Semimartingales from the Fokker-Planck equation

Dedicated to Professor Wendell H. Fleming
on the occasion of his seventy seventh birthday

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Abstract

We show the existence of a semimartingale of which one-dimensional marginal distributions are given by the solution of the Fokker-Planck equation with the p-th integrable drift vector ($p > 1$).

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1 Introduction.

Let $\mathcal{M}_1(\mathbf{R}^d)$ denote the complete separable metric space, with a weak topology, of Borel probability measures on $\mathbf{R}^d$ ($d \geq 1$).

Let $b : [0,1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ be measurable and $\{P_t(dx)\}_{0 \leq t \leq 1}, \subset \mathcal{M}_1(\mathbf{R}^d)$, satisfy the following Fokker-Planck equation: for $f \in C_b^{1,2}([0,1] \times \mathbf{R}^d)$ and $t \in [0,1]$,

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$$
\begin{aligned}
&\int_{\mathbf{R}^d} f(t,x)P_t(dx) - \int_{\mathbf{R}^d} f(0,x)P_0(dx) \qquad (1.1)\\
= &\int_0^t ds \int_{\mathbf{R}^d} \Big(\frac{\partial f(s,x)}{\partial s} + \frac{1}{2}\triangle f(s,x) + < b(s,x), D_x f(s,x) > \Big) P_s(dx),
\end{aligned}
$$

where $\triangle := \sum_{i=1}^d \partial^2/\partial x_i^2$, $D_x := (\partial/\partial x_i)_{i=1}^d$, and $< \cdot, \cdot >$ denotes the inner product in $\mathbf{R}^d$.

Inspired by Born's probabilistic interpretation of a solution to Schrödinger's equation, Nelson proposed the problem of the construction of a diffusion process $\{X(t)\}_{0\le t\le 1}$ for which the following holds (see [20]):

$$
X(t) = X(0) + \int_0^t b(s, X(s))ds + W(t) \quad (t \in [0,1]), \qquad (1.2)
$$

$$
P(X(t) \in dx) = P_t(dx) \quad (t \in [0,1]), \qquad (1.3)
$$

where $\{W(t)\}_{0\le t\le 1}$ is a $\sigma[X(s) : 0 \le s \le t]$-Wiener process.

The first result was given by Carlen [2] (see also [23]). It was generalized, by Mikami [12], to the case where the second order differential operator has a variable coefficient. The further generalization and almost complete resolution was made by Cattiaux and Léonard [3-6] (see also [1, 13-15] for the related topics). But in these papers, they assumed that

$$
\int_0^1 dt \int_{\mathbf{R}^d} |b(t,x)|^2 P_t(dx) < \infty \qquad (1.4)
$$

for some b for which (1.1) holds. This is called the **finite energy condition** for $\{P_t(dx)\}_{0\le t\le 1}$.

Remark 1.1 *It is known that b is not unique for $\{P_t(dx)\}_{0\le t\le 1}$ in (1.1) (see [12] or [3-6]).*

In this paper we consider Nelson's problem under a weaker assumption than (1.4): there exists $p > 1$ such that

$$
\int_0^1 dt \int_{\mathbf{R}^d} |b(t,x)|^p P_t(dx) < \infty \qquad (1.5)
$$

for some b for which (1.1) holds. We call (1.5) the **generalized finite energy condition** for $\{P_t(dx)\}_{0\le t\le 1}$.

Let $L(t,x;u):[0,1]\times \mathbf{R}^d\times \mathbf{R}^d \mapsto [0,\infty)$ be continuous and be convex in u. Let $\mathcal{A}$ denote the set of all $\mathbf{R}^d$-valued, continuous semimartingales $\{X(t)\}_{0\le t\le 1}$ on a complete filtered probability space such that there exists a Borel measurable $\beta_X:[0,1]\times C([0,1])\mapsto \mathbf{R}^d$ for which
(i) $\omega\mapsto\beta_X(t,\omega)$ is $\mathcal{B}(C([0,t]))_+$-measurable for all $t\in[0,1]$, where $\mathcal{B}(C([0,t]))$ denotes the Borel σ-field of $C([0,t])$ and $\mathcal{B}(C([0,t]))_+$ denotes the left hand side limit of $t\mapsto\mathcal{B}(C([0,t]))$,
(ii) $\{W_X(t):=X(t)-X(0)-\int_0^t\beta_X(s,X)ds\}_{0\le t\le 1}$ is a $\sigma[X(s):0\le s\le t]$-Wiener process.
For P_0 and $P_1\in\mathcal{M}_1(\mathbf{R}^d)$, put

$$\begin{aligned}V(P_0,P_1) \;:=\; &\inf\Big\{E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big]\Big|\\ &\qquad PX(t)^{-1}=P_t(t=0,1), X\in\mathcal{A}\Big\}, \qquad (1.6)\end{aligned}$$

$$\begin{aligned}&v(P_0,P_1) \qquad (1.7)\\ :=\;&\inf\Big\{\int_0^1\int_{\mathbf{R}^d}L(t,x;b(t,x))P(t,dx)dt\Big|P(t,dx)=P_t(dx)(t=0,1),\\ &\{P(t,dx)\}_{0\le t\le 1}\subset\mathcal{M}_1(\mathbf{R}^d),(b(t,x),P(t,dx)) \text{ satisfies (1.1)}\Big\}.\end{aligned}$$

In [12] where $u\mapsto L$ is quadratic, we proved and used the following:

$$V(P_0,P_1)=v(P_0,P_1). \qquad (1.8)$$

Remark 1.2 *As a typical case, when $L=|u|^2$, the minimizer of $V(P_0,P_1)$ is known to be the h-path process for the space-time Brownian motion (see [7, 18] and the references therein). It is known that its zero-noise limit exists and is the unique minimizer of Monge's problem (see [16, 19]).*

In this paper we prove (1.8) for a more general function L by the duality theorem for V. To make the point clearer, we describe [18] briefly. For P_0 and $P_1\in\mathcal{M}_1(\mathbf{R}^d)$, put

$$\mathcal{V}(P_0,P_1):=\sup\Big\{\int_{\mathbf{R}^d}\varphi(1,y)P_1(dy)-\int_{\mathbf{R}^d}\varphi(0,x)P_0(dx)\Big\}, \qquad (1.9)$$

where the supremum is taken over all classical solutions φ to the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \frac{\partial \varphi(t,x)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + H(t,x;D_x\varphi(t,x)) &= 0((t,x) \in (0,1) \times \mathbf{R}^d) \quad (1.10) \\ \varphi(1,\cdot) &\in C_b^\infty(\mathbf{R}^d) \end{aligned}$$

(see Lemma 3.1). Here for $(t,x,z) \in [0,1] \times \mathbf{R}^d \times \mathbf{R}^d$,

$$H(t,x;z) := \sup_{u \in \mathbf{R}^d} \{< z, u > -L(t,x;u)\}. \quad (1.11)$$

The following was proved in [18] and is called the duality theorem for the stochastic optimal control problem (1.6).

Theorem 1.1 (Duality Theorem) *Suppose that (A.1)-(A.4) in section 2 hold. Then for any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(P_0, P_1) = \mathcal{V}(P_0, P_1)(\in [0, \infty]). \quad (1.12)$$

Suppose in addition that $V(P_0, P_1)$ is finite. Then $V(P_0, P_1)$ has a minimizer and for any minimizer $\{X(t)\}_{0 \le t \le 1}$ of $V(P_0, P_1)$,

$$\beta_X(t, X) = b_X(t, X(t)) := E[\beta_X(t, X)|(t, X(t))]. \quad (1.13)$$

Remark 1.3 *(1.12) can be considered as a counterpart in the stochastic optimal control theory of the duality theorem in the Monge-Kantorovich problem (see [10, 17, 21, 22] and the references therein).*

Using a similar result to (1.8) on small time intervals $\subset [0,1]$, we prove that for $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P}), \quad (1.14)$$

where

$$\mathbf{V}(\mathbf{P}) := \inf\left\{ E\left[\int_0^1 L(t, X(t); \beta_X(t, X))dt\right] \middle| PX(t)^{-1} = P_t(0 \le t \le 1), X \in \mathcal{A} \right\}, \quad (1.15)$$

$$\mathbf{v}(\mathbf{P}) := \inf\Big\{\int_0^1 dt \int_{\mathbf{R}^d} L(t,x;b(t,x))P_t(dx)|b \text{ satisfies (1.1)}\Big\}. \qquad (1.16)$$

In particular, the existence of a minimizer of $\mathbf{V}(\mathbf{P})$ implies that of a semimartingale for which (1.2)-(1.3) hold. When $p = 2$ in (1.5), this semimartingale is Markovian. But we do not know if it is also true even when $1 < p < 2$. This is our future problem.

In section 2 we state our result which will be proved in section 4. Technical lemmas are given in section 3.

I would like to dedicate this paper to Professor Wendell H. Fleming on the occasion of his seventy seventh birthday. I would like to thank him for his constant encouragement since I was a student of his.

2 Main result.

In this section we state our result.

We state assumptions on L.
(A.1). There exists $p > 1$ such that

$$\liminf_{|u|\to\infty} \frac{\inf\{L(t,x;u) : (t,x) \in [0,1] \times \mathbf{R}^d\}}{|u|^p} > 0.$$

(A.2).

$$\Delta L(\varepsilon_1, \varepsilon_2) := \sup \frac{L(t,x;u) - L(s,y;u)}{1 + L(s,y;u)} \to 0 \quad \text{as } \varepsilon_1,\ \varepsilon_2 \to 0,$$

where the supremum is taken over all (t,x) and (s,y), $\in [0,1] \times \mathbf{R}^d$, for which $|t-s| \le \varepsilon_1$, $|x-y| < \varepsilon_2$ and all $u \in \mathbf{R}^d$.
(A.3). (i)$L(t,x;u) \in C^3([0,1] \times \mathbf{R}^d \times \mathbf{R}^d : [0,\infty))$,
(ii) $D_u^2 L(t,x;u)$ is positive definite for all $(t,x,u) \in [0,1] \times \mathbf{R}^d \times \mathbf{R}^d$,
(iii) $\sup\{L(t,x;o) : (t,x) \in [0,1] \times \mathbf{R}^d\}$ is finite,
(iv) $|D_x L(t,x;u)|/(1 + L(t,x;u))$ is bounded,
(v) $\sup\{|D_u L(t,x;u)| : (t,x) \in [0,1] \times \mathbf{R}^d, |u| \le R\}$ is finite for all $R > 0$.
(A.4). (i) $\Delta L(0,\infty)$ is finite, or (ii) $p = 2$ in (A.1).

Remark 2.1 *(i). (A.3, ii) implies that $L(t,x;u)$ is strictly convex in u. (ii). $(1 + |u|^2)^{p/2}$ $(p > 1)$ satisfies (A.1)-(A.3) and (A.4,i).*

We state that (1.8) holds.

Theorem 2.1 *Suppose that (A.1)-(A.4) hold. Then for any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(P_0, P_1) = v(P_0, P_1)(\in [0, \infty]). \tag{2.1}$$

The following is our main result (see (1.15)-(1.16) for notations).

Theorem 2.2 *Suppose that (A.1)-(A.4) hold. Then*
(i) for any $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P})(\in [0, \infty]). \tag{2.2}$$

(ii) For any $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1}, \subset \mathcal{M}_1(\mathbf{R}^d)$, for which $\mathbf{v}(\mathbf{P})$ is finite, there exist a unique minimizer $b_o(t, x)$ of $\mathbf{v}(\mathbf{P})$ and a minimizer $X \in \mathcal{A}$, of $\mathbf{V}(\mathbf{P})$. In particular, for any minimizer $X \in \mathcal{A}$, of $\mathbf{V}(\mathbf{P})$,

$$\beta_X(t, X) = b_o(t, X(t)) \tag{2.3}$$

and (1.2)-(1.3) with $b = b_o$ hold.

Remark 2.2 *If $\mathbf{v}(\mathbf{P})$ is finite, then the generalized finite energy condition (1.5) holds from (A.1).*

3 Lemmas.

In this section we give technical lemmas.

In the same way as to $\mathcal{A}$, we define the set of semimartingales $\mathcal{A}_t$ in $C([t, 1])$. We recall the following result.

Lemma 3.1 ([8, p. 210, Remark 11.2]) *Suppose that (A.1) and (A.3) hold. Then for any $f \in C_b^\infty(\mathbf{R}^d)$, the HJB equation (1.10) with $\varphi(1, \cdot) = f$ has a unique solution $\varphi \in C^{1,2}([0, 1] \times \mathbf{R}^d) \cap C_b^{0,1}([0, 1] \times \mathbf{R}^d)$, which can be written as follows:*

$$\begin{aligned} \varphi(t, x) \quad = \quad & \sup_{X \in \mathcal{A}_t} \Big\{ E[f(X(1)) | X(t) = x] \\ & -E\Big[\int_t^1 L(s, X(s); \beta_X(s, X)) ds \Big| X(t) = x \Big] \Big\}, \end{aligned} \tag{3.1}$$

where for the maximizer $X \in \mathcal{A}_t$, the following holds:

$$\beta_X(s, X) = D_z H(s, X(s); D_x \varphi(s, X(s))).$$

Fix $P_0 \in \mathcal{M}_1(\mathbf{R}^d)$. For $f \in C_b(\mathbf{R}^d)$, put

$$V^*(f) := \sup_{P \in \mathcal{M}_1(\mathbf{R}^d)} \Big\{ \int_{\mathbf{R}^d} f(x) P(dx) - V(P_0, P) \Big\}, \tag{3.2}$$

$$v^*(f) := \sup_{P \in \mathcal{M}_1(\mathbf{R}^d)} \Big\{ \int_{\mathbf{R}^d} f(x) P(dx) - v(P_0, P) \Big\}. \tag{3.3}$$

The following lemma plays a crucial role in the proof of Theorem 2.1.

Lemma 3.2 *(i) Suppose that (A.3, i, ii) hold. Then for any Q_0 and $Q_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(Q_0, Q_1) \geq v(Q_0, Q_1). \tag{3.4}$$

(ii) Suppose in addition that (A.1) and (A.3) hold. Then for any $f \in C_b^\infty(\mathbf{R}^d)$,

$$V^*(f) \geq v^*(f). \tag{3.5}$$

(Proof) We first prove (i). For $X \in \mathcal{A}$ for which $E[\int_0^1 L(t, X(t); \beta_X(t, X)) dt]$ is finite and for which $PX(t)^{-1} = Q_t$ $(t = 0, 1)$, $(b_X(t, x), P(X(t) \in dx))$ satisfies (1.1) with $(b(t, x), P_t(dx)) = (b_X(t, x), P(X(t) \in dx))$ (see (1.13) for notation). Indeed, for any $f \in C_b^{1,2}([0,1] \times \mathbf{R}^d)$ and $t \in [0,1]$, by Itô's formula,

$$\begin{aligned}
& \int_{\mathbf{R}^d} f(t, x) P(X(t) \in dx) - \int_{\mathbf{R}^d} f(0, x) P(X(0) \in dx) \\
= & E[f(t, X(t)) - f(0, X(0))] \\
= & \int_0^t ds E\Big[\frac{\partial f(s, X(s))}{\partial s} + \frac{1}{2} \triangle f(s, X(s)) + < \beta_X(s, X), D_x f(s, X(s)) > \Big] \\
= & \int_0^t ds E\Big[\frac{\partial f(s, X(s))}{\partial s} + \frac{1}{2} \triangle f(s, X(s)) + < b_X(s, X(s)), D_x f(s, X(s)) > \Big] \\
= & \int_0^t ds \int_{\mathbf{R}^d} \Big(\frac{\partial f(s, x)}{\partial s} + \frac{1}{2} \triangle f(s, x) + < b_X(s, x), D_x f(s, x) > \Big) P(X(s) \in dx).
\end{aligned} \tag{3.6}$$

Hence, from Remark 2.1, (i), by Jensen's inequality,

$$\begin{aligned}
& E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big] \qquad (3.7)\\
\geq\ & E\Big[\int_0^1 L(t,X(t);b_X(t,X(t)))dt\Big]\\
=\ & \int_0^1 dt\int_{\mathbf{R}^d} L(t,x;b_X(t,x))P(X(t)\in dx)\geq v(Q_0,Q_1).
\end{aligned}$$

Next we prove (ii). For φ in (3.1) and $\{(b(t,x),P(t,dx))\}_{0\leq t\leq 1}$ for which $\{P(t,dx)\}_{0\leq t\leq 1}\subset\mathcal{M}_1(\mathbf{R}^d)$ and (1.1) with $P(0,dx)=P_0$ holds,

$$\int_{\mathbf{R}^d} f(x)P(1,dx)-\int_{\mathbf{R}^d}\varphi(0,x)P_0(dx)\leq\int_0^1 dt\int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx). \qquad (3.8)$$

Indeed, take $\psi\in C_o^\infty(\mathbf{R}^d:[0,\infty))$ for which $\psi(x)=1$ ($|x|\leq 1$) and $\psi(x)=0$ ($|x|\geq 2$), and put $\psi_R(x):=\psi(x/R)$ for $R>0$. Then from (1.1),

$$\begin{aligned}
& \int_{\mathbf{R}^d}\psi_R(x)f(x)P(1,dx)-\int_{\mathbf{R}^d}\psi_R(x)\varphi(0,x)P(0,dx) \qquad (3.9)\\
=\ & \int_0^1 dt\int_{\mathbf{R}^d}\psi_R(x)\Big[\frac{\partial\varphi(t,x)}{\partial t}+\frac{1}{2}\triangle\varphi(t,x)+<b(t,x),D_x\varphi(t,x)>\Big]P(t,dx)\\
& +\int_0^1 dt\int_{\mathbf{R}^d}\Big[<D_x\psi_R(x),D_x\varphi(t,x)>+\frac{1}{2}\triangle\psi_R(x)\varphi(t,x)\\
& \qquad +<b(t,x),D_x\psi_R(x)>\varphi(t,x)\Big]P(t,dx).
\end{aligned}$$

Let $R\to\infty$. Then we obtain (3.8) from (1.10), (A.1) and Lemma 3.1.

Lemma 3.1 and (3.8) implies (ii). Indeed,

$$\begin{aligned}
v^*(f)\ =\ & \sup\Big\{\int_{\mathbf{R}^d} f(x)P(1,dx)-\int_0^1 dt\int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx)| \qquad (3.10)\\
& P(0,dx)=P_0(dx),\{P(t,dx)\}_{0\leq t\leq 1}\subset\mathcal{M}_1(\mathbf{R}^d),\\
& (b(t,x),P(t,dx))\text{ satisfies (1.1).}\Big\}\\
\leq\ & \int_{\mathbf{R}^d}\varphi(0,x)P_0(dx)\quad\text{(from (3.8))}
\end{aligned}$$

$$
\begin{aligned}
&= \sup\Big\{E\Big[f(X(1)) - \int_0^1 L(t, X(t); \beta_X(t, X))dt\Big]\Big| \\
&\qquad PX(0)^{-1} = P_0, X \in \mathcal{A}\Big\} \quad \text{(from Lemma 3.1)} \\
&= V^*(f). \square
\end{aligned}
$$

Let $(\Omega, \mathbf{B}, \{\mathbf{B}_t\}_{t\geq 0}, P)$ be a complete filtered probability space, X_o be a $(\mathbf{B}_0)$-adapted random variable, and $\{W(t)\}_{t\geq 0}$ denote a d-dimensional $(\mathbf{B}_t)$-Wiener process for which $W(0) = o$ (see e.g., [11]). For a $\mathbf{R}^d$-valued, $(\mathbf{B}_t)$-progressively measurable stochastic process $\{u(t)\}_{0\leq t\leq 1}$, put

$$
X^u(t) = X_o + \int_0^t u(s)ds + W(t) \quad (t \in [0, 1]). \tag{3.11}
$$

Then the following is known.

Lemma 3.3 *Suppose that $E[\int_0^1 |u(t)|dt]$ is finite. Then $\{X^u(t)\}_{0\leq t\leq 1} \in \mathcal{A}$ and*

$$
\beta_{X^u}(t, X^u) = E[u(t)|X^u(s), 0 \leq s \leq t] \tag{3.12}
$$

(see [11, p. 270]). Besides, by Jensen's inequality,

$$
E\Big[\int_0^1 L(t, X^u(t); u(t))dt\Big] \geq E\Big[\int_0^1 L(t, X^u(t); \beta_{X^u}(t, X^u))dt\Big]. \tag{3.13}
$$

For $\mathbf{P} := \{P_t(dx)\}_{0\leq t\leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and $n \geq 1$, put

$$
\begin{aligned}
V_n(\mathbf{P}) &:= \inf\Big\{E\Big[\int_0^1 L(t, X(t); \beta_X(t, X))dt\Big]\Big| \qquad (3.14) \\
&\qquad PX(t)^{-1} = P_t \left(t = \frac{i}{2^n}, i = 0, \cdots, 2^n\right), X \in \mathcal{A}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
v_n(\mathbf{P}) &:= \inf\Big\{\int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x))P(t, dx)\Big| \qquad (3.15) \\
&\qquad P(t, dx) = P_t(dx) \left(t = \frac{i}{2^n}, i = 0, \cdots, 2^n\right), \\
&\qquad \{P(t, dx)\}_{0\leq t\leq 1} \subset \mathcal{M}(\mathbf{R}^d), (b(t, x), P(t, dx)) \text{ satisfies } (1.1)\Big\}.
\end{aligned}
$$

Then we have

Lemma 3.4 *Suppose that (A.1)-(A.4) hold. Then for any* $\mathbf{P} := \{P_t(dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ *and* $n \ge 1$,

$$v_n(\mathbf{P}) = V_n(\mathbf{P}). \tag{3.16}$$

(Proof) For $i = 0, \cdots, 2^n - 1$, put

$$\begin{aligned} V_{n,i}(\mathbf{P}) \quad := \quad & \inf\Big\{E\Big[\int_0^{\frac{1}{2^n}} L(t, X(t); \beta_X(t, X))dt\Big]\Big| \\ & PX(t)^{-1} = P_{t+\frac{i}{2^n}}\ \Big(t = 0, \frac{1}{2^n}\Big), X \in \mathcal{A}\Big\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} & v_{n,i}(\mathbf{P}) \\ := \quad & \inf\Big\{\int_0^{\frac{1}{2^n}} dt \int_{\mathbf{R}^d} L(t, x; b(t, x))P(t, dx)\Big| \\ & P(t, dx) = P_{t+\frac{i}{2^n}}(dx)\ \Big(t = 0, \frac{1}{2^n}\Big), \{P(t, dx)\}_{0\le t\le \frac{1}{2^n}} \subset \mathcal{M}(\mathbf{R}^d), \\ & (b(t, x), P(t, dx)) \text{ satisfies (1.1) on } [0, 1/2^n]\Big\}. \end{aligned} \tag{3.18}$$

Then, from Theorem 2.1,

$$v_n(\mathbf{P}) = \sum_{i=0}^{2^n-1} v_{n,i}(\mathbf{P}) = \sum_{i=0}^{2^n-1} V_{n,i}(\mathbf{P}). \tag{3.19}$$

Since $V_n(\mathbf{P}) \ge v_n(\mathbf{P})$ from (3.6)-(3.7), we only have to prove the following:

$$\sum_{i=0}^{2^n-1} V_{n,i}(\mathbf{P}) \ge V_n(\mathbf{P}). \tag{3.20}$$

Suppose that the left hand side of (3.20) is finite. For $i = 0, \cdots 2^n - 1$, take a minimizer $X_{n,i}$ of $V_{n,i}(\mathbf{P})$ (see Theorem 1.1), and put

$$P_{n,i} := PX_{n,i}\Big(\cdot - \frac{i}{2^n}\Big)^{-1} \quad \text{on } (C([\tfrac{i}{2^n}, \tfrac{i+1}{2^n}] : \mathbf{R}^d), \mathcal{B}(C([\tfrac{i}{2^n}, \tfrac{i+1}{2^n}] : \mathbf{R}^d))), \tag{3.21}$$

$$P_n\Big(dX|_{C([0,1]:\mathbf{R}^d)}\Big) := P_{n,0}\Big(dX|_{C([0,\frac{1}{2^n}]:\mathbf{R}^d)}\Big) \times\Pi_{i=1}^{2^n-1}P_{n,i}\Big(dX|_{C([\frac{i}{2^n},\frac{i+1}{2^n}]:\mathbf{R}^d)}\Big|X_{n,i}\Big(\frac{i}{2^n}\Big)=X\Big(\frac{i}{2^n}\Big)\Big) \tag{3.22}$$

on $(C([0,1]:\mathbf{R}^d),\mathcal{B}(C([0,1]:\mathbf{R}^d)))$. Under the completion of this measure, the coordinate process $\{X_n(t)\}_{0\le t\le 1}$ satisfies the following:

$$X_n(t)=X_n(0)+\sum_{i=0}^{2^n-1}\int_{\min(\frac{i}{2^n},t)}^{\min(\frac{i+1}{2^n},t)}b_{n,i}\Big(s-\frac{i}{2^n},X_n(s)\Big)ds+W_{X_n}(t) \quad (0\le t\le 1), \tag{3.23}$$

where $b_{n,i}$ denotes the drift vector of $X_{n,i}$ (see Theorem 1.1). In particular, $PX_n(t)^{-1}=P_t$ $(t=i/2^n, i=0,\cdots,2^n)$, which implies (3.20). □

4 Proofs.

In this section we prove our results given in section 2.

When $L=|u|^2$, the following proof extremely simplifies that of [12, Lemma 2.5].

(Proof of Theorem 2.1). Lemma 3.2, (i) and the following complete the proof:

$$\begin{aligned} & v(P_0,P_1) \\ \ge & \sup_{f\in C_b^\infty(\mathbf{R}^d)}\Big\{\int_{\mathbf{R}^d}f(x)P_1(dx)-v^*(f)\Big\} \quad \text{(from (3.3))} \\ \ge & \sup_{f\in C_b^\infty(\mathbf{R}^d)}\Big\{\int_{\mathbf{R}^d}f(x)P_1(dx)-V^*(f)\Big\} \quad \text{(from Lemma 3.2, (ii))} \\ = & V(P_0,P_1) \quad \text{(from Theorem 1.1 (see (3.10)))}.\Box \end{aligned} \tag{4.1}$$

(Proof of Theorem 2.2). We first prove (i). From (3.6)-(3.7), $\mathbf{V}(\mathbf{P})\ge\mathbf{v}(\mathbf{P})$. Therefore we only have to show that

$$\mathbf{v}(\mathbf{P})\ge\mathbf{V}(\mathbf{P}). \tag{4.2}$$

Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then, from Lemma 3.4,

$$\mathbf{v}(\mathbf{P}) \geq v_n(\mathbf{P}) = V_n(\mathbf{P}) \tag{4.3}$$

and X_n constructed in (3.23) is a minimizer of $V_n(\mathbf{P})$.

Let b_n denote the drift vector of $\{X_n(t)\}_{0\leq t\leq 1}$. It is easy to see that $\{(X_n(t), \int_0^t b_n(s, X_n(s))ds) : t \in [0,1]\}_{n\geq 1}$ is tight in $C([0,1] : \mathbf{R}^{2d})$ from (A.1) (see [23, Theorem 3] or [9]). Take a weakly convergent subsequence $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s))ds) : t \in [0,1]\}_{k\geq 1}$ such that

$$\begin{aligned} & \liminf_{n\to\infty} E\Big[\int_0^1 L(t, X_n(t); b_n(s, X_n(s)))dt\Big] \\ = & \lim_{k\to\infty} E\Big[\int_0^1 L(t, X_{n_k}(t); b_{n_k}(s, X_{n_k}(s)))dt\Big]. \end{aligned} \tag{4.4}$$

Let $\{(X(t), A(t))\}_{t\in[0,1]}$ denote the limit of $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s))ds) : t \in [0,1]\}_{k\geq 1}$ as $k \to \infty$. Then $\{X(t)-X(0)-A(t)\}_{t\in[0,1]}$ is a $\sigma[X(s) : 0 \leq s \leq t]$-Wiener process and $\{A(t)\}_{t\in[0,1]}$ is absolutely continuous (see [23, Theorem 5] or [9]). We can also prove, in the same way as in the proof of [15, (3.17)], the following: from (4.3)-(4.4), (A.2) and (A.3, ii) (see Remark 2.1, (i)),

$$\begin{aligned} \mathbf{v}(\mathbf{P}) & \geq \liminf_{n\to\infty} E\Big[\int_0^1 L(t, X_n(t); b_n(t, X_n(t)))dt\Big] \\ & \geq E\Big[\int_0^1 L\Big(t, X(t); \frac{dA(t)}{dt}\Big)dt\Big] \\ & \geq \tilde{E}\Big[\int_0^1 L\Big(t, X(t); \beta_X(t, X)\Big)dt\Big] \quad \text{(from Lemma 3.3)} \\ & \geq \mathbf{V}(\mathbf{P}). \end{aligned} \tag{4.5}$$

Here $\tilde{E}$ denotes the mean value by the completion of $PX(\cdot)^{-1}$ and we used the fact that $P(X(t) \in dx) = P_t(dx)$ for all $t \in [0,1]$. Indeed,

$$P(X(t) \in dx) = \lim_{n\to\infty} P\Big(X\Big(\frac{[2^n t]}{2^n}\Big) \in dx\Big) \quad \text{weakly,}$$

$$P\Big(X\Big(\frac{[2^n t]}{2^n}\Big) \in dx\Big) = P_{\frac{[2^n t]}{2^n}}(dx) \to P_t(dx) \quad \text{as } n \to \infty \text{ weakly.}$$

Next we prove (ii). Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then (2.2) and (4.5) show the existence of a minimizer X of $\mathbf{V}(\mathbf{P})$. In the same way as in (3.7), Theorem 2.2, (i) and the strict convexity of $u \mapsto L(t,x;u)$ (see Remark 2.1, (i)) imply that $\beta_X(t,X) = b_X(t,X(t))$ and $b_X(t,x)$ is a minimizer of $\mathbf{v}(\mathbf{P})$.

Let b_1 and b_2 be minimizers of $\mathbf{v}(\mathbf{P})$. Then for any $\lambda \in (0,1)$, $\lambda b_1(t,x) + (1-\lambda)b_2(t,x)$ satisfies (1.1), and

$$\begin{aligned} & \mathbf{v}(\mathbf{P}) \qquad (4.6) \\ \le\ & \int_0^1 \int_{\mathbf{R}^d} L(t,x;\lambda b_1(t,x) + (1-\lambda)b_2(t,x))P_t(dx) \\ \le\ & \lambda \int_0^1 \int_{\mathbf{R}^d} L(t,x;b_1(t,x))P_t(dx) + (1-\lambda)\int_0^1 \int_{\mathbf{R}^d} L(t,x;b_2(t,x))P_t(dx) \\ =\ & \mathbf{v}(\mathbf{P}). \end{aligned}$$

The strict convexity of $u \mapsto L(t,x;u)$ implies the uniqueness of a minimizer of $\mathbf{v}(\mathbf{P})$.□

References

[1] Blumenthal RM, Corson HH (1972) On continuous collections of measures, in: Le Cam, L., Neyman, J., Scott, E.L. (Eds.), Proc. 6th Berkeley Sympos. Math. Statist. Probab. **2**, Berkeley 1970/1971. Univ. California press, Berkeley, pp. 33-40

[2] Carlen EA (1984) Conservative diffusions. Commun Math Phys 94:293-315

[3] Cattiaux P, Léonard C (1994) Minimization of the Kullback information of diffusion processes. Ann Inst H Poincar Probab Statist 30:83-132

[4] Cattiaux P, Léonard C (1995) Correction to: "Minimization of the Kullback information of diffusion processes" [Ann. Inst. H. Poincar Probab. Statist. 30 (1994), no. 1, 83–132]. Ann Inst H Poincar Probab Statist 31:705–707

[5] Cattiaux P, Léonard C (1995) Large deviations and Nelson processes. Forum Math 7:95–115

[6] Cattiaux P, Léonard C (1996) Minimization of the Kullback information for some Markov processes, in: Azema, J., Emery, M., Yor, M. (Eds.), Séminaire de Probabilités, XXX, Lecture Notes in Math., Vol. 1626. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, pp. 288–311

[7] Doob JL (1984) Classical potential theory and its probabilistic counterpart. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo

[8] Fleming WH, Soner HM (1993) Controlled Markov Processes and Viscosity Solutions. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo

[9] Jacod J, Shiryaev AN (1987) Limit theorems for stochastic processes. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo

[10] Kellerer HG (1984) Duality theorems for marginal problems. Z Wahrsch Verw Gebiete 67:399–432

[11] Liptser RS, Shiryaev AN (1977) Statistics of random processes I. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo

[12] Mikami T (1990) Variational processes from the weak forward equation. Commun Math Phys 135:19-40

[13] Mikami T (1999) Markov marginal problems and their applications to Markov optimal control, in: McEneaney, W. M., Yin, G. G., Zhang, Q. (Eds.), Stochastic Analysis, Control, Optimization and Applications, A Volume in Honor of W. H. Fleming. Birkhäuser, Boston, pp. 457-476

[14] Mikami T (2000) Dynamical systems in the variational formulation of the Fokker-Planck equation by the Wasserstein metric. Appl Math Optim 42:203-227

[15] Mikami T (2002) Optimal control for absolutely continuous stochastic processes and the mass transportation problem. Elect Comm in Probab 7:199-213

[16] Mikami T (2004) Monge's problem with a quadratic cost by the zero-noise limit of h-path processes. Probab Theory Related Fields 129:245-260

[17] Mikami T (2004) A Simple Proof of Duality Theorem for Monge-Kantorovich Problem. To appear in Kodai Math J

[18] Mikami T, Thieullen M (2004) Duality Theorem for Stochastic Optimal Control Problem. Hokkaido University preprint series, #652

[19] Mikami T, Thieullen M (2005) Optimal Transportation Problem by Stochastic Optimal Control. Hokkaido University preprint series, #690

[20] Nelson E (1967) Dynamical theories of Brownian motion. Princeton University Press, Princeton

[21] Rachev ST, Rüschendorf L (1998) Mass transportation problems, Vol. I: Theory, Vol. II: Application. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo

[22] Villani C (2003) Topics in Optimal Transportation. Amer Math Soc, Providence.

[23] Zheng WA (1985) Tightness results for laws of diffusion processes application to stochastic mechanics. Ann Inst Henri Poincaré 21:103-124