\lambda\nu}D^{(1)}(x'-x'')\ . \tag{28}$$

It is useful to simplify the complicated form of anti- and ordinary commutators in the integrand of the second term of Eq. (26),

$$[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\{\bar{\psi}^{(0)}(x'),[\bar{\psi}^{(0)}(x''),\gamma_\nu\bar{\psi}^{(0)}(x'')]\}]_1\ . \tag{29}$$

Let us rewrite the expression (29)

$$\begin{aligned}(29)=&(\gamma_\mu)_{\alpha\beta}\bar{\bar{S}}_{\beta\gamma}(x-x')(\gamma_\lambda)_{\gamma\delta}(\gamma_\nu)_{\varepsilon\zeta}\iiiint dx_1dx_2dx_3dx_4\,G(x-x_1)\,G(x'-x_2)\\&\times G(x''-x_3)\,G(x''-x_4)[\bar{\psi}_\alpha^{(0)}(x_1),\{\psi_\delta^{(0)}(x_2),[\bar{\psi}_\varepsilon^{(0)}(x_3),\psi_\zeta^{(0)}(x_4)]\}]_1\ ,\end{aligned} \tag{30}$$

where the integrand leads except the form factors to

$$\begin{aligned}&2S_{\delta\alpha}^{(1)}(x_2-x_1)[\bar{\psi}_\varepsilon^{(0)}(x_3),\psi_\zeta^{(0)}(x_4)]_1-2S_{\zeta\alpha}^{(1)}(x_4-x_1)[\bar{\psi}_\varepsilon^{(0)}(x_3),\psi_\delta^{(0)}(x_2)]_1\\&-2S_{\delta\varepsilon}^{(1)}(x_2-x_3)[\bar{\psi}_\alpha^{(0)}(x_1),\psi_\zeta^{(0)}(x_4)]_1+2S_{\zeta\varepsilon}^{(1)}(x_4-x_3)[\bar{\psi}_\alpha^{(0)}(x_1),\psi_\delta^{(0)}(x_2)]_1\ .\end{aligned} \tag{31}$$

Accordingly,

$$\begin{aligned}(29)=&2Tr(\bar{\bar{S}}^{(1)}(x'-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda)[\bar{\psi}^{(0)}(x''),\gamma_\nu\bar{\psi}^{(0)}(x'')]_1\\&-2[\bar{\psi}^{(0)}(x''),\gamma_\nu\bar{\bar{S}}^{(1)}(x''-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\psi}^{(0)}(x')]_1\\&-2[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\bar{S}}^{(1)}(x'-x'')\gamma_\nu\bar{\psi}^{(0)}(x'')]_1\\&+2[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\psi}^{(0)}(x')]_1Tr\gamma_\nu\bar{\bar{S}}^{(1)}(x''-x'')\ ,\end{aligned} \tag{32}$$

where

$$\bar{\bar{S}}^{(1)}(x''-x'')=\iint dx_3dx_4\,G(x''-x_3)\,G(x''-x_4)\,S^{(1)}(x_4-x_3)\ . \tag{33}$$

It is easily shown that the last term of Eq. (32) vanishes due to the vanishing trace factor

$$Tr\gamma_\nu\bar{\bar{S}}^{(1)}(x''-x'')=0\ , \tag{34}$$

which is a consequence of the even character of $\Delta^{(1)}$-function.

Thus we obtain for the first term of Eq. (25)

$$\begin{aligned}&\frac{i}{2}[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\psi}^{(2)}(x)]_{1,0}\\&=-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\bar{S}}(x'-x')\gamma_\nu\bar{\psi}^{(0)}(x'')]_1\delta_{\lambda\nu}\bar{\bar{D}}^{(1)}(x'-x'')dx'dx''\\&\quad+\frac{i}{4}\iint Tr(\bar{\bar{S}}^{(1)}(x'-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda)[\bar{\psi}^{(0)}(x''),\gamma_\nu\bar{\psi}^{(0)}(x'')]_1\\&\quad\times\delta_{\lambda\nu}\bar{\bar{D}}(x'-x'')dx'dx''\end{aligned} \tag{35}$$

$$-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x''),\gamma_\nu\bar{\bar{S}}^{(1)}(x''-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\psi}^{(0)}(x')]_1\delta_{\lambda\nu}\bar{\bar{D}}(x'-x'')dx'dx''$$

$$-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\bar{S}}^{(1)}(x'-x'')\gamma_\nu\bar{\psi}^{(0)}(x'')]_1\delta_{\lambda\nu}\bar{\bar{D}}(x'-x'')dx'dx''$$

$$-\frac{i}{2}\delta m^{(2)}\int[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\bar{S}}(x-x')\psi^{(0)}(x')]_1dx' .$$

The third term of Eq. (33) can be calculated in the same manner as the first term, which reads

$$\begin{aligned}
&\frac{i}{2}[\bar{\psi}^{(2)}(x),\gamma_\mu\bar{\psi}^{(0)}(x)]_{1,0}\\
&=-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x'')\gamma_\nu\bar{\bar{S}}(x''-x')\gamma_\lambda\bar{\bar{S}}(x'-x)\gamma_\mu,\bar{\psi}^{(0)}(x)]_1\delta_{\lambda\nu}\bar{\bar{D}}^{(1)}(x'-x'')dx'dx''\\
&\quad+\frac{i}{4}\iint[\bar{\psi}^{(0)}(x''),\gamma_\lambda\bar{\psi}^{(0)}(x'')]_1Tr(\gamma_\nu\bar{\bar{S}}(x'-x)\gamma_\mu\bar{\bar{S}}^{(1)}(x-x'))\delta_{\lambda\nu}\\
&\quad\times\bar{\bar{D}}(x'-x'')dx'dx'' \qquad (36)\\
&\quad-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x')\gamma_\nu\bar{\bar{S}}(x'-x)\gamma_\mu\bar{\bar{S}}^{(1)}(x-x'')\gamma_\lambda,\bar{\psi}^{(0)}(x'')]_1\delta_{\lambda\nu}\bar{\bar{D}}(x'-x'')dx'dx''\\
&\quad-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x'')\gamma_\lambda\bar{\bar{S}}^{(1)}(x''-x')\gamma_\nu\bar{\bar{S}}(x'-x)\gamma_\mu,\bar{\psi}^{(0)}(x)]_1\delta_{\lambda\nu}\bar{\bar{D}}(x'-x'')dx'dx''\\
&\quad-\frac{i}{2}\delta m^{(2)}\int[\bar{\psi}^{(0)}(x')\bar{\bar{S}}(x'-x)\gamma_\mu,\bar{\psi}^{(0)}(x)]_1dx' .
\end{aligned}$$

Finally employing the following relation

$$\begin{aligned}
[\bar{\psi}_\alpha^{(0)}(x'')A_\lambda^{(0)}(x''),A_\nu^{(0)}(x')\psi_\beta^{(0)}(x')]&=\frac{1}{2}[\bar{\psi}_\alpha^{(0)}(x''),\psi_\beta^{(0)}(x')]\{A_\lambda^{(0)}(x''),A_\nu^{(0)}(x')\}\\
&+\frac{1}{2}\{\bar{\psi}_\alpha^{(0)}(x''),\psi_\beta^{(0)}(x')\}[A_\lambda^{(0)}(x''),A_\nu^{(0)}(x')] ,
\end{aligned} \qquad (37)$$

we get

$$\begin{aligned}
&\frac{i}{2}[\bar{\psi}^{(1)}(x),\gamma_\mu\bar{\psi}^{(1)}(x)]\\
&=-\frac{i}{4}\iint dx'dx''[\bar{\psi}^{(0)}(x''),\gamma_\lambda\bar{\bar{S}}(x''-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\nu\bar{\psi}^{(0)}(x')]\\
&\quad\times\{\bar{A}_\lambda^{(0)}(x''),\bar{A}_\nu^{(0)}(x')\} \qquad (38)\\
&\quad-\frac{i}{4}\iint dx'dx''\{\bar{\psi}^{(0)}(x''),\gamma_\lambda\bar{\bar{S}}(x''-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\nu\bar{\psi}^{(0)}(x')\}\\
&\quad\times[\bar{A}_\lambda^{(0)}(x''),\bar{A}_\nu^{(0)}(x')] .
\end{aligned}$$

As the one-electron part of the second term of Eq. (38) vanishes because of the anti-commutator for the spinor fields, we obtain for the second term of Eq. (25)

$$\frac{i}{2}[\bar{\psi}^{(1)}(x,)\gamma_\mu\bar{\psi}^{(1)}(x)]_{1,0} \tag{39}$$
$$=-\frac{i}{4}dx'dx''[\bar{\psi}^{(0)}(x''),\gamma_\lambda\bar{\bar{S}}(x''-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\nu\bar{\psi}^{(0)}(x')]_1\delta_{\lambda\nu}\bar{\bar{D}}^{(1)}(x-x'').$$

From Eqs. (35), (36), and (39) we obtain

$$\begin{aligned}
&\langle j_\mu^{(2)}(x)\rangle_{1,0}\\
&=-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\{\bar{\bar{S}}(x'-x'')\bar{\bar{D}}^{(1)}(x'-x'')\\
&\quad+\bar{\bar{S}}^{(1)}(x'-x'')\bar{\bar{D}}(x'-x'')\}\gamma_\lambda\bar{\psi}^{(0)}(x'')]_1dx'dx''\\
&\quad-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x'')\gamma_\lambda\{\bar{\bar{S}}(x''-x')\bar{\bar{D}}^{(1)}(x'-x'')+\bar{\bar{S}}^{(1)}(x''-x')\bar{\bar{D}}(x'-x'')\}\\
&\quad\times\gamma_\lambda\bar{\bar{S}}(x'-x)\gamma_\mu,\bar{\psi}^{(0)}(x)]_1dx'dx''\\
&\quad-\frac{i}{2}\delta m^{(2)}\int[\bar{\psi}^{(0)}(x),\gamma_\mu\bar{\bar{S}}(x-x')\psi^{(0)}(x')]_1dx'\\
&\quad-\frac{i}{2}\delta m^{(2)}\int[\bar{\psi}^{(0)}(x')\bar{\bar{S}}(x'-x)\gamma_\mu,\bar{\psi}^{(0)}(x)]_1dx'\\
&\quad+\frac{i}{4}\iint[\bar{\psi}^{(0)}(x''),\gamma_\lambda\bar{\psi}^{(0)}(x'')]_1Tr\{\bar{\bar{S}}^{(1)}(x'-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\\
&\quad+\bar{\bar{S}}(x'-x)\gamma_\mu\bar{\bar{S}}^{(1)}(x-x')\gamma_\lambda\}\bar{\bar{D}}(x'-x'')dx'dx''\\
&\quad-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x''),\gamma_\lambda\bar{\bar{S}}^{(1)}(x''-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\bar{D}}(x'-x'')\bar{\psi}^{(0)}x')]_1dx'dx''\\
&\quad-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x''),\gamma_\lambda\bar{\bar{S}}(x''-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\bar{D}}^{(1)}(x'-x'')\bar{\psi}^{(0)}(x')]_1dx'dx''\\
&\quad-\frac{i}{4}\iint[\bar{\psi}^{(0)}(x''),\gamma_\lambda\bar{\bar{S}}(x''-x)\gamma_\mu\bar{\bar{S}}^{(1)}(x-x')\gamma_\lambda\bar{\bar{D}}(x''-x')\bar{\psi}^{(0)}(x')]_1dx'dx''.
\end{aligned} \tag{40}$$

If we put here

$$\bar{\phi}(x)=-\frac{1}{2}\gamma_\lambda\{\bar{\bar{S}}^{(1)}(x-x')\bar{\bar{D}}(x-x')+\bar{\bar{S}}(x-x')\bar{\bar{D}}^{(1)}(x-x')\}\gamma_\lambda\bar{\psi}^{(0)}(x')dx' \tag{41}$$

and

$$\bar{K}_\mu(x''-x,x-x')=\gamma_\lambda\bar{\bar{S}}^{(1)}(x''-x)\gamma_\mu\bar{\bar{S}}(x-x')\gamma_\lambda\bar{\bar{D}}(x'-x'')$$

$$+\gamma_\lambda \bar{\bar{S}}(x''-x)\gamma_\mu \bar{\bar{S}}^{(1)}(x-x')\gamma_\lambda \bar{\bar{D}}(x''-x') \tag{42}$$
$$+\gamma_\lambda \bar{\bar{S}}(x''-x)\gamma_\mu \bar{\bar{S}}(x-x')\gamma_\lambda \bar{\bar{D}}^{(1)}(x'-x''),$$

then Eq. (46) leads to

$$\langle j_\mu^{(2)}(x)\rangle_{1,0} = \frac{i}{2}\iint [\bar{\psi}^{(0)}(x), \gamma_\mu \bar{\bar{S}}(x-x')(\bar{\phi}(x')-\delta m^{(2)}\psi^{(0)}(x'))]_1 dx'$$
$$+\frac{i}{2}\iint [(\bar{\bar{\phi}}(x')-\delta m^{(2)}\bar{\psi}^{(0)}(x'))\,\bar{\bar{S}}(x'-x)\gamma_\mu,\ \bar{\psi}^{(0)}(x)]_1 dx' \tag{43}$$
$$-\frac{i}{4}\iint [\bar{\psi}^{(0)}(x'), \bar{\bar{K}}_\mu(x'-x, x-x'')\,\bar{\psi}^{(0)}(x'')]_1 dx'dx'' + vac.\ pol.,$$

where the last term arises from the vacuum polarization and is gigen by

$$vac.\ pol. = \frac{i}{4}\iint [\bar{\psi}^{(0)}(x''), \gamma_\lambda \bar{\psi}^{(0)}(x'')]_1\, Tr\,\{\bar{\bar{S}}^{(1)}(x'-x)\gamma_\mu \bar{\bar{S}}(x-x')\gamma_\lambda\ . \tag{44}$$

The third term of Eq. (43) can also be written as

$$-\frac{i}{2}\iint (\bar{\psi}^{(0)}(x')\,\bar{\bar{K}}_\mu(x'-x, x-x'')\,\bar{\psi}^{(0)}(x''))_1 dx'dx''\ . \tag{45}$$

Thus we obtain for the electron self-mass in the first approximation

$$\delta m^{(2)}\psi^{(0)}(x) = \bar{\phi}(x)\,, \tag{46}$$

or

$$\delta m^{(2)} = -\frac{1}{2}g(-m^2)\int e^{-ikx}\gamma_\lambda\{\bar{\bar{S}}^{(1)}(x)\,\bar{\bar{D}}(x)+\bar{\bar{S}}(x)\,\bar{\bar{D}}^{(1)}(x)\}\,\gamma_\lambda dx\,, \tag{47}$$

where the function g is defined by the Fourier transform

$$G(x) = \frac{1}{(2\pi)^4}\int dk\, e^{ikx} g(k^2)\ . \tag{48}$$

Similarly we have

$$F(x) = \frac{1}{(2\pi)^4}\int dk\, e^{ikx} f(k^2)\ . \tag{49}$$

From Eq. (46) or (47) it is obvious that the self-mass of the electron can be made finite provided the functional form of the form factors is appropriately chosen.

By the way we shall consider the third term of Eq. (43) or the expression (45). Employing the Fourier integral representations

$$\bar{S}(x)=\frac{1}{(2\pi)^4}\int dk\, e^{ikx}\frac{i\gamma k-m}{k^2+m^2}\ , \tag{50}$$

$$S^{(1)}(x)=\frac{1}{(2\pi)^3}\int dk\, e^{ikx}(i\gamma k-m)\,\delta(k^2+m^2)\ , \tag{51}$$

$$\bar{D}(x)=\frac{1}{(2\pi)^4}\int dk\, e^{ikx}\frac{1}{k^2}, \tag{52}$$

$$D^{(1)}(x)=\frac{1}{(2\pi)^3}\int dk\, e^{ikx}\delta(k^2)\ , \tag{53}$$

and Eqs. (48) and (49), $\widetilde{K}_\mu(x'-x, x-x'')$ is evaluated as follows:

$$\begin{aligned}
&\widetilde{K}_\mu(x'-x, x-x'')\\
&=\frac{1}{(2\pi)^{11}}\iiint dk\,dk'\,dk''\,e^{i(k+k')(x'-x)}e^{i(k+k'')(x-x'')}g(k'^2)\,g(k''^2)\\
&\quad\times f(k^2)\gamma_\nu(i\gamma k'-m)\gamma_\mu(i\gamma k''-m)\gamma_\nu\\
&\quad\times\Big[g(-m^2)\frac{\delta(k'^2+m^2)}{(k''^2+m^2)k}+g(-m^2)\frac{\delta(k''^2+m^2)}{(k'^2+m^2)k^2}\\
&\qquad\qquad+f(0)\frac{\delta(k^2)}{(k'^2+m^2)\,(k''^2+m^2)}\Big]\ .
\end{aligned} \tag{54}$$

In order to simplify the calculation we shall put

$$f(0)=g(-m^2)=1\ . \tag{55}$$

But this is justifiable: On the one hand, from Eq. (49) $f(0)$ is given by

$$f(0)=\int F(x)\,dx\ , \tag{56}$$

and $F(x)$ is a delta-like function, accordingly there will be no objection to put $f(0)=1$. On the other hand, for a free spinor field we have

$$\bar{\psi}^{(0)}(x)=g(-m^2)\psi^{(0)}(x)\ , \tag{57}$$

so that the current $j_\mu^{(0)}(x)$ in the vacuum differs from the ordinary current merely by a charge renormalization factor

$$e'=eg^2(-m^2)\ . \tag{58}$$

Hence, it will be preferable to put $g(-m^2)=1$.

Now it is convenient to perform the transformation of the integration parameters

$$p'_\mu = k_\mu + k'_\mu \quad \text{and} \quad p''_\mu = k_\mu + k''_\mu \,, \tag{59}$$

then only such values of p'_μ and p''_μ as satisfy the relation

$$p'^2 + m^2 = p''^2 + m^2 = 0 \tag{60}$$

are effective because from Eq. (57) we can readily verify that

$$(i\gamma p' + m)\,\bar{\varphi}^{(0)} = 0 \quad \text{and} \quad (i\gamma p'' + m)\,\bar{\varphi}^{(0)} = 0\,. \tag{61}$$

Thus we obtain

$$\begin{aligned}
&\widetilde{K}_\mu(x'-x, x-x'') \\
&= \frac{1}{2(2\pi)^{11}} \int_{-1}^{1} dv \int_0^1 du \iiint dk\, dp'\, dp''\, e^{ip'(x'-x)} e^{ip''(x-x'')} g((p'-k)^2) \\
&\quad \times g((p''-k)^2) f(k^2) \gamma_\nu (i\gamma(p'-k)-m)\, \gamma_\mu (i\gamma(p''-k)-m) \\
&\quad \times \gamma_\nu \delta''(k^2 - k(p'+p''+(p'-p'')v)u)\,.
\end{aligned} \tag{62}$$

We shall carry out the second step transformation

$$k_\mu \to k_\mu + (p'_\mu + p''_\mu + (p'_\mu - p''_\mu)v)\,\frac{u}{2}\,, \tag{63}$$

and obtain as effective terms of $\widetilde{K}_\mu(x'-x, x-x'')$ in view of the free Dirac equations of motion

$$\begin{aligned}
&\widetilde{K}_\mu(x'-x, x-x'') \\
&= \frac{1}{(2\pi)^{11}} \int_{-1}^{1} dv \int_0^1 u\,du \iiint dk\, dp'\, dp''\, e^{ip'(x'-x)} e^{ip''(x-x'')} \delta''(k^2 + \lambda^2 u^2) \\
&\times g\Big[\Big(p'-k-(p'+p''+(p'-p'')v)\frac{u}{2}\Big)^2\Big]\, g\Big[\Big(p''-k-(p'+p''+(p'-p'')v)\frac{u}{2}\Big)^2\Big] \\
&\times f\Big[\, k + (p'+p''+(p'-p'')v)\frac{u}{2}\Big)^2\Big]\Big[2m^2\gamma_\mu\Big(1-u-\frac{1}{2}u^2\Big) - \frac{1}{2}\gamma_\mu k^2 \\
&+ m(u-u^2)\,\sigma_{\mu\nu}(p'_\nu - p''_\nu) + (p'-p'')^2 \gamma_\mu \Big(1-u+\frac{1-v^2}{4}u^2\Big) \\
&+ (1+u)\,uvmi\,(p'_\mu - p''_\mu)\Big]\,,
\end{aligned} \tag{64}$$

where

$$\lambda^2 = m^2\Big(1 + \frac{(p'-p'')^2}{4m^2}(1-v^2)\Big) \tag{65}$$

and

$$\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu]\,. \tag{66}$$

The term that contribute to the additional spin magnetic moment is picked out from among the terms in the integrand of Eq. (64):

$$\begin{aligned}
&\widetilde{M}_\mu(x'-x, x-x'') \\
&= \frac{1}{(2\pi)^{11}}\int_{-1}^{1} dv \int_0^1 u du \int dk \int dp' \int dp'' e^{ip'(x'-x)} e^{ip''(x-x'')} \delta''(k^2+\lambda^2 u^2) \\
&\times g\Big[\Big(p'-k-(p'+p''+(p'-p'')v)\frac{u}{2}\Big)^2\Big] g\Big[\Big(p''-k-(p'+p''+(p'-p'')v)\frac{u}{2}\Big)^2\Big] \\
&\times f\Big[\Big(k+(p'+p''+(p'-p'')v)\frac{u}{2}\Big)^2\Big] m(u-u^2)\sigma_{\mu\nu}(p'_\nu - p''_\nu) .
\end{aligned} \tag{97}$$

If one can control the behavior of the factors f and g or F and G so that the contribution (67) to the anomalous magnetic moment fits for its observed value, the form factor will carry a significance. Furthermore, if a parameter λ with the dimension of length is determined so as to have a definite value as well as the functional form of the factors:

$$f \equiv f(k^2;\lambda) \quad \text{and} \quad g \equiv g(k^2;\lambda) , \tag{68}$$

then the value will describe the characteristic structure of the interaction between photon and electron, or may related to the structure of elementary particles. If this is the case, it is worthy of note that the constant λ is of Lorentz-invariant character, in reference to which it may be interesting to recall the PAULI's statement.[6]

§ 3. Final remarks

We have not followed the orthodox method, in which one derives the equations of motion from the prescribed Lagrangian by the variational principle, but employed the immediately replacing method of the ordinary interaction terms in the equations of motion with the non-localized ones. Although this may not be approved from the formalistic point of view, it seems that this is not so unsatisfactory because the whole theory is relativistically invariant. It is of course natural that such a method should meet with more justification.

If one employs, according to the RAYSKI's suggestion,

$$L' = \frac{ie}{4}\left\{[\bar{\psi}(x), \gamma_\mu \psi(x)], \widetilde{A}_\mu(x)\right\} \tag{69}$$

as a Lagrangian interaction term, then it is evident from our calculation that the electron self-energy can be made finite, but all the expressions will become a little more complicated.

References

1) J. Rayski, *Acta Physica Polonica* XI, Fasc. 1 (1951), 25.
2) H. McManus, *Proc. Roy. Soc.* A **195** (1948), 323.
3) C. N. Yang and D. Feldman, *Phys. Rev.* **79** (1950), 972.
4) G. Källén, *Arkiv för Fysik* Bd. 2, Nr. 19 (1950).
5) J. Schwinger, *Phys. Rev.* **74** (1948), 1439; **75** (1949), 651; and **76** (1949), 790.
6) W. Pauli, *Handb. d. Physik* XXIV/1, p. 246.