On Practical Methods of Conformal Representation.

By

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It is well known that conformal representations can be applied to solve the potential problem in two dimensions. One may think, however, that a labourious calculation is needed to obtain the conformal representation in general, and consequently that the conformal representation is not convenient for practical use to solve the potential problem. But in fact this is not true, if the method of treating the function of the conformal representation is suitable. In this article it is shown how to obtain the conformal representation.

If the stream and equi-potential lines are given in a domain having a simple boundary, the two families of lines can be transformed conformally in the given domain by some function of the conformal representation. Therefore one obtains first the conformal representations defined by elementary functions. Though there are many elementary functions, one needs to calculate only a few functions, since they are connected with each other in the complex domain. For example there are

$$\log(z+\sqrt{1+z^2}) = i\cos^{-1}\sqrt{1+z^2}\ , \quad \tan^{-1}z = \cos^{-1}\frac{z}{\sqrt{1-z^2}}$$

The most important elementary functions are the following

1. $Z = \log z$

2. $Z = \dfrac{1}{z}$ or $Z = z+\dfrac{1}{z}$

3. $Z = \cos^{-1}z$

where $Z = X+iY$ and $z = x+iy$.

These functions can be calculated as exactly as needed, and the loci of $X =$ const. and $Y =$ const. can be drawn as exactly as needed in the domain of the complex variable z .

Besides these elementary functions, those defined by the elliptic integrals of the first and the second kind are often used in the practical application of the conformal representation.

$$4. \quad Z = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \quad \text{or} \quad Z = F(z, k) \quad \text{or} \quad z = \mathrm{sn}(Z, k) \ ,$$

$$5. \quad Z = \int_0^z \sqrt{\frac{1-k^2z^2}{1-z^2}}\, dz \qquad \text{or} \quad Z = E(z, k) \ .$$

The loci of $X=$ const. and $Y=$ const. in the domain of a complex variable z are easily obtained by means of the elliptic integrals of the real variable, $F(x, k)$ and $E(x, k)$, since the loci can be expressed by the simple algebraic formula of the integrals F and E. On this problem the present writer has published a paper in "Scientific Papers of the Institute of Physical and Chemical Research," No. 561.

By taking advantage of these conformal representations, the stream and equi-potential lines in a domain having various boundaries can be obtained by the following method based on the character of the regular analytic function.

If $Z=f(z)$ is a function of conformal representation, that is the regular analytic function, then $Z = af(z)$, $a = re^{i\theta}$ is also a function of the conformal representation, and r is the magnification and θ is the angle of rotation of the figure conformally represented.

Therefore if one has a figure represented conformally by means of a function, the figure which is uniformly magnified or rotated can also be a conformally represented figure.

If $Z_1 = f(z)$ and $Z_2 = \varphi(z)$ are the functions of a conformal representation, then the sum $Z = Z_1 + Z_2$ is also a function of the conformal representation.

Method I. Now put the loci of $X_1 =$ const. and $Y_1 =$ const. on the loci of $X_2 =$ const, and $Y_2 =$ const. Observe the point at which $X_1 = a_1$, $X_2 = a_2$ or $Y_2 = b_1$, $Y_2 = b_2$ intersect with each other respectively and read the coordinates of the intersecting point by $X = a_1 + a_2$, $Y = b_1 + b_2$. If one connects the points of the same value of X or Y by smooth curves, then the curves denote the loci of $X =$ const. and $Y =$ const. in z-plane.

It is easily proved that if $Z = f(z)$ and $Z_1 = \varphi(Z)$ are the functions of conformal representation, then $Z_1 = \Psi(z)$ is also the function of conformal representation. Therefore one obtains the following method.

Method II. If a given figure A, which consists of the two families of curves intersecting at right angles with each other in the first domain, is transformed conformally into figure B in the second domain, and the one-to-one correspondence is perfectly established between the two figures, then any other figure C in the first domain can be transformed conformally into figure D in the second domain by the following method. Put C on A so as to take the given relative position between C and A. Observe the point, at which a curve of C intersects the curves of A. As the two families of curves of A are looked upon as the curves to indicate the curvilinear coordinates, one reads the curvilinear coordinates of the intersecting point. One marks on B the point, to which the coordinates, thus read off, correspond, and connects these points by a smooth curve. The curve thus obtained is the one transformed from C into D. Thus one transforms all the curves in the first domain conformally into figure D in the second domain.

If $Z=f(\zeta)$ and $z=\varphi(\zeta)$ and if f and φ are the functions of the conformal representation, then $z=F(Z)$ is also the function of the conformal representation. Therefore one obtains the following method.

Method III. One draws the loci $x=$ const. and $y=$ const. in ζ-plane by means of the function $z=\varphi(\zeta)$ and the loci of $X=$ const. and $Y=$ const. in ζ-plane by means of the function $Z=f(\zeta)$. One puts the loci of $x=$ const. and $y=$ const. on the loci of $X=$ const. and $Y=$ const. in the domain ζ and transforms the loci of $x=$ const. and $y=$ const. on Z-plane by method II. Then one will obtain the conformal representation:

$$z=F(Z)$$

Consequently from the functions of conformal representation $Z_1=f_1(\zeta)$, $Z_2=f_2(\zeta)$ and $z=\varphi(\zeta)$ the functions of conformal representation $z=F_1(Z_1)$ and $z=F_2(Z_2)$ are obtained. From these functions one can have again the function of conformal representation of the form.

$$z=F(Z_1+Z_2).$$

Method IV. Now suppose that the loci of $x=$ const. and $y=$ const. are given in the domain D of ζ-plane, and the loci are transformed conformally into domain A of Z_1-plane by $Z_1=f_1(\zeta)$ and in another domain B of Z_2-plane by $Z_2=f_2(\zeta)$, according to method III. Add the values of Z_1 and Z_2 which correspond to the same value of z, and draw the loci of $x=$ const. and $y=$ const. in domain C of Z-plane. Then the loci are the conformal representation by means of the function $z=F(Z_1+Z_2)$. This method can be applied in the case where the domain represented by a known function is a little different from the one in which the potential problem is investigated.

By taking advantage of the conformal representation of such functions (1) (2) (3) (4) and (5), one can obtain the stream and equi-potential lines in the domain of various boundaries by transforming the figure according to methods I, II, III, or IV. In the following articles it is proposed to show how to apply these methods to the conformal representation for practical purposes.