On the Uniqueness of Weak Weyl Representations of the Canonical Commutation Relation*

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Abstract

Let (T, H) be a weak Weyl representation of the canonical commutation relation (CCR) with one degree of freedom. Namely T is a symmetric operator and H is a self-adjoint operator on a complex Hilbert space $\mathcal{H}$ satisfying the weak Weyl relation: For all $t \in \mathbb{R}$ (the set of real numbers), $e^{-itH}D(T) \subset D(T)$ (i is the imaginary unit and $D(T)$ denotes the domain of T) and $Te^{-itH}\psi = e^{-itH}(T+t)\psi,\ \forall t \in \mathbb{R}, \forall \psi \in D(T)$. In the context of quantum theory where H is a Hamiltonian, T is called a strong time operator of H. In this paper we prove the following theorem on uniqueness of weak Weyl representations: Let $\mathcal{H}$ be separable. Assume that H is bounded below with $\varepsilon_0 := \inf \sigma(H)$ and $\sigma(T) = \{z \in \mathbb{C} | \mathrm{Im}\, z \geq 0\}$, where $\mathbb{C}$ is the set of complex numbers and, for a linear operator A on a Hilbert space, $\sigma(A)$ denotes the spectrum of A. Suppose that $\{\overline{T}, T^*, H\}$ ($\overline{T}$ is the closure of T) is irreducible. Then $(\overline{T}, H)$ is unitarily equivalent to the weak Weyl representation $(-\overline{p}_{\varepsilon_0,+}, q_{\varepsilon_0,+})$ on the Hilbert space $L^2((\varepsilon_0, \infty))$, where $q_{\varepsilon_0,+}$ is the multiplication operator by the variable $\lambda \in (\varepsilon_0, \infty)$ and $p_{\varepsilon_0,+} := -id/d\lambda$ with $D(d/d\lambda) = C_0^\infty((\varepsilon_0, \infty))$. Using this theorem, we construct a Weyl representation of the CCR from the weak Weyl representation $(\overline{T}, H)$.

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1 Introduction and Main Results

A pair (T, H) of a symmetric operator T and a self-adjoint operator H on a complex Hilbert space $\mathcal{H}$ is called a *weak Weyl representation* of the canonical commutation relation (CCR) with one degree of freedom if it obeys the *weak Weyl relation*: For all $t \in \mathbb{R}$ (the set of real numbers), $e^{-itH}D(T) \subset D(T)$ (i is the imaginary unit and $D(T)$ denotes the domain of T) and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi, \ \forall t \in \mathbb{R}, \forall \psi \in D(T). \tag{1.1}$$

This type of representations of the CCR was first discussed by Schmüdgen [13, 14] from a purely operator theoretical point of view and then by Miyamoto [8] in application to a theory of time operator in quantum theory. In the context of quantum theory where H is a Hamiltonian, T is called a *strong time operator* of H [3, 5]. A generalization of a weak Weyl relation was presented by the present author [2] to cover a wider range of applications to quantum physics including quantum field theory.

In the paper [6], a general structure for construction of a Weyl representation of the CCR (see below) from a weak Weyl representation which satisfies some additional property was discussed. In this paper we consider another important problem, i.e., the problem on uniqueness (up to unitary equivalences) of weak Weyl representations, which has not been discussed so far in the literature. This problem has an independent interest in the theory of weak Weyl representations. Before stating the main results on this problem, however, we need some preliminaries.

We denote by $\mathrm{W}(\mathcal{H})$ the set of all the weak Weyl representations on $\mathcal{H}$:

$$\mathrm{W}(\mathcal{H}) := \{(T, H) | (T, H) \text{ is a weak Weyl representation on } \mathcal{H}\}. \tag{1.2}$$

It is easy to see that, if (T, H) is in $\mathrm{W}(\mathcal{H})$, then so are $(\overline{T}, H)$ and $(-T, -H)$, where $\overline{T}$ denotes the closure of T.

For a linear operator A on a Hilbert space, $\sigma(A)$ (resp. $\rho(A)$) denotes the spectrum (resp. the resolvent set) of A (if A is closable, then $\sigma(A) = \sigma(\overline{A})$). Let $\mathbb{C}$ be the set of complex numbers and

$$\Pi_+ := \{z \in \mathbb{C} | \mathrm{Im}\, z > 0\}, \quad \Pi_- := \{z \in \mathbb{C} | \mathrm{Im}\, z < 0\}. \tag{1.3}$$

In the previous paper [4], we proved the following facts:

Theorem 1.1 [4] *Let* $(T, H) \in \mathrm{W}(\mathcal{H})$. *Then:*

(i) If H is bounded below, then either $\sigma(T) = \overline{\Pi}_+$ (the closure of Π_+) or $\sigma(T) = \mathbb{C}$.

(ii) If H is bounded above, then either $\sigma(T) = \overline{\Pi}_-$ or $\sigma(T) = \mathbb{C}$.

(iii) If H is bounded, then $\sigma(T) = \mathbb{C}$.

This theorem has to be taken into account in considering the uniqueness problem of weak Weyl representations.

A form of representations of the CCR stronger than weak Weyl representations is known as a *Weyl representation* of the CCR which is a pair (T, H) of *self-adjoint* operators on $\mathcal{H}$ obeying the *Weyl relation*

$$e^{itT}e^{isH} = e^{-its}e^{isH}e^{itT}, \quad \forall t, \forall s \in \mathbb{R}. \tag{1.4}$$

It is well known (the von Neumann uniqueness theorem [9]) that, every Weyl representation on a *separable* Hilbert space is unitarily equivalent to a direct sum of the Schrödinger representation (q, p) on $L^2(\mathbb{R})$, where q is the multiplication operator by the variable $x \in \mathbb{R}$ and $p = -iD_x$ with D_x being the generalized differential operator in x (cf. [3, §3.5], [10, Theorem 4.3.1], [11, Theorem VIII.14]).

It is easy to see that a Weyl representation is a weak Weyl representation (but the converse is not true). Therefore, as far as the Hilbert space under consideration is separable, the non-trivial case for the uniqueness problem of weak Weyl representations is the one where they are *not* Weyl representations. A general class of such weak Weyl representations (T, H) are given in the case where H is semi-bounded (bounded below or bounded above). In this case, T is not essentially self-adjoint [2, Theorem 2.8], implying Theorem 1.1.

Two simple examples in this class are constructed as follows:

Example 1.1 Let $a \in \mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}_a^+)$ with $\mathbb{R}_a^+ := (a, \infty)$. Let $q_{a,+}$ be the multiplication operator on $L^2(\mathbb{R}_a^+)$ by the variable $\lambda \in \mathbb{R}_a^+$:

$$D(q_{a,+}) := \left\{ f \in L^2(\mathbb{R}_a^+) | \int_a^\infty \lambda^2 |f(\lambda)|^2 d\lambda < \infty \right\}, \tag{1.5}$$

$$q_{a,+}f := \lambda f, \quad f \in D(q_{a,+}) \tag{1.6}$$

and

$$p_{a,+} := -i\frac{d}{d\lambda} \tag{1.7}$$

with $D(p_{a,+}) = C_0^\infty(\mathbb{R}_a^+)$, the set of infinitely differentiable functions on $\mathbb{R}_a^+$ with bounded support in $\mathbb{R}_a^+$. Then it is easy to see that $q_{a,+}$ is self-adjoint, bounded below with $\sigma(q_{a,+}) = [a, \infty)$ and $p_{a,+}$ is a symmetric operator. Moreover, $(-p_{a,+}, q_{a,+})$ is a weak Weyl representation of the CCR. Hence, as remarked above, $(-\overline{p}_{a,+}, q_{a,+})$ also is a weak Weyl representation.

Note that $p_{a,+}$ is not essentially self-adjoint and

$$\sigma(-p_{a,+}) = \sigma(-\overline{p}_{a,+}) = \overline{\Pi}_+. \tag{1.8}$$

In particular, $\pm\overline{p}_{a,+}$ are maximal symmetric, i.e., they have no non-trivial symmetric extensions (e.g., [12, §X.1, Corollary]).

Example 1.2 Let $b \in \mathbb{R}$ and consider the Hilbert space $L^2(\mathbb{R}_b^-)$ with $\mathbb{R}_b^- := (-\infty, b)$. Let $q_{b,-}$ be the multiplication operator on $L^2(\mathbb{R}_b^-)$ by the variable $\lambda \in \mathbb{R}_b^-$. and

$$p_{b,-} := -i\frac{d}{d\lambda} \tag{1.9}$$

with $D(p_{b,-}) = C_0^\infty(\mathbb{R}_b^-)$. Then $q_{b,-}$ is self-adjoint, bounded above with $\sigma(q_{b,-}) = (-\infty, b]$, $p_{b,-}$ is a symmetric operator, and $(-p_{b,-}, q_{b,-})$ is a weak Weyl representation of the CCR. As in the case of $p_{a,+}$, $p_{b,-}$ is not essentially self-adjoint and

$$\sigma(-p_{b,-}) = \overline{\Pi}_-. \tag{1.10}$$

A relation between $(-p_{a,+}, q_{a,+})$ and $(-p_{b,-}, q_{b,-})$ is given as follows. Let $U_{ab} : L^2(\mathbb{R}_a^+) \to L^2(\mathbb{R}_b^-)$ be a linear operator defined by

$$(U_{ab}f)(\lambda) := f(a + b - \lambda), \quad f \in L^2(\mathbb{R}_a^+), \text{ a.e.}\lambda \in \mathbb{R}_b^-.$$

Then U_{ab} is unitary and

$$U_{ab}q_{a,+}U_{ab}^{-1} = a + b - q_{b,-}, \quad U_{ab}p_{a,+}U_{ab}^{-1} = -p_{b,-}. \tag{1.11}$$

In view of the von Neumann uniqueness theorem for Weyl representations, the pair $(-\overline{p}_{a,+}, q_{a,+})$ (resp. $(-\overline{p}_{b,-}, q_{b,-})$) may be a reference pair in classifying weak Weyl representations (T, H) with H being bounded below (resp. bounded above).

By Theorem 1.1, we can define two subsets of $\mathrm{W}(\mathcal{H})$:

$$\mathrm{W}_+(\mathcal{H}) := \{(T, H) \in \mathrm{W}(\mathcal{H}) | H \text{ is bounded below and } \sigma(T) = \overline{\Pi}_+\}, \tag{1.12}$$

$$\mathrm{W}_-(\mathcal{H}) := \{(T, H) \in \mathrm{W}(\mathcal{H}) | H \text{ is bounded above and } \sigma(T) = \overline{\Pi}_-\}. \tag{1.13}$$

Then, as shown above, $(-p_{a,+}, q_{a,+}) \in \mathrm{W}_+(L^2(\mathbb{R}_a^+))$ and $(-p_{b,-}, q_{b,-}) \in \mathrm{W}_-(L^2(\mathbb{R}_b^-))$.

For a set $\mathcal{A}$ of linear operators on a Hilbert space $\mathcal{H}$, we set $\mathcal{A}' := \{B \in \mathsf{B}(\mathcal{H}) | BA \subset AB, \forall A \in \mathcal{A}\}$, called the strong commutant of $\mathcal{A}$ in $\mathcal{H}$, where $\mathsf{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$ with $D(B) = \mathcal{H}$. We say that $\mathcal{A}$ is irreducible if $\mathcal{A}' = \{cI | c \in \mathbb{C}\}$, where I is the identity on $\mathcal{H}$.

The main results of the present paper are as follows:

Theorem 1.2 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_+(\mathcal{H})$ with $\varepsilon_0 := \inf \sigma(H)$. Suppose that $\{\overline{T}, T^*, H\}$ is irreducible. Then there exists a unitary operator $U : \mathcal{H} \to L^2(\mathbb{R}_{\varepsilon_0}^+)$ such that*

$$U\overline{T}U^{-1} = -\overline{p}_{\varepsilon_0,+}, \quad UHU^{-1} = q_{\varepsilon_0,+}. \tag{1.14}$$

In particular

$$\sigma(H) = [\varepsilon_0, \infty). \tag{1.15}$$

Remark 1.1 It is known that, for every weak Weyl representation $(T, H) \in \mathrm{W}(\mathcal{H})$ ($\mathcal{H}$ is not necessarily separable), H is purely absolutely continuous [8, 13].

As a corollary of Theorem 1.2, we have the following result:

Theorem 1.3 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_-(\mathcal{H})$ with $b := \sup \sigma(H)$. Suppose that $\{\overline{T}, T^*, H\}$ is irreducible. Then there exists a unitary operator $V : \mathcal{H} \to L^2(\mathbb{R}_b^-)$ such that*

$$V\overline{T}V^{-1} = -\overline{p}_{b,-}, \quad VHV^{-1} = q_{b,-}. \tag{1.16}$$

In particular

$$\sigma(H) = (-\infty, b]. \tag{1.17}$$

Proof. As remarked in the second paragraph of this section, $(-T, -H) \in \mathrm{W}_+(\mathcal{H})$ with $a := \inf \sigma(-H) = -b$ and $\sigma(-T) = \overline{\Pi}_+$. Hence, we can apply Theorem 1.2 to conclude that there exists a unitary operator $U : \mathcal{H} \to L^2(\mathbb{R}_a^+)$ such that

$$U\overline{T}U^{-1} = \overline{p}_{a,+}, \quad UHU^{-1} = -q_{a,+}.$$

By Example 1.2, we have

$$U_{ab}\overline{p}_{a,+}U_{ab}^{-1} = -\overline{p}_{b,-}, \quad U_{ab}q_{a,+}U_{ab}^{-1} = -q_{b,-},$$

where we have used that $a + b = 0$. Hence, putting $V := U_{ab}U$, we obtain the desired result. ∎

Remark 1.2 In view of Theorems 1.2 and 1.3, it would be interesting to know when $\sigma(T) = \overline{\Pi}_+$ (resp. $\overline{\Pi}_-$) for $(T, H) \in \mathrm{W}(\mathcal{H})$ with H bounded below (resp. above). Concerning this problem, we have the following results [5]:

(i) Let $(T, H) \in \mathrm{W}(\mathcal{H})$ and H be bounded below. Suppose that, for some $\beta_0 > 0$, $\mathrm{Ran}(e^{-\beta_0 H}T)$ (the range of $e^{-\beta_0 H}T$) is dense in $\mathcal{H}$. Then $\sigma(T) = \overline{\Pi}_+$.

(ii) Let $(T, H) \in \mathrm{W}(\mathcal{H})$ and H be bounded above. Suppose that, for some $\beta_0 > 0$, $\mathrm{Ran}(e^{\beta_0 H}T)$ is dense in $\mathcal{H}$. Then $\sigma(T) = \overline{\Pi}_-$.

Proof of Theorem 1.2 is given in Section 2. In Section 3 we present examples. In the last section we apply Theorem 1.2 to the general theory established in [6] and obtain a class of weak Weyl representations, from which a class of Weyl representations is constructed.

2 Some Facts and Proof of Theorem 1.2

To prove Theorem 1.2, we first present some key facts.

Lemma 2.1 *Let S be a closed symmetric operator on $\mathcal{H}$ such that $\sigma(S) = \overline{\Pi}_+$. Then there exists a unique strongly continuous one-parameter semi-group $\{Z(t)\}_{t\geq 0}$ whose generator is iS. Moreover, each $Z(t)$ is an isometry:*

$$Z(t)^* Z(t) = I, \quad \forall t \geq 0. \tag{2.1}$$

Proof. This fact is probably well known. But, for completeness, we give a proof. By the assumption $\sigma(S) = \overline{\Pi}_+$, we have $\sigma(iS) = \{z \in \mathbb{C} | \mathrm{Re}\, z \leq 0\}$. Therefore the positive real axis $(0, \infty)$ is included in the resolvent set $\rho(iS)$ of iS. Since S is symmetric, it follows that

$$\|(iS - \lambda)^{-1}\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Hence, by the Hille-Yosida theorem, iS generates a strongly continuous one-parameter semi-group $\{Z(t)\}_{t\geq 0}$ of contractions. For all $\psi \in D(iS) = D(S)$, $Z(t)\psi$ is in $D(S)$ and strongly differentiable in $t \geq 0$ with

$$\frac{d}{dt}Z(t)\psi = iSZ(t)\psi = Z(t)iS\psi.$$

This equation and the symmetricity of S imply that $\|Z(t)\psi\|^2 = \|\psi\|^2, \forall t \geq 0$. Hence (2.1) follows. ∎

Lemma 2.2 *Let $(T, H) \in \mathrm{W}_+(\mathcal{H})$. Then there exists a unique strongly continuous one-parameter semi-group $\{U_T(t)\}_{t\geq 0}$ whose generator is $i\overline{T}$. Moreover, each $U_T(t)$ is an isometry and*

$$U_T(t)e^{-isH} = e^{its}e^{-isH}U_T(t), \quad t \geq 0, s \in \mathbb{R}. \tag{2.2}$$

Proof. We can apply Lemma 2.1 to $S = \overline{T}$ to conclude that $i\overline{T}$ generates a strongly continuous one-parameter semi-group $\{U_T(t)\}_{t\geq 0}$ of isometries on $\mathcal{H}$. For all $\psi \in D(\overline{T})$ and all $t \geq 0$, $U_T(t)\psi$ is in $D(\overline{T})$ and strongly differentiable in $t \geq 0$ with

$$\frac{d}{dt}U_T(t)\psi = i\overline{T}U_T(t)\psi = U_T(t)i\overline{T}\psi.$$

Let $s \in \mathbb{R}$ be fixed and $V(t) := e^{its}e^{-isH}U_T(t)e^{isH}$. Then $\{V(t)\}_{t\geq 0}$ is a strongly continuous one-parameter semi-group of isometries. Let $\psi \in D(\overline{T})$. Then $e^{-isH}\psi \in D(\overline{T})$ and

$$\overline{T}e^{-isH}\psi = e^{-isH}\overline{T}\psi + se^{-isH}\psi.$$

Hence $V(t)\psi$ is in $D(\overline{T})$ and strongly differentiable in t with

$$\frac{d}{dt}V(t)\psi = i\overline{T}V(t)\psi.$$

This implies that $V(t)\psi = U_T(t)\psi, \forall t \geq 0$. Since $D(\overline{T})$ is dense, it follows that $V(t) = U_T(t), \forall t \geq 0$, implying (2.2). ∎

Let $a \in \mathbb{R}$ be fixed. For each $t \geq 0$, we define a linear operator $U_a(t)$ on $L^2(\mathbb{R}_a^+)$ as follows: For each $f \in L^2(\mathbb{R}_a^+)$,

$$(U_a(t)f)(\lambda) := \begin{cases} f(\lambda - t) & \lambda > t + a \\ 0 & a < \lambda \leq t + a \end{cases} \tag{2.3}$$

Then it is easy to see that $\{U_a(t)\}_{t\geq 0}$ is a strongly continuous one-parameter semi-group of isometries on $L^2(\mathbb{R}_+^a)$.

Lemma 2.3 *The generator of $\{U_a(t)\}_{t\geq 0}$ is $-i\overline{p}_{a,+}$.*

Proof. Let iA be the generator of $\{U_a(t)\}_{t\geq 0}$. Then it follows from the isometry of $U_a(t)$ that A is a closed symmetric operator. It is easy to see that $-p_{a,+} \subset A$ and hence $-\overline{p}_{a,+} \subset A$. As already remarked in Example 1.1, $-\overline{p}_{a,+}$ is maximal symmetric. Hence $A = -\overline{p}_{a,+}$. ∎

We recall a result of Bracci and Picasso [7]. Let $\{U(\alpha)\}_{\alpha\geq 0}$ and $\{V(\beta)\}_{\beta\in\mathbb{R}}$ be a strongly continuous one-parameter semi-group and a strongly continuous one-parameter unitary group on $\mathcal{H}$ respectively, satisfying

$$U(\alpha)^*U(\alpha) = I, \quad \alpha \geq 0, \tag{2.4}$$

$$U(\alpha)V(\beta) = e^{i\alpha\beta}V(\beta)U(\alpha), \quad \alpha \geq 0, \beta \in \mathbb{R}. \tag{2.5}$$

Then, by the Stone theorem, there exists a unique self-adjoint operator P on $\mathcal{H}$ such that

$$V(\beta) = e^{-i\beta P}, \quad \beta \in \mathbb{R}. \tag{2.6}$$

Lemma 2.4 [7] *Let $\mathcal{H}$ be separable. Suppose that P is bounded below with $\nu := \inf \sigma(P)$ and that $\{U(\alpha), U(\alpha)^*, V(\beta) | \alpha \geq 0, \beta \in \mathbb{R}\}$ is irreducible. Then there exists a unitary operator $S : \mathcal{H} \to L^2(\mathbb{R}^+_\nu)$ such that*

$$SV(\beta)S^{-1} = e^{-i\beta q_{\nu,+}}, \beta \in \mathbb{R}, \tag{2.7}$$
$$SU(\alpha)S^{-1} = U_\nu(\alpha), \quad \alpha \geq 0. \tag{2.8}$$

We denote the generator of $\{U(\alpha)\}_{\alpha \geq 0}$ by iQ. It follows that Q is closed and symmetric.

Lemma 2.5 *Let S and ν be as in Lemma 2.4. Then*

$$SPS^{-1} = q_{\nu,+}, \tag{2.9}$$
$$SQS^{-1} = -\overline{p}_{\nu,+}. \tag{2.10}$$

In particular

$$\sigma(P) = [\nu, \infty). \tag{2.11}$$

Proof. Relations (2.6) and (2.7) imply (2.9). Simlarly (2.10) follows from (2.8) and Lemma 2.3. ∎

Lemma 2.6 *Let $(T, H) \in \mathrm{W}(\mathcal{H})$ with $\sigma(T) = \overline{\Pi}_+$. Suppose that $\{\overline{T}, T^*, H\}$ is irreducible. Then $\{U_T(t), U_T(t)^*, e^{-isH} | t \geq 0, s \in \mathbb{R}\}$ is irreducible.*

Proof. Let $B \in \mathsf{B}(\mathcal{H})$ be such that

$$BU_T(t) = U_T(t)B, \tag{2.12}$$
$$BU_T(t)^* = U_T(t)^*B, \tag{2.13}$$
$$Be^{-isH} = e^{-isH}B, \forall t \geq 0, \forall s \in \mathbb{R}. \tag{2.14}$$

Let $\psi \in D(\overline{T})$. Then, by (2.12), we have $BU_T(t)\psi = U_T(t)B\psi, \forall t \geq 0$. By Lemma 2.2, the left hand side is strongly differentiable in t with $d(BU_T(t)\psi)/dt = iB\overline{T}U_T(t)\psi$. Hence so does the right hand side and we obtain that $B\psi \in D(\overline{T})$ and $B\overline{T}\psi = \overline{T}B\psi$. Therefore $B\overline{T} \subset \overline{T}B$. Note that (2.13) implies that $U_T(t)B^* = B^*U_T(t)$. Hence it follows that $B^*\overline{T} \subset \overline{T}B^*$, which implies that $BT^* \subset T^*B$, where we have used the following general facts: for every densely defined closable linear operator A on $\mathcal{H}$ and all $C \in \mathsf{B}(\mathcal{H})$, $(CA)^* = A^*C^*, (AC)^* \supset C^*A^*, (\bar{A})^* = A^*$. Similarly (2.14) implies that $BH \subset HB$. Hence $B \in \{\overline{T}, T^*, H\}'$. Therefore $B = cI$ for some $c \in \mathbb{C}$.

Proof of Theorem 1.2

By Lemmas 2.2 and 2.6, we can apply Lemma 2.4 to the case where $V(\beta) = e^{-i\beta H}, \beta \in \mathbb{R}$ and $U(\alpha) = U_T(\alpha), \alpha \geq 0$. Then the desired results follow from Lemmas 2.4 and 2.5.

3 Examples

Example 3.1 Let $\mathbb{R}^d_{\mathbf{x}} = \{\mathbf{x} = (x_1, \cdots, x_d) | x_j \in \mathbb{R}, j = 1, \cdots, d\}$. We denote by q_j the j-th position operator on $L^2(\mathbb{R}^d_{\mathbf{x}})$ (the multiplication operator by the j-th variable x_j) and $p_j := -iD_j$ the j-th momentum operator, where D_j is the generalized partial differential operator in x_j. The free Hamiltonian for a non-relativistic quantum particle with mass $M > 0$ is given by

$$H_0 := -\frac{1}{2M}\Delta,$$

where $\Delta := \sum_{j=1}^d D_j^2$ is the generalized Laplacian on $L^2(\mathbb{R}^d_{\mathbf{x}})$. It is well known that H_0 is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^d_{\mathbf{x}})$ and absolutely continuous with $\sigma(H_0) = [0, \infty)$.

We denote by $\mathcal{F} : L^2(\mathbb{R}^d_{\mathbf{x}}) \to L^2(\mathbb{R}^d_{\mathbf{k}})$ the Fourier transform:

$$(\mathcal{F}f)(\mathbf{k}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d_{\mathbf{x}}} e^{-i\mathbf{kx}} f(\mathbf{x}) d\mathbf{x}, \quad f \in L^2(\mathbb{R}^d_{\mathbf{x}})$$

in the L^2 sense. Let

$$\mathbb{M}_j := \left\{\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{R}^d_{\mathbf{k}} | k_j \neq 0\right\} \subset \mathbb{R}^d_{\mathbf{k}}$$

For each $j = 1, \cdots, d$, we define

$$T_j^{\mathrm{AB}} := \frac{M}{2}\left(q_j p_j^{-1} + p_j^{-1} q_j\right)$$

with $D(T_j^{\mathrm{AB}}) := \mathcal{F}^{-1} C_0^\infty(\mathbb{M}_j)$. It is easy to see that (T_j^{AB}, H_0) is a weak Weyl representation of the CCR [2, 8]. The operator T_j^{AB} is called the *Aharonov-Bohm time operator* [1]. In the previous paper [5], we proved that $\sigma(T_j^{\mathrm{AB}}) = \overline{\Pi}_+$. Hence $(T_j^{\mathrm{AB}}, H_0) \in \mathrm{W}_+(L^2(\mathbb{R}^d_{\mathbf{x}}))$. Note that $\inf \sigma(H_0) = 0$.

We consider the case $d = 1$. In this case, one can directly show that $(\overline{T}_1^{\mathrm{AB}}, H_0)$ is unitarily equivalent to the two direct sum of the weak Weyl representation $(-\overline{p}_{0,+}, q_{0,+})$ on $L^2((0, \infty))$.

Example 3.2 (A relativistic time operator [2]) The free Hamiltonian for a relativistic quantum particle with mass $m \geq 0$ and spin 0 is given by

$$H_{\mathrm{rel}} := \sqrt{-\Delta + m^2}$$

acting in $L^2(\mathbb{R}^d_{\mathbf{x}})$. For each $j = 1, \cdots, d$, we define

$$T_j^{\mathrm{rel}} := \frac{1}{2}\left(H_{\mathrm{rel}} p_j^{-1} q_j + q_j p_j^{-1} H_{\mathrm{rel}}\right)$$

with $D(T_j^{\mathrm{rel}}) := \mathcal{F}^{-1} C_0^\infty(\mathbb{M}_j)$. As is shown in [2], $(T_j^{\mathrm{rel}}, H_{\mathrm{rel}})$ is a weak Weyl representtion. Moreover $\sigma(T_j^{\mathrm{rel}}) = \overline{\Pi}_+$ [4]. Hence $(T_j^{\mathrm{rel}}, H_{\mathrm{rel}}) \in \mathrm{W}_+(L^2(\mathbb{R}^d_{\mathbf{x}}))$. One has that $\inf \sigma(H_{\mathrm{rel}}) = m$.

We consider the case $d = 1$. In this case, one can directly prove that $(\overline{T}_1^{\mathrm{rel}}, H_0)$ is unitarily equivalent to the two direct sum of the weak Weyl representation $(-\overline{p}_{m,+}, q_{m,+})$ on $L^2((m, \infty))$.

4 Construction of a Weyl representation from a weak Weyl representation

In the previous paper [6], a general structure was found to construct a Weyl representation from a weak Weyl representation. Here we recall it.

Theorem 4.1 [6, Corollary 2.6] *Let (T, H) be a weak Weyl representation on a Hilbert space $\mathcal{H}$ with T closed. Then the operator*

$$L := \log |H| \tag{4.1}$$

is well-defined, self-adjoint and the operator

$$D := \frac{1}{2}(TH + \overline{HT}) \tag{4.2}$$

is a symmetric operator. Moreover, if D is essentially self-adjoint, then $(\overline{D}, L)$ is a Weyl representation of the CCR and $\sigma(|H|) = [0, \infty)$.

To apply this theorem, we need a lemma.

Lemma 4.2 *Let $a \in \mathbb{R}$ and*

$$d_a := -\frac{1}{2}(p_{a,+}q_{a,+} + \overline{q_{a,+}p_{a,+}}) \tag{4.3}$$

acting in $L^2(\mathbb{R}_a^+)$. Then d_a is essentially self-adjoint if and only if $a = 0$.

Proof. Let $a > 0$. Then the function u on $\mathbb{R}_a^+$ defined by $u(\lambda) = 1/\lambda^{3/2}$, $\lambda > a$ is in $C^\infty(\mathbb{R}_a^+) \cap L^2(\mathbb{R}_a^+)$ with $\lambda u'(\lambda) = -(3/2)u(\lambda)$. In the present case, we have $D(p_{a,+}q_{a,+}) = C_0^\infty(\mathbb{R}_a^+) = D(p_{a,+})$. Hence $D(d_a) = C_0^\infty(\mathbb{R}_a^+)$. It follows that, for all $f \in D(d_a)$, $\langle u, (d_a - i)f\rangle = 0$. This implies that $u \in \ker(d_a^* + i)$ and hence $\ker(d_a^* + i) \neq \{0\}$. Therefore d_a is not essentially self-adjoint. Thus, if d_a is essentially self-adjoint, then $a \leq 0$. Let $a < 0$ and $v \in \ker(d_a^* + i)$. Then, for all $f \in C_0^\infty(\mathbb{R}_a^+)$, $\langle v, (d_a - i)f\rangle = 0$. This implies the distribution equation $\lambda D_\lambda v(\lambda) = -(3/2)v(\lambda)$ on $\mathbb{R}_a^+$. Hence $v(\lambda) = \{c_1\chi_{[a,0]}(\lambda) + c_1'\chi_{(0,\infty)}(\lambda)\}/|\lambda|^{3/2}$ for a.e. $\lambda \in \mathbb{R}_a^+$ with constants c_1 and c_1', where χ_S is the characteristic function of the set S. Since v is in $L^2(\mathbb{R}_a^+)$, it follows that $c_1 = c_1' = 0$ and hence $v = 0$. Thus $\ker(d_a^* + i) = \{0\}$. Next, let $w \in \ker(d_a^* - i)$. Then, in the same way as in the preceding case, we have $w(\lambda) = \{c_2\chi_{[a,0]}(\lambda) + c_2'\chi_{(0,\infty)}(\lambda)\}|\lambda|^{1/2}$ with constants c_2 and c_2'. Since w is in $L^2(\mathbb{R}_a^+)$, it follows that $c_2' = 0$ and hence $w(\lambda) = c_2\chi_{[a,0]}(\lambda)|\lambda|^{1/2}$. Thus $\dim\ker(d_a^* - i) = 1$. By a general criterion on essential self-adjointness, we conclude that d_a is not essentially self-adjoint. Thus, if d_a is essentially self-adjoint, then $a = 0$.

In the same way as in the preceding case, one can easily show that, if $a = 0$, then d_a is essentially self-adjoint. ∎

Now we can prove the following theorem.

Theorem 4.3 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_+(\mathcal{H})$ with T closed and $\inf\sigma(H) = 0$. Suppose that $\{T, T^*, H\}$ is irreducible. Let L and D be as in (4.1) and (4.2) respectively. Then D is essentially self-adjoint and $(\overline{D}, L)$ is a Weyl representation of the CCR.*

Proof. Let $\hat{d}_0$ be the operator d_0 with $p_{0,+}$ replaced by $\overline{p}_{0,+}$. Then, by Theorem 1.2, D is unitarily equivalent to $\hat{d}_0$. We have $d_0 \subset \hat{d}_0$. By Lemma 4.2, d_0 is essentially self-adjoint. Hence $\hat{d}_0$ is essentially self-adjoint. Therefore it follows that D is essentially self-adjoint. The second half of the theorem follows from Theorem 4.1. ∎

Finally we remark on the case where $(T, H) \in \mathrm{W}_-(\mathcal{H})$:

Corollary 4.4 *Let $\mathcal{H}$ be separable and $(T, H) \in \mathrm{W}_-(\mathcal{H})$ with T closed and $\sup \sigma(H) = 0$. Suppose that $\{T, T^*, H\}$ is irreducible. Let L and D be as in (4.1) and (4.2) respectively. Then D is essentially self-adjoint and $(\overline{D}, L)$ is a Weyl representation of the CCR.*

Proof. We have $(-T, -H) \in \mathrm{W}_+(\mathcal{H})$ with $\inf \sigma(-H) = 0$. The operator D (resp. L) for $(-T, -H)$ is the same as that for (T, H). Hence the conclusions follow from Theorem 4.3. ∎

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