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Citation	Mathematische Zeitschrift, 252(4), 849-862 https://doi.org/10.1007/s00209-005-0890-x
Issue Date	2006-04
Doc URL	http://hdl.handle.net/2115/5783
Rights	The original publication is available at www.springerlink.com
Type	article (author version)
File Information	MZ252-4.pdf



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The automorphism groups of the vertex operator algebras V_L^+ : general case

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Abstract

In this article, we give a method of calculating the automorphism groups of the vertex operator algebras V_L^+ associated with even lattices L . For example, by using this method we determine the automorphism groups of V_L^+ for even lattices of rank one, two and three, and even unimodular lattices.

Introduction

Let L be a (positive-definite) even lattice and let V_L^+ be the fixed-points of the VOA V_L associated with L under an automorphism θ_{V_L} lifting the -1 -isometry of L . The automorphism groups $\text{Aut}(V_L^+)$ of the VOAs V_L^+ were described in [DG1] for lattices L of rank 1, in [DG2] for lattices L of rank 2, and in [Sh] for lattices L without roots. The primary purpose of this article is to generalize the method of calculating $\text{Aut}(V_L^+)$ in [Sh] to all even lattices L .

Let V be a VOA and let G be an automorphism group of V . Then the subspace V^G of points fixed by G is a subVOA. Clearly $N_{\text{Aut}(V)}(G)$ acts on V^G . Then the question arises as to whether or not any automorphism of V^G comes from $N_{\text{Aut}(V)}(G)$. Take V to be the VOA V_L and G to be the group generated by the involution θ_{V_L} . Then the quotient group H_L of $C_{\text{Aut}(V_L)}(\theta_{V_L})$ by the subgroup $\langle \theta_{V_L} \rangle$ acts faithfully on V_L^+ . In [DG2] it was shown that $\text{Aut}(V_L^+)$ coincides with H_L if L does not have vectors of norm 2 or 4 and the rank of L is greater than 1. In this article, we can obtain a definitive answer: $\text{Aut}(V_L^+)$ is larger than H_L if and only if L is obtained by Construction B or is isomorphic to the E_8 -lattice.

We recall the method of [Sh]. Let S_L denote the set of all isomorphism classes of irreducible V_L^+ -modules. Then $\text{Aut}(V_L^+)$ acts on S_L . It was shown that the stabilizer of

*The author was supported by the Japan Society for the Promotion of Science Research Fellowships for Young Scientists and COE grant of Hokkaido University.

the isomorphism class $[0]^-$ of the irreducible V_L^+ -module V_L^- is equal to the subgroup H_L . The orbit Q_L of $[0]^-$ was determined when L has no roots. Moreover, Q_L was regarded as a subset of an elementary abelian 2-group by using the fusion rules of V_L^+ . Hence there exists a group homomorphism from $\text{Aut}(V_L^+)$ to a general linear group over \mathbb{F}_2 . Then by using the kernel and image $\text{Aut}(V_L^+)$ can be described.

The main result of this article is the following: The orbits Q_L are determined for all even lattices L (Theorem 4.3). This allows us to determine the automorphism group of V_L^+ .

We explain our method of determining Q_L . Since the action of $\text{Aut}(V_L^+)$ on Q_L preserves the graded dimensions and fusion rules, we obtain some necessary conditions satisfied by elements of Q_L . For any element W of untwisted type in S_L satisfying the conditions, we will show that there exists an automorphism exchanging $[0]^-$ and W . To do this, we use a characterization of even lattices obtained by Construction B (Theorem 2.2) and certain automorphisms given in [FLM]. Thus we obtain sufficient and necessary conditions for isomorphism classes of untwisted type to belong Q_L . Moreover we will classify even lattices L such that Q_L contains isomorphism classes of twisted type. Determining isomorphism classes of twisted type in Q_L , we obtain the orbit Q_L .

Throughout this article, we will work over the field \mathbb{C} of complex numbers unless otherwise stated. We denote the set of integers by \mathbb{Z} and the rings of integers modulo p by \mathbb{Z}_p . We often identify \mathbb{Z}_2 with the field \mathbb{F}_2 of two elements. Let Ω_n denote the set $\{1, 2, \dots, n\}$ for $n \in \mathbb{Z}_{>0}$. We view the power set of Ω_n as an n -dimensional vector space over \mathbb{F}_2 naturally. For a code C and $l \in \mathbb{Z}$, let C_l denote the set of codewords of C of weight l . For a subset U of an n -dimensional vector space \mathbb{R}^n over the real field \mathbb{R} and $m \in \mathbb{R}$, let U_m denote the set of vectors in U of norm m . For a lattice L , the dual lattice of L is denoted by L^* . For a group G and its subgroup H , $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer of H in G respectively. Let V be a VOA and let (M, Y_M) be a V -module. For an automorphism g of V , let $M \circ g$ denote the V -module $(M, Y_{M \circ g})$ defined by $Y_{M \circ g}(v, z) = Y_M(gv, z)$, $v \in V$.

Acknowledgments. The author would like to thank Professor Atsushi Matsuo for valuable suggestions and helpful advice. He also thanks Professor Masahiko Miyamoto for useful comments and Professor Toshiyuki Abe for reading the manuscript.

1 Preliminaries

In this section, we recall or give some definitions and facts necessary in this article.

1.1 Construction B

In this subsection, we recall a standard method for constructing lattices from linear binary codes.

Let n be a positive integer and let $\{\alpha_i \mid i \in \Omega_n\}$ be an orthogonal basis of \mathbb{R}^n satisfying $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$. For a subset $J \subset \Omega_n$, we set $\alpha_J = \sum_{i \in J} \alpha_i$. Let C be a binary code of length n . Then

$$L_B(C) = \sum_{c \in C} \mathbb{Z} \frac{1}{2} \alpha_c + \sum_{i,j \in \Omega_n} \mathbb{Z}(\alpha_i + \alpha_j) \quad (1.1)$$

is called the *lattice obtained by Construction B* from C . We note that $L_B(C)$ is even if and only if C is doubly even. We call $\{\pm \alpha_i \mid i \in \Omega_n\}$ a *frame* of $L_B(C)$ with respect to the expression (1.1). The following lemma is easy to prove.

Lemma 1.1. $|L_B(C)_2| = 8|C_4|$.

1.2 Vertex operator algebra V_L^+

In this subsection, we review some properties of the vertex operator algebra V_L^+ . For the details of its construction, see [FLM].

Let L be a (positive-definite) even lattice and let \hat{L} be a central extension:

$$1 \rightarrow \langle \kappa_L \mid \kappa_L^2 = 1 \rangle \rightarrow \hat{L} \twoheadrightarrow L \rightarrow 1$$

such that $[a, b] = \kappa_L^{\langle \bar{a}, \bar{b} \rangle}$ for $a, b \in \hat{L}$. Let $\theta_{\hat{L}}$ be an involution of \hat{L} induced by the -1 -isometry of L . Set $K_L = \{a^{-1}\theta_{\hat{L}}(a) \mid a \in \hat{L}\}$. Then K_L is a normal subgroup of \hat{L} . Let V_L denote the VOA associated with L . The automorphism group $\text{Aut}(V_L)$ of V_L contains an involution θ_{V_L} induced by $\theta_{\hat{L}}$. Its fixed-points on V_L is denoted by V_L^+ . Then V_L^+ is a subVOA of V_L .

In [DN2, AD], it was shown that any irreducible V_L^+ -module is isomorphic to one of $V_{\lambda+L}^\pm$ ($\lambda \in L^* \cap (L/2)$), $V_{\mu+L}$ ($\mu \in L^* \setminus (L/2)$) and $V_L^{T_\chi, \pm}$, where T_χ is an irreducible \hat{L}/K_L -module with central character χ . In this article, we use the following notation: $[\mu]$, $[\lambda]^\pm$ and $[\chi]^\pm$ denote the isomorphism classes of $V_{\mu+L}$, $V_{\lambda+L}^\pm$ and $V_L^{T_\chi, \pm}$ respectively. The isomorphism classes $[\mu]$, $[\lambda]^\pm$ are called *untwisted type* and the isomorphism classes $[\chi]^\pm$ are called *twisted type*.

Note 1.2. In this article, we take an involution on $V_L^{T_\chi}$ induced by the identity operator on T_χ and consider the ± 1 -eigenspace $V_L^{T_\chi, \pm}$. However in [FLM] an involution on $V_L^{T_\chi}$ induced by the -1 -isometry on T_χ is used.

The fusion rules of V_L^+ were determined in [Ab, ADL]. In particular the following hold.

Lemma 1.3. [Ab, ADL]

- (1) Let λ be a vector in $L^* \cap (L/2)$. Then the fusion rules $[0]^- \times [\lambda]^\pm = [\lambda]^\mp$ hold.
- (2) Let λ be a vector in $L^* \cap (L/2)$ satisfying $\langle \lambda, \lambda \rangle \in \mathbb{Z}$. Then the fusion rule $[\lambda]^\varepsilon \times [\lambda]^\varepsilon = [0]^+$ holds for any $\varepsilon \in \{\pm\}$.
- (3) Let W_1 and W_2 be isomorphism classes of irreducible modules of V_L^+ . If isomorphism classes of twisted type appear in $W_1 \times W_2$ then one of W_1 and W_2 is of twisted type.

1.3 Automorphism groups of V_L and V_L^+

In this section, we review the results on automorphism groups of V_L and V_L^+ for even lattices L .

We start by recalling the automorphism group of V_L . For a lattice L , we denote by $O(L)$ the group of automorphisms of L which preserve the bilinear form. Let $O(\hat{L})$ denote the group of automorphisms of \hat{L} which preserve the bilinear form on the quotient of \hat{L} by its normal subgroup of order 2. For $g \in O(\hat{L})$, let \bar{g} denote the linear automorphism of L defined by $\bar{g}(\bar{a}) = \overline{g(a)}$, $a \in \hat{L}$. We view an element $f \in \text{Hom}(L, \mathbb{Z}_2)$ as the automorphism of \hat{L} which sends a to $\kappa_L^{f(\bar{a})}a$. Hence we obtain an embedding $\text{Hom}(L, \mathbb{Z}_2) \subset O(\hat{L})$. In Proposition 5.4.1 of [FLM], the following sequence is exact:

$$1 \rightarrow \text{Hom}(L, \mathbb{Z}_2) \hookrightarrow O(\hat{L}) \twoheadrightarrow O(L) \rightarrow 1. \quad (1.2)$$

In [DN1], the automorphism group $\text{Aut}(V_L)$ of V_L was described as follows:

Proposition 1.4. [DN1, Theorem 2.1] *Let L be an even lattice. Then $\text{Aut}(V_L) = N(V_L)O(\hat{L})$, where $N(V_L) = \langle \exp(v_0) \mid v \in (V_L)_1 \rangle$ is a normal subgroup of $\text{Aut}(V_L)$. Moreover, $\text{Aut}(V_L)/N(V_L)$ is isomorphic to a quotient group of $O(L)$.*

In [Do] it was shown that any irreducible V_L -module is isomorphic to $V_{\lambda+L}$ for some $\lambda \in L^*$. The group $\text{Aut}(V_L)$ acts on the set of isomorphism classes of irreducible V_L -modules as follows.

Lemma 1.5. (1) *Any element of $N(V_L)$ fixes all isomorphism classes of irreducible V_L -module.*

(2) *Let g be an element of $O(\hat{L})$ and let λ be a vector in L^* . Then g sends the isomorphism class of $V_{\lambda+L}$ to that of $V_{\bar{g}^{-1}(\lambda)+L}$.*

Now, let us consider automorphisms of V_L^+ . By the definition of V_L^+ , the centralizer $C_{\text{Aut}(V_L)}(\theta_{V_L})$ acts on V_L^+ . Set

$$H_L = C_{\text{Aut}(V_L)}(\theta_{V_L}) / \langle \theta_{V_L} \rangle.$$

Then H_L acts faithfully on V_L^+ , and $H_L \subset \text{Aut}(V_L^+)$. Let S_L denote the set of all isomorphism classes of irreducible V_L^+ -modules. Then H_L is characterized as follows.

Lemma 1.6. [Sh, Proposition 3.10] *The group H_L is the stabilizer of $[0]^-$ under the action of $\text{Aut}(V_L^+)$ on S_L .*

Since θ_{V_L} belongs to the center of $O(\hat{L})$, H_L contains $O(\hat{L}) / \langle \theta_{V_L} \rangle$.

Lemma 1.7. [Sh, Proposition 2.9] *For $g \in O(\hat{L}) / \langle \theta_{V_L} \rangle$, we have*

$$\begin{aligned} [\mu] \circ g &= [\bar{g}^{-1}(\mu)], \quad \mu \in L^* \setminus (L/2), \\ \{[\lambda]^\pm \circ g\} &= \{[\bar{g}^{-1}(\lambda)]^\pm\}, \quad \lambda \in L^* \cap (L/2), \\ [0]^\pm \circ g &= [0]^\pm. \end{aligned}$$

Moreover for $\lambda \in L^* \cap (L/2)$ there exists an automorphism h of V_L^+ such that $[\lambda]^+ \circ h = [\lambda]^-$.

Let Q_L denote the orbit of $[0]^-$ under the action of $\text{Aut}(V_L^+)$ on S_L . Since automorphisms of a VOA preserves the fusion rules and the graded dimensions, we have the following inclusions:

Lemma 1.8. [Sh, Lemma 3.12] *Let L be an even lattice of rank n .*

- (1) *If $n \neq 8, 16$ then $Q_L \subseteq \{[0]^-, [\lambda]^\pm \mid \lambda \in L^* \cap (L/2), |(\lambda + L)_2| = 2n + |L_2|\}$.*
- (2) *If $n = 8$ then $Q_L \subseteq \{[0]^-, [\lambda]^\pm, [\chi]^- \mid \lambda \in L^* \cap (L/2), |(\lambda + L)_2| = 2n + |L_2|\}$, where χ ranges over the central characters of \hat{L}/K_L with $\chi(\kappa K_L) = -1$.*
- (3) *If $n = 16$ then $Q_L \subseteq \{[0]^-, [\lambda]^\pm, [\chi]^+ \mid \lambda \in L^* \cap (L/2), |(\lambda + L)_2| = 2n + |L_2|\}$, where χ ranges over the central characters of \hat{L}/K_L with $\chi(\kappa K_L) = -1$.*

Recall that a lattice L is said to be *2-elementary* if $2L^* \subset L$, and said to be *totally even* if both $\sqrt{2}L^*$ and L are even.

Lemma 1.9. (1) [Sh, Proposition 3.14] *If Q_L contains isomorphism classes of twisted type then L is 2-elementary totally even.*

- (2) [Sh, Lemma 3.6] *If L is 2-elementary totally even then isomorphism classes of twisted type with the same sign are conjugate under the action of $\text{Aut}(V_L^+)$.*

1.4 Extra automorphisms of V_L^+

In this subsection we review automorphisms of V_L^+ not in H_L from [FLM].

Let C be a doubly even code of length n and let L be the lattice obtained by Construction B from C with frame $\{\alpha_i \mid i \in \Omega_n\}$. In Chapter 11 of [FLM], an automorphism σ not in H_L was constructed. This automorphism satisfies $[0]^- \circ \sigma = [\alpha_1]^+$. Lemma 1.6 shows that $\sigma \notin H_L$. By Lemma 1.7, there exists an automorphism h of V_L^+ such that $[\alpha_1]^+ \circ h = [\alpha_1]^-$.

We now assume that C contains the all-ones codeword. Let us see the action of σ on some isomorphism classes of irreducible V_L^+ -modules. Set $\beta_n = \alpha_{\Omega_n}/4$ and $\gamma_n = \alpha_{\Omega_n}/4 - \alpha_1$. By the assumption, vectors β_n and γ_n belong to $L^* \cap (L/2)$. By [FLM, Theorem 11.5.1], $\{[\beta_n]^\pm \circ \sigma, [\gamma_n]^\pm \circ \sigma\} = \{[\chi_1]^\pm, [\chi_2]^\pm\}$ for some central characters χ_i of \hat{L}/K_L . Comparing the graded dimensions, we obtain $\{[\beta_n]^\pm \circ \sigma\} = \{[\chi_1]^+, [\chi_2]^+\}$ and $\{[\gamma_n]^\pm \circ \sigma\} = \{[\chi_1]^-, [\chi_2]^-\}$.

The result is summarized in the following lemmas.

Lemma 1.10. [FLM] *Let L be an even lattice obtained by Construction B with frame $\{\alpha_i\}$. Then the orbit of $[0]^-$ contains $[\alpha_1]^\pm$. In particular the cardinality of the orbit of $[0]^-$ is greater than 1.*

Lemma 1.11. [FLM] *Let L be the even lattice obtained by Construction B from a doubly even code C containing the all-one codeword. Let $\varepsilon \in \{\pm\}$.*

- (1) The orbit of the isomorphism classes $[\beta_n]^\varepsilon$ contains isomorphism classes of twisted type with sign $+$.
- (2) The orbit of the isomorphism classes $[\gamma_n]^\varepsilon$ contains isomorphism classes of twisted type with sign $-$.

2 Characterization of even lattices obtained by Construction B

In this section, we characterize even lattices obtained by Construction B. We will later use our characterization to determine the automorphism group of V_L^+ .

Let L be a (positive-definite) even lattice of rank n . We set

$$R_L = \left\{ \lambda + L \in L^*/L \mid \lambda \in L/2, |(\lambda + L)_2| \geq 2n + |L_2| \right\}. \quad (2.1)$$

Note 2.1. The definition of R_L comes from the necessary conditions satisfied by isomorphism classes of untwisted type in Q_L (cf. Lemma 1.8).

Then even lattices obtained by Construction B are characterized as follows.

Theorem 2.2. *Let L be an even lattice of rank n . Then the following conditions are equivalent:*

- (1) L is obtained by Construction B.
- (2) The set R_L is not empty.

To prove this theorem, we need some lemmas.

Lemma 2.3. *Let L be an even lattice of rank n and let $\lambda + L$ be an element of R_L . Then $(\lambda + L)_2$ contains an orthogonal basis of \mathbb{R}^n .*

Proof. Since L is even and $\lambda \in L^*$, the norms of vectors in $\lambda + L$ are contained in $\langle \lambda, \lambda \rangle + 2\mathbb{Z}$. It follows from $(\lambda + L)_2 \neq \emptyset$ that $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$. Hence $L' = L + \mathbb{Z}\lambda$ is an even lattice and L'_2 forms a root system. In particular, the inner products of vectors in $(\lambda + L)_2$ are contained in $\{0, \pm 1, \pm 2\}$.

Let $Y_r = \{y_1, \dots, y_r\}$ be a subset of $(\lambda + L)_2$ such that $\langle y_i, y_j \rangle = 2\delta_{i,j}$. We set $\tilde{Y}_r = \{\pm y \mid y \in Y_r\}$. Then $|\tilde{Y}_r| = 2r$. To prove this lemma, we will show that if $r < n$ then there exists a vector in $(\lambda + L)_2$ orthogonal to Y_r . Define

$$X(Y_r) = \left\{ x \in (\lambda + L)_2 \mid \langle x, y_i \rangle \in \{\pm 1\} \text{ for } \exists i \in \Omega_r \right\},$$

where $\Omega_r = \{1, 2, \dots, r\}$. Clearly $\tilde{Y}_r \cap X(Y_r) = \emptyset$. For $x \in X(Y_r)$, we set

$$m(x) = \min \left\{ i \in \Omega_r \mid \langle y_i, x \rangle \in \{\pm 1\} \right\}. \quad (2.2)$$

Set $X(Y_r)^+ = \{x \in X(Y_r) \mid \langle x, y_{m(x)} \rangle = 1\}$. Then $|X(Y_r)| = 2|X(Y_r)^+|$. Since $|x - y_{m(x)}|^2 = |x|^2 + |y_{m(x)}|^2 - 2\langle x, y_{m(x)} \rangle = 2$ for $x \in X(Y_r)^+$, we consider the map $\rho : X(Y_r)^+ \rightarrow \{\{\pm v\} \mid v \in L_2\}$, $x \mapsto \{\pm(x - y_{m(x)})\}$.

Let us show that ρ is injective. First we suppose $x - y_{m(x)} = x' - y_{m(x')}$. If $m(x) = m(x')$ then $x = x'$. So we may assume that $m(x) < m(x')$. By the definition of Y_r and (2.2)

$$\langle y_{m(x)}, y_{m(x')} \rangle = \langle x', y_{m(x)} \rangle = 0. \quad (2.3)$$

Hence we have

$$2 = \langle x - y_{m(x)}, x' - y_{m(x')} \rangle = \langle x, x' - y_{m(x')} \rangle.$$

Since both x and $x' - y_{m(x')}$ belong to L'_2 , we have $x = x' - y_{m(x')}$. However it contradicts $(\lambda + L) \cap L = \phi$ since $x \in \lambda + L$ and $x' - y_{m(x')} \in L$.

Next we suppose $x - y_{m(x)} = y_{m(x')} - x'$. If $m(x) = m(x')$ then $x + x' = 2(y_{m(x)})$ and $|x|^2 = |x'|^2 = |y_{m(x)}|^2 = 2$. Hence $x = x' = y_{m(x)}$, which is a contradiction. So we may assume $m(x) < m(x')$. Then by (2.3), we have

$$2 = \langle x - y_{m(x)}, y_{m(x')} - x' \rangle = \langle x, y_{m(x')} - x' \rangle,$$

which implies that $x = y_{m(x')} - x'$. However, it contradicts $(\lambda + L) \cap L = \phi$. Hence ρ is injective. This shows that $|X(Y_r)^+| \leq |L_2|/2$, namely $|X(Y_r)| \leq |L_2|$. Since $\tilde{Y}_r \cap X(Y_r) = \phi$, we have $|\tilde{Y}_r \cup X(Y_r)| \leq 2r + |L_2|$. So, if $r < n$ then there exists $x \in (\lambda + L)_2$ such that $x \notin X(Y_r) \cup \tilde{Y}_r$, namely $\langle x, Y_r \rangle = 0$. Therefore $(\lambda + L)_2$ contains an orthogonal basis of \mathbb{R}^n . \square

Lemma 2.4. *Let L be an even lattice of rank n . For $\lambda + L \in R_L$, there exist a doubly even code C and a frame in $\lambda + L$ such that $L = L_B(C)$.*

Proof. By Lemma 2.3, $\lambda + L$ contains vectors e_1, e_2, \dots, e_n satisfying $\langle e_i, e_j \rangle = 2\delta_{i,j}$. Since $2\lambda \in L$, we have $e_i \pm e_j \in L$. Set $E = \bigoplus_{i=1}^n \mathbb{Z}e_i$ and $L' = L + \mathbb{Z}\lambda$. Then L'/E is a subspace of $E^*/E \cong \mathbb{Z}_2^n$. So we regard L'/E as a binary code C of length n . We can choose a basis B in $\{\pm e_i \mid i \in \Omega_n\}$ so that L is the lattice obtained by Construction B from C with frame B (cf. the proof of [Sh, Proposition 1.8]). \square

Proof of Theorem 2.2. Suppose (1). Let $\{\alpha_i\}$ be the frame. Then $|(\alpha_1 + L)_2| = 2n + |L_2|$, and $\alpha_1 + L \in R_L$. Hence (1) \Rightarrow (2). It follows from Lemma 2.4 that (2) \Rightarrow (1). \square

Remark 2.5. *The proof of Theorem 2.2 implies that $|(\lambda + L)_2| = 2n + |L_2|$ for any $\lambda + L \in R_L$.*

Let us show some lemmas by using Theorem 2.2. Let L be the even lattice obtained by Construction B from a doubly even code C of length n with frame $\{\pm\alpha_i \mid i \in \Omega_n\}$. Set $\beta_n = \alpha_{\Omega_n}/4$ and $\gamma_n = \alpha_{\Omega_n}/4 - \alpha_1$.

Lemma 2.6. *The following conditions are equivalent:*

- (1) $\gamma_n + L \in R_L$.
- (2) $n = 8$ and C contains the all-one codeword.

Proof. Suppose (2). Since C contains the all-one codeword, $\gamma_8 \in L/2$. Clearly $\gamma_8 \in L^*$. It is easy to see that

$$(\gamma_8 + L)_2 = \{\pm(\alpha_{\Omega_8}/4 - \alpha_i), \alpha_{\Omega_8} - \alpha_c/2 + \alpha_j, \alpha_{\Omega_8} - \alpha_c/2 - \alpha_k \mid c \in C_4, i \in \Omega_8, j \in c, k \in \Omega_8 \setminus c\}.$$

Hence $|(\gamma_8 + L)_2| = 16 + 8|C_4| = 16 + |L_2|$ by Lemma 1.1. Thus $\gamma_8 + L \in R_L$.

Conversely, we suppose (1). Since the norm of γ_n is minimal in $\gamma_n + L$ and it is $1 + n/8$, the rank n of L must be 8. Since $\gamma_8 \in L/2$, we obtain $\alpha_{\Omega_8}/2 \in L$. Hence the all-one codeword belongs to C . \square

Lemma 2.7. *The following conditions are equivalent:*

- (1) $\beta_n + L \in R_L$.
- (2) $n = 16$ and C contains a subcode isomorphic to the Reed-Muller code $RM(1, 4)$.

Proof. Let k be the dimension of C . Suppose (2). In [PLF], doubly even codes of length 16 containing the all-one codeword were classified. In particular doubly even codes of length 16 containing $RM(1, 4)$ can be classified. Hence we obtain $|C_4| = 0, 4, 12, 28$ for $k = 5, 6, 7, 8$ respectively. So $32 + |L_2| = 2^k$. On the other hand,

$$(\beta_{16} + L)_2 = \{\beta_{16} - \alpha_c/2 \mid c \in C\}. \quad (2.4)$$

Hence $|(\beta_{16} + L)_2| = 2^k$. Therefore $\beta_{16} + L \in R_L$.

Conversely we suppose (1). Since the norm of β_n is minimal in $\beta_n + L$ and it is $n/8$, the rank n of L must be 16. By Lemma 2.3, $(\beta_{16} + L)_2$ contains an orthogonal basis F . Set $\tilde{F} = \{\pm v \mid v \in F\}$. By (2.4), $\tilde{F} = \{\beta_{16} - \alpha_c/2 \mid c \in D\}$ for some subset D of C . Clearly $|D| = 32$. Let d be an element of D . Set $D^0 = d + D$. Then $\tilde{F}^0 = \{\beta_{16} - \alpha_c/2 \mid c \in D^0\}$ is a set of 32 vectors, two of which are equal, opposite, or orthogonal. Since D^0 contains the all-zero codeword, D^0 consists of the all-zero and all-one codewords and 30 codewords with weight 8. Moreover, the cardinality of any intersection of codewords with weight 8 in D^0 is 0, 4 or 8. Hence D^0 must be isomorphic to the Reed-Muller code $RM(1, 4)$. Therefore C contains a subcode isomorphic to the Reed-Muller code $RM(1, 4)$. \square

3 Automorphism groups of V_L^+ for even unimodular lattices of rank 8 and 16

In this section, we determine the automorphism groups of V_L^+ for even unimodular lattices of rank 8 and 16. In particular, we will compare $\text{Aut}(V_L^+)$ with its subgroup $H_L \cong C_{\text{Aut}(V_L)}(\theta_{V_L})/\langle \theta_{V_L} \rangle$.

Let L be an even unimodular lattice of rank 8 or 16. Then the VOA V_L^+ has exactly 4 isomorphism classes of irreducible V_L^+ -modules $[0]^\pm$ and $[\chi_0]^\pm$, where χ_0 is the unique faithful character of \hat{L}/K_L . Since the graded dimensions of $[\chi_0]^+$ and $[\chi_0]^-$ are different, the cardinality of the orbit Q_L of $[0]^-$ is 1 or 2. In the following subsections, we will determine $|Q_L|$.

3.1 Automorphism group of V_L^+ for the even unimodular lattice of rank 8

In this subsection, we study the automorphism group of $V_{E_8}^+$, where E_8 is the unique even unimodular lattice of rank 8 up to isomorphism.

Lemma 3.1. *There are automorphisms of $V_{E_8}^+$ mapping $[0]^-$ to $[\chi_0]^-$. In particular Q_{E_8} contains isomorphism classes of twisted type.*

Proof. The degree 1 subspace of $V_{E_8}^+$ forms the simple Lie algebra of type D_8 under the 0-th product and $V_{E_8}^+ \cong V_{D_8}$. By [Do], V_{D_8} has exactly 4 non-isomorphic irreducible modules $V_{\lambda+D_8}$, $\lambda + D_8 \in D_8^*/D_8$.

On the other hand, there exists an involution τ of the root lattice of type D_8 such that τ exchanges two elements of D_8^*/D_8 . By Lemma 1.5 (2), lifts of τ exchange two isomorphism classes of irreducible V_{D_8} -modules. This shows that there are automorphisms of $V_{E_8}^+$ mapping $[0]^-$ to $[\chi_0]^-$. \square

Proposition 3.2. *The group H_{E_8} is a normal subgroup of $\text{Aut}(V_{E_8}^+)$ of index 2. In particular $\text{Aut}(V_{E_8}^+)/H_{E_8} \cong \mathbb{Z}_2$.*

Proof. Lemma 3.1 shows that the cardinality of the orbit Q_{E_8} of $[0]^-$ is 2. By Lemma 1.6 H_{E_8} is a subgroup of the index 2 of $\text{Aut}(V_{E_8}^+)$. \square

3.2 Automorphism groups of V_L^+ for even unimodular lattices of rank 16

In this subsection, we study the automorphism groups of V_L^+ for even unimodular lattices L of rank 16. It is known that $E_8 \oplus E_8$ and Γ_{16} are the only even unimodular lattices of rank 16 up to isomorphism (cf. [CS]). We note that the root sublattice of Γ_{16} is of type D_{16} .

Let U be a root lattice of type $D_8 \oplus D_8$. Let N be an even overlattice of U such that $|N : U| = 2$ and $N_2 = U_2$. It is easy to check that N is unique up to isomorphism. Since the determinant of N is 4, there are three unimodular overlattices of N . In particular even unimodular lattices $E_8 \oplus E_8$ and Γ_{16} are obtained as overlattices of N .

Lemma 3.3. *Any element of $\text{Aut}(V_N)$ of V_N fixes all isomorphism class of irreducible V_N -modules.*

Proof. The automorphism group $O(N)$ of N acts on N^*/N . Since unimodular overlattices of N are non-isomorphic, $O(N)$ fixes all elements of N^*/N . By Lemma 1.5 (2), we obtain this lemma. \square

Lemma 3.4. *The VOAs $V_{\Gamma_{16}}^+$ and $V_{E_8 \oplus E_8}^+$ are isomorphic to V_N .*

Proof. First we consider the lattice $E_8 \oplus E_8$. Since $V_{E_8}^+$ is isomorphic to V_{D_8} , $V_{E_8 \oplus E_8}^+$ contains a subVOA V isomorphic to V_U . Since V_U is rational, $V_{E_8 \oplus E_8}^+ = V \oplus M$ as V -module for some V -module M . By the classification of irreducible modules of V_U ([Do]), M is isomorphic to the irreducible V_U -module $V_{\lambda+U}$, where $\lambda + U \in U^*/U$ satisfying $\langle \lambda, U \rangle = N$. Since $V_N = V_U \oplus V_{\lambda+U}$ is a simple current extension of V_U , $V_{E_8 \oplus E_8}^+$ has a unique VOA structure extending its V -module structure (cf. Proposition 5.3 in [DM]) and $V_{E_8 \oplus E_8}^+ \cong V_N$.

Next, we consider the lattice Γ_{16} . Since the root sublattice of Γ_{16} is D_{16} , $V_{\Gamma_{16}}^+$ contains $V_{D_{16}}^+$. The degree 1 subspace of $V_{D_{16}}^+$ forms a simple Lie algebra of type $D_8 \oplus D_8$ under the 0-th product and $V_{D_{16}}^+ \cong V_U$. Similarly to the argument above, we obtain $V_{\Gamma_{16}}^+ \cong V_N$. \square

This lemma shows that the cardinality of the orbit Q_L of $[0]^-$ is 1 for any even unimodular lattice L of rank 16. By Lemma 1.6 $\text{Aut}(V_L^+)$ coincides with H_L .

Proposition 3.5. *The automorphism group $\text{Aut}(V_L^+)$ of V_L^+ coincides with H_L for any even unimodular lattice L of rank 16.*

4 The orbit of the isomorphism class of V_L^-

In this section, we determine the orbit Q_L of $[0]^-$. We note that Q_L was determined in [Sh] when L has no roots.

Lemma 4.1. *The orbit Q_L contains the isomorphism class $[\lambda]^\varepsilon$ for any $\lambda \in R_L$, $\varepsilon \in \{\pm\}$.*

Proof. By Lemma 1.8 any isomorphism class of untwisted type in Q_L must be $[\lambda]^\varepsilon$ for some $\lambda \in R_L$ and $\varepsilon \in \{\pm\}$. Conversely by Lemma 1.10 and 2.4 Q_L contains $[\lambda]^\varepsilon$ for all $\lambda + L \in R_L$ and $\varepsilon \in \{\pm\}$. \square

So let us discuss the cases where Q_L contains isomorphism classes of twisted type. We consider the following three conditions on even lattices L :

- (a) L is obtained by Construction B from a doubly even code of length 8 containing the all-one codeword.
- (b) L is obtained by Construction B from a doubly even code of length 16 containing a subcode isomorphic to the first order Reed-Muller code $RM(1, 4)$ of length 16.
- (c) L is isomorphic to the E_8 -lattice.

Proposition 4.2. *The orbit Q_L contains isomorphism classes of twisted type if and only if L satisfies (a), (b) or (c).*

Proof. By Lemma 1.11, 2.6, 2.7 and 3.1 if L satisfies (a), (b) or (c) then Q_L contains isomorphism classes of irreducible V_L^+ -modules of twisted type.

So we suppose that Q_L contains an isomorphism class $[\chi]^\varepsilon$ of twisted type. Then by Lemma 1.8 the rank of L is 8 or 16, and $\varepsilon = -$ and $+$ if $n = 8$ and 16 respectively. Moreover by Lemma 1.9 (1) L is 2-elementary totally even. If L is unimodular then L is isomorphic to one of E_8 , $E_8 \oplus E_8$ and Γ_{16} . By the result of the previous section, L must be isomorphic to E_8 . Hence L satisfies (c).

We now assume that L is not unimodular. Let us show that Q_L contains $[\lambda]^\delta$ for some $\lambda \in L^* \cap (L/2)$, $\delta \in \{\pm\}$. By comparing the coefficients of q in the graded dimensions of V_L^- and $V_L^{T_{\chi, \varepsilon}}$, the theta series of L is written by the Dedekind-eta series. By using the transformation formula on theta series of lattices and their dual lattices, we can describe the theta series of L^* . In particular, $L^* \setminus L$ has vectors of norm 2 (cf. the proof of Proposition 3.14 in [Sh]). Let λ be an element of L^* such that $(\lambda + L)_2 \neq \phi$. Let g be an element of $\text{Aut}(V_L^+)$ such that $[0]^- \circ g = [\chi]^\varepsilon$. By Lemma 1.3 (1), we obtain $[\chi]^\varepsilon \times ([\lambda]^+ \circ g) = [\lambda]^- \circ g$. By Lemma 1.3 (3) one of $[\lambda]^\pm \circ g$ must be of twisted type. By comparing the graded dimensions, it has the same sign ε . By Lemma 1.9 (2), Q_L contains $[\lambda]^\delta$ for some $\delta \in \{\pm\}$. By Lemma 1.8 $\lambda + L \in R_L$. Thus L is obtained by Construction B from a code C by Theorem 2.2.

Since L is 2-elementary totally even, C contains the all-one codeword. Hence (a) holds if the rank of L is 8. Consider the case where $n = 16$. Since the theta series of L is described in terms of the weight enumerator of C , we can describe the weight enumerator of C . By using the classification of even codes of length 16 [PLF], (b) holds if the rank of L is 16. \square

By Lemma 1.8, 1.9, 4.1 and Proposition 4.2, the orbit Q_L is determined.

Theorem 4.3. *Let L be an even lattice of rank n .*

- (1) *If L satisfies (a) or (c) then $Q_L = \{[0]^-, [\lambda]^\pm, [\chi]^- \mid \lambda \in L^* \cap (L/2), |(\lambda + L)_2| = 2n + |L_2|\},$ where χ ranges over the central characters of \hat{L}/K_L with $\chi(\kappa K_L) = -1$ and $\varepsilon = +$.*
- (2) *If L satisfies (b) then $Q_L = \{[0]^-, [\lambda]^\pm, [\chi]^+ \mid \lambda \in L^* \cap (L/2), |(\lambda + L)_2| = 2n + |L_2|\},$ where χ ranges over the central characters of \hat{L}/K_L with $\chi(\kappa K_L) = -1$.*
- (3) *If L does not satisfy neither (a), (b) nor (c) then $Q_L = \{[0]^-, [\lambda]^\pm \mid \lambda \in L^* \cap (L/2), |(\lambda + L)_2| = 2n + |L_2|\}.$*

By Lemma 1.6, Theorem 2.2 and 4.3, we have the following corollary.

Corollary 4.4. *The automorphism group $\text{Aut}(V_L^+)$ of V_L^+ is greater than H_L if and only if the even lattice L satisfies one of the following:*

- (1) *L is obtained by Construction B.*
- (2) *L is isomorphic to the E_8 -lattice.*

5 A method of determining of the shape of the automorphism group of V_L^+

In this section, we give a method of determining the shape of $\text{Aut}(V_L^+)$ for an arbitrary even lattice L . This method is a generalization of that in [Sh, Section 3.4]. For the conditions (a), (b) and (c), see the previous section. If L satisfies (c) then $\text{Aut}(V_L^+)$ is determined in Section 3.1.

First we consider the lattice L satisfying neither (a), (b) nor (c). Then $Q_L = \{[0]^-, [\lambda]^\pm \mid \lambda \in R_L\}$, and $P_L = \{[0]^+\} \cup Q_L$ has an elementary abelian 2-group structure under the fusion rules (cf. [Sh, Proposition 3.17]). So we obtain a group homomorphism φ_L from $\text{Aut}(V_L^+)$ to $GL(P_L)$. Since the kernel of φ_L is a subgroup of H_L , it can be determined. Moreover the index of $\varphi(H_L)$ in $\text{Im } \varphi_L$ is equal to the cardinality of Q_L . Hence we can determine the image of φ_L . Therefore we can calculate the shape of $\text{Aut}(V_L^+)$ in principle.

Suppose that L satisfies (a) or (b). In this case, we consider the set S_L of all isomorphism classes of irreducible V_L^+ -modules. Since L is 2-elementary totally even, S_L has an elementary abelian 2-group structure under the fusion rules (cf. [Ab, ADL] and [Sh, Proposition 3.4]). Moreover S_L has a natural quadratic form associated with a non-singular symplectic form preserved by the action of $\text{Aut}(V_L^+)$ (cf. [Sh, Theorem 3.8]). Hence we obtain a group homomorphism ψ_L from $\text{Aut}(V_L^+)$ to the orthogonal group $O(S_L)$ associated with the quadratic form. Similarly to the case above, we can determine the image and kernel of ψ_L , and we can describe the shape of $\text{Aut}(V_L^+)$ in principle.

Note 5.1. For many important lattices L without roots, the shapes of $\text{Aut}(V_L^+)$ were determined in [Sh, Section 4] by using this method.

6 Automorphism groups of VOSAs V_L^+ for odd lattices

Let L be an odd lattice. In this section, we consider the vertex operator superalgebra V_L^+ . For $i \in \{0, 1\}$, set $L^i = \{v \in L \mid \langle v, v \rangle \equiv i \pmod{2}\}$. Then L^0 is an even sublattice of L . We will describe $\text{Aut}(V_L^+)$ by using $\text{Aut}(V_{L^0}^+)$.

Let $\text{Aut}(V_{L^0}^+; V_{L^1}^+)$ denote the subgroup of $\text{Aut}(V_{L^0}^+)$ fixing the isomorphism class of $V_{L^1}^+$. Let α be a vector in L^1 . Then $2\alpha \in L^0$ and $\langle \alpha, \alpha \rangle \in \mathbb{Z}$. By Lemma 1.3 (2) $[\alpha]^+ \times [\alpha]^+ = [0]^+$. Let τ denote the involution acting as $(-1)^i$ on $V_{L^i}^+$. Applying [Sh, Theorem 3.3] to our case, we obtain $C_{\text{Aut}(V_L^+)}(\tau)/\langle \tau \rangle \cong \text{Aut}(V_{L^0}^+; V_{L^1}^+)$.

On the other hand, any automorphism of V_L^+ preserves both $V_{L^0}^+$ and $V_{L^1}^+$ since the graded dimensions of $V_{L^0}^+$ and $V_{L^1}^+$ are in $\mathbb{Z}[[q]]$ and in $\mathbb{Z}q^{1/2}[[q]]$ respectively. Hence $C_{\text{Aut}(V_L^+)}(\tau) = \text{Aut}(V_L^+)$. Therefore we have the following proposition.

Proposition 6.1. *Let L be an odd lattice. Then $\text{Aut}(V_L^+) \cong \langle \tau \rangle \cdot \text{Aut}(V_{L^0}^+; V_{L^1}^+)$.*

Since the shape of $\text{Aut}(V_{L_0}^+)$ can be described by the method given in the previous section, $\text{Aut}(V_L^+)$ can be determined in principle.

7 Examples

In this section, we calculate $\text{Aut}(V_L^+)$ for some lattices by using the method of Section 5.

7.1 Even lattices of rank one, two and three

In this section, we determine $\text{Aut}(V_L^+)$ for even lattices of rank one two and three.

Let L be an even lattice L of rank n . Suppose that $n \leq 3$. By Theorem 4.3, $Q_L = \{[0]^-, [\lambda]^\pm \mid \lambda \in R_L\}$. So we consider R_L . By Theorem 2.2, let us consider even lattices obtained by Construction B. It is easy to see that a code C of length n is doubly even if and only if C consists of the all-zero codeword. Hence L is obtained by Construction B if and only if $L \cong 2A_1, \sqrt{2}(A_1 \oplus A_1)$ or $\sqrt{2}A_3$. If L is not obtained by Construction B then $\text{Aut}(V_L^+) \cong C_{\text{Aut}(V_L)}(\theta_{V_L})/\langle \theta_{V_L} \rangle$. The case where $L \cong \sqrt{2}A_3$ was done in Theorem 4.3 of [Sh]. So let us consider the automorphism groups of V_L^+ for $2A_1$ and $\sqrt{2}(A_1 \oplus A_1)$.

First we consider the case where $L \cong 2A_1$. Let γ be a generator of L . Then $R_L = \{\gamma/2 + L\}$. Hence $Q_L = \{[0]^-, [\gamma/2]^\pm\}$. Set $P_L = \{[0]^+\} \cup Q_L$. Then P_L has an elementary abelian 2-group structure under the fusion rules and, $P_L \cong \mathbb{F}_2^2$. So we obtain a group homomorphism $\varphi_L : \text{Aut}(V_L^+) \rightarrow GL(P_L)$. On the other hand, $H_L \cong \mathbb{Z}_2$ and its generator exchanges $[\gamma/2]^+$ and $[\gamma/2]^-$. Since $\text{Ker } \varphi_L$ is a subgroup of H_L , φ_L is injective. Clearly $\varphi_L(H_L)$ is a maximal subgroup of $GL(P_L) \cong S_3$. Since $\text{Aut}(V_L^+)$ contains automorphisms not in H_L (cf. Lemma 1.10), φ_L is surjective. Thus we obtain $\text{Aut}(V_L^+) \cong S_3$.

Next let us consider the case where $L \cong \sqrt{2}(A_1 \oplus A_1)$. Let $\{a_1, a_2\}$ be a basis of L satisfying $\langle a_i, a_j \rangle = 4\delta_{i,j}$. Set $a_i^* = a_i/4$ and $b = 2(a_1^* + a_2^*)$. Then $\{a_1^*, a_2^*\}$ is a basis of the dual lattice of L and $R_L = \{b + L\}$. So $Q_L = \{[0]^-, [b]^\pm\}$. Set $P_L = \{[0]^+\} \cup Q_L$. Then P_L has an elementary abelian 2-group structure under the fusion rules and $P_L \cong \mathbb{F}_2^2$. So we obtain a group homomorphism $\varphi_L : \text{Aut}(V_L^+) \rightarrow GL(P_L)$. On the other hand, H_L is isomorphic to the direct product of the dihedral group of order 8 and the group of order 2. The kernel of φ_L is isomorphic to 2^3 , and H_L contains elements exchanging $[b]^+$ and $[b]^-$. So $\varphi_L(H_L)$ is a maximal subgroup of $GL(P_L) \cong S_3$. Since $\text{Aut}(V_L^+)$ contains automorphisms not in H_L , φ_L is surjective. Therefore we obtain $\text{Aut}(V_L^+) \cong 2^3.S_3$. It is easy to check that $\text{Aut}(V_L^+) \cong S_4 \times \mathbb{Z}_2$.

The result is summarized in the following proposition.

Proposition 7.1. *Let L be an even lattice of rank one, two or three. Then*

$$\text{Aut}(V_L^+) \cong \begin{cases} S_3 & \text{if } L \cong 2A_1, \\ S_4 \times \mathbb{Z}_2 & \text{if } L \cong \sqrt{2}(A_1 \oplus A_1), \\ (2^2 : S_4).S_3 & \text{if } L \cong \sqrt{2}A_3, \\ C_{\text{Aut}(V_L)}(\theta_{V_L})/\langle \theta_{V_L} \rangle & \text{otherwise.} \end{cases}$$

Note 7.2. The automorphism groups of V_L^+ for lattices of rank one and two were already obtained in [DG1] and [DG2] respectively by using the action of $\text{Aut}(V_L^+)$ on certain homogeneous subspaces of V_L^+ . In the articles, more precise structures of $\text{Aut}(V_L^+)$ were described.

7.2 Even unimodular lattices

Let L be an even unimodular lattice. Since the determinant of any lattice obtained by Construction B is not 1, L is not obtained by Construction B. Hence $R_L = \phi$ by Theorem 2.2. By Theorem 4.3, $|Q_L| = 2$ if $L \cong E_8$, and $|Q_L| = 1$ if $L \not\cong E_8$. By Lemma 1.6 and Proposition 3.2, we obtain the following proposition.

Proposition 7.3. *Let L be an even unimodular lattice of rank n . Then*

$$\text{Aut}(V_L^+) \cong \begin{cases} (C_{\text{Aut}(V_L)}(\theta_{V_L})/\langle \theta_{V_L} \rangle) \cdot \mathbb{Z}_2 & \text{if } \text{rank} L = 8, \\ C_{\text{Aut}(V_L)}(\theta_{V_L})/\langle \theta_{V_L} \rangle & \text{if } \text{rank} L \geq 16. \end{cases}$$

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