Covariance kernel and the central limit theorem in the total variation distance

Toshio Mikami

Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: mikami@math.sci.hokudai.ac.jp

We modify and generalize the idea of covariance kernels for Borel probability measures on $\mathbf{R}^d$, and study the relation between the central limit theorem in the total variation distance and the convergence of covariance kernels.

Key Words: Covariance kernel, central limit theorem, total variation distance

1. INTRODUCTION.

Cacoullos and Papathanasiou introduced a function, called a covariance kernel or ω-function $\omega(x)$, for a probability density function $f(x)$ on $\mathbf{R}$ to study the characterization of probability distributions (see [1]). It is known that $f(x)$ is normal if and only if $\omega(x) \equiv 1$ (see [3]). Cacoullos, Papathanasiou and Utev proved that the convergence, as $n \to \infty$ in $L^1(\mathbf{R}, dx)$, of a sequence of probability density functions $\{f_n\}_{n=1}^{\infty}$ with interval supports on $\mathbf{R}$ to $g_1(x) \equiv (2\pi)^{-1/2}\exp(-x^2/2)$ is equivalent to that of $\{\omega_n f_n - f_n\}_{n=1}^{\infty}$ to 0, where ω_n denotes a ω-function of f_n (see [4]).

We generalized their result, by a different method, to the case where probability measures under consideration are Borel probability measures on $\mathbf{R}$ (see [5]).

Cacoullos and Papathanasiou introduced a covariance kernel for a probability density function $f(x)$ on $\mathbf{R}^d$ for $d \geq 2$ (see [2]). Papathanasiou used it to show that $L^1(\mathbf{R}^d, dx)$-norm of $f - g_d$ ($g_d(x) \equiv \Pi_{i=1}^d g_1(x_i)$ for $x = (x_i)_{i=1}^d \in \mathbf{R}^d$) is dominated by that of $(\omega_f^i f - f)_{i=1}^d$, where $(\omega_f^i)_{i=1}^d$ denotes a covariance kernel (vector) of f (see [7]). Papadatos and Papathanasiou studied the relation between $L^1(\mathbf{R}^d, dx)$-norm of $f_1 - f_2$ and covariance kernels of f_1 and f_2 and of their marginals for two probability density functions f_1 and f_2 on $\mathbf{R}^d$ (see [6]).

In these papers they assumed that

$$\sigma_k \equiv (\int_{\mathbf{R}^d} y_i y_j f_k(y)dy - \int_{\mathbf{R}^d} y_i f_k(y)dy \int_{\mathbf{R}^d} y_j f_k(y)dy)_{i,j=1}^d \tag{1}$$

is positive definite for $k = 1, 2$, and that the following holds:

$$(\int_{\mathbf{R}^d} (\sigma_k^{-1} y)_i f_k(y)dy)(\int_{\mathbf{R}} f_k(x)dx_i) = \int_{\mathbf{R}} (\sigma_k^{-1} x)_i f_k(x)dx_i \tag{2}$$

for all $i = 1, \cdots, d$ and $k = 1, 2$, and that f_1 and f_2 have convex supports since they used an identity in [2]. They also considered the discrete case under a similar condition.

In this paper we modify and generalize the idea of a covariance kernel for any Borel probability measure on $\mathbf{R}^d$. We also show, without such a restriction as above, that the convergence, as $n \to \infty$ in the total variation distance, of a sequence of Borel probability measures $\{P_n\}_{n\geq 1}$ on $\mathbf{R}^d$ to a standard normal distribution is equivalent to that of $\mathbf{W}(P_n) - Id \times P_n$ to 0 (see section 2 for definition), where Id denotes an $d\times d$-identity matrix. Our proof is different from that of [5], and our result in this paper generalizes it to a multi-dimensional case.

In section 2 we state our main result which will be proved in section 3. In section 4 we give a typical example.

2. MAIN RESULT.

First we give some notations.

For a Borel probability measure P on $(\mathbf{R}^d, \mathbf{B}(\mathbf{R}^d))$ and any set S and $S' \subset \{1, \cdots, d\}$ for which $S \cap S' = \emptyset$ and for which $S' \neq \{1, \cdots, d\}$, put

$$P^S_{(x_j)_{j\in S'}}(\Pi_{i\in S} dx_i) \tag{3}$$

$$\equiv \begin{cases} \int_{\{(x_j)_{j\notin S\cup S'} \in \mathbf{R}^{d-\#(S\cup S')}\}} P_{(x_j)_{j\in S'}}(\Pi_{i\notin S'} dx_i) & \text{if } S\cup S' \neq \{1,\cdots,d\} \\ & \quad \text{and if } S \neq \emptyset, \\ P_{(x_j)_{j\in S'}}(\Pi_{i\notin S'} dx_i) & \text{if } S \cup S' = \{1,\cdots,d\}, \\ 1 & \text{if } S = \emptyset, \end{cases}$$

$$P^S(\Pi_{i\in S} dx_i) \equiv \begin{cases} \int_{\{(x_j)_{j\notin S} \in \mathbf{R}^{d-\#(S)}\}} P(\Pi_{j=1}^d dx_j) & \text{if } 1 \leq \#(S) < d, \\ P(\Pi_{j=1}^d dx_j) & \text{if } \#(S) = d, \\ 1 & \text{if } \#(S) = 0, \end{cases} \tag{4}$$

Here $\#(S)$ denotes a cardinal number of the set S, and $P_{(x_j)_{j\in S'}}(\Pi_{i\notin S'}dx_i)$ denotes a regular conditional probability of P given $(x_j)_{j\in S'}$ (see [8]). When it is not confusing, we write $\{i\}\equiv i$, $(x_j)_{\{j:j<i\}}\equiv(x_j)_{j<i}$, $(x_j)_{\{j:j\neq i\}}\equiv(x_j)_{j\neq i}$, etc. for the sake of simplicity.

The following definition is a modification and a generalization of the idea of covariance kernels in [2], and generalizes that in [5] to a multi-dimensional case.

Definition 2.1. For a Borel probability measure P on $(\mathbf{R}^d,\mathbf{B}(\mathbf{R}^d))$ such that $\int_{\mathbf{R}^d}|y|^2P(dy)<\infty$, put for $i=1,\cdots,d$,

$$\begin{aligned}\mathbf{W}^i(P)(dx) &\equiv P^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)dx_i \\ &\qquad\times\int_{-\infty}^{x_i}\{\int_{\mathbf{R}} zP_{(x_j)_{j\neq i}}(dz)-y\}P_{(x_j)_{j\neq i}}(dy), \qquad (5)\\ \mathbf{W}(P)(dx) &\equiv (\delta_{ij}\mathbf{W}^i(P)(dx))_{i,j=1}^d. \qquad (6)\end{aligned}$$

Here we put $\delta_{ij}=1$ if $i=j$, and $=0$ if $i\neq j$ $(1\le i,j\le d)$. When $\mathbf{W}^i(P)(dx)$ is absolutely continuous with respect to dx, we put $\mathbf{W}^i(P)(dx)/dx\equiv W^i(P)(x)$ and $\mathbf{W}(P)(dx)/dx\equiv W(P)(x)$.

Remark 2. 1. Suppose that $\int_{\mathbf{R}^d}x_iP(dx)=0$ and $\int_{\mathbf{R}^d}|x_i|^2P(dx)=1$ for $i=1,\cdots,d$. Then $\mathbf{W}^i(P)$ is a probability measure when $d=1$ (see (22)). Suppose also that $P(dx)/dx\equiv p(x)$ exists and that $P(dx)$ is a product measure. Then the covariance kernel $\omega_p^i(x)$ in [2] is equal to $W^i(P)(x)/p(x)$.

For two finite measures P and Q on $(\mathbf{R}^d,\mathbf{B}(\mathbf{R}^d))$, let

$$\begin{aligned}\rho(P(dx),Q(dx)) \equiv \sup\{|\int_{\mathbf{R}^d}\varphi(x)(P(dx)-Q(dx))| \qquad (7)\\ :\varphi \text{ is Borel measurable from } \mathbf{R}^d \text{ to } [-1,1]\}\end{aligned}$$

denote the total variation distance between them.

Remark 2. 2. For two probability measures P and Q on $(\mathbf{R}^d,\mathbf{B}(\mathbf{R}^d))$,

$$\rho(P(dx),Q(dx))=2\sup_{A\in\mathbf{B}(\mathbf{R}^d)}|P(A)-Q(A)| \qquad (8)$$

(see [8, p. 360, Lemma 1]).

The following is our main result.

THEOREM 2.1. *Suppose that* $\{P_n\}_{n\geq 1}$ *is a sequence of Borel probability measures on* $(\mathbf{R}^d, \mathbf{B}(\mathbf{R}^d))$ *such that* $\int_{\mathbf{R}^d} |y_i|^2 P_n(dy) = 1$ $(1 \leq i \leq d, 1 \leq n)$. *Then the following (I) and (II) are equivalent.*

$$(I). \qquad \lim_{n\to\infty} \rho(P_n(dx), g_d(x)dx) = 0.$$

$$(II). \qquad \lim_{n\to\infty} \sum_{i=1}^{d} \rho(P_n(dx), W^i(P_n)(dx)) = 0.$$

Roughly speaking, Theorem 2.1 means that the central limit theorem in the total variation distance is equivalent to the convergence of nonnegative definite matrices, to an identity matrix, which are coefficinets of the second order differential operators of the second order PDEs that are satisfied by probability measures under consideration.

In fact, when $P(dx)/dx \equiv p(x)$ exists, $W(P)(x)$ is a nonnegative definite matrix and the following holds: for any $\varphi \in C_o^\infty(\mathbf{R}^d; \mathbf{R})$,

$$\begin{aligned} &\int_{\mathbf{R}^d} \sum_{i,j=1}^{d} (\delta_{ij} W^i(P)(x)/p(x))(\partial^2 \varphi(x)/\partial x_i \partial x_j) p(x) dx \qquad (9) \\ = \; &-\int_{\mathbf{R}^d} \sum_{i=1}^{d} (\int_{\mathbf{R}} z P_{(x_j)_{j\neq i}}(dz) - x_i)(\partial \varphi(x)/\partial x_i) p(x) dx. \end{aligned}$$

If (I) or (II) in Theorem 2.1 holds, then

$$\lim_{n\to\infty} \sum_{i=1}^{d} \int_{\mathbf{R}^d} | \int_{\mathbf{R}} z (P_n)_{(x_j)_{j\neq i}}(dz)|^2 P_n(dx) = 0 \qquad (10)$$

(see Lemmas 3.2 and 3.3).

3. PROOF.

Before we prove Theorem 2.1, we state and prove technical lemmas.

Lemma 3.1. *For any Borel probability measure P on $(\mathbf{R}^d, \mathbf{B}(\mathbf{R}^d))$,*

$$\begin{aligned} & \rho(P(dx), g_d(x)dx) \\ \leq & \sum_{i=1}^{d} \rho(P(dx), g_1(x_i)dx_i P^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)) \\ \leq & \ 2d\rho(P(dx), g_d(x)dx). \end{aligned} \tag{11}$$

Proof. When $d = 1$, (11) is true (see (3)). Suppose that $d > 1$. Then one can show the following by induction in d:

$$\begin{aligned} & P(dx) - g_d(x)dx \\ = & \sum_{i=1}^{d} \Pi_{1\leq k\leq i-1} g_1(x_k)dx_k (P^{(j)_{j\geq i}}(\Pi_{j\geq i}dx_j) - g_1(x_i)dx_i P^{(j)_{j>i}}(\Pi_{j>i}dx_j)), \end{aligned}$$

where we put $\Pi_{1\leq k\leq 0} g_1(x_k)dx_k \equiv 1$. This together with the following proves the first inequality in (11): for $i = 2, \cdots, d$,

$$\begin{aligned} & P^{(j)_{j\geq i}}(\Pi_{j\geq i}dx_j) - g_1(x_i)dx_i P^{(j)_{j>i}}(\Pi_{j>i}dx_j) \\ = & \int_{\{(x_j)_{j<i}\in\mathbf{R}^{i-1}\}} (P(dx) - g_1(x_i)dx_i P^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)). \end{aligned}$$

The second inequality in (11) can be shown by the following: for $i = 1, \cdots, d$,

$$\begin{aligned} & P(dx) - g_1(x_i)dx_i P^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j) \\ = & \ P(dx) - g_d(x)dx + g_1(x_i)dx_i \int_{\{x_i\in\mathbf{R}\}} (g_d(x)dx - P(dx)). \end{aligned}$$

▮

Lemma 3.2. *Suppose that $d > 1$, and that a sequence of Borel probability measures $\{P_n\}_{n\geq 1}$ on $(\mathbf{R}^d, \mathbf{B}(\mathbf{R}^d))$ satisfies the following: for some $i \in \{1, \cdots, d\}$,*

$$\lim_{n\to\infty} \rho(P_n(dx), g_1(x_i)dx_i (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)) = 0, \tag{12}$$

and $\int_{\mathbf{R}^d} |x_i|^2 P_n(dx) = 1$ *for all* $n \geq 1$. *Then the following holds:*

$$\lim_{n\to\infty} \int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i} dx_j) | \int_{\mathbf{R}} x_i (P_n)^i_{(x_j)_{j\neq i}}(dx_i)|^2 = 0. \tag{13}$$

Proof. For $R > 0$,

$$\begin{aligned}
&\int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i} dx_j) | \int_{\mathbf{R}} x_i (P_n)^i_{(x_j)_{j\neq i}}(dx_i)|^2 \\
\leq\ & 2 \int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i} dx_j) | \int_{-R}^{R} x_i (P_n)^i_{(x_j)_{j\neq i}}(dx_i)|^2 \\
& \quad +2 \int_{\{x\in\mathbf{R}^d : |x_i|\geq R\}} |x_i|^2 P_n(dx).
\end{aligned} \tag{14}$$

The first part of the right hand side of (14) can be shown to converge to zero as $n \to \infty$, by the following:

$$| \int_{-R}^{R} x_i (P_n)^i_{(x_j)_{j\neq i}}(dx_i)| \leq R, \quad (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i} dx_j) - a.s., \tag{15}$$

and

$$\begin{aligned}
&\int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i} dx_j) | \int_{-R}^{R} x_i (P_n)^i_{(x_j)_{j\neq i}}(dx_i)| \\
=\ & \sum_{k=1}^{2} (-1)^k \int_{A_{n,k}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i} dx_j) \\
& \quad \times \{R((P_n)^i_{(x_j)_{j\neq i}}((-\infty, R]) - \int_{-\infty}^{R} g_1(x_i) dx_i \\
& \quad +(P_n)^i_{(x_j)_{j\neq i}}((-\infty, -R]) - \int_{-\infty}^{-R} g_1(x_i) dx_i) \\
& \quad - \int_{-R}^{R} ((P_n)^i_{(x_j)_{j\neq i}}((-\infty, x_i]) - \int_{-\infty}^{x_i} g_1(y) dy) dx_i\} \\
\leq\ & 4R\rho(P_n(dx), g_1(x_i) dx_i (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i} dx_j)),
\end{aligned} \tag{16}$$

where we put

$$A_{n,1} \equiv \{(x_j)_{j\neq i} \in \mathbf{R}^{d-1} : \int_{-R}^{R} x_i (P_n)^i_{(x_j)_{j\neq i}}(dx_i) < 0\},$$

$$A_{n,2} \equiv \{(x_j)_{j\neq i} \in \mathbf{R}^{d-1} : \int_{-R}^{R} x_i (P_n)^i_{(x_j)_{j\neq i}}(dx_i) \geq 0\}$$

(see Remark 2.2). In (16) we used the following:

$$\int_{-R}^{R} x_i g_1(x_i) dx_i = 0.$$

The second part of the right hand side of (14) can be shown to converge to zero as $n \to \infty$, by the following: by (12),

$$\int_{\{x\in\mathbf{R}^d : |x_i|\geq R\}} |x_i|^2 P_n(dx) \overset{n\to\infty}{\longrightarrow} 1 - \int_{-R}^{R} |y|^2 g_1(y) dy \overset{R\to\infty}{\longrightarrow} 0. \quad (17)$$

$\blacksquare$

LEMMA 3.3. *Suppose that $d > 1$, and that a sequence of Borel probability measures $\{P_n\}_{n\geq 1}$ on $(\mathbf{R}^d, \mathbf{B}(\mathbf{R}^d))$ satisfies the following: for some $i \in \{1, \cdots, d\}$,*

$$\lim_{n\to\infty} \rho(W^i(P_n)(dx), P_n(dx)) = 0, \quad (18)$$

and that $\int_{\mathbf{R}^d} |x_i|^2 P_n(x) dx = 1$ for all $n \geq 1$. Then the following holds:

$$\lim_{n\to\infty} \int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}} (\Pi_{j\neq i} dx_j) | \int_{\mathbf{R}} y (P_n)^i_{(x_j)_{j\neq i}}(dy)|^2 = 0, \quad (19)$$

and for any $R > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \sup_{A\in\mathbf{B}(\mathbf{R}^d)} | \int_{\{((x_j)_{j\neq i}, y)\in A : |y|\leq R\}} (P_n)^{(j)_{j\neq i}} (\Pi_{j\neq i} dx_j) g_1(y) dy \quad (20) \\ &\qquad \times (\int_{-\infty}^{y} -z (P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(y) - \int_{-\infty}^{0} -z (P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(0))| \\ &= 0, \end{aligned}$$

and

$$\lim_{n\to\infty}\sup_{A\in\mathbf{B}(\mathbf{R}^{d-1})}|\int_A (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j) \tag{21}$$
$$\times(\int_{-\infty}^0 -z(P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(0)-1)|=0.$$

Proof. (19) can be proved by (18) and by the following:

$$\begin{aligned} &\int_{\mathbf{R}^d} W^i(P_n)(dx) \\ = &\int_{\mathbf{R}^{d-1}}(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j) \\ &\times\{\int_{-\infty}^0(-y)(\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)-y)(P_n)^i_{(x_j)_{j\neq i}}(dy) \\ &-\int_0^\infty y(\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)-y)(P_n)^i_{(x_j)_{j\neq i}}(dy)\} \\ = &-\int_{\mathbf{R}^{d-1}}(P_n)^{(j)_{j\neq i}}((x_j)_{j\neq i})\Pi_{j\neq i}dx_j|\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)|^2+1, \end{aligned} \tag{22}$$

where we used the following:

$$\int_{\mathbf{R}}(\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)-y)(P_n)^i_{(x_j)_{j\neq i}}(dy)=0. \tag{23}$$

(20) can be proved by (18)-(19) and by the following: for $y\in[-R,R]$,

$$\begin{aligned} &\int_{-\infty}^y -z(P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(y)-\int_{-\infty}^0 -z(P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(0) \\ = &\int_0^y x_i g_1(x_i)^{-1}(\int_{-\infty}^{x_i} -z(P_n)^i_{(x_j)_{j\neq i}}(dz)dx_i-(P_n)^i_{(x_j)_{j\neq i}}(dx_i)), \end{aligned} \tag{24}$$

and

$$(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)\int_{-\infty}^{x_i} -z(P_n)^i_{(x_j)_{j\neq i}}(dz)dx_i-P_n(dx) \tag{25}$$

$$
\begin{aligned}
&= -(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)(P_n)^i_{(x_j)_{j\neq i}}((-\infty,x_i])dx_i \\
&\quad +W^i(P_n)(dx)-P_n(dx).
\end{aligned}
$$

(21) can be proved by (17)-(20) and by the following: for $R>0$,

$$
\begin{aligned}
&(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)(1-\int_{-\infty}^0 -z(P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(0)) \qquad (26)\\
&= (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)\{\int_{\{x_i\in\mathbf{R}:|x_i|>R\}}((P_n)^i_{(x_j)_{j\neq i}}(dx_i) \\
&\quad -(\int_{-\infty}^0 -z(P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(0))g_1(x_i)dx_i) \\
&\quad +\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)\int_{-R}^R (P_n)^i_{(x_j)_{j\neq i}}((-\infty,x_i])dx_i \\
&\quad +\int_{-R}^R(\int_{-\infty}^y -z(P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(y) \\
&\quad -\int_{-\infty}^0 -z(P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(0))g_1(y)dy\} \\
&\quad +\int_{\{x_i\in\mathbf{R}:|x_i|\leq R\}}(P_n(dx)-W^i(P_n)(dx)).
\end{aligned}
$$

▮

Finally we prove Theorem 2.1.

Proof (Proof of Theorem 2.1). We only have to prove the case where $d>1$. In fact, when $d=1$, the proof is done from the case where $d>1$, by considering probability measures $\{P_n(dx)\times g_1(y)dy\}_{n\geq 1}$ on $(\mathbf{R}^2,\mathbf{B}(\mathbf{R}^2))$.

We assume that $d>1$ from here on.

Suppose that (I) in Theorem 2.1 holds. Then the following which will be proved later holds: for any $i\in\{1,\cdots,d\}$

$$
\lim_{n\to\infty}\rho(W^i(P_n)(dx),(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)g_1(x_i)dx_i)=0, \qquad (27)
$$

which implies (II) by Lemma 3.1.

(27) is true. Indeed, by (23),

$$
W^i(P_n)(dx)-(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)g_1(x_i)dx_i \qquad (28)
$$

$$
\begin{aligned}
= \ & (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)dx_i \\
& \times\{1_{(-\infty,0]}(x_i)(\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)(P_n)^i_{(x_j)_{j\neq i}}((-\infty,x_i]) \\
& \qquad -\int_{(-\infty,x_i]} y((P_n)^i_{(x_j)_{j\neq i}}(dy)-g_1(y)dy)) \\
& -1_{(0,\infty)}(x_i)(\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)(P_n)^i_{(x_j)_{j\neq i}}((x_i,\infty)) \\
& \qquad -\int_{(x_i,\infty)} y((P_n)^i_{(x_j)_{j\neq i}}(dy)-g_1(y)dy))\},
\end{aligned}
$$

where $1_A(x)$ denotes an indicator function of the set A. We also have

$$
\begin{aligned}
& \int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)|\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)| \\
& \times(\int_{-\infty}^0 dx_i(P_n)^i_{(x_j)_{j\neq i}}((-\infty,x_i])+\int_0^\infty dx_i(P_n)^i_{(x_j)_{j\neq i}}((x_i,\infty))) \\
= \ & \int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)|\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)|\int_{\mathbf{R}}|y|(P_n)^i_{(x_j)_{j\neq i}}(dy) \\
\leq \ & (\int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)|\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)|^2)^{1/2}\to 0,
\end{aligned}
\tag{29}
$$

as $n\to\infty$ by Lemma 3.2. For $R>0$,

$$
\begin{aligned}
& \{(x_i,y):-\infty<y\leq x_i\leq 0\} \\
= \ & \{(x_i,y):-\infty<y\leq x_i\leq -R\}\cup\{(x_i,y):-\infty<y\leq -R<x_i\leq 0\} \\
& \quad \cup\{(x_i,y):-R<y\leq x_i\leq 0\}, \\
& \{(x_i,y):0<x_i<y<\infty\} \\
= \ & \{(x_i,y):R\leq x_i<y<\infty\}\cup\{(x_i,y):0<x_i<R\leq y<\infty\} \\
& \quad \cup\{(x_i,y):0<x_i<y<R\}.
\end{aligned}
$$

Hence the following (30)-(31) completes the proof of (27).

$$
\begin{aligned}
& \int_{\mathbf{R}^{d-1}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j) \\
& \quad \times\{\int_{-\infty}^{-R} dx_i\int_{-\infty}^{x_i}|y|((P_n)^i_{(x_j)_{j\neq i}}(dy)+g_1(y)dy) \\
& \quad +\int_R^\infty dx_i\int_{x_i}^\infty |y|((P_n)^i_{(x_j)_{j\neq i}}(dy)+g_1(y)dy)
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&+R(\int_{-\infty}^{-R} |y|((P_n)^i_{(x_j)_{j\neq i}}(dy)+g_1(y)dy) \\
&+\int_R^{\infty} |y|((P_n)^i_{(x_j)_{j\neq i}}(dy)+g_1(y)dy))\} \\
= & \int_{\{x\in \mathbf{R}^d:|x_i|\geq R\}} |x_i|^2 P_n(dx) + \int_{\{y\in \mathbf{R}:|y|\geq R\}} |y|^2 g_1(y)dy \to 0,
\end{aligned}
$$

as $n\to\infty$ and then $R\to\infty$ (see (17)), and for any Borel measurable $\varphi : \mathbf{R}^d \mapsto [-1,1]$,

$$
\begin{aligned}
&|\int_{\mathbf{R}^d} \varphi(x)(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)dx_i \qquad (31)\\
&\times\{1_{(-R,0]}(x_i)\int_{(-R,x_i]} y(-(P_n)^i_{(x_j)_{j\neq i}}(dy)+g_1(y)dy) \\
&\quad +1_{(0,R)}(x_i)\int_{(x_i,R)} y((P_n)^i_{(x_j)_{j\neq i}}(dy)-g_1(y)dy)\}| \\
\leq\ & 2R^2\rho(P_n(dx),(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)g_1(x_i)dx_i).
\end{aligned}
$$

Suppose that (II) in Theorem 2.1 holds. Then the following which will be proved later holds: for all $i\in\{1,\cdots,d\}$,

$$
\lim_{n\to\infty}\rho(P_n(dx),(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)g_1(x_i)dx_i)=0, \qquad (32)
$$

which implies (I) in Theorem 2.1 by Lemma 3.1.

We prove (32) to complete the proof. For any $R>0$, by Chebychev's inequality,

$$
\begin{aligned}
&\int_{\{x\in\mathbf{R}^d:|x_i|\geq R\}} (P_n(dx)+(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)g_1(x_i)dx_i) \qquad (33)\\
\leq\ & R^{-2}(\int_{\{x\in\mathbf{R}^d:|x_i|\geq R\}} |x_i|^2P_n(dx)+\int_{\{y\in\mathbf{R}:|y|\geq R\}}|y|^2g_1(y)dy) < 2/R^2,
\end{aligned}
$$

and for any $A\in\mathbf{B}(\mathbf{R}^d)$

$$
\int_{\{((x_j)_{j\neq i},x_i)\in A:|x_i|\leq R\}} (P_n(dx)-(P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)g_1(x_i)dx_i) \qquad (34)
$$

$$
\begin{aligned}
= &\int_{\{((x_j)_{j\neq i},x_i)\in A:|x_i|\leq R\}} (P_n(dx) - W^i(P_n)(dx)) \\
&\quad + \int_{\{((x_j)_{j\neq i},x_i)\in A:|x_i|\leq R\}} (P_n)^{(j)_{j\neq i}}(\Pi_{j\neq i}dx_j)dx_i \\
&\quad \times\{\int_{\mathbf{R}} z(P_n)^i_{(x_j)_{j\neq i}}(dz)\int_{-\infty}^{x_i}(P_n)^i_{(x_j)_{j\neq i}}(dz) \\
&\quad +(\int_{-\infty}^{x_i} -z(P_n)^i_{(x_j)_{j\neq i}}(dz)/g_1(x_i)-1)g_1(x_i)\},
\end{aligned}
$$

which completes the proof of (32) by Lemma 3.3. ▮

4. A TYPICAL EXAMPLE.

In this section we give a typical example.

Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables such that

$$P(X_1 \in dx) = \frac{1}{8\pi}1_{\{y\in\mathbf{R}^2|1<|y|<3\}}(x)dx. \tag{35}$$

Put

$$S_n := \sqrt{\frac{2}{5n}}\sum_{k=1}^{n} X_k, \quad \begin{pmatrix} S_{n,1} \\ S_{n,2} \end{pmatrix} := S_n, \quad P_n(dx) := P(S_n \in dx). \tag{36}$$

Then the dispersion matrix of S_n is an identity matrix, and $p_n(x) := P_n(dx)/dx$ exists, and for any $n \geq 2$,

$$p_n(x) > 0 \quad \text{if and only if} \quad |x| < 3n\sqrt{\frac{2}{5n}}. \tag{37}$$

For $i = 1$, 2,

$$W^i(P_n)(x)dx = P(S_{n,j} \in dx_j)dx_i\int_{-\infty}^{x_i} -yP(S_{n,i}\in dy|S_{n,j}=x_j) \tag{38}$$

($j = 1$, 2, $j \neq i$), where $(x_1, x_2) := x$.

Replace $E[w_1^i(X_1)g_i(S_n)]$ in [2, (3.1)] and $E[w_1(T_1)g_i(S_n)]$ in [2, (3.2)] by

$$\int_{\mathbf{R}^2} W^i(P_1)(x)dxE\Big[g_i\Big(\sqrt{\frac{1}{n}}x + \sqrt{\frac{n-1}{n}}S_{n-1}\Big)\Big].$$

Then one can show, in the same way as in [2, Theorem 3.1], that the following holds: for $n \geq 2$ and $i = 1, 2$,

$$\rho(W^i(P_n)(x)dx, p_n(x)dx) = \int_{\mathbf{R}^2} \left|\frac{W^i(P_n)(x)}{p_n(x)} - 1\right| p_n(x)dx \tag{39}$$
$$\leq \int_{\mathbf{R}^2} \left|\frac{W^i(P_n)(x)}{p_n(x)} - 1\right|^2 p_n(x)dx = \int_{\mathbf{R}^2} \left|\frac{W^i(P_n)(x)}{p_n(x)}\right|^2 p_n(x)dx - 1$$
$$\to 0 \quad (\text{as } n \to \infty).$$

This together with Theorem 2.1 implies that the following holds:

$$\lim_{n\to\infty} \rho(P_n(dx), g_d(x)dx) = 0. \tag{40}$$

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