数理解析研究所講究録
1581 巻 2008 年 103-108

SURFACE SYMMETRIES, HOMOLOGY REPRESENTATIONS, AND GROUP COHOMOLOGY

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Given a finite group G of automorphisms of a compact Riemann surface, we discuss a relation between Mumford-Morita-Miller classes of odd indices and the homology representation of G. Since most participants were group theorists rather than topologists, I separate the algebraic and the topological ingredients and explain the former in detail.

1. SURFACE SYMMETRIES

1.1. **The Grieder group of a finite group.** Let G be a finite group and $\hat{\gamma}$ the conjugacy class of $\gamma \in G$. We denote by $\langle\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_q\rangle$ an unordered q-tuple ($q \geq 0$) of conjugacy classes of nontrivial elements of G satisfying $\gamma_1\gamma_2\cdots\gamma_q \in [G,G]$, and $\mathcal{M}_G$ the set of all such q-tuples. We can define an abelian monoid structure on $\mathcal{M}_G$ by

$$\langle\hat{\gamma}_1,\ldots,\hat{\gamma}_q\rangle + \langle\hat{\gamma}_{q+1},\ldots,\hat{\gamma}_r\rangle = \langle\hat{\gamma}_1,\ldots,\hat{\gamma}_q,\hat{\gamma}_{q+1},\ldots,\hat{\gamma}_r\rangle.$$

The identity element is the empty tuple $\langle\rangle$. We call $\mathcal{M}_G$ the *Grieder monoid* of G. Now let $\mathcal{M}'_G$ be the submonoid generated by $\langle\hat{\gamma},\hat{\gamma}^{-1}\rangle$ ($\gamma \in G$) and set $\mathcal{A}_G := \mathcal{M}_G/\mathcal{M}'_G$. The quotient $\mathcal{A}_G$ is an abelian group. The inverse element is given by

$$-\langle\hat{\gamma}_1,\ldots,\hat{\gamma}_q\rangle = \langle\hat{\gamma}_1^{-1},\ldots,\hat{\gamma}_q^{-1}\rangle \text{ in } \mathcal{A}_G.$$

We call $\mathcal{A}_G$ the *Grieder group* of G. As the names suggest, $\mathcal{M}_G$ and $\mathcal{A}_G$ were introduced and studied by Grieder [5, 6] to study surface symmetries. First of all, $\mathcal{A}_G$ is finitely generated:

Proposition 1 ([5]). *$\mathcal{A}_G \cong \mathbb{Z}^m \oplus \mathbb{Z}_2^n$ for some $m,n \geq 0$.*

A homomorphism $f : H \to G$ of groups induces a homomorphism $f_* : \mathcal{A}_H \to \mathcal{A}_G$ of abelian groups by $f_*(\langle\hat{\gamma}_1,\ldots,\hat{\gamma}_q\rangle) = \langle f(\hat{\gamma}_1),\ldots,f(\hat{\gamma}_q)\rangle$ so that the assignment $G \mapsto \mathcal{A}_G$ is a covariant functor. In addition, for an inclusion $i : H \hookrightarrow G$, one can also define the restriction $i^* : \mathcal{A}_G \to \mathcal{A}_H$ via surface symmetries. Grieder [5] verified the double coset formula and hence proved the following proposition:

Proposition 2. *The assignment $G \mapsto \mathcal{M}_G$ is a Mackey functor.*

1.2. **Ramification data.** By a *surface symmetry* we mean a pair (G,C), where C is a compact Riemann surface of genus $g \geq 2$, and G is a finite group of automorphisms of C. For each $x \in C$, let G_x be the isotropy subgroup at x. Note that G_x is necessary cyclic. Set $\mathcal{S} = \{x \in C \mid G_x \neq 1\}$, and let $\mathcal{S}/G = \{x_1, x_2, \ldots, x_q\}$ be a set of representatives of G-orbits of elements of $\mathcal{S}$. For each $x_i \in \mathcal{S}/G$, choose a generator γ_i of G_{x_i} such that γ_i acts on the holomorphic tangent space $T_{x_i}C$ by $z \mapsto \exp(2\pi\sqrt{-1}/|G_{x_i}|)z$ with respect to a suitable local coordinate z at x_i. The *ramification data* of (G,C), abbreviated by $\delta(G,C)$, is the unordered q-tuple $\langle\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_q\rangle$. It satisfies $\gamma_1\gamma_2\cdots\gamma_q \in [G,G]$, and hence $\delta(G,C)$ is an element of the Grieder monoid $\mathcal{M}_G$. Conversely, we have the following proposition.

Proposition 3 (see [5]). *For any element $\mu \in \mathcal{M}_G$, there exists a surface symmetry (G,C) whose ramification data coincides with μ.*

2. Group cohomology

2.1. **The first Chern class.** Let $\langle\gamma\rangle$ be a cyclic group of order m generated by γ and $\rho_\gamma : \langle\gamma\rangle \to \mathbb{C}^\times$ a linear character defined by $\gamma \mapsto \exp(2\pi i/m)$. For any finite group G, we have natural isomorphisms

$$\mathrm{Hom}(G,\mathbb{C}^\times) \cong H^1(G,\mathbb{C}^\times) \cong H^2(G,\mathbb{Z}).$$

Here, the latter isomorphism is the connecting homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$. Define $c(\gamma) \in H^2(\langle\gamma\rangle,\mathbb{Z})$ to be the image of ρ_γ under the isomorphism $\mathrm{Hom}(\langle\gamma\rangle,\mathbb{C}^\times) \cong H^2(\langle\gamma\rangle,\mathbb{Z})$. The cohomology class $c(\gamma)$ is sometimes called the *first Chern class* of ρ_γ.

2.2. **MMM classes (algebra).** For each element $\mu = \langle\hat{\gamma}_1, \ldots, \hat{\gamma}_q\rangle$ of $\mathcal{M}_G$, define a series of cohomology classes $e_k(\mu) \in H^{2k}(G,\mathbb{Z})$ $(k \geq 1)$ by

$$e_k(\mu) := \sum_{i=1}^{q} \mathrm{Tr}_{\langle\gamma_i\rangle}^G (c(\gamma_i)^k) \in H^{2k}(G,\mathbb{Z}),$$

where $\mathrm{Tr}_{\langle\gamma\rangle}^G : H^*(\langle\gamma\rangle,\mathbb{Z}) \to H^*(G,\mathbb{Z})$ is the transfer. We call $e_k(\mu)$ the *k-th Mumford-Morita-Miller class* of μ (MMM class in short). The definition of $e_k(\mu)$ is motivated by topology, as will be explained in the next subsection. Observe that the assignment $\mu \mapsto e_k(\mu)$ defines a well-defined homomorphism $\mathcal{M}_G \to H^{2k}(G,\mathbb{Z})$ of abelian monoids. For k odd, it induces a well-defined homomorphism $\mathcal{A}_G \to H^{2k}(G,\mathbb{Z})$ of abelian groups, for we have $c(\gamma^{-1}) = -c(\gamma)$. In addition, we can prove the following proposition:

Proposition 4. *For odd $k \geq 1$, the homomorphism $\mathcal{A}_G \to H^{2k}(G,\mathbb{Z})$ is a natural transformation of Mackey functors.*

2.3. **MMM classes (topology).** The definition of $e_k(\mu)$ is inspired by a result of Kawazumi and Uemura [8] concerning of characteristic classes of oriented surface bundles. Let Σ_g be the closed oriented surface of genus $g \geq 2$. Let $\pi : E \to B$ an oriented Σ_g-bundle, $T^v E$ the tangent bundle along the fiber of π, and $e \in H^2(E;\mathbb{Z})$ the Euler class of $T^v E$. Define $e_k^{\mathrm{top}}(\pi) \in H^{2k}(B;\mathbb{Z})$ by $e_k^{\mathrm{top}}(\pi) := \pi_!(e^{k+1})$ where $\pi_! : H^*(E;\mathbb{Z}) \to H^{*-2}(B;\mathbb{Z})$ is the Gysin homomorphism (the superscript "top" stands for "topology"). $e_k^{\mathrm{top}}(\pi)$ is called the *k-th Mumford-Morita-Miller class* of π, as it was introduced in [11, 10, 9].

Now let (G,C) be a surface symmetry as in Section 1.2. Associated with (G,C), there is an oriented surface bundle $\pi : EG \times_G C \to BG$ called the *Borel construction*, where $EG \to BG$ is the universal G-bundle. We denote by $e_k^{\mathrm{top}}(G,C) \in H^{2k}(G,\mathbb{Z})$ the k-th MMM class of the Borel construction π. A result of Kawazumi and Uemura [8] implies the following result:

Theorem 5. *We have $e_k^{\mathrm{top}}(G,C) = e_k(\delta(G,C))$ where $\delta(G,C)$ is the ramification data of (G,C).*

3. Homology representations

3.1. **Algebra.** In what follows, we denote by $R(G)$ the complex representation ring (or the character ring) of a finite group G. Let $\langle\gamma\rangle$ be a cyclic group of order m generated by γ and $\rho_\gamma : \langle\gamma\rangle \to \mathbb{C}^\times$ a linear character as in Section 2.1. Define $\Delta_\gamma \in R(\langle\gamma\rangle) \otimes \mathbb{Q}$ by

$$\Delta_\gamma := 2\sum_{k=1}^{m-1}\left(\frac{k}{m} - \frac{1}{2}\right)\rho_\gamma^{\otimes k} = \frac{2}{m}\sum_{k=1}^{m-1} k\rho_\gamma^{\otimes k} - r_{\langle\gamma\rangle} + 1_{\langle\gamma\rangle},$$

where $r_{\langle\gamma\rangle}$ is the regular representation and $1_{\langle\gamma\rangle}$ is the trivial 1-dimensional representation of $\langle\gamma\rangle$. Now, for each element $\mu = \langle\hat{\gamma}_1, \ldots, \hat{\gamma}_q\rangle$ of $\mathcal{M}_G$, define the *G-signature* $\sigma(\mu)$ of μ by

$$\sigma(\mu) := \sum_{k=1}^{q} \mathrm{Ind}_{\langle\gamma_k\rangle}^{G}(\Delta_{\gamma_k}) \in R(G) \otimes \mathbb{Q}.$$

Proposition 6. $\sigma(\mu) \in R(G)$ *for every G and $\mu \in \mathcal{M}_G$.*

See the next section for the proof. Note that, in case $\mu \in \mathcal{M}_G$ consists of a single conjugacy class ($\mu = \langle\hat{\gamma}\rangle$ for $\gamma \in [G,G]$), Proposition 6 was proved by T. Yoshida [13].

The assignment $\mu \mapsto \sigma(\mu)$ yields a homomorphism $\mathcal{M}_G \to R(G)$ of monoids, which induces a well-defined homomorphism $\mathcal{A}_G \to R(G)$ of abelian groups. In addition, we can prove the following proposition:

Proposition 7. *$\mathcal{A}_G \to R(G)$ is a natural transformation of Mackey functors.*

3.2. **Topology.** Let (G,C) be a surface symmetry, and H_C the space of holomorphic 1-forms on C. Note that $\dim_{\mathbb{C}} H_C = g$ where g is the genus of the Riemann surface C. Then G acts on H_C and hence H_C is a complex representation of G. A virtual representation $\sigma^{\mathrm{top}}(G,C) := H_C - \overline{H}_C \in R(G)$ is called the *G-signature* of (G,C), where $\overline{H}_C$ is the complex conjugate.

Proposition 8. *We have $\sigma^{\mathrm{top}}(G,C) = \sigma(\delta(G,C))$ where $\delta(G,C)$ is the ramification data of (G,C).*

The character of $\sigma^{\mathrm{top}}(G,C)$ is given by the Eichler trace formula (see [4] for instance). The proposition can be verified by comparing characters of $\sigma^{\mathrm{top}}(G,C)$ and $\sigma(\delta(G,C))$. An alternative proof was given by N. Kawazumi (unpublished manuscript). Since every $\mu \in \mathcal{M}_G$ can be realized as a ramification data of a surface symmetry, Proposition 6 follows from the last proposition. The following fact is an easy consequence of Proposition 8.

Corollary 9. *If all the complex characters of G are $\mathbb{R}$-valued, then $\sigma(\mu) = 0$ for all $\mu \in \mathcal{M}_G$.*

Proof. Choose a surface symmetry (G,C) with $\delta(G,C) = \mu$. Then we have $\sigma(\mu) = \sigma^{\mathrm{top}}(G,C) = H_C - \overline{H}_C = 0$ since $H_C = \overline{H}_C$ by the assumption. $\square$

4. A RELATION OF $e_k(\mu)$ AND $\sigma(\mu)$

Theorem 10. *Let G be a finite group and $\mu, \nu \in \mathcal{M}_G$.*

(1) *If $\sigma(\mu) = \sigma(\nu)$ then $e_k(\mu) = e_k(\nu)$ for all odd $k \geq 1$.*
(2) *If $\sigma(\mu) = 0$ then $e_k(\mu) = 0$ for all odd $k \geq 1$.*

Since $R(G)$ is free as an abelian group, the homomorphism $\mathcal{A}_G \to R(G)$ in Section 3.1 induces $\phi_1 : \mathcal{A}_G/\mathrm{Tor}(\mathcal{A}_G) \to R(G)$, where $\mathrm{Tor}(\mathcal{A}_G)$ is the torsion subgroup of $\mathcal{A}_G$. For odd $k \geq 1$, let $\phi_2 : \mathrm{Tor}(\mathcal{A}_G) \to H^{2k}(G,\mathbb{Z})$ be the restriction of the homomorphism $\mathcal{A}_G \to H^{2k}(G,\mathbb{Z})$ in Section 2.2. The proof of Theorem 10 is based on the following two facts:

Theorem 11. *For any finite group G and any odd $k \geq 1$,*

(1) *The homomorphism $\phi_1 : \mathcal{A}_G/\mathrm{Tor}(\mathcal{A}_G) \to R(G)$ is injective.*
(2) *The homomorphism $\phi_2 : \mathrm{Tor}(\mathcal{A}_G) \to H^{2k}(G,\mathbb{Z})$ is trivial.*

The first statement is proved by using a result of Edmonds and Ewing [3], while the second statement is proved by considering the cohomology of metacyclic 2-groups. The detail will appear elsewhere. Theorem 10 and Corollary 9 imply the following corollary:

Corollary 12. *If all the complex characters of G are $\mathbb{R}$-valued, then $e_k(\mu) = 0$ for all $\mu \in \mathcal{M}_G$ and odd $k \geq 1$.*

Define $\mathcal{R}_G$ to be the image of $\phi_1 : \mathcal{A}_G/\mathrm{Tor}(\mathcal{A}_G) \to R(G)$. In view of Theorem 11, there exists a series of homomorphisms $\Phi_k : \mathcal{R}_G \to H^{2k}(G,\mathbb{Z})$ (k odd) which assigns $e_k(\mu)$ to $\sigma(\mu)$. Let $c : \mathrm{Hom}(G,\mathbb{C}^\times) \to H^2(G,\mathbb{Z})$ be the natural isomorphism as in Section 2.1 and $\det : R(G) \to \mathrm{Hom}(G,\mathbb{C}^\times)$ the determinant homomorphism (see [13] for precise). Then the homomorphism Φ_1 is determined by the following proposition:

Proposition 13. $e_1(\mu) = 6 \cdot c(\det(\sigma(\mu))$ *for all* $\mu \in \mathcal{M}_G$.

The proposition follows from the Grothendieck-Riemann-Roch theorem and a result of Harer [7] (see also [1, Proposition 6]). Proposition 13 can be generalized to larger k, provided G is cyclic. Recall that, for every finite group G, there is a series of homomorphisms $s_k : R(G) \to H^{2k}(G,\mathbb{Z})$ $(k \geq 0)$ of abelian groups, which satisfies the following properties:

(1) $s_1(\rho) = c(\det \rho)$ for all $\rho \in R(G)$.
(2) If ρ is a linear character, then $s_k(\rho) = c(\rho)^k$.

$s_k(\rho)$ is called the *k-th Newton class* of $\rho \in R(G)$. See [12] for further details. Let B_{2k} be the $2k$-th Bernoulli number and N_{2k}, D_{2k} coprime integers satisfying $B_{2k}/k = N_k/D_k$. Then a result of the author and Kawazumi [2] implies the following result:

Theorem 14. *If G is cyclic, then $N_{2k} \cdot e_{2k-1}(\mu) = D_{2k} \cdot s_{2k-1}(\sigma(\mu))$ holds for all $\mu \in \mathcal{M}_G$ and $k \geq 1$.*

Now let G be a cyclic group of order m, and suppose that N_{2k} is prime to m. Choose an integer N_{2k}^* satisfying $N_{2k} \cdot N_{2k}^* \equiv 1 \pmod{m}$. Under these assumptions, we have

$$e_{2k-1}(\mu) = N_{2k}^* D_{2k} \cdot s_{2k-1}(\sigma(\mu))$$

for all $\mu \in \mathcal{M}_G$, and hence determining Φ_{2k-1} for these cases. In particular, we have $e_1(\mu) = 6 \cdot s_1(\sigma(\mu))$, $e_3(\mu) = -60 \cdot s_3(\sigma(\mu))$, $e_5(\mu) = 126 \cdot s_5(\sigma(\mu))$, $e_7(\mu) = -120 \cdot s_7(\sigma(\mu))$ for *any* cyclic group G and $\mu \in \mathcal{M}_G$, since $N_{2k} = 1$ for $1 \leq k \leq 4$.

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