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COMPACT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS ON WEIGHTED BERGMAN SPACES

TAKAHIKO NAKAZI* and RIKIO YONEDA

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan

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Abstract. Let $L_a^2(D, d\sigma d\theta/2\pi)$ be a complete weighted Bergman space on the open unit disc D, where $d\sigma$ is a positive finite Borel measure on $[0, 1)$. We show the following : when ϕ is a continuous function on the closed unit disc $\bar{D}$, T_ϕ is compact if and only if $\phi = 0$ on ∂D.

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Let D be the open unit disc and $d\sigma$ a positive finite Borel measure on $[0, 1)$. Let $L_a^2 = L_a^2(D, d\sigma d\theta/2\pi)$ be a weighted Bergman space on D; that is, L_a^2 consists of analytic functions f in D with

$$\|f\|_2^2 = \int_D |f(re^{i\theta})|^2 d\sigma d\theta/2\pi < \infty.$$

When L_a^2 is closed, P denotes the orthogonal projection from $L^2 = L^2(D, d\sigma d\theta/2\pi)$ onto L_a^2. For ϕ in $L^\infty = L^\infty(D, d\sigma d\theta/2\pi)$, we consider the Toeplitz operator $T_\phi : L_a^2 \to L_a^2$ defined by $T_\phi f = P(\phi f), f \in L_a^2$. We prove the following theorem in this paper. For the Bergman space (that is, $d\sigma = 2rdr$), the Theorem is well known; see [**5**, p. 107] and [**1**]. When $d\sigma = (1 - r^2)^\alpha dr(-1 < \alpha < \infty)$, the Theorem is also true; see [**3**] and [**4**]. However, that argument does not work for the general situation. We need a new idea in order to prove the Theorem. Let $H = H(D)$ denote the set of all analytic functions on D.

THEOREM. *Suppose that $L_a^2 = L_a^2(D, d\sigma d\theta/2\pi)$ is complete. When ϕ is a continuous function on the closed unit disc $\bar{D}$, T_ϕ is compact if and only if $\phi = 0$ on ∂D.*

In order to prove the Theorem, we need three lemmas.

LEMMA 1. *L_a^2 is complete if and only if $\sigma([\varepsilon, 1)) > 0$ for some ε with $0 \le \varepsilon < 1$.*

Proof. For $a \in D$, put

$$s(\mu, a) = \inf\left\{ \int_D |f|^2 d\mu; f \in H \text{ and } f(a) = 1 \right\},$$

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where H is the set of all analytic functions on D and $d\mu = d\sigma d\theta/2\pi$. Statement (1) of Corollary 1 in [**2**] is valid for $s(\mu, a)$ instead of $S(\mu, a)$. When $(\mathrm{supp}\mu) \cap D$ is a uniqueness set for H, by Statement (1) of Theorem 8 in [**2**], L_a^2 is complete if and only if, for all compact sets K in D, $\int_K \log s(\mu, a) r dr d\theta/\pi > -\infty$. If σ is not a zero measure, then $(\mathrm{supp}\mu) \cap D$ is a uniqueness set for H. These statements suffice to prove the Lemma.

LEMMA 2. *If* $\sigma([\varepsilon, 1)) > 0$ *for every* ε *with* $0 \le \varepsilon < 1$, *then*

$$\lim_{n\to\infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0 \quad (0 \le \varepsilon < 1).$$

Proof. When δ is a positive constant with $\varepsilon + \delta < 1$, the following inequality holds.

$$\begin{aligned} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} &\le \frac{\sigma([0, \varepsilon])}{\int_\varepsilon^1 \left(\frac{r}{\varepsilon}\right)^n d\sigma} \le \frac{\sigma([0, \varepsilon])}{\int_{\varepsilon+\delta}^1 \left(\frac{r}{\varepsilon}\right)^n d\sigma} \\ &\le \frac{\sigma([0, \varepsilon])}{\left(\frac{\varepsilon+\delta}{\varepsilon}\right)^n \sigma([\varepsilon+\delta, 1])} \quad (0 < \varepsilon < 1). \end{aligned}$$

Since they are positive and $\lim_{n\to\infty} \{(\varepsilon + \delta)/\varepsilon\}^n = \infty$, we have

$$\lim_{n\to\infty} \left(\int_0^\varepsilon r^n d\sigma \Big/ \int_\varepsilon^1 r^n d\sigma \right) = 0.$$

LEMMA 3. *If for every* ε *with* $0 \le \varepsilon < 1$, *we have*

$$\int_\varepsilon^1 r^n d\sigma > 0 \text{ and } \lim_{n\to\infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0,$$

then for any non-negative ℓ

$$\lim_{n\to\infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} = 1.$$

Proof. For every ε with $0 \le \varepsilon < 1$, the following inequality holds.

$$
\begin{aligned}
1 \ge \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} &= \frac{\int_0^\varepsilon r^{n+\ell} d\sigma + \int_\varepsilon^1 r^{n+\ell} d\sigma}{\int_0^\varepsilon r^n d\sigma + \int_\varepsilon^1 r^n d\sigma} \\
&\ge \frac{\varepsilon^\ell \int_\varepsilon^1 r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma + \int_0^\varepsilon r^n d\sigma} \\
&= \varepsilon^\ell \left(1 + \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} \right)^{-1}
\end{aligned}
$$

because $\int_\varepsilon^1 r^n d\sigma > 0$ and $\ell \ge 0$. Thus $\lim_{n\to\infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} \ge \varepsilon^\ell$. Let $\varepsilon \to 1$ to prove the lemma.

Proof. Suppose that $\phi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ is continuous on $\bar{D}$, where

$$\phi_j(r) = \int_0^{2\pi} \phi(re^{i\theta}) e^{-ij\theta} d\theta / 2\pi$$

for $j = 0, \pm 1, \pm 2, \cdots$. Then $\phi_j(r)$ is continuous on [0,1] for any j. Put

$$
\begin{aligned}
e_n(re^{i\theta}) &= a_n r^n e^{in\theta} \\
&= r^n e^{in\theta} \Big/ \sqrt{\int_0^1 r^{2n} d\sigma}
\end{aligned}
$$

for $n \ge 0$, then $\{e_n\}$ is an orthonormal basis in L_a^2. For each j, put

$$\Phi_j(re^{i\theta}) = r^{|j|} e^{-ij\theta} \phi(re^{i\theta}).$$

Then $T_{\Phi_j} = T_{r^{|j|}e^{-ij\theta}} T_\phi$ for $j \ge 0$ and $T_{\Phi_j} = T_\phi T_{r^{|j|}e^{-ij\theta}}$ for $j < 0$. If T_ϕ is compact, then T_{Φ_j} is also compact for any j. For each j, if $n \ge 0$, then

$$|\langle T_{\Phi_j} e_n, e_n \rangle| \le \|T_{\Phi_j} e_n\|_2 \|e_n\|_2 = \|T_{\Phi_j} e_n\|_2.$$

Since T_{Φ_j} is compact for each j and $e_n \to 0 (n \to \infty)$ weakly, $\|T_{\Phi_j} e_n\|_2 \to 0$ $(n \to \infty)$ and so $\langle T_{\Phi_j} e_n, e_n \rangle \to 0$ $(n \to \infty)$. For each j,

$$\begin{aligned}\langle T_{\Phi_j} e_n, e_n\rangle &= \int_0^{2\pi}\int_0^1 \phi(re^{i\theta}) r^{|j|} e^{-ij\theta} a_n^2 r^{2n} d\sigma d\theta/2\pi \\ &= a_n^2 \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma\end{aligned}$$

and then $\lim_{n\to\infty} a_n^2 \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = 0$. By Lemma 1, $\sigma([\varepsilon, 1)) > 0$ for some ε with $0 \le \varepsilon < 1$ and hence $\sigma([\varepsilon, 1)) > 0$ for every $\varepsilon < 1$. Hence, by Lemma 2, we have

$$\lim_{n\to\infty} \frac{\int_0^{\varepsilon} r^{2n} d\sigma}{\int_{\varepsilon}^1 r^{2n} d\sigma} = 0 \text{ for } (0 \le \varepsilon < 1).$$

Then, by Lemma 3, for any integer j we have

$$\lim_{n\to\infty} a_n^2 \int_0^1 r^{|j|+2n} d\sigma = 1.$$

Since $\phi_j(r)$ is continuous on [0,1], we can approximate $\phi_j(r)$ uniformly by polynomials $\sum_{t=0}^{k} c_t r^t$. Since $\lim_{n\to\infty} a_n^2 \int_0^1 r^{|j|+2n} d\sigma = 1$ for any j, we obtain

$$\lim_{n\to\infty} a_n^2 \int_0^1 \left(\sum_{t=0}^{k} c_t r^t\right) r^{|j|+2n} d\sigma = \sum_{t=0}^{k} c_t$$

and so

$$\lim_{n\to\infty} a_n^2 \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = \phi_j(1).$$

Thus $\phi_j(1) = 0$ for any j because $\lim_{n\to\infty} a_n^2 \int_0^1 \phi_j(r) r^{|j|+2n} d\sigma = 0$, and hence $\phi = 0$ on ∂D.

Conversely suppose that $\phi = 0$ on ∂D. Then we may assume that the support set of ϕ is compact in D. In order to show the compactness of T_ϕ, it is sufficient to show that if $h_n \to 0$ weakly ($n \to \infty$) in L_a^2 then $h_n \to 0$ uniformly on supp ϕ. By hypothesis on σ, any point $z \in D$ has a bounded point evaluation for L_a^2 because Statement (1) of Corollary 1 in **[2]** is valid for $s(\mu, a)$ instead of $S(\mu, a)$ and $r(\mu, a)s(\mu, a) = 1 (a \in D)$. Hence $h_n(z) \to 0$. By the boundedness of analytic functions on supp ϕ and the uniform boundedness principle, $h_n \to 0$ uniformly on supp ϕ.

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