Elasto-Plastic Analysis for Shallow Shells by Use of Laminated Element

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Abstract

The majority of elasto-plastic analyses of a shell are concerned with the direction wherein its plastic deformation proceeds not normal to but only in parallel with the middle surface of the structure. In the present report a finite element method of analysis of shallow shells is developed using a laminated element that makes relevant analysis possible in both directions. This approach also enables demonstrating such transitory behaviors of the structure that occur wherever a deformation change from elastic to elasto-plastic takes place.

1. Introduction

The present report deals with the elasto-plastic behavior of a thin homogenous isotropic shallow shell subjected to incremental loading.

The analysis is conducted by means of a finite element method using a laminated element given as a multiple set of thin layers so that the plastic region developing in the direction normal to its middle surface may be treated.

By way of a numerical example a shallow helicoidal shell under a gradually increasing gravity load will be analyzed in detail.

2. General Theory

2.1 Assumption

The analysis employs the following assumptions.

1) Kirchhoff's hypothesis that an initial plane section remains plane afterwards holds for both elastic and plastic deformation.

2) The shell under consideration is definable as consisting of a system of laminated elements.

3) The stress-strain relationship follows Hooke's law or the theory of Prandtle-Reuss in a elastic or plastic range, respectively.

4) A functinal of incremental type based on mixed variational principles is adopted in the analysis to be performed by the finite element method.

2.2 Incremental strains and incremental curvatures

Referring to the relations given in Fig. 1 incremental displacements ΔU, ΔV

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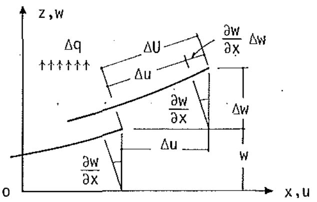


Fig. 1 Incremental displacements

and ΔW for the shallow shell may be approximated by the following equations.

$$\left.\begin{aligned} \Delta U(x, y, z) &= \Delta u(x, y) + \frac{\partial w}{\partial x}\Delta w - z\frac{\partial \Delta w}{\partial x} \\ \Delta V(x, y, z) &= \Delta v(x, y) + \frac{\partial w}{\partial y}\Delta w - z\frac{\partial \Delta w}{\partial y} \\ \Delta W(x, y, z) &= \Delta w(x, y) \end{aligned}\right\} \tag{1}$$

From Eqs. (1) both incremental strains and curvatures are derived as Eqs. (2) and (3), respectively.

$$\{\Delta\varepsilon_0\} = \begin{Bmatrix} \Delta\varepsilon_{x0} \\ \Delta\varepsilon_{y0} \\ \Delta\gamma_{yx0} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \Delta u}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial \Delta w}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\Delta w \\ \dfrac{\partial \Delta v}{\partial y} + \dfrac{\partial w}{\partial y}\dfrac{\partial \Delta w}{\partial y} + \dfrac{\partial^2 w}{\partial y^2}\Delta w \\ \dfrac{\partial \Delta u}{\partial y} + \dfrac{\partial \Delta v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial \Delta w}{\partial y} + \dfrac{\partial w}{\partial y}\dfrac{\partial \Delta w}{\partial x} + 2\dfrac{\partial^2 w}{\partial x \partial y}\Delta w \end{Bmatrix} \tag{2}$$

$$\{\Delta\phi\} = \begin{Bmatrix} \Delta\phi_x \\ \Delta\phi_y \\ \Delta\phi_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 \Delta w}{\partial x^2} \\ -\dfrac{\partial^2 \Delta w}{\partial y^2} \\ -2\dfrac{\partial^2 \Delta w}{\partial x \partial y} \end{Bmatrix} \tag{3}$$

2.3 Incremental functional

Taking as basic independent variables incremental displacements Δu, Δv and Δw and incremental moments ΔM_x, ΔM_y and ΔM_{xy} in such a way as to assure the conditions of compatibility of Eqs. (4) on the element boundary leads to the incremental functional of Eq. (5).

$$\left.\begin{aligned} &\Delta u = \Delta\bar{u}, \quad \Delta v = \Delta\bar{v}, \quad \Delta w = \Delta\bar{w} \\ &\Delta M_n = \Delta\bar{M}_n \end{aligned}\right\} \tag{4}$$

$$\begin{aligned} \Delta\Pi = \iint\Big[& \frac{E\cdot h}{2(1-\nu^2)}\Big\{\Big(\frac{\partial \Delta u}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial \Delta w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\Delta w\Big)^2 + \Big(\frac{\partial \Delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \Delta w}{\partial y} + \frac{\partial^2 w}{\partial y^2}\Delta w\Big)^2 \\ & + 2\nu\Big(\frac{\partial \Delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \Delta w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\Delta w\Big)\Big(\frac{\partial \Delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \Delta w}{\partial y} + \frac{\partial^2 w}{\partial y^2}\Delta w\Big) \\ & + \frac{1-\nu}{2}\Big(\frac{\partial \Delta u}{\partial y} + \frac{\partial \Delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \Delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \Delta w}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\Delta w\Big)^2\Big\} \\ & + \frac{\partial \Delta M_x}{\partial x}\frac{\partial \Delta w}{\partial x} + \frac{\partial \Delta M_y}{\partial y}\frac{\partial \Delta w}{\partial y} + \frac{\partial \Delta M_{xy}}{\partial x}\frac{\partial \Delta w}{\partial y} + \frac{\partial \Delta M_{xy}}{\partial y}\frac{\partial \Delta w}{\partial x} \\ & - \frac{6}{Eh^3}\{\Delta M_x^2 + \Delta M_y^2 - 2\nu\Delta M_x\Delta M_y + 2(1+\nu)\Delta M_{xy}^2\} - \Delta q\cdot\Delta w\Big]dxdy \end{aligned}$$

(Cont.)

$$-\oint \Delta M_{ns}\frac{\partial \Delta w}{\partial s}ds-\int_{s_{\bar{p}}}[\Delta \bar{N}_{xn}\Delta u+\Delta \bar{N}_{yn}\Delta v+\Delta \bar{V}_n\Delta w]ds-\int_{s_{\bar{u}}}\Delta M_n\frac{\partial \Delta \bar{w}}{\partial n}ds \tag{5}$$

where E=Young's Modulus; ν=Poisson's ratio; h=thickness of shell; $s_{\bar{p}}$=part of boundary on which loads are prescribed; $s_{\bar{u}}$=part of boundary on which displacements are prescribed; and $\Delta V_n=\partial\Delta M_n/\partial_n+2\partial\Delta M_{ns}/\partial s$.

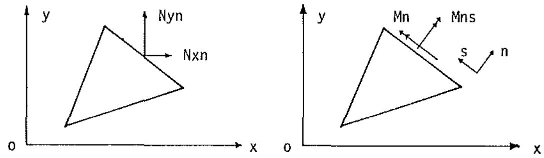


Fig. 2 Notation for stresses

And incremental membrane forces $\{\Delta N\}$ follow incremental strains $\{\Delta\varepsilon_0\}$ in the middle surface to yield Eqs. (6).

$$\{\Delta N\}=\begin{Bmatrix}\Delta N_x\\ \Delta N_y\\ \Delta N_{xy}\end{Bmatrix}=\frac{E\cdot h}{(1-\nu^2)}\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 0.5(1-\nu)\end{pmatrix}\begin{Bmatrix}\Delta\varepsilon_{x_0}\\ \Delta\varepsilon_{y_0}\\ \Delta\gamma_{xy_0}\end{Bmatrix} \tag{6}$$

Now for each incremental displacement Δu, Δv and Δw and incremental moments ΔM_x, ΔM_y and ΔM_{xy}, taking the stationary condition for Eq. (5) and rearranging by the relations of Eqs. (6) result in the following governing equation in an incremental form for the shallow shell.

Within the element

$$\left.\begin{aligned}&\frac{\partial \Delta N_x}{\partial x}+\frac{\partial \Delta N_{xy}}{\partial y}=0\\ &\frac{\partial \Delta N_y}{\partial y}+\frac{\partial \Delta N_{xy}}{\partial x}=0\\ &\frac{\partial^2 \Delta M_x}{\partial x^2}+2\frac{\partial^2 \Delta M_{xy}}{\partial x\partial y}+\frac{\partial^2 \Delta M_y}{\partial y^2}-\frac{\partial^2 w}{\partial y^2}\Delta N_x-\frac{\partial^2 w}{\partial y^2}\Delta N_y-2\frac{\partial^2 w}{\partial x\partial y}\Delta N_{xy}+\Delta q=0\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}&\frac{\partial^2 \Delta w}{\partial x^2}+\frac{12}{Eh^3}(\Delta M_x-\nu\Delta M_y)=0\\ &\frac{\partial^2 \Delta w}{\partial y^2}+\frac{12}{Eh^3}(\Delta M_y-\nu\Delta M_x)=0\\ &\frac{\partial^2 \Delta w}{\partial x\partial y}+\frac{6(1+\nu)}{Eh^3}\Delta M_{xy}=0\end{aligned}\right\} \tag{8}$$

On the boundary

$$\Delta N_{xn}=\Delta\bar{N}_{xn},\qquad \Delta N_{yn}=\Delta\bar{N}_{yn},\qquad \Delta V_n=\Delta\bar{V}_n \tag{9}$$

$$\frac{\partial \Delta w}{\partial n}=\frac{\partial \Delta\bar{w}}{\partial n} \tag{10}$$

3. Finite Element Solution

3.1 Incremental membrane forces, moments and functional for a laminated element

With the strain distribution assumed to obey Kirchhoff's hypothesis, incremental strains at a distance z_i from the middle surface as shown Fig. 3 are expressed as follows.

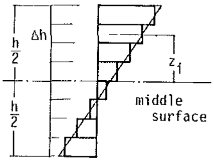


Fig. 3 Strain distribution across layers of element

$$\{\Delta\varepsilon_i\}=\{\Delta\varepsilon_0\}+z_i\{\Delta\phi\} \tag{11}$$

Hence incremental membrane force $\{\Delta N\}$ and incremental moment $\{\Delta M\}$ are given as

$$\left.\begin{aligned}\{\Delta N\}&=(\sum\Delta h[D_i])\{\Delta\varepsilon_0\}+(\sum\Delta h\cdot z_i[D_i]\{\Delta\phi\}\\ \{\Delta M\}&=(\sum\Delta h\cdot z_i[D_i])\{\Delta\varepsilon_0\}+(\sum\Delta h\cdot z_i^2[D_i])\{\Delta\phi\}\end{aligned}\right\} \tag{12}$$

with $[D_i]$ to be taken as $[_eD_i]$ and $[_pD_i]$ for the elastic and plastic range, respectively.

From Eqs. (12) incremental curvatures and incremental membrance forces result as

$$\left.\begin{aligned}\{\Delta\phi\}&=(\sum\Delta h\cdot z_i^2[D_i]^{-1}\{\Delta M\}-(\sum\Delta h\cdot z_i^2[D_i]^{-1}(\sum\Delta h\cdot z_i[D_i]^{\mathrm{T}}\{\Delta\varepsilon_0\}\\ &=[\alpha]\{\Delta M\}-[\beta]\{\Delta\varepsilon_0\}\\ \{\Delta N\}&=(\sum\Delta h\cdot z_i])(\sum\Delta h\cdot z_i^2[D_i])^{-1}\{\Delta M\}\\ &\quad+[(\sum\Delta h[D_i])-(\sum\Delta h\cdot z_i[D_i])(\sum\Delta h\cdot z_i^2[D_i]^{-1}(\sum\Delta h\cdot z_i[D_i])^{\mathrm{T}}]\{\Delta\varepsilon_0\}\\ &=[\beta]^{\mathrm{T}}\{\Delta M\}+[\Psi]\{\Delta\varepsilon_0\}\end{aligned}\right\} \tag{13}$$

Rewriting Eq. (5) using Eqs. (13) gives an incremental form of functional for elasto-plastic analysis of a shallow shell, that is

$$\begin{aligned}\Delta\Pi=\iint\Big[&\frac{1}{2}\{\Delta\varepsilon_0\}^{\mathrm{T}}[\Psi]\{\Delta\varepsilon_0\}+\{\Delta\varepsilon_0\}^{\mathrm{T}}[\beta]^{\mathrm{T}}\{\Delta M\}\\ &+\frac{\partial\Delta M_x}{\partial x}\frac{\partial\Delta w}{\partial x}+\frac{\partial\Delta M_y}{\partial y}\frac{\partial\Delta w}{\partial y}+\frac{\partial\Delta M_{xy}}{\partial x}\frac{\partial\Delta w}{\partial y}+\frac{\partial\Delta M_{xy}}{\partial y}\frac{\partial\Delta w}{\partial x}\\ &-\frac{1}{2}\{\Delta M\}^{\mathrm{T}}[\alpha]\{\Delta M\}-\Delta q\cdot\Delta w\Big]dxdy-\oint\Delta M_{ns}\frac{\partial\Delta w}{\partial s}ds\\ &-\int_{s_p}[\Delta\bar{N}_{xn}\Delta u+\Delta\bar{N}_{yn}\cdot\Delta v+\Delta\bar{V}_n\Delta w]ds-\int_{s_u}\Delta M_n\frac{\partial\Delta\bar{w}}{\partial n}ds\end{aligned} \tag{14}$$

3.2 Application of theory to finite element method

Throughout this finite element analysis both displacement and stress function are assumed as follows for the adopted triangular element (Fig. 4).

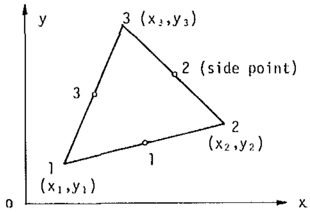


Fig. 4 Geometry of element

$$\{\Delta u, \Delta v, \Delta w\} = [1, x, y][A]\begin{pmatrix} \Delta u_1, & \Delta v_1, & \Delta w_1 \\ \Delta u_2, & \Delta v_2, & \Delta w_2 \\ \Delta u_3, & \Delta v_3, & \Delta w_3 \end{pmatrix} \tag{15}$$

$$\{\Delta M_x, \Delta M_y, \Delta M_{xy}\} = [C_1, C_2, C_3] \tag{16}$$

where C_i = constants.

Further, incremental moments ΔM_x, ΔM_y and ΔM_{xy} in Eqs. (16) are denoted in terms of incremental moments on the element boundary ΔM_{n_1}, ΔM_{n_2} and ΔM_{n_3}, respectively, namely

$$\begin{Bmatrix} \Delta M_x \\ \Delta M_y \\ \Delta M_{xy} \end{Bmatrix} = \begin{pmatrix} \cos^2\theta_1, & \sin^2\theta_1, & 2\sin\theta_1\cos\theta_1 \\ \cos^2\theta_2, & \sin^2\theta_2, & 2\sin\theta_2\cos\theta_2 \\ \cos^2\theta_3, & \sin^2\theta_3, & 2\sin\theta_3\cos\theta_3 \end{pmatrix}^{-1} \begin{Bmatrix} \Delta M_{n_1} \\ \Delta M_{n_2} \\ \Delta M_{n_3} \end{Bmatrix} = [T]\{\Delta M_n\} \tag{17}$$

where

$$\sin\theta_i = (x_i - x_j)/l_i, \qquad \cos\theta_i = (y_j - y_i)/l_i$$

$$l_i = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \qquad i = 1, 2, 3 \rightarrow j = 2, 3, 1$$

Substituting Eqs. (15) and (17) into Eq. (14) and taking the variations of incremental displacements Δu, Δv and Δw and incremental moment ΔM_n end in the following variational set of equations for elasto-plastic analysis of a shallow shell.

$$\begin{Bmatrix} \delta\Delta u \\ \delta\Delta v \\ \delta\Delta w \\ \delta\Delta M_n \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12}^{\mathrm{T}} & K_{22} & K_{23} & K_{24} \\ K_{13}^{\mathrm{T}} & K_{23}^{\mathrm{T}} & K_{33} & K_{34} \\ K_{14}^{\mathrm{T}} & K_{24}^{\mathrm{T}} & K_{34}^{\mathrm{T}} & K_{44} \end{bmatrix} \begin{Bmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \Delta M_n \end{Bmatrix} - \begin{Bmatrix} \Delta P_n \\ \Delta P_v \\ \Delta P_w \\ \Delta P_m \end{Bmatrix} \right) = 0 \tag{18}$$

where

$$[K_{11}] = \frac{1}{4\lambda}[\psi_{11}\{a_i\}\{a_j\} + \psi_{13}(\{a_i\}\{b_j\} + \{b_i\}\{a_j\}) + \psi_{33}\{b_i\}\{b_j\}]$$

$$[K_{12}] = \frac{1}{4\lambda}[\psi_{12}\{a_i\}\{b_j\} + \psi_{13}\{a_i\}\{a_j\} + \psi_{23}\{b_i\}\{b_j\} + \psi_{33}\{b_i\}\{a_j\}]$$

$$[K_{12}] = \frac{1}{4\lambda}[(\alpha_x\psi_{11} + \alpha_y\psi_{13})\{a_i\}\{a_j\} + (\alpha_y\psi_{23} + \alpha_x\psi_{33})\{b_i\}\{b_j\}$$
$$+ (\alpha_y\psi_{12} + \alpha_x\psi_{13})\{a_i\}\{b_j\} + (\alpha_x\psi_{13} + \alpha_y\psi_{33})\{b_i\}\{a_j\}]$$

$$[K_{14}] = \frac{1}{2}[\{a_i\}, 0, \{b_i\}][\beta]^{\mathrm{T}}[T]$$

$$[K_{22}] = \frac{1}{4\lambda}[\psi_{22}\{b_i\}\{b_j\} + \psi_{23}(\{a_i\}\{b_j\} + \{b_i\}\{a_j\}) + \psi_{33}\{a_i\}\{a_j\}]$$

$$[K_{23}] = \frac{1}{4\lambda}[(\alpha_x\psi_{13} + \alpha_y\psi_{33})\{a_i\}\{a_j\} + (\alpha_y\psi_{22} + \alpha_x\psi_{33})\{b_i\}\{b_j\}$$
$$+ (\alpha_y\psi_{23} + \alpha_x\psi_{33})\{a_i\}\{b_j\} + (\alpha_x\psi_{12} + \alpha_y\psi_{23})\{b_i\}\{a_j\}]$$

$$[K_{24}] = \frac{1}{2}[0, \{b_i\}, \{a_i\}][\beta]^{\mathrm{T}}[T]$$

$$[K_{33}] = \frac{1}{4\lambda}[(\alpha_x^2\psi_{11} + 2\alpha_x\alpha_y\psi_{13} + \alpha_y^2\psi_{33})\{a_i\}\{a_j\}$$
$$+ (\alpha_y^2\psi_{22} + 2\alpha_x\alpha_y\psi_{23} + \alpha_x^2\psi_{33})\{b_i\}\{b_j\}$$
$$+ (\alpha_x\alpha_y\psi_{12} + \alpha_x^2\psi_{13} + \alpha_y^2\psi_{23} + \alpha_x\alpha_y\psi_{33})(\{a_i\}\{b_j\} + \{b_i\}\{a_j\})]$$

$$[K_{34}]=\frac{1}{2}[\{a_i\},\ \{b_i\}][B][\beta]^{\mathrm{T}}[T]-[C][F][T]$$

$$[K_{44}]=-\lambda[T]^{\mathrm{T}}[\alpha][T]$$

$$\{\varDelta P_u\}=[E]\{\varDelta \bar{N}_{xn}\}$$

$$\{\varDelta P_v\}=[E]\{\varDelta \bar{N}_{yn}\}$$

$$\{\varDelta P_w\}=\frac{\lambda}{3}\varDelta q\begin{Bmatrix}1\\1\\1\end{Bmatrix}$$

$$\{\varDelta P_m\}=[T]^{\mathrm{T}}\left\{l_i\left(\frac{\partial\varDelta\bar{w}}{\partial n}\right)_i\right\}$$

$$[B]=\begin{bmatrix}\alpha_x, & 0, & \alpha_y\\ 0, & \alpha_y, & \alpha_x\end{bmatrix}$$

$$[C]=-\{a_i\}\{l_j\sin\theta_j\}+\{b_i\}\{l_j\cos\theta_j\}$$

$$[F]=[-\sin\theta_i\cos\theta_i,\ \sin\theta_i\cos\theta_i,\ \cos^2\theta_i-\sin^2\theta_i]$$

$$[E]=\frac{1}{2}\begin{Bmatrix}l_1, & 0, & l_3\\ l_1, & l_2, & 0\\ 0, & l_2, & l_3\end{Bmatrix}$$

$$\alpha_x=\frac{1}{2\lambda}\{a_j\}\{w_i\}$$

$$\alpha_y=\frac{1}{2\lambda}\{b_j\}\{w_i\}$$

$$\{a_j\}=\{y_2-y_3,\ y_3-y_1,\ y_1-y_2\}=\{a_i\}^{\mathrm{T}}$$

$$\{b_j\}=\{x_3-x_2,\ x_1-x_3,\ x_2-x_1\}=\{b_i\}^{\mathrm{T}}$$

and

$$\lambda=\frac{1}{2}\{x_2(y_3-y_1)+x_1(y_2-y_3)+x_3(y_1-y_2)\}$$

4. Numerical Example

Numerical analysis was performed on a shallow helicoidal shell subjected to a uniformly distributed load with its surface of helicoid considered as that of the initial deformation.

The structure in plan view divided breadthways and lengthways into a net of respectively 3 by 10 meshes was diagonally subdivided into triangles for the elements that were five-layered.

The adopted yield condition and stress-strain relationship in the plastic range are of von Mises and Yamada, respectively.

Fig. 5 and 6 are curves of load-deflection relationship at respective points A and B on the axis of symmetry for the shell. Therein the load at the elastic limit for the structure increases with the increased slope of its surface. And then the deflection at B shows a greater increase than at A where the displacement of the shell remains very small.

Figs. 7 through to 9 illustrate the development of plastic regions with increasing load intensity.

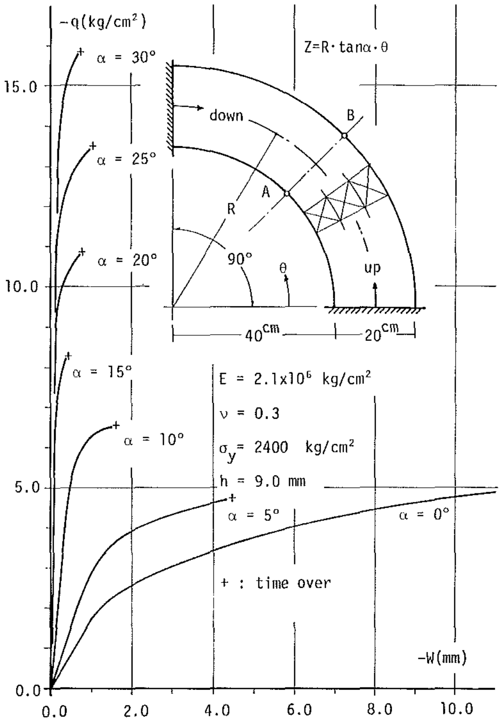


Fig. 5 Load-deflection curve at point A

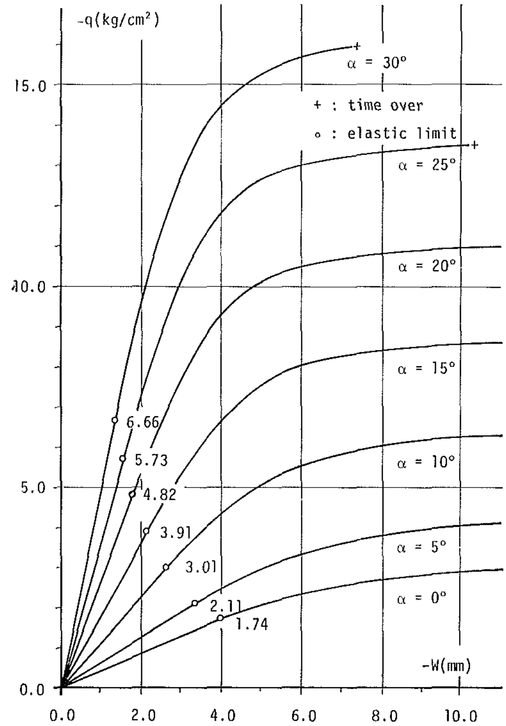
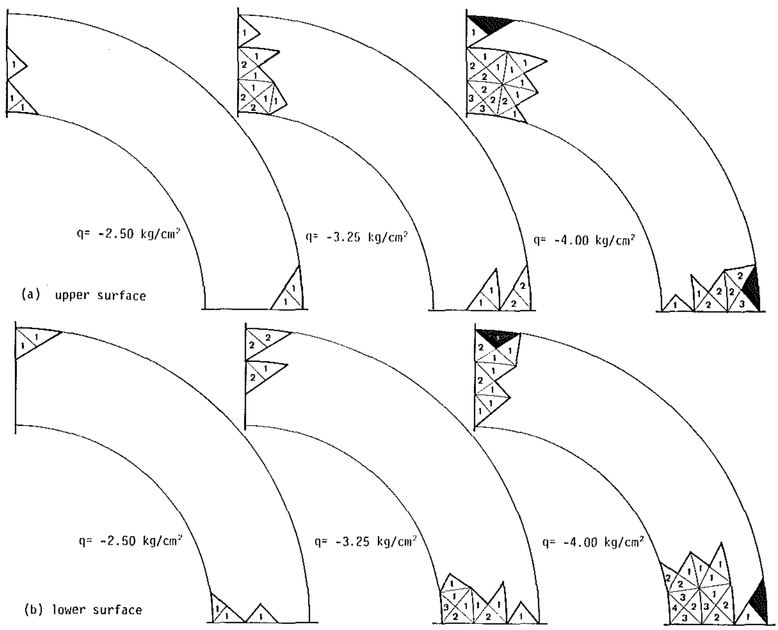


Fig. 6 Load-deflection curve at point B



Legend: Hatched are elements with all their layers yielded. Each number on element of its yielded layers stands for depth of its plastic deformation.

Fig. 7 Development of plastic regions for structure with angle of inclination of surface $\alpha=5°$

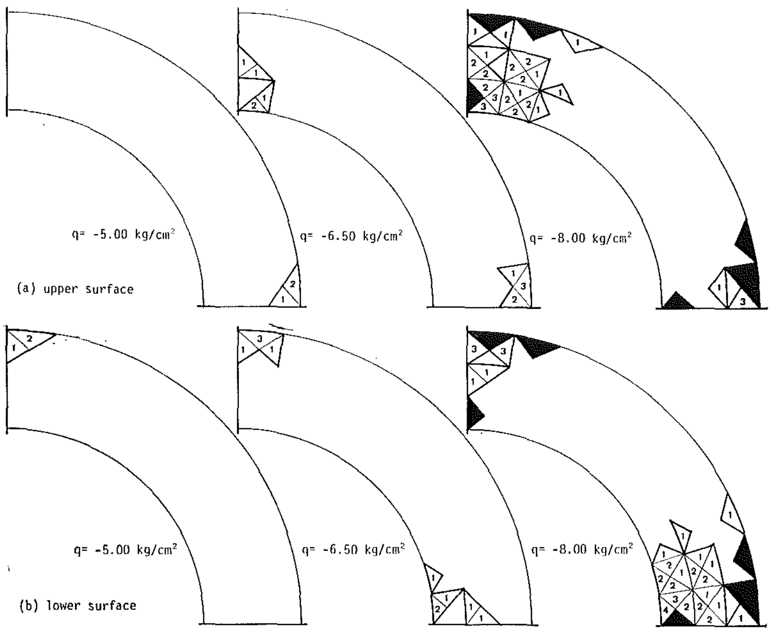


Fig. 8 Development of plastic regions for structure with angle of inclination of surface $\alpha=15°$

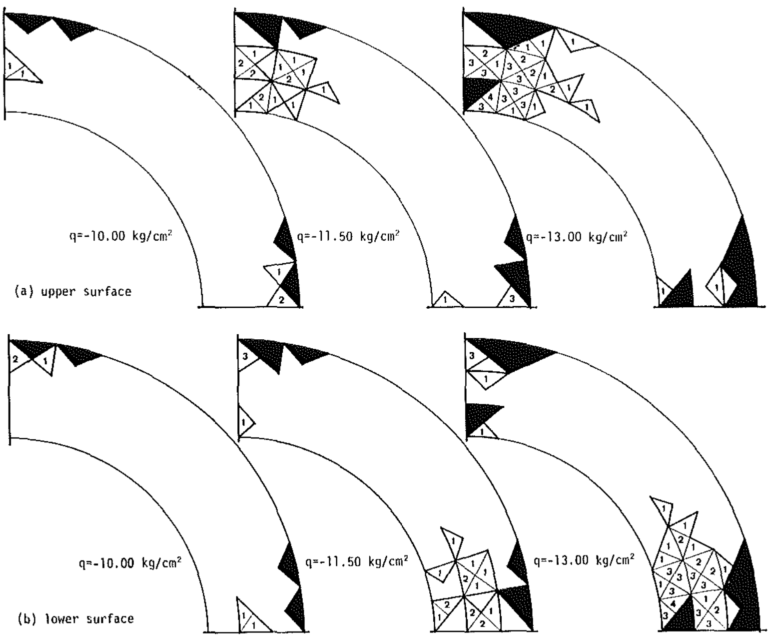


Fig. 9 Development of plastic regions for structure with angle of inclination of surface $\alpha=25°$

5. Conclusion

The presented method was shown to ensure a follow-up of plastic domains as they evolve in the directions conforming to the curvatures of the structure as well as in the normal directions to them. The procedures herein used for a helicoidal shell is readily applicable to other types of shallow shells.

References

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