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COMPACT TOEPLITZ OPERATORS WITH CONTINUOUS SYMBOLS ON WEIGHTED BERGMAN SPACES

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Abstract. Let $L_a^2(D, d\sigma d\theta/2\pi)$ be a complete weighted Bergman space on the open unit disc D , where $d\sigma$ is a positive finite Borel measure on $[0, 1)$. We show the following : when ϕ is a continuous function on the closed unit disc \bar{D} , T_ϕ is compact if and only if $\phi = 0$ on ∂D .

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Let D be the open unit disc and $d\sigma$ a positive finite Borel measure on $[0, 1)$. Let $L_a^2 = L_a^2(D, d\sigma d\theta/2\pi)$ be a weighted Bergman space on D ; that is, L_a^2 consists of analytic functions f in D with

$$\|f\|_2^2 = \int_D |f(re^{i\theta})|^2 d\sigma d\theta/2\pi < \infty.$$

When L_a^2 is closed, P denotes the orthogonal projection from $L^2 = L^2(D, d\sigma d\theta/2\pi)$ onto L_a^2 . For ϕ in $L^\infty = L^\infty(D, d\sigma d\theta/2\pi)$, we consider the Toeplitz operator $T_\phi : L_a^2 \rightarrow L_a^2$ defined by $T_\phi f = P(\phi f)$, $f \in L_a^2$. We prove the following theorem in this paper. For the Bergman space (that is, $d\sigma = 2rdr$), the Theorem is well known; see [5, p. 107] and [1]. When $d\sigma = (1 - r^2)^\alpha dr$ ($-1 < \alpha < \infty$), the Theorem is also true; see [3] and [4]. However, that argument does not work for the general situation. We need a new idea in order to prove the Theorem. Let $H = H(D)$ denote the set of all analytic functions on D .

THEOREM. Suppose that $L_a^2 = L_a^2(D, d\sigma d\theta/2\pi)$ is complete. When ϕ is a continuous function on the closed unit disc \bar{D} , T_ϕ is compact if and only if $\phi = 0$ on ∂D .

In order to prove the Theorem, we need three lemmas.

LEMMA 1. L_a^2 is complete if and only if $\sigma([\varepsilon, 1)) > 0$ for some ε with $0 \leq \varepsilon < 1$.

Proof. For $a \in D$, put

$$s(\mu, a) = \inf \left\{ \int_D |f|^2 d\mu; f \in H \text{ and } f(a) = 1 \right\},$$

where H is the set of all analytic functions on D and $d\mu = d\sigma d\theta/2\pi$. Statement (1) of Corollary 1 in [2] is valid for $s(\mu, a)$ instead of $S(\mu, a)$. When $(\text{supp}\mu) \cap D$ is a uniqueness set for H , by Statement (1) of Theorem 8 in [2], L_a^2 is complete if and only if, for all compact sets K in D , $\int_K \log s(\mu, a) r dr d\theta/\pi > -\infty$. If σ is not a zero measure, then $(\text{supp}\mu) \cap D$ is a uniqueness set for H . These statements suffice to prove the Lemma.

LEMMA 2. *If $\sigma([\varepsilon, 1)) > 0$ for every ε with $0 \leq \varepsilon < 1$, then*

$$\lim_{n \rightarrow \infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0 \quad (0 \leq \varepsilon < 1).$$

Proof. When δ is a positive constant with $\varepsilon + \delta < 1$, the following inequality holds.

$$\begin{aligned} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} &\leq \frac{\sigma([0, \varepsilon])}{\int_\varepsilon^1 \left(\frac{r}{\varepsilon}\right)^n d\sigma} \leq \frac{\sigma([0, \varepsilon])}{\int_{\varepsilon+\delta}^1 \left(\frac{r}{\varepsilon}\right)^n d\sigma} \\ &\leq \frac{\sigma([0, \varepsilon])}{\left(\frac{\varepsilon + \delta}{\varepsilon}\right)^n \sigma([\varepsilon + \delta, 1])} \quad (0 < \varepsilon < 1). \end{aligned}$$

Since they are positive and $\lim_{n \rightarrow \infty} \{(\varepsilon + \delta)/\varepsilon\}^n = \infty$, we have

$$\lim_{n \rightarrow \infty} \left(\int_0^\varepsilon r^n d\sigma / \int_\varepsilon^1 r^n d\sigma \right) = 0.$$

LEMMA 3. *If for every ε with $0 \leq \varepsilon < 1$, we have*

$$\int_\varepsilon^1 r^n d\sigma > 0 \text{ and } \lim_{n \rightarrow \infty} \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} = 0,$$

then for any non-negative ℓ

$$\lim_{n \rightarrow \infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} = 1.$$

Proof. For every ε with $0 \leq \varepsilon < 1$, the following inequality holds.

$$\begin{aligned}
 1 &\geq \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} = \frac{\int_0^\varepsilon r^{n+\ell} d\sigma + \int_\varepsilon^1 r^{n+\ell} d\sigma}{\int_0^\varepsilon r^n d\sigma + \int_\varepsilon^1 r^n d\sigma} \\
 &\geq \frac{\varepsilon^\ell \int_\varepsilon^1 r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma + \int_0^\varepsilon r^n d\sigma} \\
 &= \varepsilon^\ell \left(1 + \frac{\int_0^\varepsilon r^n d\sigma}{\int_\varepsilon^1 r^n d\sigma} \right)^{-1}
 \end{aligned}$$

because $\int_\varepsilon^1 r^n d\sigma > 0$ and $\ell \geq 0$. Thus $\lim_{n \rightarrow \infty} \frac{\int_0^1 r^{n+\ell} d\sigma}{\int_0^1 r^n d\sigma} \geq \varepsilon^\ell$. Let $\varepsilon \rightarrow 1$ to prove the lemma.

Proof. Suppose that $\phi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} \phi_j(r) e^{ij\theta}$ is continuous on \bar{D} , where

$$\phi_j(r) = \int_0^{2\pi} \phi(re^{i\theta}) e^{-ij\theta} d\theta / 2\pi$$

for $j = 0, \pm 1, \pm 2, \dots$. Then $\phi_j(r)$ is continuous on $[0, 1]$ for any j . Put

$$\begin{aligned}
 e_n(re^{i\theta}) &= a_n r^n e^{in\theta} \\
 &= r^n e^{in\theta} / \sqrt{\int_0^1 r^{2n} d\sigma}
 \end{aligned}$$

for $n \geq 0$, then $\{e_n\}$ is an orthonormal basis in L_a^2 . For each j , put

$$\Phi_j(re^{i\theta}) = r^{|j|} e^{-ij\theta} \phi(re^{i\theta}).$$

Then $T_{\Phi_j} = T_{r^{|j|} e^{-ij\theta}} T_\phi$ for $j \geq 0$ and $T_{\Phi_j} = T_\phi T_{r^{|j|} e^{-ij\theta}}$ for $j < 0$. If T_ϕ is compact, then T_{Φ_j} is also compact for any j . For each j , if $n \geq 0$, then

$$|\langle T_{\Phi_j} e_n, e_n \rangle| \leq \|T_{\Phi_j} e_n\|_2 \|e_n\|_2 = \|T_{\Phi_j} e_n\|_2.$$

Since T_{Φ_j} is compact for each j and $e_n \rightarrow 0$ ($n \rightarrow \infty$) weakly, $\|T_{\Phi_j} e_n\|_2 \rightarrow 0$ ($n \rightarrow \infty$) and so $\langle T_{\Phi_j} e_n, e_n \rangle \rightarrow 0$ ($n \rightarrow \infty$). For each j ,

$$\begin{aligned}
\langle T_{\Phi_j} e_n, e_n \rangle &= \int_0^{2\pi} \int_0^1 \phi(re^{i\theta}) r^{|\bar{j}|} e^{-ij\theta} a_n^2 r^{2n} d\sigma d\theta / 2\pi \\
&= a_n^2 \int_0^1 \phi_j(r) r^{|\bar{j}|+2n} d\sigma
\end{aligned}$$

and then $\lim_{n \rightarrow \infty} a_n^2 \int_0^1 \phi_j(r) r^{|\bar{j}|+2n} d\sigma = 0$. By Lemma 1, $\sigma([\varepsilon, 1]) > 0$ for some ε with $0 \leq \varepsilon < 1$ and hence $\sigma([\varepsilon, 1]) > 0$ for every $\varepsilon < 1$. Hence, by Lemma 2, we have

$$\lim_{n \rightarrow \infty} \frac{\int_0^\varepsilon r^{2n} d\sigma}{\int_\varepsilon^1 r^{2n} d\sigma} = 0 \text{ for } (0 \leq \varepsilon < 1).$$

Then, by Lemma 3, for any integer j we have

$$\lim_{n \rightarrow \infty} a_n^2 \int_0^1 r^{|\bar{j}|+2n} d\sigma = 1.$$

Since $\phi_j(r)$ is continuous on $[0, 1]$, we can approximate $\phi_j(r)$ uniformly by polynomials $\sum_{t=0}^k c_t r^t$. Since $\lim_{n \rightarrow \infty} a_n^2 \int_0^1 r^{|\bar{j}|+2n} d\sigma = 1$ for any j , we obtain

$$\lim_{n \rightarrow \infty} a_n^2 \int_0^1 \left(\sum_{t=0}^k c_t r^t \right) r^{|\bar{j}|+2n} d\sigma = \sum_{t=0}^k c_t$$

and so

$$\lim_{n \rightarrow \infty} a_n^2 \int_0^1 \phi_j(r) r^{|\bar{j}|+2n} d\sigma = \phi_j(1).$$

Thus $\phi_j(1) = 0$ for any j because $\lim_{n \rightarrow \infty} a_n^2 \int_0^1 \phi_j(r) r^{|\bar{j}|+2n} d\sigma = 0$, and hence $\phi = 0$ on ∂D .

Conversely suppose that $\phi = 0$ on ∂D . Then we may assume that the support set of ϕ is compact in D . In order to show the compactness of T_ϕ , it is sufficient to show that if $h_n \rightarrow 0$ weakly ($n \rightarrow \infty$) in L_a^2 then $h_n \rightarrow 0$ uniformly on $\text{supp } \phi$. By hypothesis on σ , any point $z \in D$ has a bounded point evaluation for L_a^2 because Statement (1) of Corollary 1 in [2] is valid for $s(\mu, a)$ instead of $S(\mu, a)$ and $r(\mu, a)s(\mu, a) = 1$ ($a \in D$). Hence $h_n(z) \rightarrow 0$. By the boundedness of analytic functions on $\text{supp } \phi$ and the uniform boundedness principle, $h_n \rightarrow 0$ uniformly on $\text{supp } \phi$.

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