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INVARIANT SUBSPACES OF FINITE CODIMENSION AND UNIFORM ALGEBRAS

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Abstract. Let A be a uniform algebra on a compact Hausdorff space X and m a probability measure on X. Let $H^p(m)$ be the norm closure of A in $L^p(m)$ with $1 \leq p < \infty$ and $H^\infty(m)$ the weak $*$ closure of A in $L^\infty(m)$. In this paper, we describe a closed ideal of A and exhibit a closed invariant subspace of $H^p(m)$ for A that is of finite codimension.

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1. Introduction. Let A be a uniform algebra on a compact Hausdorff space X. $M(A)$ denotes the maximal ideal space of A. Let m be a probability measure on X. $H^p(m)$ denotes the norm closure of A in $L^p(m)$ with $1 \leq p < \infty$ and $H^\infty(m)$ denotes the weak $*$ closure of A in $L^\infty(m)$. $H^p(m)$ is called an *abstract Hardy space*. When A is a disc algebra, if m is the normalized Lebesgue measure on the unit circle, $H^p(m)$ is the usual Hardy space and if m is the normalized area measure on the unit disc, $H^p(m)$ is the usual Bergman space.

Let I be a closed ideal of A. In this paper, we are interested in I with $\dim A/I < \infty$. Then A/I is called a Q-*algebra*. Two dimensional Q-algebras can be described easily; that is, $I = \{f \in A;\ \phi_1(f) = \phi_2(f) = 0\}$, where $\phi_j \in M(A)\ \ (j = 1, 2)$, or $I = \{f \in A;\ \phi(f) = D_\phi(f) = 0\}$, where $\phi \in M(A)$ and D_ϕ is a bounded point derivation at ϕ. One of the authors [**3**] showed that a two dimensional operator algebra on a Hilbert space is a Q-algebra. It seems to be worthwile to describe a finite dimensional Q-algebra. In Section 2, we describe an ideal I with $\dim A/I < \infty$ using a theorem of T. W. Gamelin [**2**]. As a result, a finite dimensional Q-algebra is described.

When M is a closed subspace of $H^p(m)$ and $AM \subset M$, M is called an *invariant subspace for A*. In this paper, we are interested in M with $\dim H^p(m)/M < \infty$. When A is the polydisc algebra on T^n and m is the normalized Lebesgue measure on T^n, a finite codimensional invariant subspace M in $H^p(m)$ was described by P. Ahern and D. N. Clark [**1**] using the ideals in the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ of finite codimension whose zero sets are contained in the polydisk D^n. In Section 3, for an

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arbitrary uniform algebra A we describe a finite codimensional invariant subspace M in $H^p(m)$ using the result in Section 2.

2. Finite codimensional ideal. Let $\phi \in M(A)$. A closed subalgebra H of A is a (ϕ, k)-*subalgebra* if there is a sequence of closed subalgebras $A = A_0 \supset A_1 \supset \cdots \supset A_k = H$ such that A_j is the kernel of a continuous point derivation D_j of A_{j-1} at ϕ. If H is a (ϕ, k)-subalgebra of A, then H has finite codimension in A and $M(H) = M(A)$ by [**2**, Lemma 9.1].

If I is a closed ideal of A and A/I is of finite dimension, $B = \mathbb{C} + I$ is a closed subalgebra of A, and A/B is of finite dimension. By a theorem of T. W. Gamelin [**2**, Theorem 9.8], we can describe B and so also I. Since B is a special closed subalgebra of A we can describe I more explicitly.

THEOREM 1. *If I is a closed ideal of A and A/I is of finite dimension, then there exists a closed subalgebra $E = E(I)$ of A such that $E = \{f \in A : \phi_1(f) = \cdots = \phi_n(f)\}$, $1 \leq n < \infty$, $\{\phi_j\} \subset M(A)$ and*

$$I = H^E_\phi \cap \ker \phi,$$

where $\phi = \phi_j|E$, $1 \leq j \leq n$ and H^E_ϕ is a (ϕ, k)-subalgebra with respect to E for some k.

Proof. Put $H = I + \mathbb{C}$; then A/H is of finite dimension. By a theorem of T. W. Gamelin [**2**, Theorem 9.8], H can be obtained from A in two steps.

(i) There exist pairs of points ψ_j, ψ'_j, $1 \leq j \leq \ell$, in $M(A)$ such that if E consists of the $f \in A$ such that $\psi_j(f) = \psi'_j(f)$, $1 \leq j \leq \ell$, then $H \subset E \subset A$.

(ii) There exist distinct points $\theta_j \in M(E)$ and θ_j-subalgebras H_j of E, $1 \leq j \leq k$, such that $H = H_1 \cap \cdots \cap H_k$.

Put $\tilde{\psi}_j = \psi_j|E = \psi'_j|E$ for $1 \leq j \leq \ell$; then $\tilde{\psi}_j$ belongs to $M(E)$. Since I is an ideal of A, $I \subset \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j$. To see this, let $f \in A$ such that $\psi_j(f) \neq \psi'_j(f)$. If $g \in I$, then $fg \in I$ but $\psi_j(fg) \neq \psi'_j(fg)$ when $\tilde{\psi}_j(g) \neq 0$. This contradicts the fact that $fg \in E$. Thus $\tilde{\psi}_j(g) = 0$. Hence $I \subset \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j$ and so $H \subseteq \bigcap_{j=1}^{\ell} \ker \tilde{\psi}_j + \mathbb{C}$. By the definition of E, $\tilde{\psi}_1 = \cdots = \tilde{\psi}_\ell$. Therefore E has the form $E = \{f \in A; \phi_1(f) = \cdots = \phi_n(f)\}$, $1 \leq n < \infty$, and $\{\phi_j\} \subset M(A)$.

For each j with $1 \leq j \leq k$, H_j is a θ_j-subalgebra of E for $\theta_j \in M(E)$. Hence there is a sequence of closed subalgebras $E = E_{j0} \supset E_{j1} \supset \cdots \supset E_{j\ell_j} = H_j$ such that E_{jt} is the kernel of a continuous point derivation D_{jt} of E_{jt-1} at θ_j. We shall write $E_{j\ell_j} = \ker D_{\theta_j}$, where D_{θ_j} is a derivation on $E_{j(\ell_j-1)}$. Then $H = \bigcap_{j=1}^{k} \ker D_{\theta_j}$ and so $I = \{\bigcap_{j=1}^{k} \ker D_{\theta_j}\} \cap \ker \theta$, for some $\theta \in M(H)$. Suppose that g is an arbitrary function in I. For any $j (1 \leq j \leq k)$, there exists a function $f \in E_{j(\ell_j-1)}$ such that $f \notin E_{j\ell_j} = \ker D_{\theta_j}$. Since $fg \in I$ and $D_{\theta_j}(g) = 0$, $D_{\theta_j}(fg) = \theta_j(g) D_{\theta_j}(f) = 0$ because D_{θ_j} is a derivation on $E_{j(\ell_j-1)}$. This implies that $\theta_j(g) = 0$. Hence $I \subset \bigcap_{j=1}^{k} \ker \theta_j$. Therefore by the definition of E, $\theta_1 = \cdots = \theta_k \in M(E)$, and so $H_1 = \cdots = H_k$. Thus $\theta_1|H = \theta$ and $I = (\ker D_{\theta_1}) \cap \ker \theta_1$. Since $I \subset \bigcap_{j=1}^{n} \ker \phi_j$, $I \subset (\ker \phi_1) \cap (\ker D_{\theta_1}) \cap \ker \theta_1$ and so $\phi_1|E = \theta_1$.

COROLLARY 1. *If I is a closed ideal of A and A/I is of finite dimension 2, then $I = \{f \in A;\ \phi_1(f) = \phi_2(f) = 0\}$, where $\phi_j \in M(A)$ $(j = 1, 2)$ and $\phi_1 \neq \phi_2$, or $I = \{f \in A;\ \phi(f) = D_\phi(f) = 0\}$, where $\phi \in M(A)$, and D_ϕ is a bounded point derivation at ϕ.*

Proof. When $\dim A/I = 2$, by Theorem 1, $E = A$ or $E = \{f \in A;\ \phi_1(f) = \phi_2(f)\}$. If $E = A$, then $H^E_\phi = \{f \in A;\ D_\phi(f) = 0\}$ and if $E = \{f \in A;\ \phi_1(f) = \phi_2(f)\}$, then $H^E_\phi = E$, since $\dim A/H^E_\phi = 1$ because $H^E_\phi = I + \mathbb{C}$. This implies the corollary.

COROLLARY 2. *If B is a finite dimensional Q-algebra and $B_0 = \operatorname{rad} B$ is its radical, then there exist subalgebras $B_1, B_2, \ldots, B_{k+1}$ in B_0 such that $B_{k+1} = \{0\}$, $\dim B_j/B_{j+1} = 1$ and B_{j+1} is an ideal of B_j for $j = 0, 1, \ldots, k$. Hence* rad B *has a basis $\{f_0, f_1, \ldots, f_k\}$ such that $(f_j)^{2((k+1)-j)} = 0$ for $j = 0, 1, \ldots, k$.*

Proof. Since B is a Q-algebra, $B = A/I$ for some uniform algebra A and some closed ideal I of A. Also, since B is of finite dimension, we can apply Theorem 1 to A and I. In the notation of Theorem 1, rad $B = \{f \in E;\ \phi(f) = 0\}/I$. Since H^E_ϕ is a ϕ-subalgebra with respect to E, there exists a sequence of closed subalgebras $E = E_0 \underset{\neq}{\supset} E_2 \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} E_{k+1} = H^E_\phi$ such that E_j is the kernel of a continuous point derivation D_j of E_{j-1} at ϕ. Hence $E_{j+1} \cap \ker\phi$ is an ideal of $E_j \cap \ker\phi$ and $\dim\{E_j \cap \ker\phi/E_{j+1} \cap \ker\phi\} = 1$. Put $B_j = (E_j \cap \ker\phi)/I$. Then $\dim B_j/B_{j+1} = 1$ and B_{j+1} is an ideal of B_j, for $j = 0, 1, \ldots, k$, and $B_{k+1} = \{0\}$. For each j, there exists f_j such that $B_j = \langle f_j\rangle + B_{j+1}$ and then $\{f_0, f_1, \ldots, f_k\}$ is a basis of rad $B = B_0$. Observe that f_j^2 belongs to B_{j+1} because $E_{j+1} = \ker D_{j+1}$. Thus $(f_j)^{2(k+1-j)} = 0$.

3. Finite codimensional invariant subspace. For a subset S of $H^p(m)$, $[S]_p$ denotes the closure of S in $H^p(m)$.

THEOREM 2. *If M is an invariant subspace of H^p with $\dim H^p/M = n < \infty$, then there exists a closed ideal of A such that $\dim A/I = n$, $[I]_p = M$ and $I = M \cap A$. If H^E_ϕ is a (ϕ, k)-subalgebra with respect to $E = E(I)$, then $[E_j]_p \underset{\neq}{\supset} [E_{j+1}]_p$ for any $j (0 \le j \le k-1)$ and $\dim H^p/[E]_p = \dim A/E$. Conversely, if $\dim A/I = n < \infty$, then $\dim H^p/[I]_p \le n$. If $[E_j]_p \underset{\neq}{\supset} [E_{j+1}]_p$, for any j with $0 \le j \le k-1$ and $\dim H^p/[E]_p = \dim A/E$, then $\dim H^p/[I]_p = n$ and $[I]_p \cap A = I$.*

Proof. Suppose that M is an invariant subspace of $H^p(m)$ and $\dim H^p(m)/M = n < \infty$. Then there exist n linearly independent linear functionals $\psi_1, \ldots, \psi_n$ in $(H^p)^*$ such that $\psi_j = 0$ on M for $1 \le j \le n$. Put $\phi_j = \psi_j|A$ for $1 \le j \le n$ and $I = M \cap A$. Then $I = \bigcap_{j=1}^n \ker\phi_j$ and so $\dim A/I = n$. For $\phi_1, \ldots, \phi_n$ are independent linear functionals in A^* because A is dense in $H^p(m)$. If $M \underset{\neq}{\supset} [I]_p$, then there exists $\psi_{n+1} \in H^p(m)^*$ such that $\psi_{n+1} = 0$ on $[I]_p$ and $\psi_1, \ldots, \psi_n, \psi_{n+1}$ are independent linear functionals in $(H^p)^*$. If we put $\phi_{n+1} = \psi_{n+1}|A$, then $\phi_1, \ldots, \phi_n, \phi_{n+1}$ are independent linear functionals in A^* and $I \subseteq \bigcap_{j=1}^{n+1} \ker\phi_j$. This contradiction implies that $M = [I]_p$. Note that $\dim H^p/[E_k] = \dim H^p/[I]_p - 1 = \dim A/I - 1 = \dim A/E_k$. If $\dim H^p/[E_0]_p \underset{\neq}{<} \dim A/E_0$ where $E_0 = E$ or $[E_j]_p = [E_{j+1}]_p$, for some j $(0 \le j \le k-1)$, then this contradicts the fact that $\dim H^p/[E_k]_p = \dim A/E_k$. The converse is clear.

COROLLARY 3. *If M is an invariant subspace of H^p with $\dim H^p/M = 2$, then $M = \{f \in H^p;\ \Phi_1(f) = \Phi_2(f) = 0\}$, where $\Phi_j \in (H^p)^*$, and $\Phi_j(fg) = \Phi_j(f)\Phi_j(g)$, for $f \in H^p$ and $g \in A$, or $M = \{f \in H^p;\ \Phi(f) = D_\phi(f) = 0\}$, where $\Phi, D_\Phi \in (H^p)^*$, $\Phi(fg) = \Phi(f)\Phi(g)$ and $D_\Phi(fg) = \Phi(f)D_\Phi(g) + \Phi(g)D_\Phi(f)$ for $f \in H^p$ and $g \in A$.*

Proof. This follows from Corollary 1 and Theorem 2.

COROLLARY 4. *If M is an invariant subspace of H^p with* $\dim H^p/M = n < \infty$*, then there exist* $f_1, \dots, f_n$ *in* A *such that* $\{f_j + M\}_{j=1}^n$ *is a basis in* H^p/M.

Proof. By Theorem 2, if $I = M \cap A$, then $\dim A/I = n$ and $M = [I]_p$. Hence there exist $f_1, \dots, f_n$ in A such that $\{f_j + I\}_{j=1}^n$ is a basis in A/I. If f_j belongs to M, then f_j also belongs to $M \cap A = I$ and so f_j does not belong to M. This proves the corollary.

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