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Semimartingales from the Fokker-Planck equation

Dedicated to Professor Wendell H. Fleming on the occasion of his seventy seventh birthday

> Toshio Mikami^{*} Hokkaido University

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Abstract

We show the existence of a semimartingale of which one-dimensional marginal distributions are given by the solution of the Fokker-Planck equation with the *p*-th integrable drift vector (p > 1).

AMS (MOS) SUBJECT CLASSIFICATION NUMBERS: 93E20 A shortened version of the title: Semimartingales from the FP equation

1 Introduction.

Let $\mathcal{M}_1(\mathbf{R}^d)$ denote the complete separable metric space, with a weak topology, of Borel probability measures on \mathbf{R}^d $(d \ge 1)$.

Let $b : [0,1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ be measurable and $\{P_t(dx)\}_{0 \le t \le 1}, \subset \mathcal{M}_1(\mathbf{R}^d),$ satisfy the following Fokker-Planck equation: for $f \in C_b^{1,2}([0,1] \times \mathbf{R}^d)$ and $t \in [0,1],$

^{*}Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan; mikami@math.sci.hokudai.ac.jp; phone & fax no. 81/11/706/3444; partially supported by the Grant-in-Aid for Scientific Research, No. 15340047, 15340051 and 16654031, JSPS.

$$\int_{\mathbf{R}^d} f(t,x) P_t(dx) - \int_{\mathbf{R}^d} f(0,x) P_0(dx)$$

$$= \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{\partial f(s,x)}{\partial s} + \frac{1}{2} \Delta f(s,x) + \langle b(s,x), D_x f(s,x) \rangle \right) P_s(dx),$$
(1.1)

where $\Delta := \sum_{i=1}^{d} \partial^2 / \partial x_i^2$, $D_x := (\partial / \partial x_i)_{i=1}^d$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbf{R}^d .

Inspired by Born's probabilistic interpretation of a solution to Schrödinger's equation, Nelson proposed the problem of the construction of a diffusion process $\{X(t)\}_{0 \le t \le 1}$ for which the following holds (see [20]):

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + W(t) \quad (t \in [0, 1]), \quad (1.2)$$

$$P(X(t) \in dx) = P_t(dx) \quad (t \in [0, 1]),$$
(1.3)

where $\{W(t)\}_{0 \le t \le 1}$ is a $\sigma[X(s) : 0 \le s \le t]$ -Wiener process.

The first result was given by Carlen [2] (see also [23]). It was generalized, by Mikami [12], to the case where the second order differential operator has a variable coefficient. The further generalization and almost complete resolution was made by Cattiaux and Léonard [3-6] (see also [1, 13-15] for the related topics). But in these papers, they assumed that

$$\int_{0}^{1} dt \int_{\mathbf{R}^{d}} |b(t,x)|^{2} P_{t}(dx) < \infty$$
(1.4)

for some b for which (1.1) holds. This is called the **finite energy condition** for $\{P_t(dx)\}_{0 \le t \le 1}$.

Remark 1.1 It is known that b is not unique for $\{P_t(dx)\}_{0 \le t \le 1}$ in (1.1) (see [12] or [3-6]).

In this paper we consider Nelson's problem under a weaker assumption than (1.4): there exists p > 1 such that

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t,x)|^p P_t(dx) < \infty$$
(1.5)

for some b for which (1.1) holds. We call (1.5) the generalized finite energy condition for $\{P_t(dx)\}_{0 \le t \le 1}$.

Let $L(t, x; u) : [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto [0, \infty)$ be continuous and be convex in u. Let \mathcal{A} denote the set of all \mathbf{R}^d -valued, continuous semimartingales $\{X(t)\}_{0 \le t \le 1}$ on a complete filtered probability space such that there exists a Borel measurable $\beta_X : [0, 1] \times C([0, 1]) \mapsto \mathbf{R}^d$ for which (i) $\omega \mapsto \beta_X(t, \omega)$ is $\mathcal{B}(C([0, t]))_+$ -measurable for all $t \in [0, 1]$, where $\mathcal{B}(C([0, t]))$ denotes the Borel σ -field of C([0, t]) and $\mathcal{B}(C([0, t]))_+$ denotes the left hand side limit of $t \mapsto \mathcal{B}(C([0, t]))$,

(ii) $\{W_X(t) := X(t) - X(0) - \int_0^t \beta_X(s, X) ds\}_{0 \le t \le 1}$ is a $\sigma[X(s) : 0 \le s \le t]$ -Wiener process. For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, put

$$V(P_0, P_1) := \inf \left\{ E\left[\int_0^1 L(t, X(t); \beta_X(t, X)) dt\right] \right|$$
$$PX(t)^{-1} = P_t(t = 0, 1), X \in \mathcal{A} \right\},$$
(1.6)

$$v(P_{0}, P_{1})$$

$$:= \inf \left\{ \int_{0}^{1} \int_{\mathbf{R}^{d}} L(t, x; b(t, x)) P(t, dx) dt \middle| P(t, dx) = P_{t}(dx)(t = 0, 1), \\ \{P(t, dx)\}_{0 \le t \le 1} \subset \mathcal{M}_{1}(\mathbf{R}^{d}), (b(t, x), P(t, dx)) \text{ satisfies } (1.1) \right\}.$$

$$(1.7)$$

In [12] where $u \mapsto L$ is quadratic, we proved and used the following:

$$V(P_0, P_1) = v(P_0, P_1).$$
(1.8)

Remark 1.2 As a typical case, when $L = |u|^2$, the minimizer of $V(P_0, P_1)$ is known to be the h-path process for the space-time Brownian motion (see [7, 18] and the references therein). It is known that its zero-noise limit exists and is the unique minimizer of Monge's problem (see [16, 19]).

In this paper we prove (1.8) for a more general function L by the duality theorem for V. To make the point clearer, we describe [18] briefly. For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, put

$$\mathcal{V}(P_0, P_1) := \sup\left\{\int_{\mathbf{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx)\right\}, \qquad (1.9)$$

where the supremum is taken over all classical solutions φ to the following Hamilton-Jacobi-Bellman equation:

$$\frac{\partial \varphi(t,x)}{\partial t} + \frac{1}{2} \Delta \varphi(t,x) + H(t,x; D_x \varphi(t,x)) = 0((t,x) \in (0,1) \times \mathbf{R}^d) (1.10)$$
$$\varphi(1,\cdot) \in C_b^{\infty}(\mathbf{R}^d)$$

(see Lemma 3.1). Here for $(t, x, z) \in [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d$,

$$H(t, x; z) := \sup_{u \in \mathbf{R}^d} \{ \langle z, u \rangle - L(t, x; u) \}.$$
 (1.11)

The following was proved in [18] and is called the duality theorem for the stochastic optimal control problem (1.6).

Theorem 1.1 (Duality Theorem) Suppose that (A.1)-(A.4) in section 2 hold. Then for any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,

$$V(P_0, P_1) = \mathcal{V}(P_0, P_1) (\in [0, \infty]). \tag{1.12}$$

Suppose in addition that $V(P_0, P_1)$ is finite. Then $V(P_0, P_1)$ has a minimizer and for any minimizer $\{X(t)\}_{0 \le t \le 1}$ of $V(P_0, P_1)$,

$$\beta_X(t,X) = b_X(t,X(t)) := E[\beta_X(t,X)|(t,X(t))].$$
(1.13)

Remark 1.3 (1.12) can be considered as a counterpart in the stochastic optimal control theory of the duality theorem in the Monge-Kantorovich problem (see [10, 17, 21, 22] and the references therein).

Using a similar result to (1.8) on small time intervals $\subset [0, 1]$, we prove that for $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d),$

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P}),\tag{1.14}$$

where

$$\mathbf{V}(\mathbf{P}) := \inf \left\{ E\left[\int_0^1 L(t, X(t); \beta_X(t, X)) dt\right] \middle| PX(t)^{-1} = P_t(0 \le t \le 1), X \in \mathcal{A} \right\},$$
(1.15)

$$\mathbf{v}(\mathbf{P}) := \inf \left\{ \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P_t(dx) | b \text{ satisfies } (1.1) \right\}.$$
(1.16)

In particular, the existence of a minimizer of $\mathbf{V}(\mathbf{P})$ implies that of a semimartingale for which (1.2)-(1.3) hold. When p = 2 in (1.5), this semimartingale is Markovian. But we do not know if it is also true even when 1 .This is our future problem.

In section 2 we state our result which will be proved in section 4. Technical lemmas are given in section 3.

I would like to dedicate this paper to Professor Wendell H. Fleming on the occasion of his seventy seventh birthday. I would like to thank him for his constant encouragement since I was a student of his.

2 Main result.

In this section we state our result.

We state assumptions on L.

(A.1). There exists p > 1 such that

$$\liminf_{|u| \to \infty} \frac{\inf \{ L(t, x; u) : (t, x) \in [0, 1] \times \mathbf{R}^d \}}{|u|^p} > 0$$

(A.2).

$$\Delta L(\varepsilon_1, \varepsilon_2) := \sup \frac{L(t, x; u) - L(s, y; u)}{1 + L(s, y; u)} \to 0 \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0,$$

where the supremum is taken over all (t, x) and $(s, y), \in [0, 1] \times \mathbf{R}^d$, for which $|t-s| \leq \varepsilon_1, |x-y| < \varepsilon_2$ and all $u \in \mathbf{R}^d$. (A.3). (i) $L(t, x; u) \in C^3([0, 1] \times \mathbf{R}^d \times \mathbf{R}^d : [0, \infty))$, (ii) $D_u^2 L(t, x; u)$ is positive definite for all $(t, x, u) \in [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d$, (iii) $\sup\{L(t, x; o) : (t, x) \in [0, 1] \times \mathbf{R}^d\}$ is finite, (iv) $|D_x L(t, x; u)|/(1 + L(t, x; u))$ is bounded, (v) $\sup\{|D_u L(t, x; u)| : (t, x) \in [0, 1] \times \mathbf{R}^d, |u| \leq R\}$ is finite for all R > 0. (A.4). (i) $\Delta L(0, \infty)$ is finite, or (ii) p = 2 in (A.1).

Remark 2.1 (i). (A.3, ii) implies that L(t, x; u) is strictly convex in u. (ii). $(1 + |u|^2)^{p/2}$ (p > 1) satisfies (A.1)-(A.3) and (A.4,i).

We state that (1.8) holds.

Theorem 2.1 Suppose that (A.1)-(A.4) hold. Then for any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,

$$V(P_0, P_1) = v(P_0, P_1) (\in [0, \infty]).$$
(2.1)

The following is our main result (see (1.15)-(1.16) for notations).

Theorem 2.2 Suppose that (A.1)-(A.4) hold. Then (i) for any $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d),$

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P}) (\in [0, \infty]). \tag{2.2}$$

(ii) For any $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1}, \subset \mathcal{M}_1(\mathbf{R}^d)$, for which $\mathbf{v}(\mathbf{P})$ is finite, there exist a unique minimizer $b_o(t, x)$ of $\mathbf{v}(\mathbf{P})$ and a minimizer $X \in \mathcal{A}$, of $\mathbf{V}(\mathbf{P})$. In particular, for any minimizer $X \in \mathcal{A}$, of $\mathbf{V}(\mathbf{P})$,

$$\beta_X(t, X) = b_o(t, X(t)) \tag{2.3}$$

and (1.2)-(1.3) with $b = b_o$ hold.

Remark 2.2 If $\mathbf{v}(\mathbf{P})$ is finite, then the generalized finite energy condition (1.5) holds from (A.1).

3 Lemmas.

In this section we give technical lemmas.

In the same way as to \mathcal{A} , we define the set of semimartingales \mathcal{A}_t in C([t, 1]). We recall the following result.

Lemma 3.1 ([8, p. 210, Remark 11.2]) Suppose that (A.1) and (A.3) hold. Then for any $f \in C_b^{\infty}(\mathbf{R}^d)$, the HJB equation (1.10) with $\varphi(1, \cdot) = f$ has a unique solution $\varphi \in C^{1,2}([0,1] \times \mathbf{R}^d) \cap C_b^{0,1}([0,1] \times \mathbf{R}^d)$, which can be written as follows:

$$\varphi(t,x) = \sup_{X \in \mathcal{A}_t} \left\{ E[f(X(1))|X(t) = x] - E\left[\int_t^1 L(s,X(s);\beta_X(s,X))ds \middle| X(t) = x\right] \right\},$$
(3.1)

where for the maximizer $X \in A_t$, the following holds:

$$\beta_X(s,X) = D_z H(s,X(s); D_x \varphi(s,X(s)))$$

Fix $P_0 \in \mathcal{M}_1(\mathbf{R}^d)$. For $f \in C_b(\mathbf{R}^d)$, put

$$V^{*}(f) := \sup_{P \in \mathcal{M}_{1}(\mathbf{R}^{d})} \left\{ \int_{\mathbf{R}^{d}} f(x) P(dx) - V(P_{0}, P) \right\},$$
(3.2)

$$v^{*}(f) := \sup_{P \in \mathcal{M}_{1}(\mathbf{R}^{d})} \left\{ \int_{\mathbf{R}^{d}} f(x) P(dx) - v(P_{0}, P) \right\}.$$
 (3.3)

The following lemma plays a crucial role in the proof of Theorem 2.1.

Lemma 3.2 (i) Suppose that (A.3, i, ii) hold. Then for any Q_0 and $Q_1 \in \mathcal{M}_1(\mathbf{R}^d)$,

$$V(Q_0, Q_1) \ge v(Q_0, Q_1). \tag{3.4}$$

(ii) Suppose in addition that (A.1) and (A.3) hold. Then for any $f \in C_b^{\infty}(\mathbf{R}^d)$,

$$V^*(f) \ge v^*(f).$$
 (3.5)

(Proof) We first prove (i). For $X \in \mathcal{A}$ for which $E[\int_0^1 L(t, X(t); \beta_X(t, X))dt]$ is finite and for which $PX(t)^{-1} = Q_t$ (t = 0, 1), $(b_X(t, x), P(X(t) \in dx))$ satisfies (1.1) with $(b(t, x), P_t(dx)) = (b_X(t, x), P(X(t) \in dx))$ (see (1.13) for notation). Indeed, for any $f \in C_b^{1,2}([0, 1] \times \mathbf{R}^d)$ and $t \in [0, 1]$, by Itô's formula,

$$\int_{\mathbf{R}^d} f(t,x) P(X(t) \in dx) - \int_{\mathbf{R}^d} f(0,x) P(X(0) \in dx)$$

$$= E[f(t,X(t)) - f(0,X(0))]$$

$$= \int_0^t ds E \Big[\frac{\partial f(s,X(s))}{\partial s} + \frac{1}{2} \Delta f(s,X(s)) + \langle \beta_X(s,X), D_x f(s,X(s)) \rangle \Big]$$

$$= \int_0^t ds E \Big[\frac{\partial f(s,X(s))}{\partial s} + \frac{1}{2} \Delta f(s,X(s)) + \langle b_X(s,X(s)), D_x f(s,X(s)) \rangle \Big]$$

$$= \int_0^t ds \int_{\mathbf{R}^d} \Big(\frac{\partial f(s,x)}{\partial s} + \frac{1}{2} \Delta f(s,x) + \langle b_X(s,x), D_x f(s,x) \rangle \Big) P(X(s) \in dx).$$
(3.6)

Hence, from Remark 2.1, (i), by Jensen's inequality,

$$E\left[\int_{0}^{1} L(t, X(t); \beta_{X}(t, X))dt\right]$$

$$\geq E\left[\int_{0}^{1} L(t, X(t); b_{X}(t, X(t)))dt\right]$$

$$= \int_{0}^{1} dt \int_{\mathbf{R}^{d}} L(t, x; b_{X}(t, x))P(X(t) \in dx) \geq v(Q_{0}, Q_{1}).$$
(3.7)

Next we prove (ii). For φ in (3.1) and $\{(b(t, x), P(t, dx))\}_{0 \le t \le 1}$ for which $\{P(t, dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and (1.1) with $P(0, dx) = P_0$ holds,

$$\int_{\mathbf{R}^{d}} f(x)P(1,dx) - \int_{\mathbf{R}^{d}} \varphi(0,x)P_{0}(dx) \le \int_{0}^{1} dt \int_{\mathbf{R}^{d}} L(t,x;b(t,x))P(t,dx).$$
(3.8)

Indeed, take $\psi \in C_o^{\infty}(\mathbf{R}^d : [0, \infty))$ for which $\psi(x) = 1$ ($|x| \le 1$) and $\psi(x) = 0$ ($|x| \ge 2$), and put $\psi_R(x) := \psi(x/R)$ for R > 0. Then from (1.1),

$$\int_{\mathbf{R}^{d}} \psi_{R}(x) f(x) P(1, dx) - \int_{\mathbf{R}^{d}} \psi_{R}(x) \varphi(0, x) P(0, dx)$$
(3.9)
$$= \int_{0}^{1} dt \int_{\mathbf{R}^{d}} \psi_{R}(x) \Big[\frac{\partial \varphi(t, x)}{\partial t} + \frac{1}{2} \Delta \varphi(t, x) + \langle b(t, x), D_{x} \varphi(t, x) \rangle \Big] P(t, dx)$$
$$+ \int_{0}^{1} dt \int_{\mathbf{R}^{d}} \Big[\langle D_{x} \psi_{R}(x), D_{x} \varphi(t, x) \rangle + \frac{1}{2} \Delta \psi_{R}(x) \varphi(t, x)$$
$$+ \langle b(t, x), D_{x} \psi_{R}(x) \rangle \varphi(t, x) \Big] P(t, dx).$$

Let $R \to \infty$. Then we obtain (3.8) from (1.10), (A.1) and Lemma 3.1.

Lemma 3.1 and (3.8) implies (ii). Indeed,

$$v^{*}(f) = \sup \left\{ \int_{\mathbf{R}^{d}} f(x) P(1, dx) - \int_{0}^{1} dt \int_{\mathbf{R}^{d}} L(t, x; b(t, x)) P(t, dx) | (3.10) \right. \\ \left. P(0, dx) = P_{0}(dx), \{P(t, dx)\}_{0 \le t \le 1} \subset \mathcal{M}_{1}(\mathbf{R}^{d}), \\ \left. (b(t, x), P(t, dx)) \text{ satisfies } (1.1). \right\} \right\} \\ \le \int_{\mathbf{R}^{d}} \varphi(0, x) P_{0}(dx) \quad (\text{from } (3.8))$$

$$= \sup \left\{ E \left[f(X(1)) - \int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] \right|$$
$$PX(0)^{-1} = P_0, X \in \mathcal{A} \right\} \quad \text{(from Lemma 3.1)}$$
$$= V^*(f).\square$$

Let $(\Omega, \mathbf{B}, {\mathbf{B}_t}_{t\geq 0}, P)$ be a complete filtered probability space, X_o be a (\mathbf{B}_0) -adapted random variable, and ${W(t)}_{t\geq 0}$ denote a d-dimensional (\mathbf{B}_t) -Wiener process for which W(0) = o (see e.g., [11]). For a \mathbf{R}^d -valued, (\mathbf{B}_t) -progressively measurable stochastic process ${u(t)}_{0\leq t\leq 1}$, put

$$X^{u}(t) = X_{o} + \int_{0}^{t} u(s)ds + W(t) \quad (t \in [0, 1]).$$
(3.11)

Then the following is known.

Lemma 3.3 Suppose that $E[\int_0^1 |u(t)|dt]$ is finite. Then $\{X^u(t)\}_{0 \le t \le 1} \in \mathcal{A}$ and

$$\beta_{X^{u}}(t, X^{u}) = E[u(t)|X^{u}(s), 0 \le s \le t]$$
(3.12)

(see [11, p. 270]). Besides, by Jensen's inequality,

$$E\left[\int_{0}^{1} L(t, X^{u}(t); u(t))dt\right] \ge E\left[\int_{0}^{1} L(t, X^{u}(t); \beta_{X^{u}}(t, X^{u}))dt\right].$$
 (3.13)

For $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d) \text{ and } n \ge 1$, put

$$V_{n}(\mathbf{P}) := \inf \left\{ E\left[\int_{0}^{1} L(t, X(t); \beta_{X}(t, X)) dt\right] \right|$$

$$PX(t)^{-1} = P_{t} \left(t = \frac{i}{2^{n}}, i = 0, \cdots, 2^{n}\right), X \in \mathcal{A} \right\},$$
(3.14)

$$v_{n}(\mathbf{P}) := \inf \left\{ \int_{0}^{1} dt \int_{\mathbf{R}^{d}} L(t, x; b(t, x)) P(t, dx) \right|$$

$$P(t, dx) = P_{t}(dx) \left(t = \frac{i}{2^{n}}, i = 0, \cdots, 2^{n} \right),$$

$$\{P(t, dx)\}_{0 \le t \le 1} \subset \mathcal{M}(\mathbf{R}^{d}), (b(t, x), P(t, dx)) \text{ satisfies } (1.1) \right\}.$$
(3.15)

Then we have

Lemma 3.4 Suppose that (A.1)-(A.4) hold. Then for any $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d) \text{ and } n \ge 1,$

$$v_n(\mathbf{P}) = V_n(\mathbf{P}). \tag{3.16}$$

(Proof) For $i = 0, \dots, 2^n - 1$, put

$$V_{n,i}(\mathbf{P}) := \inf \left\{ E\left[\int_0^{\frac{1}{2^n}} L(t, X(t); \beta_X(t, X)) dt \right] \right|$$
$$PX(t)^{-1} = P_{t + \frac{i}{2^n}} \left(t = 0, \frac{1}{2^n} \right), X \in \mathcal{A} \right\}, \quad (3.17)$$

$$v_{n,i}(\mathbf{P})$$
(3.18)
:= $\inf \left\{ \int_{0}^{\frac{1}{2^{n}}} dt \int_{\mathbf{R}^{d}} L(t,x;b(t,x))P(t,dx) \right|$
 $P(t,dx) = P_{t+\frac{i}{2^{n}}}(dx) \left(t = 0, \frac{1}{2^{n}}\right), \{P(t,dx)\}_{0 \le t \le \frac{1}{2^{n}}} \subset \mathcal{M}(\mathbf{R}^{d}),$
 $(b(t,x), P(t,dx)) \text{ satisfies (1.1) on } [0,1/2^{n}] \}.$

Then, from Theorem 2.1,

$$v_n(\mathbf{P}) = \sum_{i=0}^{2^n - 1} v_{n,i}(\mathbf{P}) = \sum_{i=0}^{2^n - 1} V_{n,i}(\mathbf{P}).$$
 (3.19)

Since $V_n(\mathbf{P}) \ge v_n(\mathbf{P})$ from (3.6)-(3.7), we only have to prove the following:

$$\sum_{i=0}^{2^{n}-1} V_{n,i}(\mathbf{P}) \ge V_{n}(\mathbf{P}).$$
(3.20)

Suppose that the left hand side of (3.20) is finite. For $i = 0, \dots 2^n - 1$, take a minimizer $X_{n,i}$ of $V_{n,i}(\mathbf{P})$ (see Theorem 1.1), and put

$$P_{n,i} := PX_{n,i} \left(\cdot - \frac{i}{2^n} \right)^{-1} \quad \text{on } (C([\frac{i}{2^n}, \frac{i+1}{2^n}] : \mathbf{R}^d), \mathcal{B}(C([\frac{i}{2^n}, \frac{i+1}{2^n}] : \mathbf{R}^d))),$$
(3.21)

$$P_n\left(dX|_{C([0,1]:\mathbf{R}^d)}\right) := P_{n,0}\left(dX|_{C([0,\frac{1}{2^n}]:\mathbf{R}^d)}\right)$$

$$\times \Pi_{i=1}^{2^n-1} P_{n,i}\left(dX|_{C([\frac{i}{2^n},\frac{i+1}{2^n}]:\mathbf{R}^d)} \middle| X_{n,i}\left(\frac{i}{2^n}\right) = X\left(\frac{i}{2^n}\right)\right)$$
(3.22)

on $(C([0,1]: \mathbf{R}^d), \mathcal{B}(C([0,1]: \mathbf{R}^d)))$. Under the completion of this measure, the coordinate process $\{X_n(t)\}_{0 \le t \le 1}$ satisfies the following:

$$X_n(t) = X_n(0) + \sum_{i=0}^{2^n - 1} \int_{\min(\frac{i}{2^n}, t)}^{\min(\frac{i+1}{2^n}, t)} b_{n,i}\left(s - \frac{i}{2^n}, X_n(s)\right) ds + W_{X_n}(t) \quad (0 \le t \le 1),$$
(3.23)

where $b_{n,i}$ denotes the drift vector of $X_{n,i}$ (see Theorem 1.1). In particular, $PX_n(t)^{-1} = P_t$ $(t = i/2^n, i = 0, \dots, 2^n)$, which implies (3.20). \Box

4 Proofs.

In this section we prove our results given in section 2.

When $L = |u|^2$, the following proof extremely simplifies that of [12, Lemma 2.5].

(Proof of Theorem 2.1). Lemma 3.2, (i) and the following complete the proof:

$$v(P_{0}, P_{1})$$

$$\geq \sup_{f \in C_{b}^{\infty}(\mathbf{R}^{d})} \left\{ \int_{\mathbf{R}^{d}} f(x) P_{1}(dx) - v^{*}(f) \right\}$$
(from (3.3))
$$\geq \sup_{f \in C_{b}^{\infty}(\mathbf{R}^{d})} \left\{ \int_{\mathbf{R}^{d}} f(x) P_{1}(dx) - V^{*}(f) \right\}$$
(from Lemma 3.2, (ii))
$$= V(P_{0}, P_{1})$$
(from Theorem 1.1 (see (3.10))).

(Proof of Theorem 2.2). We first prove (i). From (3.6)-(3.7), $\mathbf{V}(\mathbf{P}) \geq \mathbf{v}(\mathbf{P})$. Therefore we only have to show that

$$\mathbf{v}(\mathbf{P}) \ge \mathbf{V}(\mathbf{P}). \tag{4.2}$$

Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then, from Lemma 3.4,

$$\mathbf{v}(\mathbf{P}) \ge v_n(\mathbf{P}) = V_n(\mathbf{P}) \tag{4.3}$$

and X_n constructed in (3.23) is a minimizer of $V_n(\mathbf{P})$.

Let b_n denote the drift vector of $\{X_n(t)\}_{0 \le t \le 1}$. It is easy to see that $\{(X_n(t), \int_0^t b_n(s, X_n(s))ds) : t \in [0, 1]\}_{n \ge 1}$ is tight in $C([0, 1] : \mathbf{R}^{2d})$ from (A.1) (see [23, Theorem 3] or [9]). Take a weakly convergent subsequence $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s))ds) : t \in [0, 1]\}_{k \ge 1}$ such that

$$\lim_{n \to \infty} E\left[\int_{0}^{1} L(t, X_{n}(t); b_{n}(s, X_{n}(s)))dt\right]$$
(4.4)
=
$$\lim_{k \to \infty} E\left[\int_{0}^{1} L(t, X_{n_{k}}(t); b_{n_{k}}(s, X_{n_{k}}(s)))dt\right].$$

Let $\{(X(t), A(t))\}_{t \in [0,1]}$ denote the limit of $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s))ds) : t \in [0,1]\}_{k\geq 1}$ as $k \to \infty$. Then $\{X(t)-X(0)-A(t)\}_{t\in [0,1]}$ is a $\sigma[X(s): 0 \leq s \leq t]$ -Wiener process and $\{A(t)\}_{t\in [0,1]}$ is absolutely continuous (see [23, Theorem 5] or [9]). We can also prove, in the same way as in the proof of [15, (3.17)], the following: from (4.3)-(4.4), (A.2) and (A.3, ii) (see Remark 2.1, (i)),

$$\mathbf{v}(\mathbf{P}) \geq \liminf_{n \to \infty} E\left[\int_{0}^{1} L(t, X_{n}(t); b_{n}(t, X_{n}(t)))dt\right]$$

$$\geq E\left[\int_{0}^{1} L\left(t, X(t); \frac{dA(t)}{dt}\right)dt\right]$$

$$\geq \tilde{E}\left[\int_{0}^{1} L\left(t, X(t); \beta_{X}(t, X)\right)dt\right]$$
(from Lemma 3.3)
$$\geq \mathbf{V}(\mathbf{P}).$$

$$(4.5)$$

Here \tilde{E} denotes the mean value by the completion of $PX(\cdot)^{-1}$ and we used the fact that $P(X(t) \in dx) = P_t(dx)$ for all $t \in [0, 1]$. Indeed,

$$P(X(t) \in dx) = \lim_{n \to \infty} P\left(X\left(\frac{[2^n t]}{2^n}\right) \in dx\right) \quad \text{weakly,}$$
$$P\left(X\left(\frac{[2^n t]}{2^n}\right) \in dx\right) = P_{\frac{[2^n t]}{2^n}}(dx) \to P_t(dx) \quad \text{as } n \to \infty \text{ weakly.}$$

Next we prove (ii). Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then (2.2) and (4.5) show the existence of a minimizer X of $\mathbf{V}(\mathbf{P})$. In the same way as in (3.7), Theorem 2.2, (i) and the strict convexity of $u \mapsto L(t, x; u)$ (see Remark 2.1, (i)) imply that $\beta_X(t, X) = b_X(t, X(t))$ and $b_X(t, x)$ is a minimizer of $\mathbf{v}(\mathbf{P})$.

Let b_1 and b_2 be minimizers of $\mathbf{v}(\mathbf{P})$. Then for any $\lambda \in (0, 1)$, $\lambda b_1(t, x) + (1 - \lambda)b_2(t, x)$ satisfies (1.1), and

$$\mathbf{v}(\mathbf{P})$$
(4.6)
$$\leq \int_{0}^{1} \int_{\mathbf{R}^{d}} L(t, x; \lambda b_{1}(t, x) + (1 - \lambda) b_{2}(t, x)) P_{t}(dx)$$
(4.6)
$$\leq \lambda \int_{0}^{1} \int_{\mathbf{R}^{d}} L(t, x; b_{1}(t, x)) P_{t}(dx) + (1 - \lambda) \int_{0}^{1} \int_{\mathbf{R}^{d}} L(t, x; b_{2}(t, x)) P_{t}(dx)$$

$$= \mathbf{v}(\mathbf{P}).$$

The strict convexity of $u \mapsto L(t, x; u)$ implies the uniqueness of a minimizer of $\mathbf{v}(\mathbf{P}).\Box$

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