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On the commutativity of fundamental groups of complements to plane curves

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Introduction

In this paper, we prove the following:

THEOREM. *Let $C \subset \mathbb{P}^3$ be a complex reduced irreducible space curve which is non-degenerate (i.e., C is not contained in any plane). Let $p: C \to \mathbb{P}^2$ be a general projection. Then the topological fundamental group $\pi_1(\mathbb{P}^2 \setminus p(C))$ is abelian.*

When C is non-singular, the image of the general projection has only nodes as its singularities, and hence the complement has an abelian fundamental group thanks to Fulton–Deligne's theorem on Zariski conjecture ([**1**, **2**]). The point of our theorem is that we make no assumptions on the singularities of the space curve C.

We apply this theorem to the problem of commutativity of fundamental groups of complements to plane curves of a given degree with prescribed numbers of nodes and cusps. Here a cusp means a germ of curve singularity which is analytically isomorphic to a small neighborhood of the singular point of the affine curve $x^2 + y^3 = 0$. The strongest result concerned with this problem is Nori's result [**5**, proposition 6·5]. (Nori considered not only plane curves with only nodes and cusps but curves on arbitrary surfaces with any kind of singularities.) Let $C \subset \mathbb{P}^2$ be a reduced irreducible plane curve of degree d whose singular locus consists of n nodes and k cusps.

THEOREM (Nori). *Suppose that $2n + 6k < d^2$. Then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.*

We will prove a proposition which enlarges slightly the region of (d, n, k) on which the commutativity of $\pi_1(\mathbb{P}^2 \setminus C)$ is guaranteed.

PROPOSITION 1. *Suppose that $2n \geqslant d^2 - 5d + 8$. Then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.*

1. *Proof of Theorem*

Let G denote the Grassmannian variety Grass $(\mathbb{P}^1, \mathbb{P}^3)$ of all lines in $\mathbb{P}^3$, and let $(\mathbb{P}^3)^*$ be the dual projective space Grass $(\mathbb{P}^2, \mathbb{P}^3)$ of $\mathbb{P}^3$. For a plane $\Pi \in (\mathbb{P}^3)^*$, we put

$$\Sigma(\Pi) := \{l \in G;\ \ l \subset \Pi\}.$$

We also put

$$U := \{l \in G;\ \ C \cap l = \emptyset\}.$$

When $\Pi \in (\mathbb{P}^3)^*$ is general, no three points among $C \cap \Pi$ are co-linear. Indeed, let $B \subset (\mathbb{P}^3)^*$ be the Zariski open dense subset consisting of the planes which intersect C

at its non-singular points transversely. Let $\Pi_0 \in B$ be a plane which serves as a base point of B. Since C is irreducible, the monodromy group of the action of $\pi_1(B, \Pi_0)$ on the set $C \cap \Pi_0$ is the full symmetric group, because the action is 2-transitive and the monodromy group contains a simple transposition (cf. [**4**, Uniform Position Lemma]). Hence if there exist co-linear three points among $C \cap \Pi$ for a general $\Pi \in (\mathbb{P}^3)^*$, then all points in $C \cap \Pi$ are on a line for a general Π. Note that this condition is closed for the choice of Π in $(\mathbb{P}^3)^*$ because C is non-degenerate. Thus all points in $C \cap \Pi$ are on a line for every Π, which is absurd. Consequently, $\Sigma(\Pi) \setminus U$ is a union of lines on $\Sigma(\Pi) \cong \mathbb{P}^2$, which is a normal crossing divisor when the plane Π is generally chosen. Hence $\pi_1(\Sigma(\Pi) \cap U)$ is abelian for a general Π because of Fulton–Deligne's theorem ([**1**–**3**]). Let $I \subset (\mathbb{P}^3)^* \times G$ be the closed subvariety $\{(\Pi, l);\ \Pi \supset l\}$. The second projection $\rho_2 \colon I \to G$ is a fibre bundle with fibres isomorphic to $\mathbb{P}^1$. Hence ρ_2 induces an isomorphism $\pi_1(\rho_2^{-1}(U)) \cong \pi_1(U)$. Consider the projection $\rho_1 \colon \rho_2^{-1}(U) \to (\mathbb{P}^3)^*$. The fibre $\rho_1^{-1}(\Pi) \subset \rho_2^{-1}(U)$ over $\Pi \in (\mathbb{P}^3)^*$ is isomorphic to $\Sigma(\Pi) \cap U$, which is non-empty and non-singular for every Π. We apply Nori's Lemma [**5**, lemma 1·5 (C)] to conclude that the inclusion of the fibre induces a surjection $\pi_1(\Sigma(\Pi) \cap U) \to \to \pi_1(\rho_2^{-1}(U))$ for a general Π. Hence $\pi_1(\rho_2^{-1}(U))$ is abelian, and thus $\pi_1(U)$ is also abelian.

We fix a plane $\Pi_0 \subset \mathbb{P}^3$ and denote by A the complement $\mathbb{P}^3 \setminus \Pi_0$. Let $J_A \subset A \times G$ be the closed subvariety $\{(P, l);\ P \in l\}$. Consider the projection $\tau_2 \colon J_A \to G$. If $l \not\subset \Pi_0$, then $\tau_2^{-1}(l)$ is isomorphic to $\mathbb{A}^1$, while if $l \subset \Pi_0$, then $\tau_2^{-1}(l)$ is empty. Thus τ_2 induces an isomorphism $\pi_1(\tau_2^{-1}(U)) \cong \pi_1(U \setminus \Sigma(\Pi_0))$. Since $\Sigma(\Pi_0)$ is of codimension 2 in G, we have $\pi_1(U \setminus \Sigma(\Pi_0)) \cong \pi_1(U)$. Therefore, $\pi_1(\tau_2^{-1}(U))$ is isomorphic to $\pi_1(U)$. We have an isomorphism $J_A \cong A \times \Pi_0$ given by $(P, l) \mapsto (P, l \cap \Pi_0)$. Let $\tilde{\tau}_2 \colon A \times \Pi_0 \to G$ be the morphism corresponding to $\tau_2 \colon J_A \to G$ via this isomorphism. Let $D \subset G$ be the Zariski closed subset $G \setminus U$. We provide it with the reduced structure. For $P \in A$, consider the scheme theoretic intersection

$$D_P := (\{P\} \times \Pi_0) \cap \tilde{\tau}_2^{-1}(D),$$

which is considered as a sub-scheme of Π_0. If $P \in C$, then $D_P = \Pi_0$. Suppose that $P \notin C$. Let $p\langle P \rangle \colon C \to \Pi_0$ denote the projection with the centre P. Then we see that the reduced part $(D_P)_{\mathrm{red}}$ of D_P coincides with the reduced part $p\langle P \rangle(C)_{\mathrm{red}}$ of the image of the projection $p\langle P \rangle$. Hence D_P is a reduced divisor of Π_0 if $\deg\ p\langle P \rangle(C) = \deg C$.

Claim. There are at most finitely many points $P \in A \setminus C$ such that the degree of $p\langle P \rangle(C)_{\mathrm{red}}$ is less than the degree of C.

Since $C \cap A \subset A$ is of codimension 2, Claim implies that the locus of all points $P \notin A$ such that the scheme theoretic intersection D_P is *not* a reduced divisor of Π_0 is of codimension 2 in A. Let $P \in A$ be generally chosen. Applying [**6**, theorem 1], we get $\pi_1(\Pi_0 \setminus p\langle P \rangle(C)) \cong \pi_1(\tilde{\tau}_2^{-1}(U)) \cong \pi_1(U)$. Therefore $\pi_1(\Pi_0 \setminus p\langle P \rangle(C))$ is abelian, and Theorem is proved.

Proof of Claim. Suppose that there exists an irreducible curve $\Xi \subset A \setminus C$ such that $\deg\ p\langle P \rangle(C)_{\mathrm{red}} < \deg C$ for all $P \in \Xi$. Let P_0 be a general point of Ξ and Q_0 a general point of C. Since P_0 is not on C, the projection $p\langle P_0 \rangle \colon C \to \Pi_0$ with the centre P_0 must be of mapping degree $\geqslant 2$ onto its image. Thus there exists a point $R_0 \in C$ such that $Q_0 \neq R_0$ and $p\langle P_0\rangle(Q_0) = p\langle P_0\rangle(R_0)$. Since Q_0 is chosen generally, $p\langle P_0\rangle(Q_0)$ is a non-singular point of $p\langle P_0\rangle(C)_{\mathrm{red}}$, and the morphism $p\langle P_0\rangle\colon C \to p\langle P_0\rangle(C)_{\mathrm{red}}$ is étale at Q_0 and at R_0. This implies that the tangent lines $l(Q_0)$ and $l(R_0)$ of C at Q_0 and at R_0, respectively, intersect each other in $\mathbb{P}^3$. Since P_0 is also chosen generally on Ξ, there exist small open neighbourhoods (in the sense of complex analytic geometry) $V \subset \Xi$ of P_0 in Ξ and $W \subset C$ of R_0 in C, and an isomorphism $f\colon V \xrightarrow{\sim} W$ such that $p\langle P\rangle(Q_0) = p\langle P\rangle(f(P))$ for all $P \in V$. This implies that the tangent line $l(R)$ of C at R intersects $l(Q_0)$ for all $R \in W$. Therefore $l(R) \cap l(Q_0) \neq \emptyset$ for every non-singular point R of C. However, consider the projection $\lambda\colon C \to \mathbb{P}^1$ with the centre $l(Q_0)$. Since C is non-degenerate, λ is surjective, and hence, by Sard's theorem, λ is smooth at a general point $R \in C$. This implies that $l(R) \cap l(Q_0) = \emptyset$ for a general $R \in C$, and we get a contradiction.

2. *Proof of Proposition* 1

First note that if $2n \geqslant d^2 - 5d + 8$ and $2n + 2k < d^2 - 4d + 3$, then $2n + 6k$ is automatically less than d^2. Therefore, taking Nori's theorem into account, we may assume that the inequality $2n + 2k \geqslant d^2 - 4d + 3$ also holds.

Let $\nu\colon \tilde{C} \to C$ be the normalization of C. The genus of $\tilde{C}$ is given by

$$g = (d-1)(d-2)/2 - n - k.$$

The assumed inequalities imply that

$$d \geqslant \max\,(2g+1, g+3), \quad \text{and} \quad k \leqslant d - g - 3. \tag{2·1}$$

Let L be the pull-back of $\mathcal{O}_C(1)$ by ν, which is an invertible sheaf of degree d on $\tilde{C}$. By the first inequality of (2·1), L is very ample and $\dim|L| = d - g$. Let C' be the image of the embedding of $\tilde{C}$ in $\mathbb{P}^{d-g}$ by L. Then C is obtained by a certain projection $C' \to \mathbb{P}^2$. Let $\Gamma \subset \mathbb{P}^{d-g}$ be the centre of this projection, which is a linear subspace of codimension 3. Let $T \subset \mathbb{P}^{d-g}$ be the union of all lines tangent to C'. Then Γ intersects T at distinct k points $P_1, \ldots, P_k$, each of which corresponds to a cusp of C bijectively. By the second inequality of (2·1), there exists a linear subspace $\Lambda \subset \Gamma$ of codimension 1 which contains $P_1, \ldots, P_k$. Let $p\langle\Lambda\rangle\colon C' \to \mathbb{P}^3$ be the projection with the centre Λ. The image $C'' \subset \mathbb{P}^3$ is a non-degenerate curve, which has at least k singular points $Q_1, \ldots, Q_k$ corresponding to $P_1, \ldots, P_k \in \Lambda \cap T$. The projection $p\langle\Gamma\rangle\colon C' \to C$ with the centre Γ factors as $C' \to C'' \to C$, and $C'' \to C$ maps $Q_1, \ldots, Q_k$ to the cusps of C bijectively. Therefore, C'' has k cusps and some nodes (possibly none) as its only singularities. Thus the image $p(C'')$ of the general projection of C'' to $\mathbb{P}^2$ has the same numbers and types of singular points as the original curve C. Note that C and $p(C'')$ are contained in an irreducible equisingular family of plane curves, every member of which is obtained as an image of a projection from $C'' \subset \mathbb{P}^3$ to $\mathbb{P}^2$. Thus our main theorem implies that $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.

3. *Examples*

Let $\mathscr{A}$ denote the set of triples (d, n, k) with $d \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}_{\geqslant 0}$ and $k \in \mathbb{Z}_{\geqslant 0}$ for which the following hold:

(i) There exists a reduced irreducible plane curve $C \subset \mathbb{P}^2$ of degree d with n nodes and k cusps as its only singularities.

(ii) For any plane curve C as in (i), $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian.

Nori's result says that if (d, n, k) satisfies (i) and $2n + 6k < d^2$, then $(d, n, k) \in \mathscr{A}$. Using Proposition 1, we shall enlarge the known region of $\mathscr{A}$.

Let $C \subset \mathbb{P}^n$ be a non-degenerate irreducible curve of degree d whose singular locus consists of only nodes and cusps. Let k be the number of cusps. We write by $\mathrm{Tan}\,(C) \subset \mathbb{P}^n$ the union of all lines in $\mathbb{P}^n$ tangent to C at its non-singular points. The following proposition is elementary:

PROPOSITION 2. *If P is a general point of* $\mathrm{Tan}\,(C)$, *then the image of the projection $C \to \mathbb{P}^{n-1}$ with the centre P is of degree d, has only nodes and cusps as its singularities, and the number of cusps is* $k + 1$.

Let C be a non-singular curve of genus g, and L a line bundle on C of degree $d \geqslant \max\,(2g + 1, g + 3)$. We embed C into $\mathbb{P}^{d-g}$ by $|L|$. Applying Proposition 2 to $C \subset \mathbb{P}^{d-g}$ repeatedly, we obtain a non-degenerate space curve $C' \subset \mathbb{P}^3$ of degree d with certain number of nodes (possibly none) and $d - g - 3$ cusps as its only singularities. Let $C'' \subset \mathbb{P}^2$ be the image of a general projection $C' \to \mathbb{P}^2$. Then C'' is an irreducible plane curve of degree d with

$$n = (d^2 - 5d + 8)/2 \text{ nodes} \quad \text{and} \quad k = d - g - 3 \text{ cusps}$$

as its only singularities. Since $2n \geqslant d^2 - 5d + 8$, Proposition 1 implies that $(d, n, k) \in \mathscr{A}$. Thus, choosing a line bundle of degree $\geqslant 6g + 10$, we obtain a region of $\mathscr{A}$ not covered by Nori's theorem.

Note that there is another known region of $\mathscr{A}$, which is given by the following theorem due to Zariski [**7**]. Let $R(d, k) \subset \mathbb{P}_*(\Gamma(\mathbb{P}^2, \mathscr{O}(d)))$ be the locus of all *rational* plane curves C of degree d with n nodes and k cusps as its only singularities. Since the genus of the normalization of C is 0, we have $n = (d - 1)(d - 2)/2 - k$.

THEOREM (Zariski). (1) *The locus $R(d, k)$ is non-empty if and only if $k \leqslant 3(d-2)/2$.* (2) *For a member C of $R(d, k)$, $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian unless d is even and $k = 3(d-2)/2$.*

Thus, if $n + k = (d - 1)(d - 2)/2$ and $k \leqslant (3d - 7)/2$, we have $(d, n, k) \in \mathscr{A}$.

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