Anisotropic convexified Gauss curvature flow of bounded open sets:
stochastic approximation, weak solution and viscosity solution

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1 Introduction

Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach (see [2]).

In [3] we proposed and studied a two dimensional random crystalline algorithm for the curvature flow of smooth simple closed convex curves.

In [4] we studied a convexified Gauss curvature flow of compact sets by the level set approach in the theory of viscosity solutions.

In this talk we discuss a random crystalline algorithm of and PDE on an anisotropic convexified Gauss curvature flow of bounded open sets in $\mathbf{R}^N$ for **any** $N \geq 2$ (see [5]).

We introduce an assumption and a notation before we describe the PDE under consideration.

(A.1). $R \in L^1(\mathbf{S}^{N-1} : [0,\infty), d\mathcal{H}^{N-1})$, and $||R||_{L^1(\mathbf{S}^{N-1})} = 1$.

For $p \in \mathbf{R}^N$ and a $N \times N$-symmetric real matrix X, put $G(o, X) := 0$ and

$$G(p, X) := |p| \det{}_+ \left(-(I - \frac{p}{|p|} \otimes \frac{p}{|p|}) \frac{X}{|p|} (I - \frac{p}{|p|} \otimes \frac{p}{|p|}) + \frac{p}{|p|} \otimes \frac{p}{|p|} \right)$$

if $p \neq o$.

We study a weak solution and a viscosity solution of the following PDE in this talk:

$$\partial_t u(t, x) = \sigma^+(u, Du(t, x), t, x) R\Big(\frac{Du(t, x)}{|Du(t, x)|}\Big) G(Du(t, x), D^2 u(t, x)) \quad (1.1)$$

$((t, x) \in (0, \infty) \times \mathbf{R}^N)$. Here

$$\sigma^+(u, p, t, x) := \begin{cases} 1 & \text{if } u(t, \cdot) \le u(t, x) \text{ on } H(p, x) \text{ and } p \in \mathbf{R}^N \setminus \{o\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$H(p, x) := \{y \in \mathbf{R}^N \setminus \{x\} | < y - x, p > \le 0\}.$$

To introduce the notion of a weak solution to (1.1), we give several notations.

Let F be a closed convex subset of $\mathbf{R}^N$. For $x \in \partial F$, put

$$N_F(x) := \{p \in \mathbf{S}^{N-1} | F \subset \{y | < y - x, p > \le 0\}\}.$$

Definition 1 *Suppose that (A.1) holds. Let* $u : \mathcal{D}(u)(\subset \mathbf{R}^N) \mapsto \mathbf{R}$ *be bounded and* $r \in \mathbf{R}$*. For any* $B \in B(\mathbf{R}^N)$*, put*

$$\omega_r(R,u,B) := \int_{N_{(co\ u^{-1}([r,\infty)))^-}(B\cap\partial(co\ u^{-1}([r,\infty))))} R(p)d\mathcal{H}^{N-1}(p),$$

$$\mathbf{w}(R,u,B) := \int_{\mathbf{R}} dr\omega_r(R,u,B),$$

provided the right hand side is well defined.

Definition 2 (Weak Solutions) *Suppose that (A.1) holds.*

(i) A family of bounded open sets $\{D(t)\}_{t\ge 0}$ in $\mathbf{R}^N$ is called an anisotropic convexified Gauss curvature flow if

$$D(t) = \begin{cases} (co\ D(t)) \cap D(0) & \text{for } t \in [0, Vol(D)), \\ \emptyset & \text{for } t \ge Vol(D) \end{cases} \tag{1.2}$$

; and for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \ge 0$,

$$\int_{\mathbf{R}^N} \varphi(x)(I_{D(0)}(x) - I_{D(t)}(x))dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x)\omega_1(I_{D(s)}(\cdot), dx). \tag{1.3}$$

(ii) $u \in C_b([0,\infty) \times \mathbf{R}^N)$ is called the weak solution to (1.1) if the following holds: for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \ge 0$,

$$\int_{\mathbf{R}^N} \varphi(x)(u(0,x) - u(t,x))dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x)\mathbf{w}(u(s,\cdot), dx). \tag{1.4}$$

Before we introduce the notion of a viscosity solution to (1.1), we introduce notations

$f \in \mathcal{F}$ iff $f \in C^2([0,\infty))$, $f''(r) > 0$ on $(0,\infty)$, and $f(r)/r^N \to 0$ as $r \to 0$.

Let Ω be an open subset of $(0,\infty)\times\mathbf{R}^N$. $f\in\mathcal{A}(\Omega)$ iff $\varphi\in C^2(\Omega)$, and for any $(\hat{t},\hat{x})\in\Omega$ for which $D\varphi$ vanishes, there exists $f\in\mathcal{F}$ such that

$$|\varphi(t,x)-\varphi(\hat{t},\hat{x})-\partial_t\varphi(\hat{t},\hat{x})(t-\hat{t})|\le f(|x-\hat{x}|)+o(|t-\hat{t}|)\quad \text{as } (t,x)\to(\hat{t},\hat{x}).$$

Definition 3 (Viscosity solution) *(see [6]).*

Let $0<T\le\infty$ and set $\Omega:=(0,T)\times\mathbf{R}^N$.

(i). A function $u\in USC(\Omega)$ is called a viscosity subsolution of (1.1) in Ω if whenever $\varphi\in\mathcal{A}(\Omega)$, $(s,y)\in\Omega$, and $u-\varphi$ attains a local maximum at (s,y), then

$$\partial_t\varphi(s,y)+\sigma^-(u,D\varphi(s,y),s,y)R\Big(\frac{D\varphi(s,y)}{|D\varphi(s,y)|}\Big)G(D\varphi(s,y),D^2\varphi(s,y))\le 0, \tag{1.5}$$

where

$$\sigma^-(u,p,s,y):=\begin{cases}1 & \text{if } u(s,\cdot)<u(s,y) \text{ on } H(p,y) \text{ and } p\in\mathbf{R}^N\setminus\{o\},\\ 0 & \text{otherwise.}\end{cases}$$

(ii). A function $u\in LSC(\Omega)$ is called a viscosity supersolution of (1.1) in Ω if whenever $\varphi\in\mathcal{A}(\Omega)$, $(s,y)\in\Omega$, and $u-\varphi$ attains a local minimum at (s,y), then

$$\partial_t\varphi(s,y)+\sigma^+(u,D\varphi(s,y),s,y)R\Big(\frac{D\varphi(s,y)}{|D\varphi(s,y)|}\Big)G(D\varphi(s,y),D^2\varphi(s,y))\ge 0. \tag{1.6}$$

(iii). A function $u\in C(\Omega)$ is called a viscosity solution of (1.1) in Ω if it is both a viscosity subsolution and a supersolution of (1.1) in Ω.

Next we introduce a class of stochastic processes of which continuum limit becomes an anisotropic convexified Gauss curvature flow.

The following is an assumption on the initial set.

(A.2). D is a bounded open set in $\mathbf{R}^N$ such that $\mathrm{Vol}(\partial D) = 0$.

Take $K > 0$ so that $co\ D \subset [-K+1, K-1]^N$. Put

$$\mathcal{S}_n := \{I_A : [-K,K]^N \cap (\mathbf{Z}^N/n) \mapsto \{0,1\} | A \subset \mathbf{Z}^N/n\}.$$

For x, $z \in \mathbf{Z}^N/n$ and $v \in \mathcal{S}_n$, put

$$v_{n,z}(x) := \begin{cases} v(x) & \text{if } x \neq z, \\ 0 & \text{if } x = z \end{cases}$$

; and for a bounded $f : \mathcal{S}_n \mapsto \mathbf{R}$, put

$$A_n f(v) := n^N \sum_{z \in [-K,K]^N \cap (\mathbf{Z}^N/n)} \omega_1(R, v, \{z\}) \{f(v_{n,z}) - f(v)\}.$$

Let $\{Y_n(t,\cdot)\}_{t \geq 0}$ be a Markov process on $\mathcal{S}_n$ $(n \geq 1)$, with the generator A_n, such that $Y_n(0,z) = I_{D^c \cap (\mathbf{Z}^N/n)}(z)$.

For $(t,x) \in [0,\infty) \times [-K,K]^N$, put also

$$D_n(t) := (co\ Y_n(t,\cdot)^{-1}(1))^o \cap D. \tag{1.7}$$

$$X_n(t,x) := I_{D_n(t)}(x). \tag{1.8}$$

Then $\{X_n(t,\cdot)\}_{t \geq 0}$ is a stochastic process on

$$\mathcal{S} := \{f \in L^2([-K,K]^N) : ||f||_{L^2([-K,K]^N)} \leq (2K)^N\}$$

which is a complete separable metric space by the metric

$$d(f,g) := \sum_{k=1}^{\infty} \frac{\max(| < f-g, e_k >_{L^2([-K,K]^N)} |, 1)}{2^k}.$$

Here $\{e_k\}_{k\geq 1}$ denotes a complete orthonomal basis of $L^2([-K,K]^N)$.

By definition, the following holds.

(1) $D_n(0) \to D$ in Hausdorf metric as $n \to \infty$.

(2) $\sum_{z\in(\mathbf{Z}^N/n)\cap[-K,K]^N} |I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 0$ or 1 for all $t \geq 0$.

(3)If $|I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 1$, then $z \in \partial(co\ D_n(t-))$.

(4) $\sum_{z\in(\mathbf{Z}^N/n)\cap[-K,K]^N} |I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 1$ if and only if $t = \sigma_{n,i}$ for some i, where $0 < \sigma_{n,1} < \sigma_{n,1} < \cdots$ are random variables such that $\{\sigma_{n,i+1} - \sigma_{n,i}\}_{i>0}$ are independent and that

$$P(\sigma_{n,i+1} - \sigma_{n,i} \in dt) = n^N \exp(-n^N t)dt.$$

(5) $P(I_{D_n(\sigma_{n,i})}(z) - I_{D_n(\sigma_{n,i}-)}(z) = 1) = E[\omega_1(R, I_{D_n(\sigma_{n,i}-)}, \{z\})]$.

2 Main reslut

In this section we give our main result from [5].

The following theorem implies that D_n is a random crystalline approximation of an anisotropic convexified Gauss curvature flow.

Theorem 1 *Suppose that (A.1)-(A.2) hold. Then there exists a unique anisotropic convexified Gauss curvature flow $\{D(t)\}_{t\geq 0}$ with $D(0) = D$, and for any $\gamma > 0$,*

$$\lim_{n\to\infty} P(\sup_{0\le t} ||X_n(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \ge \gamma) = 0. \tag{2.1}$$

Suppose in addition that D is convex. Then for any $T \in [0, Vol(D))$ and $\gamma > 0$,

$$\lim_{n\to\infty} P(\sup_{0\le t\le T} d_H(D_n(t), D(t)) \ge \gamma) = 0, \tag{2.2}$$

where d_H denotes Hausdorff metric.

We introduce an additional assumption.

(A.3). $h \in C_b(\mathbf{R}^N)$ and for any $r \in \mathbf{R}$, the set $h^{-1}((r,\infty))$ is bounded or $\mathbf{R}^N$.

The following corollary implies that a level set of a continuous weak solution to (1.1) is determined by that at $t = 0$.

Corollary 1 *Suppose that (A.1) and (A.3) hold. Then there exists a unique bounded continuous weak solution $\{u(t,\cdot)\}_{t\ge 0}$ to (1.1) and for any $r \in \mathbf{R}$, $\{u(t,\cdot)^{-1}((r,\infty))\}_{t\ge 0}$ is a unique anisotropic convexified Gauss curvature flow with initial data $u(0,\cdot)^{-1}((r,\infty))$.*

We state properties of anisotropic convexified Gauss curvature flows.

Theorem 2 *Suppose that (A.1)-(A.2) hold. Let $\{D(t)\}_{t\ge 0}$ be a unique anisotropic convexified Gauss curvature flow $\{D(t)\}_{t\ge 0}$ with $D(0) = D$. Then*

(a) $t \mapsto D(t)$ is nonincreasing on $[0,\infty)$.

(b) For any $t \le T^* := Vol(D(0))$,

$$Vol(D(0) \backslash D(t)) = t. \tag{2.3}$$

(c) Let $\{D_1(t)\}_{t \geq 0}$ be an anisotropic convexified Gauss curvature flow such that $D_1(0)$ is a bounded, convex, open set which contains D. Then

$$D(t) \subset D_1(t) \quad \textit{for all } t \geq 0, \tag{2.4}$$

where the equality holds if and only if $D(0) = D_1(0)$.

We give an additional assumption and state the result on viscosity solutions to (1.1).
(A.4). $R \in C(S^{N-1} : [0, \infty))$.

Theorem 3 *Suppose that (A.2) and (A.4) hold. Let $\{D(t)\}_{t \geq 0}$ be a unique anisotropic convexified Gauss curvature flow $\{D(t)\}_{t \geq 0}$ with $D(0) = D$. Then $I_{D(t)}(x)$ and $I_{D(t)^-}(x)$ are a viscosity supersolution and a viscosity subsolution to (1.1), respectively.*

The followng results imply that $u \in C_b([0, \infty) \times \mathbf{R}^N)$ is a weak solution to (1.1) if and only if it is a viscosity solution to (1.1), in case R is constant.

Corollary 2 *Suppose that (A.3)-(A.4) hold. Then a unique weak solution $u \in C_b([0, \infty) \times \mathbf{R}^N)$ to (1.1) is a viscosity solution to it.*

Corollary 3 *(see [4]) Suppose that (A.3) holds and that R is constant. Then a continuous viscosity solution to (1.1) is unique and is a weak solution to it.*

3 Sketch of Proof

(Idea of Proof of Theorem 1). We first show that $\{X_n(t,\cdot)\}_{t\geq 0}$ is tight in $D([0,\infty) : \mathcal{S})$. By the weak convergence result on ω_1 by Bakelman [1], we show that any weak limit point of $\{X_n(t,\cdot)\}_{t\geq 0}$ is a weak solution to (1.1).

The following lemma implies the uniqueness, and hence completes the proof.

Lemma 1 *Suppose that (A.1) hold. If $\{I_{D_i(t)}\}_{t\geq 0}$ $(i = 1, 2)$ are weak solutions to (1.1) for which $D_1(0) \subset D_2(0)$, then $D_1(t) \subset D_2(t)$ for all $t \geq 0$. In particular,*

$$d(D_1(t), D_2(t)^c) \geq d(D_1(0), D_2(0)^c), \tag{3.1}$$

for $t \leq Vol(D_1(0))$.

(Sketch of Proof of Corollary 1). For $r \in \mathbf{R}$, let $\{I_{D_r(t)}\}_{t\geq 0}$ denote a unique weak solution of (1.1) with $D_r(0) = h^{-1}((r,\infty))$.

Put

$$u(t,x) := \sup\{r \in \mathbf{R} | x \in D_r(t)\}.$$

Then u is continuous. In particular, for all $t \geq 0$ and $r \in \mathbf{R}$,

$$u(t,\cdot)^{-1}((r,\infty)) = D_r(t).$$

For $n \geq 1$, put $k_{n,1} := [n \sup\{h(y)|y \in \mathbf{R}^N\}]$ and $k_{n,0} := [n \inf\{h(y)|y \in \mathbf{R}^N\}]$. Then for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \geq 0$,

$$
\begin{aligned}
&\int_{\mathbf{R}^N} \varphi(x)[\sum_{k_{n,0}\le k\le k_{n,1}} \frac{k}{n}(I_{D_{\frac{k}{n}}(t)^c}(x) - I_{D_{\frac{k+1}{n}}(t)^c}(x)) \\
&\qquad - \sum_{k_{n,0}\le k\le k_{n,1}} \frac{k}{n}(I_{D_{\frac{k}{n}}(0)^c}(x) - I_{D_{\frac{k+1}{n}}(0)^c}(x))]dx \\
= &\int_0^t ds[\sum_{k_{n,0}<k\le k_{n,1}} \frac{1}{n}\int_{\mathbf{R}^N} \varphi(x)\omega_0(R, I_{D_{\frac{k}{n}}(s)^c}(\cdot), dx)].
\end{aligned}
$$

Letting $n \to \infty$ in (3.4), u is shown to be a weak solution to (1.1).

The uniqueness of u follows from that of $D_r(\cdot)$ for all r. In fact we can show that for a continuous weak solution v to (1.1), $\{v(t,\cdot)^{-1}((r,\infty))\}_{t\ge 0}$ is an anisotropic convexified Gauss curvature flow.

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We omit the proof of Theorems 2 and 3.

(Sketch of Proof of Corollary 2)

Let u be a weak solution to (1.1).

We first show that u is a viscosity supersolution to (1.1). Suppose that u is smooth in Ω and that $\varphi \in \mathcal{A}(\Omega)$, $(s,y) \in \Omega$, and $u - \varphi$ attains a local maximum at (s,y). Then

$$\partial_s(u-\varphi)(s,y) \ge 0.$$

$$\partial_s(u-\varphi^{\varepsilon})(s,y) \ge 0$$

($\varepsilon > 0$), where $\varphi^{\varepsilon} := \varphi - \varepsilon$. Hence formally, we have, in some neighborhood of (s,y),

$$
\begin{aligned}
& \partial_s \varphi^\varepsilon(t,x) \\
\leq\ & \partial_s u(t,x) = -\mathbf{w}(u(t,\cdot),dx)/dx \\
\leq\ & -\mathbf{w}(\varphi^\varepsilon(t,\cdot),dx)/dx = -R\Big(\frac{D\varphi(t,x)}{|D\varphi(t,x)|}\Big)G(D\varphi(t,x),D^2\varphi(t,x)).
\end{aligned}
$$

Then following lemma completes the proof.

Take $\varphi \in C^2(\mathbf{R}^N : \mathbf{R})$ for which $D\varphi(x_o) \neq 0$ for some $x_o \in \mathbf{R}^N$. For $i = 1, \cdots, N$, put

$$
y_i(x) := \Big(-(1-\delta_{ij})\frac{\partial_j\varphi(x)}{|D\varphi(x)|} + \delta_{ij}\varphi(x)\Big)_{j=1}^N.
$$

Then

Lemma 2 *Suppose that all eigenvalues of $-D(D\varphi(x_o)/|D\varphi(x_o)|)$ are non-negative. Then, for $i = 1, \cdots, N$,*

$$
\frac{\partial_i\varphi(x_o)}{|D\varphi(x_o)|}G(D\varphi(x_o),D^2\varphi(x_o)) = \det(Dy_i(x_o)). \tag{3.2}
$$

Similarly one can show that u is a viscosity subsolution to (1.1).

Q. E. D.

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