The automorphism groups of the vertex operator algebras V_L^+: general case

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Abstract

In this article, we give a method of calculating the automorphism groups of the vertex operator algebras V_L^+ associated with even lattices L. For example, by using this method we determine the automorphism groups of V_L^+ for even lattices of rank one, two and three, and even unimodular lattices.

Introduction

Let L be a (positive-definite) even lattice and let V_L^+ be the fixed-points of the VOA V_L associated with L under an automorphism θ_{V_L} lifting the -1-isometry of L. The automorphism groups $\mathrm{Aut}(V_L^+)$ of the VOAs V_L^+ were described in [DG1] for lattices L of rank 1, in [DG2] for lattices L of rank 2, and in [Sh] for lattices L without roots. The primary purpose of this article is to generalize the method of calculating $\mathrm{Aut}(V_L^+)$ in [Sh] to all even lattices L.

Let V be a VOA and let G be an automorphism group of V. Then the subspace V^G of points fixed by G is a subVOA. Clearly $N_{\mathrm{Aut}(V)}(G)$ acts on V^G. Then the question arises as to whether or not any automorphism of V^G comes from $N_{\mathrm{Aut}(V)}(G)$. Take V to be the VOA V_L and G to be the group generated by the involution θ_{V_L}. Then the quotient group H_L of $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$ by the subgroup $\langle\theta_{V_L}\rangle$ acts faithfully on V_L^+. In [DG2] it was shown that $\mathrm{Aut}(V_L^+)$ coincides with H_L if L does not have vectors of norm 2 or 4 and the rank of L is greater than 1. In this article, we can obtain a definitive answer: $\mathrm{Aut}(V_L^+)$ is larger than H_L if and only if L is obtained by Construction B or is isomorphic to the E_8-lattice.

We recall the method of [Sh]. Let S_L denote the set of all isomorphism classes of irreducible V_L^+-modules. Then $\mathrm{Aut}(V_L^+)$ acts on S_L. It was shown that the stabilizer of

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the isomorphism class $[0]^-$ of the irreducible V_L^+-module V_L^- is equal to the subgroup H_L. The orbit Q_L of $[0]^-$ was determined when L has no roots. Moreover, Q_L was regarded as a subset of an elementary abelian 2-group by using the fusion rules of V_L^+. Hence there exists a group homomorphism from $\mathrm{Aut}(V_L^+)$ to a general linear group over $\mathbb{F}_2$. Then by using the kernel and image $\mathrm{Aut}(V_L^+)$ can be described.

The main result of this article is the following: The orbits Q_L are determined for all even lattices L (Theorem 4.3). This allows us to determine the automorphism group of V_L^+.

We explain our method of determining Q_L. Since the action of $\mathrm{Aut}(V_L^+)$ on Q_L preserves the graded dimensions and fusion rules, we obtain some necessary conditions satisfied by elements of Q_L. For any element W of untwisted type in S_L satisfying the conditions, we will show that there exists an automorphism exchanging $[0]^-$ and W. To do this, we use a characterization of even lattices obtained by Construction B (Theorem 2.2) and certain automorphisms given in [FLM]. Thus we obtain sufficient and necessary conditions for isomorphism classes of untwisted type to belong Q_L. Moreover we will classify even lattices L such that Q_L contains isomorphism classes of twisted type. Determining isomorphism classes of twisted type in Q_L, we obtain the orbit Q_L.

Throughout this article, we will work over the field $\mathbb{C}$ of complex numbers unless otherwise stated. We denote the set of integers by $\mathbb{Z}$ and the rings of integers modulo p by $\mathbb{Z}_p$. We often identify $\mathbb{Z}_2$ with the field $\mathbb{F}_2$ of two elements. Let Ω_n denote the set $\{1, 2, \dots, n\}$ for $n \in \mathbb{Z}_{>0}$. We view the power set of Ω_n as an n-dimensional vector space over $\mathbb{F}_2$ naturally. For a code C and $l \in \mathbb{Z}$, let C_l denote the set of codewords of C of weight l. For a subset U of an n-dimensional vector space $\mathbb{R}^n$ over the real field $\mathbb{R}$ and $m \in \mathbb{R}$, let U_m denote the set of vectors in U of norm m. For a lattice L, the dual lattice of L is denoted by L^*. For a group G and its subgroup H, $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer of H in G respectively. Let V be a VOA and let (M, Y_M) be a V-module. For an automorphism g of V, let $M \circ g$ denote the V-module $(M, Y_{M\circ g})$ defined by $Y_{M\circ g}(v, z) = Y_M(gv, z)$, $v \in V$.

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1 Preliminaries

In this section, we recall or give some definitions and facts necessary in this article.

1.1 Construction B

In this subsection, we recall a standard method for constructing lattices from linear binary codes.

Let n be a positive integer and let $\{\alpha_i |\ i \in \Omega_n\}$ be an orthogonal basis of $\mathbb{R}^n$ satisfying $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$. For a subset $J \subset \Omega_n$, we set $\alpha_J = \sum_{i \in J} \alpha_i$. Let C be a binary code of length n. Then

$$L_B(C) = \sum_{c \in C} \mathbb{Z}\frac{1}{2}\alpha_c + \sum_{i,j \in \Omega_n} \mathbb{Z}(\alpha_i + \alpha_j) \tag{1.1}$$

is called the *lattice obtained by Construction B* from C. We note that $L_B(C)$ is even if and only if C is doubly even. We call $\{\pm\alpha_i |\ i \in \Omega_n\}$ a *frame* of $L_B(C)$ with respect to the expression (1.1). The following lemma is easy to prove.

Lemma 1.1. $|L_B(C)_2| = 8|C_4|$.

1.2 Vertex operator algebra V_L^+

In this subsection, we review some properties of the vertex operator algebra V_L^+. For the details of its construction, see [FLM].

Let L be a (positive-definite) even lattice and let $\hat{L}$ be a central extension:

$$1 \to \langle \kappa_L |\ \kappa_L^2 = 1 \rangle \to \hat{L} \xrightarrow{-} L \to 1$$

such that $[a, b] = \kappa_L^{\langle \bar{a}, \bar{b} \rangle}$ for $a, b \in \hat{L}$. Let $\theta_{\hat{L}}$ be an involution of $\hat{L}$ induced by the -1-isometry of L. Set $K_L = \{a^{-1}\theta_{\hat{L}}(a) |\ a \in \hat{L}\}$. Then K_L is a normal subgroup of $\hat{L}$. Let V_L denote the VOA associated with L. The automorphism group $\mathrm{Aut}(V_L)$ of V_L contains an involution θ_{V_L} induced by $\theta_{\hat{L}}$. Its fixed-points on V_L is denoted by V_L^+. Then V_L^+ is a subVOA of V_L.

In [DN2, AD], it was shown that any irreducible V_L^+-module is isomorphic to one of $V_{\lambda+L}^{\pm}$ ($\lambda \in L^* \cap (L/2)$), $V_{\mu+L}$ ($\mu \in L^* \setminus (L/2)$) and $V_L^{T_\chi,\pm}$, where T_χ is an irreducible $\hat{L}/K_L$-module with central character χ. In this article, we use the following notation: $[\mu]$, $[\lambda]^{\pm}$ and $[\chi]^{\pm}$ denote the isomorphism classes of $V_{\mu+L}$, $V_{\lambda+L}^{\pm}$ and $V_L^{T_\chi,\pm}$ respectively. The isomorphism classes $[\mu]$, $[\lambda]^{\pm}$ are called *untwisted type* and the isomorphism classes $[\chi]^{\pm}$ are called *twisted type*.

Note 1.2. In this article, we take an involution on $V_L^{T_\chi}$ induced by the identity operator on T_χ and consider the ±1-eigenspace $V_L^{T,\pm}$. However in [FLM] an involution on $V_L^{T_\chi}$ induced by the -1-isometry on T_χ is used.

The fusion rules of V_L^+ were determined in [Ab, ADL]. In particular the following hold.

Lemma 1.3. [Ab, ADL]

(1) *Let λ be a vector in $L^* \cap (L/2)$. Then the fusion rules $[0]^- \times [\lambda]^{\pm} = [\lambda]^{\mp}$ hold.*

(2) *Let λ be a vector in $L^* \cap (L/2)$ satisfying $\langle \lambda, \lambda \rangle \in \mathbb{Z}$. Then the fusion rule $[\lambda]^{\varepsilon} \times [\lambda]^{\varepsilon} = [0]^+$ holds for any $\varepsilon \in \{\pm\}$.*

(3) *Let W_1 and W_2 be isomorphism classes of irreducible modules of V_L^+. If isomorphism classes of twisted type appear in $W_1 \times W_2$ then one of W_1 and W_2 is of twisted type.*

1.3 Automorphism groups of V_L and V_L^+

In this section, we review the results on automorphism groups of V_L and V_L^+ for even lattices L.

We start by recalling the automorphism group of V_L. For a lattice L, we denote by $O(L)$ the group of automorphisms of L which preserve the bilinear form. Let $O(\hat{L})$ denote the group of automorphisms of $\hat{L}$ which preserve the bilinear form on the quotient of $\hat{L}$ by its normal subgroup of order 2. For $g \in O(\hat{L})$, let $\bar{g}$ denote the linear automorphism of L defined by $\bar{g}(\bar{a}) = \overline{g(a)}$, $a \in \hat{L}$. We view an element $f \in \mathrm{Hom}(L, \mathbb{Z}_2)$ as the automorphism of $\hat{L}$ which sends a to $\kappa_L^{f(\bar{a})}a$. Hence we obtain an embedding $\mathrm{Hom}(L, \mathbb{Z}_2) \subset O(\hat{L})$. In Proposition 5.4.1 of [FLM], the following sequence is exact:

$$1 \to \mathrm{Hom}(L, \mathbb{Z}_2) \hookrightarrow O(\hat{L}) \xrightarrow{-} O(L) \to 1. \tag{1.2}$$

In [DN1], the automorphism group $\mathrm{Aut}(V_L)$ of V_L was described as follows:

Proposition 1.4. [DN1, Theorem 2.1] *Let L be an even lattice. Then $\mathrm{Aut}(V_L) = N(V_L)O(\hat{L})$, where $N(V_L) = \langle \exp(v_0) |\ v \in (V_L)_1 \rangle$ is a normal subgroup of $\mathrm{Aut}(V_L)$. Moreover, $\mathrm{Aut}(V_L)/N(V_L)$ is isomorphic to a quotient group of $O(L)$.*

In [Do] it was shown that any irreducible V_L-module is isomorphic to $V_{\lambda+L}$ for some $\lambda \in L^*$. The group $\mathrm{Aut}(V_L)$ acts on the set of isomorphism classes of irreducible V_L-modules as follows.

Lemma 1.5. (1) *Any element of $N(V_L)$ fixes all isomorphism classes of irreducible V_L-module.*

(2) *Let g be an element of $O(\hat{L})$ and let λ be a vector in L^*. Then g sends the isomorphism class of $V_{\lambda+L}$ to that of $V_{\bar{g}^{-1}(\lambda)+L}$.*

Now, let us consider automorphisms of V_L^+. By the definition of V_L^+, the centralizer $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$ acts on V_L^+. Set

$$H_L = C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle \theta_{V_L} \rangle.$$

Then H_L acts faithfully on V_L^+, and $H_L \subset \mathrm{Aut}(V_L^+)$. Let S_L denote the set of all isomorphism classes of irreducible V_L^+-modules. Then H_L is characterized as follows.

Lemma 1.6. [Sh, Proposition 3.10] *The group H_L is the stabilizer of $[0]^-$ under the action of $\mathrm{Aut}(V_L^+)$ on S_L.*

Since θ_{V_L} belongs to the center of $O(\hat{L})$, H_L contains $O(\hat{L})/\langle \theta_{V_L} \rangle$.

Lemma 1.7. [Sh, Proposition 2.9] *For $g \in O(\hat{L})/\langle \theta_{V_L} \rangle$, we have*

$$\begin{aligned} [\mu] \circ g &= [\bar{g}^{-1}[(\mu)], \ \mu \in L^* \setminus (L/2), \\ \{[\lambda]^{\pm} \circ g\} &= \{[\bar{g}^{-1}(\lambda)]^{\pm}\}, \ \lambda \in L^* \cap (L/2), \\ [0]^{\pm} \circ g &= [0]^{\pm}. \end{aligned}$$

Moreover for $\lambda \in L^* \cap (L/2)$ there exists an automorphism h of V_L^+ such that $[\lambda]^+ \circ h = [\lambda]^-$.

Let Q_L denote the orbit of $[0]^-$ under the action of $\mathrm{Aut}(V_L^+)$ on S_L. Since automorphisms of a VOA preserves the fusion rules and the graded dimensions, we have the following inclusions:

Lemma 1.8. [Sh, Lemma 3.12] *Let L be an even lattice of rank n.*

(1) *If $n \neq 8, 16$ then $Q_L \subseteq \{[0]^-, [\lambda]^\pm |\ \lambda \in L^* \cap (L/2),\ |(\lambda + L)_2| = 2n + |L_2|\}$.*

(2) *If $n = 8$ then $Q_L \subseteq \{[0]^-, [\lambda]^\pm,\ [\chi]^- |\ \lambda \in L^* \cap (L/2),\ |(\lambda+L)_2| = 2n+|L_2|\}$, where χ ranges over the central characters of $\hat{L}/K_L$ with $\chi(\kappa K_L) = -1$.*

(3) *If $n = 16$ then $Q_L \subseteq \{[0]^-, [\lambda]^\pm,\ [\chi]^+ |\ \lambda \in L^* \cap (L/2),\ |(\lambda + L)_2| = 2n + |L_2|\}$, where χ ranges over the central characters of $\hat{L}/K_L$ with $\chi(\kappa K_L) = -1$.*

Recall that a lattice L is said to be 2-*elementary* if $2L^* \subset L$, and said to be *totally even* if both $\sqrt{2}L^*$ and L are even.

Lemma 1.9. (1) [Sh, Proposition 3.14] *If Q_L contains isomorphism classes of twisted type then L is 2-elementary totally even.*

(2) [Sh, Lemma 3.6] *If L is 2-elementary totally even then isomorphism classes of twisted type with the same sign are conjugate under the action of $\mathrm{Aut}(V_L^+)$.*

1.4 Extra automorphisms of V_L^+

In this subsection we review automorphisms of V_L^+ not in H_L from [FLM].

Let C be a doubly even code of length n and let L be the lattice obtained by Construction B from C with frame $\{\alpha_i |\ i \in \Omega_n\}$. In Chapter 11 of [FLM], an automorphism σ not in H_L was constructed. This automorphism satisfies $[0]^- \circ \sigma = [\alpha_1]^+$. Lemma 1.6 shows that $\sigma \notin H_L$. By Lemma 1.7, there exists an automorphism h of V_L^+ such that $[\alpha_1]^+ \circ h = [\alpha_1]^-$.

We now assume that C contains the all-ones codeword. Let us see the action of σ on some isomorphism classes of irreducible V_L^+-modules. Set $\beta_n = \alpha_{\Omega_n}/4$ and $\gamma_n = \alpha_{\Omega_n}/4 - \alpha_1$. By the assumption, vectors β_n and γ_n belong to $L^* \cap (L/2)$. By [FLM, Theorem 11.5.1], $\{[\beta_n]^\pm \circ \sigma, [\gamma_n]^\pm \circ \sigma\} = \{[\chi_1]^\pm, [\chi_2]^\pm\}$ for some central characters χ_i of $\hat{L}/K_L$. Comparing the graded dimensions, we obtain $\{[\beta_n]^\pm \circ \sigma\} = \{[\chi_1]^+, [\chi_2]^+\}$ and $\{[\gamma_n]^\pm \circ \sigma\} = \{[\chi_1]^-, [\chi_2]^-\}$.

The result is summarized in the following lemmas.

Lemma 1.10. [FLM] *Let L be an even lattice obtained by Construction B with frame $\{\alpha_i\}$. Then the orbit of $[0]^-$ contains $[\alpha_1]^\pm$. In particular the cardinality of the orbit of $[0]^-$ is greater than 1.*

Lemma 1.11. [FLM] *Let L be the even lattice obtained by Construction B from a doubly even code C containing the all-one codeword. Let $\varepsilon \in \{\pm\}$.*

(1) *The orbit of the isomorphism classes* $[\beta_n]^\varepsilon$ *contains isomorphism classes of twisted type with sign* $+$.

(2) *The orbit of the isomorphism classes* $[\gamma_n]^\varepsilon$ *contains isomorphism classes of twisted type with sign* $-$.

2 Characterization of even lattices obtained by Construction B

In this section, we characterize even lattices obtained by Construction B. We will later use our characterization to determine the automorphism group of V_L^+.

Let L be a (positive-definite) even lattice of rank n. We set

$$R_L = \Big\{\lambda + L \in L^*/L \Big| \ \lambda \in L/2,\ |(\lambda+L)_2| \geq 2n + |L_2|\Big\}. \tag{2.1}$$

Note 2.1. The definition of R_L comes from the necessary conditions satisfied by isomorphism classes of untwisted type in Q_L (cf. Lemma 1.8).

Then even lattices obtained by Construction B are characterized as follows.

Theorem 2.2. *Let L be an even lattice of rank n. Then the following conditions are equivalent:*

(1) *L is obtained by Construction B.*

(2) *The set R_L is not empty.*

To prove this theorem, we need some lemmas.

Lemma 2.3. *Let L be an even lattice of rank n and let $\lambda + L$ be an element of R_L. Then $(\lambda + L)_2$ contains an orthogonal basis of $\mathbb{R}^n$.*

Proof. Since L is even and $\lambda \in L^*$, the norms of vectors in $\lambda + L$ are contained in $\langle\lambda,\lambda\rangle + 2\mathbb{Z}$. It follows from $(\lambda+L)_2 \neq \phi$ that $\langle\lambda,\lambda\rangle \in 2\mathbb{Z}$. Hence $L' = L + \mathbb{Z}\lambda$ is an even lattice and L'_2 forms a root system. In particular, the inner products of vectors in $(\lambda+L)_2$ are contained in $\{0,\pm1,\pm2\}$.

Let $Y_r = \{y_1,\dots,y_r\}$ be a subset of $(\lambda+L)_2$ such that $\langle y_i,y_j\rangle = 2\delta_{i,j}$. We set $\tilde{Y}_r = \{\pm y \mid y \in Y_r\}$. Then $|\tilde{Y}_r| = 2r$. To prove this lemma, we will show that if $r < n$ then there exists a vector in $(\lambda+L)_2$ orthogonal to Y_r. Define

$$X(Y_r) = \Big\{x \in (\lambda+L)_2 \Big| \ \langle x, y_i\rangle \in \{\pm1\} \text{ for } {}^\exists i \in \Omega_r\Big\},$$

where $\Omega_r = \{1,2,\dots,r\}$. Clearly $\tilde{Y}_r \cap X(Y_r) = \phi$. For $x \in X(Y_r)$, we set

$$m(x) = \min\Big\{i \in \Omega_r \Big| \ \langle y_i, x\rangle \in \{\pm1\}\Big\}. \tag{2.2}$$

Set $X(Y_r)^+ = \{x \in X(Y_r) |\ \langle x, y_{m(x)}\rangle = 1\}$. Then $|X(Y_r)| = 2|X(Y_r)^+|$. Since $|x - y_{m(x)}|^2 = |x|^2 + |y_{m(x)}|^2 - 2\langle x, y_{m(x)}\rangle = 2$ for $x \in X(Y_r)^+$, we consider the map $\rho : X(Y_r)^+ \to \{\{\pm v\}|\ v \in L_2\}$, $x \mapsto \{\pm(x - y_{m(x)})\}$.

Let us show that ρ is injective. First we suppose $x - y_{m(x)} = x' - y_{m(x')}$. If $m(x) = m(x')$ then $x = x'$. So we may assume that $m(x) < m(x')$. By the definition of Y_r and (2.2)

$$\langle y_{m(x)}, y_{m(x')}\rangle = \langle x', y_{m(x)}\rangle = 0. \tag{2.3}$$

Hence we have

$$2 = \langle x - y_{m(x)}, x' - y_{m(x')}\rangle = \langle x, x' - y_{m(x')}\rangle.$$

Since both x and $x' - y_{m(x')}$ belong to L_2', we have $x = x' - y_{m(x')}$. However it contradicts $(\lambda + L) \cap L = \phi$ since $x \in \lambda + L$ and $x' - y_{m(x')} \in L$.

Next we suppose $x - y_{m(x)} = y_{m(x')} - x'$. If $m(x) = m(x')$ then $x + x' = 2(y_{m(x)})$ and $|x|^2 = |x'|^2 = |y_{m(x)}|^2 = 2$. Hence $x = x' = y_{m(x)}$, which is a contradiction. So we may assume $m(x) < m(x')$. Then by (2.3), we have

$$2 = \langle x - y_{m(x)}, y_{m(x')} - x'\rangle = \langle x, y_{m(x')} - x'\rangle,$$

which implies that $x = y_{m(x')} - x'$. However, it contradicts $(\lambda + L) \cap L = \phi$. Hence ρ is injective. This shows that $|X(Y_r)^+| \le |L_2|/2$, namely $|X(Y_r)| \le |L_2|$. Since $\tilde{Y}_r \cap X(Y_r) = \phi$, we have $|\tilde{Y}_r \cup X(Y_r)| \le 2r + |L_2|$. So, if $r < n$ then there exists $x \in (\lambda + L)_2$ such that $x \notin X(Y_r) \cup \tilde{Y}_r$, namely $\langle x, Y_r\rangle = 0$. Therefore $(\lambda + L)_2$ contains an orthogonal basis of $\mathbb{R}^n$. □

Lemma 2.4. *Let L be an even lattice of rank n. For $\lambda + L \in R_L$, there exist a doubly even code C and a frame in $\lambda + L$ such that $L = L_B(C)$.*

Proof. By Lemma 2.3, $\lambda + L$ contains vectors $e_1, e_2, \dots, e_n$ satisfying $\langle e_i, e_j\rangle = 2\delta_{i,j}$. Since $2\lambda \in L$, we have $e_i \pm e_j \in L$. Set $E = \oplus_{i=1}^n \mathbb{Z}e_i$ and $L' = L + \mathbb{Z}\lambda$. Then L'/E is a subspace of $E^*/E \cong \mathbb{Z}_2^n$. So we regard L'/E as a binary code C of length n. We can choose a basis B in $\{\pm e_i|\ i \in \Omega_n\}$ so that L is the lattice obtained by Construction B from C with frame B (cf. the proof of [Sh, Proposition 1.8]). □

Proof of Theorem 2.2. Suppose (1). Let $\{\alpha_i\}$ be the frame. Then $|(\alpha_1 + L)_2| = 2n + |L_2|$, and $\alpha_1 + L \in R_L$. Hence (1) ⇒ (2). It follows from Lemma 2.4 that (2) ⇒ (1). □

Remark 2.5. *The proof of Theorem 2.2 implies that $|(\lambda + L)_2| = 2n + |L_2|$ for any $\lambda + L \in R_L$.*

Let us show some lemmas by using Theorem 2.2. Let L be the even lattice obtained by Construction B from a doubly even code C of length n with frame $\{\pm\alpha_i|\ i \in \Omega_n\}$. Set $\beta_n = \alpha_{\Omega_n}/4$ and $\gamma_n = \alpha_{\Omega_n}/4 - \alpha_1$.

Lemma 2.6. *The following conditions are equivalent:*

(1) $\gamma_n + L \in R_L$.

(2) $n = 8$ *and* C *contains the all-one codeword.*

Proof. Suppose (2). Since C contains the all-one codeword, $\gamma_8 \in L/2$. Clearly $\gamma_8 \in L^*$. It is easy to see that

$$(\gamma_8+L)_2 = \{\pm(\alpha_{\Omega_8}/4-\alpha_i), \alpha_{\Omega_8}-\alpha_c/2+\alpha_j, \alpha_{\Omega_8}-\alpha_c/2-\alpha_k | \ c \in C_4, i \in \Omega_8, j \in c, k \in \Omega_8\backslash c\}.$$

Hence $|(\gamma_8 + L)_2| = 16 + 8|C_4| = 16 + |L_2|$ by Lemma 1.1. Thus $\gamma_8 + L \in R_L$.

Conversely, we suppose (1). Since the norm of γ_n is minimal in γ_n+L and it is $1+n/8$, the rank n of L must be 8. Since $\gamma_8 \in L/2$, we obtain $\alpha_{\Omega_8}/2 \in L$. Hence the all-one codeword belongs to C. □

Lemma 2.7. *The following conditions are equivalent:*

(1) $\beta_n + L \in R_L$.

(2) $n = 16$ *and* C *contains a subcode isomorphic to the Reed-Muller code* $RM(1, 4)$.

Proof. Let k be the dimension of C. Suppose (2). In [PLF], doubly even codes of length 16 containing the all-one codeword were classified. In particular doubly even codes of length 16 containing $RM(1, 4)$ can be classified. Hence we obtain $|C_4| = 0, 4, 12, 28$ for $k = 5, 6, 7, 8$ respectively. So $32 + |L_2| = 2^k$. On the other hand,

$$(\beta_{16} + L)_2 = \{\beta_{16} - \alpha_c/2| \ c \in C\}. \tag{2.4}$$

Hence $|(\beta_{16} + L)_2| = 2^k$. Therefore $\beta_{16} + L \in R_L$.

Conversely we suppose (1). Since the norm of β_n is minimal in $\beta_n + L$ and it is $n/8$, the rank n of L must be 16. By Lemma 2.3, $(\beta_{16} + L)_2$ contains an orthogonal basis F. Set $\tilde{F} = \{\pm v| \ v \in F\}$. By (2.4), $\tilde{F} = \{\beta_{16}-\alpha_c/2| \ c \in D\}$ for some subset D of C. Clearly $|D| = 32$. Let d be an element of D. Set $D^0 = d + D$. Then $\tilde{F}^0 = \{\beta_{16} - \alpha_c/2| \ c \in D^0\}$ is a set of 32 vectors, two of which are equal, opposite, or orthogonal. Since D^0 contains the all-zero codeword, D^0 consists of the all-zero and all-one codewords and 30 codewords with weight 8. Moreover, the cardinality of any intersection of codewords with weight 8 in D^0 is 0, 4 or 8. Hence D^0 must be isomorphic to the Reed-Muller code $RM(1, 4)$. Therefore C contains a subcode isomorphic to the Reed-Muller code $RM(1, 4)$. □

3 Automorphism groups of V_L^+ for even unimodular lattices of rank 8 and 16

In this section, we determine the automorphism groups of V_L^+ for even unimodular lattices of rank 8 and 16. In particular, we will compare $\mathrm{Aut}(V_L^+)$ with its subgroup $H_L \cong C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle$.

Let L be an even unimodular lattice of rank 8 or 16. Then the VOA V_L^+ has exactly 4 isomorphism classes of irreducible V_L^+-modules $[0]^\pm$ and $[\chi_0]^\pm$, where χ_0 is the unique faithful character of $\hat{L}/K_L$. Since the graded dimensions of $[\chi_0]^+$ and $[\chi_0]^-$ are different, the cardinality of the orbit Q_L of $[0]^-$ is 1 or 2. In the following subsections, we will determine $|Q_L|$.

3.1 Automorphism group of V_L^+ for the even unimodular lattice of rank 8

In this subsection, we study the automorphism group of $V_{E_8}^+$, where E_8 is the unique even unimodular lattice of rank 8 up to isomorphism.

Lemma 3.1. *There are automorphisms of $V_{E_8}^+$ mapping $[0]^-$ to $[\chi_0]^-$. In particular Q_{E_8} contains isomorphism classes of twisted type.*

Proof. The degree 1 subspace of $V_{E_8}^+$ forms the simple Lie algebra of type D_8 under the 0-th product and $V_{E_8}^+ \cong V_{D_8}$. By [Do], V_{D_8} has exactly 4 non-isomorphic irreducible modules $V_{\lambda+D_8}$, $\lambda + D_8 \in D_8^*/D_8$.

On the other hand, there exists an involution τ of the root lattice of type D_8 such that τ exchanges two elements of D_8^*/D_8. By Lemma 1.5 (2), lifts of τ exchange two isomorphism classes of irreducible V_{D_8}-modules. This shows that there are automorphisms of $V_{E_8}^+$ mapping $[0]^-$ to $[\chi_0]^-$. □

Proposition 3.2. *The group H_{E_8} is a normal subgroup of* $\mathrm{Aut}(V_{E_8}^+)$ *of index* 2. *In particular* $\mathrm{Aut}(V_{E_8}^+)/H_{E_8} \cong \mathbb{Z}_2$.

Proof. Lemma 3.1 shows that the cardinality of the orbit Q_{E_8} of $[0]^-$ is 2. By Lemma 1.6 H_{E_8} is a subgroup of the index 2 of $\mathrm{Aut}(V_{E_8}^+)$. □

3.2 Automorphism groups of V_L^+ for even unimodular lattices of rank 16

In this subsection, we study the automorphism groups of V_L^+ for even unimodular lattices L of rank 16. It is known that $E_8 \oplus E_8$ and Γ_{16} are the only even unimodular lattices of rank 16 up to isomorphism (cf. [CS]). We note that the root sublattice of Γ_{16} is of type D_{16}.

Let U be a root lattice of type $D_8 \oplus D_8$. Let N be an even overlattice of U such that $|N : U| = 2$ and $N_2 = U_2$. It is easy to check that N is unique up to isomorphism. Since the determinant of N is 4, there are three unimodular overlattices of N. In particular even unimodular lattices $E_8 \oplus E_8$ and Γ_{16} are obtained as overlattices of N.

Lemma 3.3. *Any element of* $\mathrm{Aut}(V_N)$ *of V_N fixes all isomorphism class of irreducible V_N-modules.*

Proof. The automorphism group $O(N)$ of N acts on N^*/N. Since unimodular overlattices of N are non-isomorphic, $O(N)$ fixes all elements of N^*/N. By Lemma 1.5 (2), we obtain this lemma. □

Lemma 3.4. *The VOAs $V_{\Gamma_{16}}^+$ and $V_{E_8\oplus E_8}^+$ are isomorphic to V_N.*

Proof. First we consider the lattice $E_8 \oplus E_8$. Since $V_{E_8}^+$ is isomorphic to V_{D_8}, $V_{E_8\oplus E_8}^+$ contains a subVOA V isomorphic to V_U. Since V_U is rational, $V_{E_8\oplus E_8}^+ = V \oplus M$ as V-module for some V-module M. By the classification of irreducible modules of V_U ([Do]), M is isomorphic to the irreducible V_U-module $V_{\lambda+U}$, where $\lambda + U \in U^*/U$ satisfying $\langle \lambda, U\rangle = N$. Since $V_N = V_U \oplus V_{\lambda+U}$ is a simple current extension of V_U, $V_{E_8\oplus E_8}^+$ has a unique VOA structure extending its V-module structure (cf. Proposition 5.3 in [DM]) and $V_{E_8\oplus E_8}^+ \cong V_N$.

Next, we consider the lattice Γ_{16}. Since the root sublattice of Γ_{16} is D_{16}, $V_{\Gamma_{16}}^+$ contains $V_{D_{16}}^+$. The degree 1 subspace of $V_{D_{16}}^+$ forms a simple Lie algebra of type $D_8 \oplus D_8$ under the 0-th product and $V_{D_{16}}^+ \cong V_U$. Similarly to the argument above, we obtain $V_{\Gamma_{16}}^+ \cong V_N$. □

This lemma shows that the cardinality of the orbit Q_L of $[0]^-$ is 1 for any even unimodular lattice L of rank 16. By Lemma 1.6 $\mathrm{Aut}(V_L^+)$ is coincides with H_L.

Proposition 3.5. *The automorphism group $\mathrm{Aut}(V_L^+)$ of V_L^+ coincides with H_L for any even unimodular lattice L of rank 16.*

4 The orbit of the isomorphism class of V_L^-

In this section, we determine the orbit Q_L of $[0]^-$. We note that Q_L was determined in [Sh] when L has no roots.

Lemma 4.1. *The orbit Q_L contains the isomorphism class $[\lambda]^\varepsilon$ for any $\lambda \in R_L$, $\varepsilon \in \{\pm\}$.*

Proof. By Lemma 1.8 any isomorphism class of untwisted type in Q_L must be $[\lambda]^\varepsilon$ for some $\lambda \in R_L$ and $\varepsilon \in \{\pm\}$. Conversely by Lemma 1.10 and 2.4 Q_L contains $[\lambda]^\varepsilon$ for all $\lambda + L \in R_L$ and $\varepsilon \in \{\pm\}$. □

So let us discuss the cases where Q_L contains isomorphism classes of twisted type. We consider the following three conditions on even lattices L:

(a) L is obtained by Construction B from a doubly even code of length 8 containing the all-one codeword.

(b) L is obtained by Construction B from a doubly even code of length 16 containing a subcode isomorphic to the first order Reed-Muller code $RM(1,4)$ of length 16.

(c) L is isomorphic to the E_8-lattice.

Proposition 4.2. *The orbit Q_L contains isomorphism classes of twisted type if and only if L satisfies (a), (b) or (c).*

Proof. By Lemma 1.11, 2.6, 2.7 and 3.1 if L satisfies (a), (b) or (c) then Q_L contains isomorphism classes of irreducible V_L^+-modules of twisted type.

So we suppose that Q_L contains an isomorphism class $[\chi]^\varepsilon$ of twisted type. Then by Lemma 1.8 the rank of L is 8 or 16, and $\varepsilon = -$ and $+$ if $n = 8$ and 16 respectively. Moreover by Lemma 1.9 (1) L is 2-elementary totally even. If L is unimodular then L is isomorphic to one of E_8, $E_8 \oplus E_8$ and Γ_{16}. By the result of the previous section, L must be isomorphic to E_8. Hence L satisfies (c).

We now assume that L is not unimodular. Let us show that Q_L contains $[\lambda]^\delta$ for some $\lambda \in L^* \cap (L/2)$, $\delta \in \{\pm\}$. By comparing the coefficients of q in the graded dimensions of V_L^- and $V_L^{T_\chi,\varepsilon}$, the theta series of L is written by the Dedekind-eta series. By using the transformation formula on theta series of lattices and their dual lattices, we can describe the theta series of L^*. In particular, $L^* \setminus L$ has vectors of norm 2 (cf. the proof of Proposition 3.14 in [Sh]). Let λ be an element of L^* such that $(\lambda + L)_2 \neq \phi$. Let g be an element of $\mathrm{Aut}(V_L^+)$ such that $[0]^- \circ g = [\chi]^\varepsilon$. By Lemma 1.3 (1), we obtain $[\chi]^\varepsilon \times ([\lambda]^+ \circ g) = [\lambda]^- \circ g$. By Lemma 1.3 (3) one of $[\lambda]^\pm \circ g$ must be of twisted type. By comparing the graded dimensions, it has the same sign ε. By Lemma 1.9 (2), Q_L contains $[\lambda]^\delta$ for some $\delta \in \{\pm\}$. By Lemma 1.8 $\lambda + L \in R_L$. Thus L is obtained by Construction B from a code C by Theorem 2.2.

Since L is 2-elementary totally even, C contains the all-one codeword. Hence (a) holds if the rank of L is 8. Consider the case where $n = 16$. Since the theta series of L is described in terms of the weight enumerator of C, we can describe the weight enumerator of C. By using the classification of even codes of length 16 [PLF], (b) holds if the rank of L is 16. □

By Lemma 1.8, 1.9, 4.1 and Proposition 4.2, the orbit Q_L is determined.

Theorem 4.3. *Let L be an even lattice of rank n.*

(1) *If L satisfies (a) or (c) then $Q_L = \{[0]^-, [\lambda]^\pm,\ [\chi]^- |\ \lambda \in L^* \cap (L/2),\ |(\lambda + L)_2| = 2n + |L_2|\}$, where χ ranges over the central characters of $\hat{L}/K_L$ with $\chi(\kappa K_L) = -1$ and $\varepsilon = +$.*

(2) *If L satisfies (b) then $Q_L = \{[0]^-, [\lambda]^\pm,\ [\chi]^+ |\ \lambda \in L^* \cap (L/2),\ |(\lambda + L)_2| = 2n + |L_2|\}$, where χ ranges over the central characters of $\hat{L}/K_L$ with $\chi(\kappa K_L) = -1$.*

(3) *If L does not satisfy neither (a), (b) nor (c) then $Q_L = \{[0]^-, [\lambda]^\pm |\ \lambda \in L^* \cap (L/2),\ |(\lambda + L)_2| = 2n + |L_2|\}$.*

By Lemma 1.6, Theorem 2.2 and 4.3, we have the following corollary.

Corollary 4.4. *The automorphism group $\mathrm{Aut}(V_L^+)$ of V_L^+ is greater than H_L if and only if the even lattice L satisfies one of the following:*

(1) *L is obtained by Construction B.*

(2) *L is isomorphic to the E_8-lattice.*

5 A method of determining of the shape of the automorphism group of V_L^+

In this section, we give a method of determining the shape of $\mathrm{Aut}(V_L^+)$ for an arbitrary even lattice L. This method is a generalization of that in [Sh, Section 3.4]. For the conditions (a), (b) and (c), see the previous section. If L satisfies (c) then $\mathrm{Aut}(V_L^+)$ is determined in Section 3.1.

First we consider the lattice L satisfying neither (a), (b) nor (c). Then $Q_L = \{[0]^-, [\lambda]^\pm |\ \lambda \in R_L\}$, and $P_L = \{[0]^+\} \cup Q_L$ has an elementary abelian 2-group structure under the fusion rules (cf. [Sh, Proposition 3.17]). So we obtain a group homomorphism φ_L from $\mathrm{Aut}(V_L^+)$ to $GL(P_L)$. Since the kernel of φ_L is a subgroup of H_L, it can be determined. Moreover the index of $\varphi(H_L)$ in $\mathrm{Im}\ \varphi_L$ is equal to the cardinality of Q_L. Hence we can determine the image of φ_L. Therefore we can calculate the shape of $\mathrm{Aut}(V_L^+)$ in principle.

Suppose that L satisfies (a) or (b). In this case, we consider the set S_L of all isomorphism classes of irreducible V_L^+-modules. Since L is 2-elementary totally even, S_L has an elementary abelian 2-group structure under the fusion rules (cf. [Ab, ADL] and [Sh, Proposition 3.4]). Moreover S_L has a natural quadratic form associated with a non-singular symplectic form preserved by the action of $\mathrm{Aut}(V_L^+)$ (cf. [Sh, Theorem 3.8]). Hence we obtain a group homomorphism ψ_L from $\mathrm{Aut}(V_L^+)$ to the orthogonal group $O(S_L)$ associated with the quadratic form. Similarly to the case above, we can determine the image and kernel of ψ_L, and we can describe the shape of $\mathrm{Aut}(V_L^+)$ in principle.

Note 5.1. For many important lattices L without roots, the shapes of $\mathrm{Aut}(V_L^+)$ were determined in [Sh, Section 4] by using this method.

6 Automorphism groups of VOSAs V_L^+ for odd lattices

Let L be an odd lattice. In this section, we consider the vertex operator superalgebra V_L^+. For $i \in \{0,1\}$, set $L^i = \{v \in L |\ \langle v,v\rangle \equiv i\ (2)\}$. Then L^0 is an even sublattice of L. We will describe $\mathrm{Aut}(V_L^+)$ by using $\mathrm{Aut}(V_{L^0}^+)$.

Let $\mathrm{Aut}(V_{L^0}^+; V_{L^1}^+)$ denote the subgroup of $\mathrm{Aut}(V_{L_0}^+)$ fixing the isomorphism class of $V_{L^1}^+$. Let α be a vector in L^1. Then $2\alpha \in L^0$ and $\langle \alpha,\alpha\rangle \in \mathbb{Z}$. By Lemma 1.3 (2) $[\alpha]^+ \times [\alpha]^+ = [0]^+$. Let τ denote the involution acting as $(-1)^i$ on $V_{L^i}^+$. Applying [Sh, Theorem 3.3] to our case, we obtain $C_{\mathrm{Aut}(V_L^+)}(\tau)/\langle\tau\rangle \cong \mathrm{Aut}(V_{L^0}^+; V_{L^1}^+)$.

On the other hand, any automorphism of V_L^+ preserves both $V_{L^0}^+$ and $V_{L^1}^+$ since the graded dimensions of $V_{L^0}^+$ and $V_{L^1}^+$ are in $\mathbb{Z}[[q]]$ and in $\mathbb{Z}q^{1/2}[[q]]$ respectively. Hence $C_{\mathrm{Aut}(V_L^+)}(\tau) = \mathrm{Aut}(V_L^+)$. Therefore we have the following proposition.

Proposition 6.1. *Let L be an odd lattice. Then* $\mathrm{Aut}(V_L^+) \cong \langle\tau\rangle.\mathrm{Aut}(V_{L^0}^+; V_{L^1}^+)$.

Since the shape of $\mathrm{Aut}(V_{L^0}^+)$ can be described by the method given in the previous section, $\mathrm{Aut}(V_L^+)$ can be determined in principle.

7 Examples

In this section, we calculate $\mathrm{Aut}(V_L^+)$ for some lattices by using the method of Section 5.

7.1 Even lattices of rank one, two and three

In this section, we determine $\mathrm{Aut}(V_L^+)$ for even lattices of rank one two and three.

Let L be an even lattice L of rank n. Suppose that $n \leq 3$. By Theorem 4.3, $Q_L = \{[0]^-, [\lambda]^\pm |\ \lambda \in R_L\}$. So we consider R_L. By Theorem 2.2, let us consider even lattices obtained by Construction B. It is easy to see that a code C of length n is doubly even if and only if C consists of the all-zero codeword. Hence L is obtained by Construction B if and only if $L \cong 2A_1$, $\sqrt{2}(A_1 \oplus A_1)$ or $\sqrt{2}A_3$. If L is not obtained by Construction B then $\mathrm{Aut}(V_L^+) \cong C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle$. The case where $L \cong \sqrt{2}A_3$ was done in Theorem 4.3 of [Sh]. So let us consider the automorphism groups of V_L^+ for $2A_1$ and $\sqrt{2}(A_1 \oplus A_1)$.

First we consider the case where $L \cong 2A_1$. Let γ be a generator of L. Then $R_L = \{\gamma/2 + L\}$. Hence $Q_L = \{[0]^-, [\gamma/2]^\pm\}$. Set $P_L = \{[0]^+\} \cup Q_L$. Then P_L has an elementary abelian 2-group structure under the fusion rules and, $P_L \cong \mathbb{F}_2^2$. So we obtain a group homomorphism $\varphi_L : \mathrm{Aut}(V_L^+) \to GL(P_L)$. On the other hand, $H_L \cong \mathbb{Z}_2$ and its generator exchanges $[\gamma/2]^+$ and $[\gamma/2]^-$. Since Ker φ_L is a subgroup of H_L, φ_L is injective. Clearly $\varphi_L(H_L)$ is a maximal subgroup of $GL(P_L) \cong S_3$. Since $\mathrm{Aut}(V_L^+)$ contains automorphisms not in H_L (cf. Lemma 1.10), φ_L is surjective. Thus we obtain $\mathrm{Aut}(V_L^+) \cong S_3$.

Next let us consider the case where $L \cong \sqrt{2}(A_1 \oplus A_1)$. Let $\{a_1, a_2\}$ be a basis of L satisfying $\langle a_i, a_j\rangle = 4\delta_{i,j}$. Set $a_i^* = a_i/4$ and $b = 2(a_1^* + a_2^*)$. Then $\{a_1^*, a_2^*\}$ is a basis of the dual lattice of L and $R_L = \{b + L\}$. So $Q_L = \{[0]^-, [b]^\pm\}$. Set $P_L = \{[0]^+\} \cup Q_L$. Then P_L has an elementary abelian 2-group structure under the fusion rules and $P_L \cong \mathbb{F}_2^2$. So we obtain a group homomorphism $\varphi_L : \mathrm{Aut}(V_L^+) \to GL(P_L)$. On the other hand, H_L is isomorphic to the direct product of the dihedral group of order 8 and the group of order 2. The kernel of φ_L is isomorphic to 2^3, and H_L contains elements exchanging $[b]^+$ and $[b]^-$. So $\varphi_L(H_L)$ is a maximal subgroup of $GL(P_L) \cong S_3$. Since $\mathrm{Aut}(V_L^+)$ contains automorphisms not in H_L, φ_L is surjective. Therefore we obtain $\mathrm{Aut}(V_L^+) \cong 2^3.S_3$. It is easy to check that $\mathrm{Aut}(V_L^+) \cong S_4 \times \mathbb{Z}_2$.

The result is summarized in the following proposition.

Proposition 7.1. *Let L be an even lattice of rank one, two or three. Then*

$$\mathrm{Aut}(V_L^+) \cong \begin{cases} S_3 & \text{if } L \cong 2A_1, \\ S_4 \times \mathbb{Z}_2 & \text{if } L \cong \sqrt{2}(A_1 \oplus A_1), \\ (2^2 : S_4).S_3 & \text{if } L \cong \sqrt{2}A_3, \\ C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle & \text{otherwise.} \end{cases}$$

Note 7.2. The automorphism groups of V_L^+ for lattices of rank one and two were already obtained in [DG1] and [DG2] respectively by using the action of $\mathrm{Aut}(V_L^+)$ on certain homogeneous subspaces of V_L^+. In the articles, more precise structures of $\mathrm{Aut}(V_L^+)$ were described.

7.2 Even unimodular lattices

Let L be an even unimodular lattice. Since the determinant of any lattice obtained by Construction B is not 1, L is not obtained by Construction B. Hence $R_L = \phi$ by Theorem 2.2. By Theorem 4.3, $|Q_L| = 2$ if $L \cong E_8$, and $|Q_L| = 1$ if $L \not\cong E_8$. By Lemma 1.6 and Proposition 3.2, we obtain the following proposition.

Proposition 7.3. *Let L be an even unimodular lattice of rank n. Then*

$$\mathrm{Aut}(V_L^+) \cong \begin{cases} (C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle).\mathbb{Z}_2 & \text{if } \mathrm{rank}L = 8, \\ C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle & \text{if } \mathrm{rank}L \geq 16. \end{cases}$$

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