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The one-cocycle property for shifts

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Abstract. The two-sided shift on the infinite tensor product of copies of the $n \times n$ matrix algebra has the so-called Rohlin property, which entails the one-cocycle property, useful in analyzing cocycle-conjugacy classes. In the case $n = 2$, the restriction of the shift to the gauge-invariant CAR algebra also has the one-cocycle property. We extend the latter result to an arbitrary $n \geq 2$. As a corollary it follows that the flow α on the Cuntz algebra $\mathcal{O}_n = C^*(s_0, s_1, \ldots, s_{n-1})$ defined by $\alpha_t(s_j) = e^{ip_jt}s_j$ has the Rohlin property (for flows) if and only if $p_0, \ldots, p_{n-1}$ generate $\mathbb{R}$ as a closed sub-semigroup. Note that then such flows are all cocycle-conjugate to each other.

1. *Introduction*

For an integer n greater than 1 we denote by M_n the $n \times n$ matrix algebra over the complex numbers $\mathbb{C}$. For each integer $m \in \mathbb{Z}$ we assign a copy $M_n^{(m)}$ of M_n and take the infinite tensor product $B_n = \bigotimes_{m \in \mathbb{Z}} M_n^{(m)}$. The shift automorphism σ of B_n is defined by sending an element of $M_n^{(m)}$ to the corresponding element in $M_n^{(m+1)}$.

In [**4**] it was shown that σ has the Rohlin property (see below) in the case $n = 2$. The proof is based on a known connection between such a σ and a certain quasi-free automorphism of the C^*-algebra associated with the canonical anti-commutation relations, or the CAR algebra.

This prompts us to attempt to generalize it. A further exploitation of the CAR algebra formalism was done in [**2, 3**]. A full generalization for any $n \geq 2$ was done in [**15, 16**]. Also, an extension to the shift on $B = \bigotimes_{\mathbb{Z}}(M_2 \oplus M_3)$ was done in [**18**]. In this generality the Rohlin property for the shift σ reads as follows: for any $N \in \mathbb{N}$ and $\epsilon > 0$ there is a family $\{e_{1i} \mid i = 0, 1, \ldots, N-1\} \cup \{e_{2i} \mid i = 0, 1, \ldots, N\}$ of projections in B such that $\sum_i e_{1i} + \sum_{2i} e_{2i} = 1$ and

$$\|\sigma(e_{1,i}) - e_{1,i+1}\| < \epsilon$$

for $i = 0, \ldots, N-2$ and

$$\|\sigma(e_{2,i}) - e_{2,i+1}\| < \epsilon$$

for $i = 0, 1, \ldots, N-1$. (Hence, it follows that $\sigma(e_{1,N-1} + e_{2,N}) \approx e_{1,0} + e_{2,0}$.)

A useful consequence of this property is the one-cocycle property, i.e. for any unitary u and $\epsilon > 0$ there is a unitary v such that $\|u - v\sigma(v)^*\| < \epsilon$, which is the property we actually need for applications. We refer the reader to [**7, 10**] for the Rohlin property and more applications.

There is yet another attempt on the restriction of the shift to a certain C^*-subalgebra of $\bigotimes_{\mathbb{Z}} M_2$, which corresponds to the gauge-invariant CAR algebra, based on the CAR algebra formalism [**19**]. In this case the shift (on this subalgebra) cannot have the Rohlin property, but can have an approximate version of Rohlin property. The result is that the shift has the one-cocycle property (appropriately formulated).

In this note we extend the above result to the case of general $n > 2$. We now formulate the problem more precisely below.

Let $\mathcal{U}_n$ be the unitary group of M_n and define an action β of $\mathcal{U}_n$ on $B_n = \bigotimes_{m\in\mathbb{Z}} M_n^{(m)}$ by

$$\beta_u = \bigotimes_{m\in\mathbb{Z}} \operatorname{Ad} u^{(m)},$$

where $u^{(m)}$ is the copy of u in $M_n^{(m)}$. Note that β_u commutes with σ.

Denoting by $\mathbb{T}$ the group of complex numbers of modulus one, we regard $\mathbb{T}^{n-1}$ as a subgroup of $\mathcal{U}_n$ by

$$(z_1, \ldots, z_{n-1}) \mapsto \begin{pmatrix} 1 & & & \\ & z_1 & & \\ & & \ddots & \\ & & & z_{n-1} \end{pmatrix}$$

and define an action γ of $\mathbb{T}^{n-1}$ by $\beta|\mathbb{T}^{n-1}$. We call γ the gauge action (of $\mathbb{T}^{n-1}$) on B_n.

Let $\{e_{ij}\}$ denote the matrix units for M_n and let

$$v_m = \sum_{ij} e_{ij}^{(m)} e_{ji}^{(m+1)}.$$

Then it follows that v_m is a self-adjoint unitary in the fixed-point algebra B_n^β and satisfies $\operatorname{Ad} v_m(x^{(m)}y^{(m+1)}) = y^{(m)}x^{(m+1)}$ for $x, y \in M_n$. If we set $V_m = v_{-m}v_{-m+1}\cdots v_0\cdots v_m$, then $V_m \in B_n^\beta$ satisfies that $\sigma(x) = \lim \operatorname{Ad} V_m(x)$ for $x \in B_n$.

We set $A_n = B_n^\gamma$, the fixed-point algebra of B_n under $\gamma = \beta|\mathbb{T}^{n-1}$. Note that σ restricts to A_n, which is denoted by $\sigma|A_n$ or simply σ. Our purpose is to prove that $\sigma|A_n$ has the one-cocycle property for all n.

To present a precise statement we should note that the approximately finite-dimensional C^*-algebra A_n is prime and not simple. It has n maximal ideals of codimension one.

For $i = 0, 1, \ldots, n-1$, let I_i be the (closed, two-sided) ideal of A_n generated by $e_{jj}^{(m)}$, $j \neq i$, $m \in \mathbb{Z}$. We then note that $\left(\bigotimes_{-M}^{M} M_n\right) \cap I_i$ is orthogonal to $\Pi_{-M}^{M} e_{ii}^{(m)}$. It follows that the quotient A_n/I_i is isomorphic to $\mathbb{C}$; let φ_i be the corresponding character on A_n.

THEOREM 1.1. *Let $n = 2, 3, \ldots$ and let $B_n, A_n, \sigma, \varphi_i$ be as above. Then the shift σ on A_n has the following properties.*

(1) *For any unitary $u \in A_n$ such that $\varphi_i(u) = 1$ for $i = 0, \ldots, n-1$, there is a sequence (v_k) of unitaries in A_n such that $\|u - v_k\sigma(v_k)^*\| \to 0$.*

(2) *If (u_k) is a sequence in the unitary group $\mathcal{U}(A_n)$ such that $\varphi_i(u_k) = 1$ for $i = 0, \dots, n-1$ and $\|[u_k, x]\| \to 0$ for all $x \in B_n$, then there is a sequence (v_k) in $\mathcal{U}(A_n)$ such that $\|u_k - v_k\sigma(v_k)\| \to 0$ and $\|[v_k, x]\| \to 0$ for all $x \in B_n$.*

(3) *If (u_k) is a sequence in $\mathcal{U}(A_n)$ such that $\varphi_i(u_k) = 1$ for $i = 0, \dots, n-1$ and $\|[u_n, x]\| \to 0$ for all $x \in A_n$, then there is a sequence (v_k) in $\mathcal{U}(A_n)$ such that $\|u_k - v_k\sigma(v_k)\| \to 0$ and $\|[v_k, x]\| \to 0$ for all $x \in A_n$.*

The proof of this theorem occupies the following sections.

Note that A_n is an approximately finite-dimensional C^*-algebra, i.e. the closure of the union of an increasing sequence of finite-dimensional C^*-algebras [**1**]. Since we have to compare projections in such a C^*-algebra, we need a nice description of the dimension group $K_0(A_n)$ of A_n, i.e. the ordered group generated by equivalence classes of projections [**6**]. If $n = 2$, A_2 is already presented in [**1**], in terms of *Bratteli* diagram, and $K_0(A_2)$ is identified, by Renault [**24**], with the integer-coefficient polynomials $\mathbb{Z}[x]$, with the positivity defined by strict positivity on the open interval $(0, 1)$, which was good enough in the discussions in [**19**]. In general, Handelman [**9**] shows that $K_0(A_n)$ is identified with $\mathbb{Z}[x_1, \dots, x_{n-1}]$ with the order defined as follows: p is positive if p is expressed as

$$p = \sum_{|v|=K} c_v(1 - x_1 - \cdots - x_{n-1})^{v_0} x_1^{v_1} \cdots x_{n-1}^{v_{n-1}}$$

with non-negative coefficients $c_v \geq 0$, where the sum is taken over all $v \in \mathbb{Z}_+^n$ with $|v| = \sum_{i=0}^{n-1} v_i = K$. However, there seem to be no clear criteria for positivity as the Renault's result for $n = 2$, because the strict positivity on the interior of the $(n-1)$-simplex $\Lambda_n = \left\{(x_1, \dots, x_{n-1}) \mid x_i > 0, \sum_i x_i \leq 1\right\}$ with some boundary conditions is not enough.

By introducing a notion of *vanguard* for $p \in \mathbb{Z}[x_1, \dots, x_{n-1}]$ we give a sufficient condition for positivity, which is useful enough in the following arguments. If p is expressed as above and $i = 0, 1, \dots, n-1$, the i-vanguard of p on the level K is a certain subset of v with $|v| = K$ such that $c_v \neq 0$ and there is no w between v and Ke_i with $c_w \neq 0$. We see that the i-vanguard of p is essentially independent of K as well as the coefficients c_v; hence, if p is positive, those coefficients c_v must be positive, which does not follow from the positivity of p as a function on Λ_n. The sufficient condition states in Proposition 2.5 that if p is strictly positive on Λ_n possibly except for the vertices and the above-mentioned condition is satisfied for all the vanguards, then p is positive as an element of $\mathbb{Z}[x_1, \dots, x_{n-1}]$.

Note that A_n has quotients which are of type I. We consider irreducible representations of A_n whose images contain the compact operators, which are necessarily σ-covariant since σ is approximately inner. In Lemma 4.4 we show that the shift on a certain quotient of A_n of type I has the properties as in the theorem. This gives the first step of the proof by induction on n.

Then we show how to *locally embed* A_n into A_{n+1}, almost intertwining the shifts. This process will show, with the known result for $n = 2$ shown by the CAR algebra formalism (4.3), that the shift on A_n has the *approximate Rohlin property* for all n (see Lemma 4.6); by this property (Definition 4.2) we essentially mean that there is a sequence $(e_i^{(k)})$ in the orthogonal family of N projections in A_n for arbitrarily large N such that $\|\sigma(e_i^{(k)}) - e_{i+1}^{(k)}\| \to 0$ as $k \to \infty$ and $N[e_0^{(k)}](x)$ converges to 1 uniformly in x on every compact subset of Λ_n except for the vertices, as well as the lower estimate $[e_0^{(k)}](x) \geq C(x_0x_1\cdots x_{n-1})^S$ for some $S \in \mathbb{N}$ and $C > 0$ independent of k, where $x_0 = 1 - x_1 - \cdots - x_{n-1}$, and some estimates on the vanguards of $[e_0^{(k)}]$ independent of k. We see that the additional conditions on $[e_0^{(k)}]$ play an important role in comparison with other projections.

With some explicit estimates, in $K_0(A_n)$, of the projections involved, we then proceed just as in [**19**]. The crucial induction step is given in Lemma 4.12.

In the above theorem the first property is weaker than the third.

Remark 1.2. Let α be an approximately inner automorphism of A_n. If α has the third property of Theorem 1.1, then it also has the first property. This follows because then α is almost conjugate to the shift σ, i.e. there is a sequence (ϕ_m) of approximately inner automorphisms of A_n such that $\|\alpha - \phi_m\sigma\phi_m^{-1}\| \to 0$, as follows from Theorem 4.1 of [**8**]. If $u \in \mathcal{U}(A_n)$ such that $\varphi_i(u) = 1$, we have a sequence $(v_{m,k})$ in $\mathcal{U}(A_n)$ for each $m \in \mathbb{N}$ such that $\|\phi_m(u) - v_{m,k}\sigma(v_{m,k})^*\| \to 0$ as $k \to \infty$. Then we can choose a sequence (v_m) from $(\phi^{-1}(v_{m,k}))$ such that $\|u - v_m\alpha(v_m)^*\| \to 0$. (Since ϕ_m may not extend to an automorphism of B_n, this method does not show that α also has the second property. If α does not extend to an automorphism of B_n, α is unlikely to have the second property.)

The following is a corollary of Theorem 1.1(2).

COROLLARY 1.3. *Let n, B_n, A_n, σ be as above. Then the shift σ on B_n has a sequence (U_m) in $\mathcal{U}(A_n)$ such that $\sigma(x) = \lim_{m\to\infty} \operatorname{Ad} U_m(x)$ for $x \in B_n$ and $\lim_{m\to\infty} \|\sigma(U_m) - U_m\| = 0$.*

Proof. We have defined $V_m = v_{-m}v_{-m+1}\cdots v_0\cdots v_m \in A_n$, which satisfies

$$\sigma(x) = \lim \operatorname{Ad} V_m(x), \quad x \in B_n$$

but does not satisfy $\lim \|\sigma(V_m) - V_m\| = 0$. (By calculation, it follows that the spectrum of $V_m\sigma(V_m^*)$ is $\{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$ and hence that $\|\sigma(V_m) - V_m\| = \sqrt{3}$ for $m \geq 1$.) Since $V_m\sigma(V_m)^* \in A_n$, $\varphi_i(V_m\sigma(V_m^*)) = 1$, and $(V_m\sigma(V_m^*))$ is a central sequence in $\mathcal{U}(B_n)$, the previous theorem gives us a sequence (W_m) in $\mathcal{U}(A_n)$ such that $\|V_m\sigma(V_m^*) - W_m\sigma(W_m^*)\| \to 0$ and $\|[W_m, x]\| \to 0$ for $x \in B_n$. Let $U_m = W_m^*V_m \in \mathcal{U}(A_n)$. Then (U_m) satisfies the required conditions. $\square$

In connection with the above corollary a question arises naturally of whether (U_m) can be chosen from the smaller fixed-point algebra B_n^β.

When ℓ is a positive integer, we denote by $\sigma^{(\ell)}$ the cyclic shift on $\bigotimes_{m=0}^{\ell-1} M_n^{(m)}$ defined by $x^{(m)} \mapsto x^{(m+1)}$ with $x^{(\ell)} = x^{(0)}$. We also denote by $\gamma^{(\ell)}$ the action of $\mathbb{T}^{n-1}$ on $\bigotimes_{m=0}^{\ell-1} M_n^{(m)}$ by $\bigotimes_{m=0}^{\ell-1} \operatorname{Ad} z^{(m)}$, $z \in \mathbb{T}^{n-1}$.

COROLLARY 1.4. *For the shift σ on B_n and $\epsilon > 0$ there exists an increasing sequence (ℓ_k) in $\mathbb{N}$, an automorphism ϕ of B_n, and a unitary $w \in A_n$ such that $\phi\gamma_z = \gamma_z\phi$ for $z \in \mathbb{T}^{n-1}$, $\|w - 1\| < \epsilon$, and the pair of $\operatorname{Ad} w\phi\sigma\phi^{-1}$ and γ is conjugate to the pair of $\bigotimes_{k=1}^{\infty} \sigma^{\ell_k}$ and $\bigotimes_{k=0}^{\infty} \gamma^{(\ell_k)}$ on $\bigotimes_{k=1}^{\infty} \big(\bigotimes_{m=0}^{\ell_k-1} M_n^{(m)}\big) \cong B_n$.*

Proof. For $a, b \in \mathbb{N}$ with $a < b$ we define

$$u^{(a,b)} = \sum_{i,j} e_{ij}^{(a)} e_{ji}^{(b)}.$$

Then $\mathrm{Ad}\, u^{(a,b)}\sigma$ acts on $\bigotimes_{m=a}^{b-1} M_n^{(m)}$ as the cyclic shift.

By using Theorem 1.1 we can derive the following: for any finite subset $\mathcal{F}$ of $\mathbb{Z}$ and $\epsilon > 0$ there is a finite subset $\mathcal{G}$ of $\mathbb{Z}$ such that for any unitary $u \in A_n \cap \bigotimes_{m\notin\mathcal{G}} M_n^{(m)}$ there is a unitary $v \in A_n \cap \bigotimes_{m\notin\mathcal{F}} M_n^{(m)}$ such that $\|u - v\sigma(v^*)\| < \epsilon$.

By using this fact we define inductively a decreasing sequence (a_k), an increasing sequence (b_k), and sequences (v_k) and (w_k) of unitaries in A_n such that $a_1 = 0$, $b_1 = 1$, $w_k v_k \sigma(v_k)^* = u^{(a_k,b_k)}$, $\|w_k - 1\| < 2^{-k}\epsilon$, and $v_k, w_k \in \bigotimes_{Z_k} M_n$ for mutually disjoint family (Z_k) of finite subsets of $\mathbb{Z}$. Then $\phi = \lim_k \mathrm{Ad}(v_1 v_2 \cdots v_k)$ is well defined as an automorphism of B_n, as well as $w = \lim_k w_1 w_2 \cdots w_k$ as a unitary in A_n. Then we check that $\mathrm{Ad}\, w\phi\sigma\phi^{-1}$ leaves $\bigotimes_{m=a_k}^{b_k-1} M_n$ invariant and acts as a cyclic shift on $\bigotimes_{m=a_k}^{a_{k-1}-1} M_n \otimes \bigotimes_{m=b_{k-1}}^{b_k-1} M_n$ for each $k \geq 1$ with $a_0 = b_0 = 0$. Then with $\ell_k = b_k - b_{k-1} - a_k + a_{k-1}$ we can conclude the proof. □

The above corollary shows that the action of $\mathbb{T}^{n-1} \times \mathbb{Z}$ on B_n defined by $(z, k) \mapsto \gamma_z \sigma^k$, which does not leave any finite-dimensional C^*-subalgebra of B_n invariant, is cocycle conjugate to an action which is of product type.

Remark 1.5. Let $z \in \mathbb{T}^{n-1}$ be such that $z^m \neq 1$ for any non-zero $m \in \mathbb{Z}$ and define an action $\mathbb{Z}^2$ on B_n by $\alpha_{(a,b)} = \gamma_z^a \sigma^b$. Then α is cocycle conjugate to the action α' defined by

$$\alpha'_{(a,b)} = \phi \circ (\gamma_z)^a (\mathrm{Ad}\,\phi^{-1}(w)\sigma)^b \circ \phi^{-1} = \gamma_z^a (\mathrm{Ad}\, w\phi\sigma\phi^{-1})^b$$

in the notation of the above corollary, which is of product type. We should note that such an action with the Rohlin property (i.e. each $\alpha_{(a,b)}$ has the Rohlin property for $(a, b) \neq 0$) is unique up to cocycle-conjugacy [**23**]. Hence, since α has the Rohlin property, α is cocycle conjugate to, e.g., the action α'' defined by

$$\alpha''_{(a,b)} = \gamma_{z_1}^a \gamma_{z_2}^b,$$

where $z_1, z_2 \in \mathbb{T}^{n-1}$ is any pair which generates a copy of $\mathbb{Z}^2$ in $\mathbb{T}^{n-1}$. It seems that we still do not know if there is an action of $\mathbb{Z}^2$ on B_n with the Rohlin property which is not cocycle conjugate to an action of product type.

Another corollary extends what is stated in [**19**] for $n = 2$.

COROLLARY 1.6. *Let n be an integer greater than 1 and $\mathcal{O}_n$ be the Cuntz algebra generated by n isometries $s_0, s_1, \ldots, s_{n-1}$ satisfying the relation $\sum_{j=0}^{n-1} s_j s_j^* = 1$ (see [**5**]). For $p = (p_0, p_1, \ldots, p_{n-1}) \in \mathbb{R}^n$ define a flow α on $\mathcal{O}_n$ by $\alpha_t(s_j) = e^{ip_j t} s_j$ for $j = 0, \ldots, n-1$. Then the following conditions are equivalent:*

(1) *$\{p_0, \ldots, p_{n-1}\}$ generates $\mathbb{R}$ as a closed sub-semigroup;*
(2) *the crossed product $\mathcal{O}_n \times_\alpha \mathbb{R}$ is purely infinite and simple;*
(3) *α has the Rohlin property, i.e. for any $\lambda \in \mathbb{R}$ there is a central sequence (u_m) in $\mathcal{U}(\mathcal{O}_n)$ such that $\|\alpha_t(u_m) - e^{i\lambda t} u_m\| \to 0$ uniformly in t on every compact subset of $\mathbb{R}$.*

Proof. First we note that $\mathcal{O}_n$ is a purely infinite, simple, nuclear C^*-algebra [**5**] and such a class is now well studied [**12, 13**].

The equivalence of (1) and (2) follows from [**14, 22**]. That (3) implies (2) follows from [**17**]. What is left to prove is that (1) implies (3). If (1) holds, it is shown by combinatorial arguments in [**18**] that for any $\lambda \in \mathbb{R}$ there is a sequence (u_m) of unitaries in the *-subalgebra generated by $s_0, \ldots, s_{n-1}$ such that $\|\alpha_t(u_m) - e^{i\lambda t}u_m\| \to 0$ uniformly in t on every compact subset of $\mathbb{R}$. As in the proof of Proposition 3.2 of [**19**], with the property of $\sigma \mid A_n$ as given in Theorem 1.1(1), there is a sequence (ϕ_k) of unital endomorphisms of $\mathcal{O}_n$ such that $[\phi_k(x), y] \to 0$ for any $x, y \in \mathcal{O}_n$ and $\alpha_t\phi_k = \phi_k\alpha_t$. Then we can choose a central sequence from $\{\phi_k(u_m) \mid k, m \in \mathbb{N}\}$ which satisfies the required condition for $\lambda \in \mathbb{R}$. $\square$

We would like to add that the flows α on $\mathcal{O}_n$ satisfying the conditions in the above corollary are cocycle conjugate to each other [**19**], i.e. for any flows α and α' on $\mathcal{O}_n$ of the above form there is an automorphism ϕ of $\mathcal{O}_n$ and an α-cocycle u such that $\operatorname{Ad} u_t\alpha_t = \phi\alpha'\phi^{-1}$. See [**20, 21**] for more on Rohlin flows.

2. *The dimension group of A_n*

For $n = 2, 3, \ldots$ we denote by $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ the abelian group of integer coefficient polynomials in $x_1, \ldots, x_{n-1}$. For $v = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}_+^n$ we let

$$x^v = x_0^{v_0}x_1^{v_1}\cdots x_{n-1}^{v_{n-1}},$$

where $x_0 = 1 - x_1 - \cdots - x_{n-1}$ and $\mathbb{Z}_+$ is the set of non-negative integers. The polynomials of degree less than or equal to N are linearly spanned by x^v, $|v| = \sum_{i=0}^{n-1} v_i = N$, which are linearly independent.

We denote by $\mathbb{Z}^+[x_1, \ldots, x_{n-1}]$ the cone generated by x^v, $v \in \mathbb{Z}_+^n$. By using this as a positive cone we define an order on $\mathbb{Z}[x_1, \ldots, x_{n-1}]$, i.e. $p_1 \geq p_2$ if $p_1 - p_2 \in \mathbb{Z}^+[x_1, \ldots, x_{n-1}]$. We call an element of $\mathbb{Z}^+[x_1, \ldots, x_{n-1}]$ *positive*.

We define

$$\Lambda_n = \left\{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in [0, 1]^{n-1} \,\middle|\, \sum_{j=1}^{n-1}\lambda_j \leq 1\right\}.$$

If $\lambda \in \Lambda_n$, then λ_0 denotes $1 - \lambda_1 - \cdots - \lambda_{n-1}$.

If $p \geq 0$, then p is non-negative on Λ_n and moreover satisfies that if $p(\lambda) = 0$ for some $\lambda \in \Lambda_n$ then $p = 0$ on the face of Λ_n generated by λ. (This is because each x^v satisfies this condition.) However, the converse does not follow if $n > 2$ (compare this with the case $n = 2$ in [**24**]).

For example, if $p \in \mathbb{Z}[x_1, x_2]$ is given by

$$p(x_1, x_2) = x_1(x_1^2 + (1 - x_1 - 2x_2)^2),$$

then $p = 0$ on the face $\Lambda' = \{x \in \Lambda_3 \mid x_1 = 0\}$, but $p > 0$ on $\Lambda_3 \setminus \Lambda'$. If p is expressed as

$$p = \sum_{k+\ell+m=N} a_{k,\ell,m}x_0^k x_1^\ell x_2^m,$$

for some $N \in \mathbb{N}$ with $a_{k,\ell,m} \in \mathbb{Z}_+$, then that $\ell = 0$ implies that $a_{k,\ell,m} = 0$. Hence, we should have that

$$x_1^2 + (1 - x_1 - 2x_2)^2 = \sum_{k+\ell+m=N-1} a_{k,\ell+1,m} x_0^k x_1^\ell x_2^m.$$

If $x_1 = 0$, the left-hand side vanishes for $x_2 = \frac{1}{2}$ but not for $x_2 \neq \frac{1}{2}$. This is not possible for the right-hand side.

Another example can be given by $p(x_1, x_2) = x_1^2 + x_2^2 - x_1 x_2$. This is strictly positive on Λ_3 except for $x_0 = 1$ or $x_1 = 0 = x_2$. If we express p as $p = \sum_{k+\ell+m=N} a_{k,\ell,m} x_0^k x_1^\ell x_2^m$, then the highest order in x_0 must be $N-2$ and the sum of terms which include x_0^{N-2} is $x_0^{N-2}(x_1^2 + x_2^2 - x_1 x_2) = x_0^{N-2} p(x)$; thus p is not positive.

Note that if $p \in \mathbb{Z}[x_1, \dots, x_{n-1}]$ is positive and Λ' is a face of Λ_n with $\Lambda' \cong \Lambda_m$, $1 < m < n$, then $p|\Lambda'$ can be understood as a positive element (or zero) of $\mathbb{Z}[x_1, \dots, x_{m-1}]$ by taking $m-1$ *free variables* in a certain order.

The following is due to Handelman [**9**].

PROPOSITION 2.1. *The dimension group $K_0(A_n)$ is isomorphic to $\mathbb{Z}[x_1, \dots, x_{n-1}]$, where the isomorphism is defined by*

$$\left[\prod_{m=-M}^{M} e_{i_m i_m}^{(m)}\right] \mapsto \prod_{m=-M}^{M} x_{i_m}$$

for $i_m = 0, \dots, n-1$ and $M \in \mathbb{N}$, with $x_0 = 1 - x_1 - \cdots - x_{n-1}$.

The positive cone $C = \mathbb{Z}^+[x_1, \dots, x_{n-1}]$ is the cone with the following properties:

(1) $x_i \in C$ *for $i = 0, \dots, n-1$, where $x_0 = 1 - x_1 - \cdots - x_{n-1}$;*

(2) $pq \in C$ *if $p, q \in C$;*

(3) $p \in C$ *if $p \in \mathbb{Z}[x_1, \dots, x_{n-1}]$ is strictly positive as a function on Λ_n.*

Proof. In this proof we write A_n as A omitting the subscript n.

Let (Z_k) be an increasing sequence of finite subsets of $\mathbb{Z}$ such that the number $|Z_k|$ of elements in Z_k is k and $\bigcup_k Z_k = \mathbb{Z}$, and let $A_0 = \mathbb{C}1$ and

$$A_k = \left(\bigotimes_{m \in Z_k} M_n^{(m)}\right)^\gamma.$$

Then (A_k) is an increasing sequence of finite-dimensional C^*-subalgebras of A such that $A = \overline{\bigcup_k A_k}$.

Let $e_0, \dots, e_{n-1}$ be the canonical basis for $\mathbb{Z}^n$ and let, for each $k \in \mathbb{N}$,

$$V_k = \left\{ v \in \mathbb{Z}_+^n \,\middle|\, |v| = \sum_{i=0}^{n-1} v_i = k \right\},$$

where $\mathbb{Z}_+ = \{0, 1, 2, \dots, \}$. For $v \in V_k$ we denote by $N(v)$ the number of sequences $(e_{i_1}, \dots, e_{i_k})$ in $\{e_0, \dots, e_{n-1}\}$ such that $v = \sum_{j=1}^k e_{i_j}$, i.e.

$$N(v) = |v|!/v! = |v|!(v_0! v_1! \cdots v_{n-1}!)^{-1}.$$

Define

$$E_v = \sum \Pi_{m \in Z_k} e_{i_m, i_m}^{(m)} \in A_k,$$

where the sum is taken over all the map $i : Z_k \to \{0, \ldots, n-1\}$ such that $\sum_{m \in Z_k} e_{i_m} = v$. It follows that E_v is a minimal central projection of A_k and $A_k E_v \cong M_{N(v)}$. The embedding of A_k into A_{k+1} is given as follows: for $v \in V_k$ and $w \in V_{k+1}$, $A_k E_v$ is mapped into $A_{k+1} E_w$ with multiplicity one if and only if $w = v + e_i$ for some i or $w \geq v$ in the sense that $w_i \geq v_i$ for all i.

We define a map ψ_k of $K_0(A_k)$ into $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ by the following. If e is a minimal projection in $A_k E_v$, then

$$\psi_k([e]) = x^v = x_0^{v_0} x_1^{v_1} \cdots x_{n-1}^{v_{n-1}},$$

where $v = (v_0, \ldots, v_{n-1})$ and $x_0 = 1 - x_1 - \cdots - x_{n-1}$. Since $\psi_{k+1} \circ \iota_*$ coincides with ψ_k on $K_0(A_k)$, where ι is the embedding of A_k into A_{k+1}, we can define a homomorphism ψ of $K_0(A)$ into $\mathbb{Z}[x_1, \ldots, x_{n-1}]$. Since the range of ψ_k is the polynomials of order less than or equal to k, it follows that ψ is surjective. Since ψ_k is injective, it also follows that ψ is injective. It is easy to check that ψ is defined as indicated in the statement and that the range of ψ on the projections in A generate the positive cone $\mathbb{Z}^+[x_1, \ldots, x_{n-1}]$. Therefore, ψ is indeed the required isomorphism. Note that $\psi([1]) = 1$.

We have to show the statement on the positive cone. It is immediate that conditions (1) and (2) are valid and are enough to generate $\mathbb{Z}^+[x_1, \ldots, x_{n-1}]$. We now show the validity of condition (3), which has been known for some time at least for $n = 2$ (see, e.g., [**11**, p. 126]).

First of all note that for each $\lambda \in \Lambda_n$ there is a unique tracial state τ_λ on A such that for a minimal projection $e \in A_k E_v$ with $v \in V_k$,

$$\tau_\lambda(e) = \lambda^v = \lambda_0^{v_0} \lambda_1^{v_1} \cdots \lambda_{n-1}^{v_{n-1}}.$$

Moreover, τ_λ is factorial and all the factorial tracial states of A are of this form, as shown by the following lemma.

Let $p \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ be such that p is strictly positive on Λ_n. Since ψ is a group isomorphism, there are projections e, f in $M_m \otimes A_k$ for some m, k such that $p = \psi([e] - [f])$. What we have to show is that $[e] \geq [f]$, i.e. f is equivalent to a subprojection of e.

Suppose the contrary, i.e. there is a sequence $(v_\ell)_{\ell > k}$ with $v_\ell \in V_\ell$ such that

$$\operatorname{rank}(e(1_m \otimes E_{v_\ell})) < \operatorname{rank}(f(1 \otimes E_{v_\ell}))$$

in $M_m \otimes A_\ell E_{v_\ell}$. We extend the state on A_ℓ defined by

$$x \mapsto N(v_\ell)^{-1} \operatorname{Tr}(x E_{v_\ell})$$

to a state φ_ℓ on A, where Tr is the trace on $A_\ell E_{v_\ell} \cong M_{N(v_\ell)}$. We take a weak* limit point τ of $(\varphi_\ell)_{\ell > k}$. Then it follows that τ is a tracial state on A for which

$$\operatorname{Tr}_m \otimes \tau(e) \leq \operatorname{Tr}_m \otimes \tau(f),$$

where Tr_m is the trace on M_m. Since τ belongs to the closed convex hull of τ_λ, $\lambda \in \Lambda_n$, this contradicts the assumption that $\operatorname{Tr}_m \otimes \tau_\lambda(e) > \operatorname{Tr}_m \otimes \tau_\lambda(f)$ for all $\lambda \in \Lambda_n$. This concludes the proof of condition (3). (We see in Lemma 2.3 that the above τ is actually τ_λ for some $\lambda \in \Lambda_n$.) $\square$

LEMMA 2.2. *Let τ be a factorial tracial state on A_n. Then there is a $\lambda \in \Lambda_n$ such that $\tau = \tau_\lambda$.*

Proof. Since τ is a tracial state on $A = A_n$, there is map $\varphi_k : V_k \to [0,1]$ such that $\tau(e) = \varphi_k(v)$ for a minimal projection e in $A_k E_v$. We should note that (φ_k) is independent of the choice of (Z_k) by which A_k is defined as $A_k = \big(\bigotimes_{m\in Z_k} M_n^{(m)}\big)^\gamma$. By choosing $Z_{k+1} = Z_k \cup \{N\}$ for a large N and by using

$$\varphi_{k+1}(v + e_i) = \tau(ee_{ii}^{(N)}),$$

with e a minimal projection in $A_k E_v$, we get that

$$\varphi_{k+1}(v + e_i) = \varphi_k(v)\varphi_1(e_i)$$

for $v \in V_k$. Here we have used the assumption that τ is factorial and that $(e_{ii}^{(N)})_N$ is a central sequence with $\tau(e_{ii}^{(N)}) = \varphi_1(e_i)$. Setting $\lambda_i = \varphi_1(e_i)$ for $i = 0, \dots, n-1$, we get that $\sum_{i=0}^{n-1} \lambda_i = 1$ and $\varphi_k(v) = \lambda^v = \lambda_0^{v_0} \cdots \lambda_{n-1}^{v_{n-1}}$. Hence, $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \Lambda_n$ and $\tau = \tau_\lambda$. □

In the following lemma we adopt the notation in the proof of the above proposition. In particular, we simply denote by A the C^*-algebra A_n and we express A as the closure of $\bigcup_k A_k$, where $A_k = \big(\bigotimes_{Z_k} M_n\big)^\gamma$ and $|Z_k| = k$.

LEMMA 2.3. *Let (m_ℓ) be an increasing sequence of integers, (v_ℓ) a sequence in $\bigcup_m V_m$, and $\lambda \in \Lambda_n$ such that $v_\ell \in V_{m_\ell}$, $m_\ell \to \infty$, and*

$$\frac{v_{\ell,i}}{m_\ell} \to \lambda_i$$

as $\ell \to \infty$ for $i = 0, 1, \dots, n-1$, where $v_\ell = \sum_{i=0}^{n-1} v_{\ell,i} e_i$. Let φ_ℓ be a state of A such that $\varphi_\ell | E_{v_\ell} A_{m_\ell} = N(v_\ell)^{-1}\,\mathrm{Tr}$, where Tr is the trace on $M_{N(v_\ell)} \cong E_{v_\ell} A_{m_\ell}$. Then φ_ℓ converges to τ_λ as $\ell \to \infty$.

Proof. Let $k \in \mathbb{N}$. We evaluate $\varphi_\ell | A_k$ for $m_\ell \gg k$. Note that $\varphi_\ell | A_k$ is a tracial state; hence, it is expressed as

$$\varphi_\ell | A_k = \sum_{w \in V_k} d_w N(w)^{-1}\,\mathrm{Tr} | E_w A_k,$$

for non-negative constants d_w with $\sum_w d_w = 1$.

The multiplicity with which $E_w A_k$ is embedded into $E_{v_\ell} A_{m_\ell}$ is

$$N(w, v_\ell) = \frac{(m_\ell - k)!}{(v_{\ell,0} - w_0)!(v_{\ell,1} - w_1)! \cdots (v_{\ell,n-1} - w_{n-1})!}$$

if $w \le v_\ell$; otherwise it is zero. (For $w \le v_\ell$, $N(w, v_\ell)$ is the coefficient of $x^{v_\ell - w}$ in the expansion of $(x_0 + x_1 + \cdots + x_{n-1})^{m_\ell - k}$.) Hence, we have that if d_w is non-zero, then

$$d_w = \varphi_\ell(E_w) = N(v_\ell)^{-1} N(w) N(w, v_\ell).$$

Suppose that all λ_i are positive. If m_ℓ is so large that $v_{\ell,i} \gg w_i$, then it follows that

$$d_w = N(w) \frac{(m_\ell - k)! v_{\ell,0}! \cdots v_{\ell,n-1}!}{m_\ell!(v_{\ell,0} - w_0)! \cdots (v_{\ell,n-1} - w_{n-1})!}$$

is approximately equal to

$$N(w)\lambda_0^{w_0}\lambda_1^{w_1}\cdots\lambda_{n-1}^{w_{n-1}}.$$

This shows that $\varphi_\ell|A_k$ converges to $\tau_\lambda|A_k$.

Next we consider the case where some of λ_i are zero. For simplicity suppose that $\lambda_i = 0$ for $i = j, \ldots, n-1$ with some $j > 0$. Then we have that for $w \in V_k$ with $w_i = 0$, $i = j, \ldots, n-1$,

$$d_w \approx N(w)\lambda_0^{w_0}\lambda_1^{w_1}\cdots\lambda_{j-1}^{w_{j-1}},$$

and for $w \in V_k$ with $\sum_{i=j}^{n-1} w_i = s > 0$, d_w is of the order of $m_\ell^{-s} v_j^{w_j}\cdots v_{n-1}^{w_{n-1}}$, which converges to zero. Thus, in this case too one can conclude that $\varphi_\ell|A_k$ converges to $\tau_\lambda|A_k$. Since k is arbitrary, it follows that (φ_ℓ) converges to τ_λ in the weak* topology. □

Suppose that $p \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ is strictly positive as a function on Λ_n except for some extreme points. If the degree of p is less than or equal to k, then p can be expressed as

$$p(x_1, \ldots, x_{n-1}) = \sum_{w \in V_k} c_w x^w$$

in a unique way, where the coefficients c_w are integers and $x^w = x_0^{w_0} x_1^{w_1} \cdots x_{n-1}^{w_{n-1}}$ with $x_0 = 1 - x_1 - \cdots - x_{n-1}$. To obtain a similar expression in terms of x^v, $v \in V_{k+1}$, we just have to multiply the right-hand side with $x_0 + x_1 + \cdots + x_{n-1}$ and expand it.

Let $a_0 = \max\{w_0 \mid c_w \neq 0\}$ and define

$$q_0(x_1, \ldots, x_{n-1}) = \sum_{w \in V_k,\, w_0 = a_0} c_w x_1^{w_1} \cdots x_{n-1}^{w_{n-1}}.$$

We note that q_0 does not depend on k (if k increases by one, then so does a_0); unique to the vertex $x_0 = 1$. If $p(0, 0, \ldots, 0) = c_{(k,0,\ldots,0)} = c \neq 0$, then $q_0 = c$. Similarly we can define q_i as a polynomial in x_j, $j \neq i$. If p is positive in $\mathbb{Z}[x_1, \ldots, x_{n-1}]$, it follows that q_i, $i = 0, \ldots, n-1$ are positive (or zero) in $\mathbb{Z}[x_1, \ldots, x_{n-1}]$.

More generally we define the 0-*vanguard* V_k^0 of p on the level k as the set of $w \in V_k$ which satisfies that $c_w \neq 0$ and that if v is *in front of* w in the direction to the zeroth vertex, i.e. $v = w + se_0 - e_{i_1} - e_{i_2} - \cdots - e_{i_s} \in V_k$ with $i_j \neq 0$ for some $s > 0$, then $c_v = 0$.

Then the map $w \mapsto w + e_0$ from V_k into V_{k+1} restricts to a bijection from V_k^0 onto V_{k+1}^0 preserving the coefficients c_w.

To prove this, let $w \in V_k^0$. If $v' \in V_{k+1}$ is in front of $w + e_0$, then $v' - e_0 \in V_k$ is in front of w and hence $c_{v'-e_0} = 0$. If $v' = w' + e_i$ for some $i \neq 0$, then $w' = v' - e_i$ is in front of $w + e_0 - e_i$, which is in front of w, and hence $c_{w'} = 0$. This way we can conclude that $c_{v'} = 0$ as $c_{v'}$ is the sum of $c_{v'-e_i}$ with $v' - e_i \in V_k$. If $w + e_0 = w' + e_i$ for some $w' \neq w$, then $w' = w + e_0 - e_i$ is in front of w and hence $c_{w'} = 0$. This implies that $c_{w+e_0} = c_w \neq 0$. Hence, it follows that $w + e_0 \in V_{k+1}^0$ and $c_{w+e_0} = c_w$.

On the other hand let $v \in V_{k+1}^0$; if $w = v - e_0 \notin V_k^0$, then it follows that $c_w = 0$ or there is a $w' (\neq w)$ in front of w such that $c_{w'} \neq 0$. Since $c_v \neq 0$, there is i such that $c_{v-e_i} \neq 0$. If $i = 0$, then it means that $c_w \neq 0$. If $i \neq 0$, then $v - e_i = w + e_0 - e_i$ is in front of w and $c_{w+e_0-e_i} \neq 0$. Thus, in either case there is a w' in front of w such that $c_{w'} \neq 0$. We choose such a w' of maximal w'_0, i.e. $w' \in V_k^0$. Then $w' + e_0$ is in front of $w + e_0 = v$, which implies that $c_{w'+e_0} = 0$. If $w' + e_0 = u + e_i$ for some $i \neq 0$, $u = w' + e_0 - e_i$ is in front of w' and hence $c_u = 0$. Since $c_{w'+e_0} = c_{w'} + \sum_{i=1}^{n-1} c_{w'+e_0-e_i}$, this would imply that $c_{w'+e_0} = c_{w'}$, which is a contradiction. This proves that $v - e_0 \in V_k^0$ for $v \in V_{k+1}^0$. From the first part it also follows that $c_{v-e_0} = c_v$. Hence, the above assertion is now shown.

One can similarly define the i-vanguard V_k^i of p on the level k for $i = 1, \ldots, n-1$ and we have a similar bijection from V_k^i onto V_{k+1}^i with the property $c_{w+e_i} = c_w$, $w \in V_k^i$.

Definition 2.4. Let $p \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ and $i = 0, 1, \ldots, n-1$. The i-vanguard $V^i(p)$ of p is the set of $v : \{0, 1, \ldots, n-1\} \setminus \{i\} \to \mathbb{Z}$ such that $v^{(k)} \in V_k^i(p)$, where $V_k^i(p)$ is the ith vanguard of p on the level k and $v^{(k)}$ is defined by $v_j^{(k)} = v_j$ for $j \neq i$ and $v_i^{(k)} = k - \sum_j v_j$. When $p = \sum_{v \in V_k} c_v x^v$ and $v \in V^i(p)$ for some i, we set $p(v) = c_v$.

The following gives a sufficient condition for positivity.

PROPOSITION 2.5. *Let $p \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ be such that p is strictly positive on Λ_n except for some extreme points. Then p is positive if and only if $p(v) > 0$ for all $v \in V^i(p)$, $i = 0, 1, \ldots, n-1$.*

Proof. We write p as

$$p(x_1, \ldots, x_{n-1}) = \sum_{w \in V_k} c_w x^w,$$

where k is greater than or equal to the degree of p. We have to show that for a sufficiently large k, all c_w are non-negative by using the assumption that p is strictly positive on Λ_n except for the vertices and that $c_w > 0$ for $w \in \bigcup_{i=0}^{n-1} V_k^i(p)$; the other implication is obvious.

If the assertion is false, there is a sequence (v_ℓ) such that $v_\ell \in V_\ell$ and $c_{v_\ell} < 0$. We may suppose that $v_{\ell,i}/\ell \to \lambda_i$ for each $i = 0, 1, \ldots, n-1$. Then the above lemma shows that

$$c_{v_\ell} N(v_\ell)^{-1} \to p(\lambda).$$

where $N(v_\ell) = \ell!(v_\ell!)^{-1}$ and $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \Lambda_n$. (To make the correspondence with the lemma clearer, let us find two projections e, f in (a matrix-tensored) A such that $p = [e] - [f]$ and let φ_ℓ be a state of A such that $\varphi_\ell | E_{v_\ell} A_\ell$ is a tracial state. Then it is shown that $\varphi_\ell(e - f) = c_{v_\ell} N(v_\ell)^{-1}$, which is negative, converges to $\tau_\lambda(e - f) = p(\lambda)$.) If λ is not a vertex of Λ_n, then $p(\lambda) > 0$, which implies that λ must be a vertex; we may thus assume that $v_{\ell,0}/\ell \to 1$ and $v_{\ell,i}/\ell \to 0$ for $i = 1, \ldots, n-1$.

Let $w \in V_k$. If $w \leq v_\ell$, $E_v A_k$ is embedded into $E_{v_\ell} A_{v_\ell}$ with the multiplicity $N(w, v_\ell)$ defined before. Hence, we have that

$$c_{v_\ell} = \sum_{w \in V_k, w \leq v_\ell} c_w N(w, v_\ell).$$

If $c_w < 0$ for some $w \in V_k$ with $w \leq v_\ell$, then there is a $w' \in V_k^0(p)$ such that w' is in front of w (and $c_w > 0$). Then we can argue that c_{v_ℓ} must be positive for large ℓ by showing that the contribution of $c_w N(w, v_\ell)$ to c_{v_ℓ}, which is negative, is overshadowed by a tiny portion of $c_{w'}N(w', v_\ell)$ which is positive, as follows. Since

$$N(w, v_\ell) = \frac{(\ell - k)!}{(v_{\ell,0} - w_0)! \cdots (v_{\ell,n-1} - w_{n-1})!},$$

and $w'_0 > w_0$ and $w'_i \le w_i$ for $i > 0$, we have that $w' \le v_\ell$ for large ℓ and

$$\frac{N(w', v_\ell)}{N(w, v_\ell)} = \frac{(v_{\ell,0} - w_0)!}{(v_{\ell,0} - w'_0)!} \frac{(v_{\ell,1} - w_1)! \cdots (v_{\ell,n-1} - w_{n-1})!}{(v_{\ell,1} - w'_1)! \cdots (v_{\ell,n-1} - w'_{n-1})!}$$

is approximately equal to or more than

$$\frac{\ell^{w'_0 - w_0}}{v_{\ell,1}^{w_1 - w'_1} \cdots v_{\ell,n-1}^{w_{n-1} - w'_{n-1}}}$$

which tends to infinity as $\ell \to \infty$. Hence, we can conclude that $c_{v_\ell} > 0$ for a sufficiently large ℓ, which is a contradiction. □

LEMMA 2.6. *Let p and q be positive elements of $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ and $i = 0, 1, \ldots, n-1$. Then the i-vanguard $V^i(pq)$ of pq is contained in $\{v + w \mid v \in V^i(p),\ w \in V^i(q)\}$.*

Proof. Let $p = \sum_{v \in V_K} c_v x^v$ and $q = \sum_{v \in V_K} d_v x^v$ for some $K \in \mathbb{N}$. Suppose that there are $v, w \in V_K$ such that $c_v > 0$, $d_w > 0$, and $v + w$ is in the i-vanguard of pq on the level $2K$. If $v \notin V^i_K(p)$, then there is a $v' \in V_K$ such that v' is in front of v toward the ith vertex and $c_{v'} \neq 0$. Then $v' + w$ is in front of $v + w$ toward the ith vertex and the coefficient of $x^{v'+w}$ in the expansion of pq in x^μ, $\mu \in V_{2K}$ is greater than or equal to $c_{v'} d_w$, i.e. does not vanish, which is a contradiction. Similarly we can conclude that $w \in V^i_K(q)$. If $u \in V_{2K}$ satisfies that if $u = v + w$ with $v, w \in V_K$ then $c_v = 0$ or $d_w = 0$, then u cannot belong to the vanguards of pq. This concludes the proof. □

Later we use a more elaborate form of the following.

COROLLARY 2.7. *Let $p \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ be a positive element such that p is strictly positive on Λ_n except for the vertices. Then there exists a $K \in \mathbb{N}$ such that $p - q$ is positive for any $q \in (x_0 x_1 \cdots x_{n-1})^K \mathbb{Z}[x_1, \ldots, x_{n-1}]$ if $p - q$ is strictly positive on Λ_n except for the vertices.*

Proof. Suppose that the degree of p is less than K. Let $q \in (x_0 \cdots x_{n-1})^K \mathbb{Z}[x_1, \ldots, x_{n-1}]$. We express p and q as

$$p = \sum_{v \in V_L} c_v x^v, \quad q = \sum_{v \in V_L} d_v x^v$$

for some $L \ge K$. We consider the i-vanguard $V^i_L(p - q)$ of $p - q$ on the level L. Note that the i-vanguard $V^i_L(q)$ is confined to $\{v \mid v_j \ge K \text{ for } j \neq i\}$.

Let $v \in V^i_L(p - q)$ and suppose that $v_i \le L - K$. Since the i-vanguard of p on the level L is $V^i_{K-1}(p) + (L - K + 1)e_i$, we have that $d_v \neq 0$. (If $d_v = 0$, then we must have that $c_v \neq 0$; since v does not belong to $V^i_L(p)$, there is $w \in V^i_L(p)$ in front of v toward the ith vertex such that $c_w \neq 0$ and $w_i > L - K$. Since $d_w = 0$ for such a w, we have that $c_w - d_w = c_w \neq 0$, which is a contradiction.) Since $d_v \neq 0$, we have that all $v_j \ge K$. If $w \in V_L$ satisfies that $w_i = L - K + 1$, then $w_i - v_i > 0$ and $w_j - v_j < 0$

for $j \neq i$, i.e. we have that all $w \in V_L$ with $w_i = L - K + 1$ are in front of v towards the ith vertex. Thus, all of the points $w \in V_L$ with $w_i > L - K$ are in front of v; hence, v cannot belong to $V_L^i(p-q)$. This implies that if $v \in V_L^i(p-q)$, then $v_i > L - K$; hence, $V_L^i(p-q) = V_{K-1}^i(p) + L - K + 1$. Note that $c_w - d_w = c_w > 0$ for $w \in V_L^i(p-q)$. Since $p - q$ is strictly positive on $\Lambda_n \setminus \mathrm{Ex}(\Lambda_n)$, the conclusion follows from the previous proposition. □

To estimate the vanguards we use the following lemma.

LEMMA 2.8. *Let p be a non-negative element of $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ such that*

$$p(x_1, \ldots, x_{n-1}) \geq c \sum_{i \neq j} (x_i x_j)^S$$

on Λ_n for some $c > 0$ and $S \in \mathbb{N}$. Then the vanguards $V^i(p)$ of p are confined in the S-neighborhood of vertices; more precisely, if $\omega \in V^i(p)$ for some $i = 0, 1, \ldots, n-1$, then there is a $j \neq i$ such that $\omega_k = 0$ for $k \neq i, j$ and $\omega_j \leq S$.

Proof. Let $p = \sum_{v \in V_K} c_v x^v$ and let $i = 0, 1, \ldots, n-1$ and $j \neq i$. If $x \in \Lambda_n$ satisfies that $x_k = 0$ for $k \neq i, j$, i.e. if x is on the edge $[i, j]$ between the vertices $x_i = 1$ and $x_j = 1$, then $p(x) \geq c(x_i x_j)^S$. Since for $x_i \approx 1$ (and $x_j = 1 - x_i \approx 0$),

$$p(x) = \sum_{w \in V_k^i(p)} c_w x^w + \cdots,$$

there must be an $\omega \in V_k^i(p)$ such that $\omega_k = 0$ for $k \neq i, j$ and $\omega_j \leq S$. Hence, $V_K^i(p)$ contains $\omega^{(j)}$ for each $j \neq i$ such that $\omega_j^{(j)} \leq S$ and $\omega_k^{(j)} = 0$ for $k \neq i, j$ (and $\omega_i^{(j)} = K - \omega_j^{(j)}$). This implies that if $\omega \in V_K$ has $j \neq i$ such that $\omega_j \geq S$, then $\omega^{(j)}$ is in front of ω (towards the ith vertex), i.e. $\omega \notin V_K^i(p)$ unless $\omega = \omega^{(j)}$. This proves the assertion. □

LEMMA 2.9. *Let $k = (k_0, k_1, \ldots, k_{n-1}) \in \mathbb{Z}_+^n$. Then the order ideal of $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ generated by $x^k = x_0^{k_0} \cdots x_{n-1}^{k_{n-1}}$ is equal to $x^k \mathbb{Z}[x_1, \ldots, x_{n-1}]$, where $x_0 = 1 - x_1 - \cdots - x_{n-1}$.*

Proof. Let p be a non-negative element of $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ such that $Nx^k - p \geq 0$ for some $N \in \mathbb{N}$. Then for a sufficiently large $L \in \mathbb{N}$, we have $c, d : V_L \to \mathbb{Z}_+$ such that

$$Nx^k = \sum_{w \in V_L} c_w x^w, \quad p = \sum_{w \in V_L} d_w x^w$$

and $c_w \geq d_w \geq 0$ for all $w \in V_L$. Since $c_w > 0$ if and only if $w \geq k$, this implies that if $d_w > 0$ then $w \geq k$ and that

$$p = x^k \bigg(\sum_{w \in V_L, w \geq k} d_w x^{w-k} \bigg).$$

Since the other inclusion is obvious, this concludes the proof. □

For $k \in \mathbb{Z}_+^n$ we denote by $I(k)$ the ideal of A_n corresponding to the order ideal of $K_0(A_n) \cong \mathbb{Z}[x_1, \dots, x_{n-1}]$ generated by $x^k = x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$. The ideal I_i which is used to define the character φ_i in the main Theorem 1.1 is given as

$$I_i = \sum_{j \neq i} I(e_j).$$

We note that $I(k) \cap I(\ell) = I(k \vee \ell)$ for $k, \ell \in \mathbb{Z}_+^n$, where $(k \vee \ell)_i = \max(k_i, \ell_i)$.

3. *Covariant irreducible representations*

In this section we define some covariant irreducible representations of (A_n, σ), which are an extension of what is well known for the case $n = 2$ and will be used in the proof of Lemma 4.4, the very first step of the induction.

Let $k = (k_1, \dots, k_{n-1}) \in \mathbb{Z}_+^{n-1}$ such that $|k| = \sum_i k_i \neq 0$ and let

$$\Omega_k = \{(S_1, \dots, S_{n-1}) \in \mathcal{P}(\mathbb{Z})^{n-1} \mid |S_i| = k_i, i \neq j \Rightarrow S_i \cap S_j = \emptyset\}.$$

For $S, T \in \Omega_k$ let

$$e(S, T) = \prod_{i=1}^{n-1} \left(\prod_{m \in S_i} e_{i0}^{(m)} \prod_{m \in T_i} e_{0i}^{(m)} \right) \in A_n.$$

Note that $e(S, T) \in I(0, k_1, \dots, k_{n-1}) = I(0, k)$. On the Hilbert space $\ell^2(\Omega_k)$ we define a representation π_k of $I(0, k)$ such that $\operatorname{Ker} \pi_k \supset I((0, k) + e_i)$ for $i = 1, 2, \dots, n-1$ and

$$\pi_k(e(S, T))\xi_U = \delta_{T,U} \xi_S$$

for $S, T, U \in \Omega_k$, where $(\xi_U)_{U \in \Omega_k}$ is the canonical basis of $\ell^2(\Omega_k)$. (In the notation of the proof of Proposition 2.1, $I(0, k) \cap A_\ell$ is the ideal of $A_\ell = \left(\bigotimes_{Z_\ell} M_n\right)^\gamma$ generated by E_v, $v \in V_\ell$ with $v \geq (0, k)$ and $\pi_k(x) = \pi_k(xE_{\hat{k}})$, $x \in A_\ell$, where $\hat{k} = (\ell - |k|, k) \in V_\ell$. For $x \in A_\ell$ it follows that there is $c(S, T) \in \mathbb{C}$ for each pair $S, T \in \Omega_k$ with $\cup S, \cup T \subset Z_\ell$ such that $e(S, S)xE_{\hat{k}}e(T, T) = c(S, T)e(S, T)E_{\hat{k}}$ and $\sum e(S, S)xE_{\hat{k}}e(T, T) = xE_{\hat{k}}$, where the sum is taken over all pairs S, T with $\cup S, \cup T \subset Z_\ell$.) Note that $\pi_k(I(0, k))$ is $\mathcal{K} = \mathcal{K}(\ell^2(\Omega_k))$, the C^*-algebra of compact operators on $\ell^2(\Omega_k)$ and that $\pi_k(I(m, k)) = \mathcal{K}(\ell^2(\Omega_k))$ for any $m \in \mathbb{N}$ (because $I(m, k)$ is an ideal of $I(0, k)$ not contained in the kernel of π_k). Since $I(0, k)$ is an ideal of A_n, π_k naturally extends to an irreducible representation of A_n, which again is denoted by π_k. For $k = 0 = (0, \dots, 0)$, we let π_0 denote the character φ_0 on A_n.

LEMMA 3.1. *For $k \in \mathbb{Z}_+^{n-1}$ define an irreducible representation π_k of A_n as above. Then*

$$\operatorname{Ker} \pi_k = \sum_{i=1}^{n-1} I((k_i + 1)e_i).$$

Proof. If $k = 0$, then $\pi_0 = \varphi_0$ is a character. We have already noted this at the end of the previous section.

If $k \neq 0$, then $\pi_k(I(0, k)) = \mathcal{K}$ and $\operatorname{Ker}(\pi_k | I(0, k)) = \sum_{i=1}^{n-1} K_0(I(0, k))x_i$, because $K_0(I(0, k))$ includes the latter as a maximal ideal. Hence, $\operatorname{Ker} \pi_k$ is the largest ideal

J of A_n such that $K_0(J) \cap K_0(I(0,k)) = \sum_{i=1}^{n-1} K_0(I(0,k))x_i$. It is obvious that $J \supset \sum_{i=1}^{n-1} I((k_i+1)e_i)$.

If $J \not\subset \sum_{i=1}^{n-1} I((k_i+1)e_i)$, then there is a positive $p \in K_0(J)$ such that

$$p \notin \sum_{i=1}^{n-1} K_0(A_n)x_i^{k_i+1}.$$

We express p as

$$p = \sum_{v \in V_\ell} c_v x^v$$

for some ℓ such that $c_v \in \mathbb{Z}_+$. Then if $c_v \neq 0$, then $x^v \in J$. Hence, there must be a $v \in V_\ell$ such that $c_v \neq 0$ and $v_i \leq k_i$ for $i = 1, 2, \ldots, n-1$. Then $q = x_0^{v_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} \in J \cap I(0,k)$ but $q \notin \sum_{i=1}^{n-1} K_0(I(0,k))x_i$, which is a contradiction. This concludes the proof. $\square$

We note that π_k is a σ-covariant representation. To show this define a unitary U on $\ell^2(\Omega_k)$ by

$$U\xi_S = \xi_{S+1}, \quad s \in \Omega_k,$$

where $S+1 = (S_0+1, S_1+1, \ldots, S_{n-1}+1)$ for $S = (S_0, \ldots, S_{n-1})$ and $S_i + 1 = \{m+1 \mid m \in S_i\}$. Since

$$U\pi_k(e(S,T))U^*\xi_V = \delta_{T,V-1}U\xi_S = \delta_{T+1,V}\xi_{S+1} = \pi_k(e(S+1,T+1))\xi_V$$

for $S, T, V \in \Omega_k$, it follows that $\operatorname{Ad} U\pi_k(x) = \pi_k\sigma(x)$, $x \in A_n$. If $|k| = 1$, Ω_k is identified with $\mathbb{Z}$ and U with the shift unitary S on $\ell^2(\mathbb{Z})$. If $|k| > 1$, then U is unitarily equivalent to the shift unitary with infinite multiplicity.

From now on we denote by $\pi_{(\infty,k_1,\ldots,k_{n-1})}$ the above representation π_k with $k = (k_1, \ldots, k_{n-1})$. By assigning the role played by the index 0 to another index, we define an irreducible representation π_k for $k \in (\mathbb{Z}_+ \cup \{\infty\})^n$ such that $k_i = \infty$ for a unique $i \in \{0, 1, \ldots, n-1\}$. For such an index k, we define $\tilde{k} \in \mathbb{Z}_+^n$ by $\tilde{k}_i = k_i$ if $k_i < \infty$ and by $\tilde{k}_i = 0$ if $k_i = \infty$. We then have that $\pi_k(I(\tilde{k})) = \mathcal{K}$ and

$$\operatorname{Ker} \pi_k = \sum_{i=0}^{n-1} I((k_i+1)e_i),$$

where $I((\infty+1)e_i) = I(\infty\, e_i) = \{0\}$. Thus, we obtain the following lemma.

LEMMA 3.2. *Let $k \in (\mathbb{Z}_+ \cup \{\infty\})^n$ be such that $k_i = \infty$ for a unique i and let $\tilde{k}$ be as above, and define a σ-covariant irreducible representation π_k of A_n with an implementing unitary U as above. If $0 < k_j < \infty$ for some j, $\pi_k(I(\tilde{k}))$ is isomorphic to the C^*-algebra of compact operators and U is unitarily equivalent to $S \otimes 1$ on the Hilbert space $\ell^2(\mathbb{Z}) \otimes \mathcal{H}$, where S is the shift unitary on $\ell^2(\mathbb{Z})$ and $\mathcal{H}$ is a separable Hilbert space, and $\mathcal{H}$ is one dimensional if $|\tilde{k}| = 1$ and infinite dimensional if $|\tilde{k}| > 1$.*

In the following lemma, for a finite subset X of $\mathbb{Z} \times \mathbb{Z}$, we denote by P_X the projection onto the subspace generated by ξ_m, $m \in X$ in $\ell^2(\mathbb{Z} \times \mathbb{Z}) = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$.

LEMMA 3.3. *Let S be the shift unitary on $\ell^2(\mathbb{Z})$; $S\xi_m = \xi_{m+1}$, $m \in \mathbb{Z}$. For any finite subset X of $\mathbb{Z} \times \mathbb{Z}$ and $\epsilon > 0$, there is a finite subset Y of $\mathbb{Z} \times \mathbb{Z}$ such that if $u \in \mathcal{K}(\ell^2(\mathbb{Z} \times \mathbb{Z})) + 1$ with $uP_Y = P_Y$, then there is a unitary $v \in \mathcal{K}(\ell^2(\mathbb{Z} \times \mathbb{Z})) + 1$ such that $vP_X = P_X$ and*

$$\|u - v(S \otimes 1)v^*(S^* \otimes 1)\| < \epsilon.$$

Moreover, if X is empty, then Y can also be set to be empty.

Proof. This is Lemma 2.4 of [**19**] when $X = \emptyset = Y$. Strengthening it to the above assertion is easy, but we present a proof below.

We may replace $\mathbb{Z} \otimes \mathbb{Z}$ by $\mathbb{Z} \otimes \Lambda$ with Λ a finite non-empty set, and X by $X' \times \Lambda$ with X' a finite subset of $\mathbb{Z}$. We choose Y as $Y' \otimes \Lambda$ with Y' a finite subset of $\mathbb{Z}$.

We choose Y' so large that if u is a unitary in $\mathcal{K}(\ell^2(\mathbb{Z} \times \Lambda)) + 1$ with $uP_Y = P_Y$, then there is a projection $f \in \mathcal{K}(\ell^2(\mathbb{Z}))$ such that $\|SfS^* - f\| < \epsilon$, $fP_{X'} = 0$, and

$$\|u - ((f \otimes 1)u(f \otimes 1) + (1 - f) \otimes 1)\| < \epsilon/2.$$

There is a sequence (g_n) of projections in $\mathcal{K}(\ell^2(\mathbb{Z}))$ such that $g_n(f + P_{X'}) = 0$, $\|Sg_nS^* - g_n\| \to 0$, $\|u(g_n \otimes 1) - g_n \otimes 1\| \to 0$, and $\mathrm{rank}(g_n) \to \infty$. Letting $f_n = f + g_n$, we have that $\|f_nSf_nS^*f_n - f_n\| < \epsilon$ for all sufficiently large n and that

$$\|(f_n \otimes 1)u(S \otimes 1)(f_n \otimes 1)(S^* \otimes 1)u^*(f_n \otimes 1) - f_n \otimes 1\| < 2\epsilon$$

for all sufficiently large n. It follows that $f_nSf_n \otimes 1 \approx fSf^* \otimes 1 + g_nSg_n \otimes 1$ and $(f_n \otimes 1)u(S \otimes 1)(f_n \otimes 1) \approx (f \otimes 1)u(S \otimes 1)(f \otimes 1) + g_nSg_n \otimes 1$ are approximated by unitaries on $f_n\ell^2(\mathbb{Z} \times \Lambda)$, up to the order ϵ, whose spectra are almost uniformly distributed over $\mathbb{T}$ (due to the contribution from $g_nSg_n \otimes 1$). Hence, we find a unitary $v \in \mathcal{K}(\ell^2(\mathbb{Z} \times \Lambda)) + 1$ such that $v = v(f_n \otimes 1) + (1 - f_n) \otimes 1$ for a sufficiently large n and $\|(f_n \otimes 1)u(S \otimes 1)(f_n \otimes 1) - v(f_n \otimes 1)(S \otimes 1)(f_n \otimes 1)v^*\|$ is at most of order ϵ. This implies that $\|u(S \otimes 1) - v(S \otimes 1)v^*\|$ is at most of order ϵ, concluding the proof. □

4. *The one-cocycle and the approximate Rohlin properties*

For a subset F of $\mathbb{Z}$ we set

$$A_n(F) = \Big(\bigotimes_F M_n\Big)^\gamma$$

as a C^*-subalgebra of A_n. If $m \in \mathbb{Z}_+$ and $k \in \mathbb{N}^m$, let Q be the quotient map of A_{n+m} onto

$$B = A_{n+m} \Big/ \sum_{i=0}^{m-1} I(k_ie_{n+i}).$$

For a subset F of $\mathbb{Z}$ we set $B(F) = Q(A_{n+m}(F))$, i.e. the *local structure* of B is defined by $\{B(F) \mid F \subset \mathbb{Z}\}$.

Definition 4.1. Let $n = 2, 3, \ldots$. We say that the n-shift has the *one-cocycle property* if the following conditions are satisfied.

(1) For any $\epsilon > 0$ and any unitary $u \in A_n$ such that $\varphi_i(u) = 1$ for $i = 0, \ldots, n-1$, there is a unitary $v \in A_n$ such that $\|u - v\sigma(v)^*\| < \epsilon$.

(2) If (u_k) is a sequence in the unitary group $\mathcal{U}(A_n)$ and (F_k) is an increasing sequence of finite subsets of $\mathbb{Z}$ such that $\varphi_i(u_k) = 1$ for $i = 0, \dots, n-1$, $\bigcup_k F_k = \mathbb{Z}$, and $u_k \in A_n(\mathbb{Z} \setminus F_k)$, then there is a sequence (v_k) in $\mathcal{U}(A_n)$ and an increasing sequence (G_k) of finite subsets of $\mathbb{Z}$ such that $\|u_k - v_k\sigma(v_k)\| \to 0$, $\bigcup_k G_k = \mathbb{Z}$, and $v_k \in A_n(\mathbb{Z} \setminus G_k)$.

(3) If (u_k) is a sequence in $\mathcal{U}(A_n)$ such that $\varphi_i(u_k) = 1$ for $i = 0, \dots, n-1$ and $\|[u_n, x]\| \to 0$ for all $x \in A_n$, then there is a sequence (v_k) in $\mathcal{U}(A_n)$ such that $\|u_k - v_k\sigma(v_k)\| \to 0$ and $\|[v_k, x]\| \to 0$ for all $x \in A_n$.

We say that the n-shift has the *stable one-cocycle property* if the above conditions are satisfied for the shift σ on the quotient $B = A_{n+m}/\sum_{i=0}^{m-1} I(k_i e_{n+i})$ for any $m \in \mathbb{Z}_+$ and $k \in \mathbb{N}^m$.

By the definition if the n-shift has the stable one-cocycle property then it has the one-cocycle property. If the n-shift has the one-cocycle property for all $n \geq 2$, then the n-shift has the stable one-cocycle property. Since each of the properties in Theorem 1.1 is equivalent to the corresponding one of the above definition, the main Theorem 1.1 states that the n-shift has the stable one-cocycle property for all n.

Let $\mathrm{Ex}(\Lambda_n)$ be the set of extreme points of Λ_n, i.e. $\mathrm{Ex}(\Lambda_n)$ consists of n vertices. We call an N*-cycle* an orthogonal family of projections indexed by $\mathbb{Z}/N\mathbb{Z}$.

Definition 4.2. Let $n = 2, 3, \dots$. We say that the n-shift has the approximate Rohlin property if for any $N \in \mathbb{N}$ and $\epsilon > 0$, there exist an $N_1, S \in \mathbb{N}$ with $N_1 \geq N$ and a constant $C > 0$ such that there is a sequence $(e_i^{(k)})$ of N_1-cycles in A_n with $E_k = \sum_i e_i^{(k)}$ satisfying the following:

$$\max_i \|\sigma(e_i^{(k)}) - e_{i+1}^{(k)}\| \to 0, \quad k \to \infty,$$

$$[E_k](x) \geq C(x_0 x_1 \cdots x_{n-1})^S, \quad x \in \Lambda_n,$$

$$[E_k](x) \geq c_k \sum_{i \neq j} (x_i x_j)^S, \quad x \in \Lambda_n$$

for some $c_k > 0$, and $[E_k](x)$ converges to 1 uniformly in x on every compact subset of $\Lambda_n \setminus \mathrm{Ex}(\Lambda_n)$.

Note that when we define a Rohlin property we usually impose a centrality condition such as $\|[e_i^{(k)}, x]\| \to 0$ as $k \to \infty$ for all x. Since we are dealing with the shift here, such a condition follows automatically.

To see that the lower bound estimates on $[E_k]$ are not redundant, we can construct an automorphism α of A_n which has an approximate Rohlin property without the lower bound estimates $[E_k]$ above, but with a proper centrality condition. For example, we choose a sequence (F_k) of finite subsets of $\mathbb{Z}$ which are mutually disjoint such that $|F_k|/k \to \infty$ and choose a sequence (U_k) of unitaries in A_n such that $U_k \in A_n(F_k)$ and $U_k E_v = E_v$ for $v \in V_{|F_k|}$ with $\min_i v_i < k$ and the spectrum of $U_k E_v$ is equally distributed otherwise, where E_v is the minimal central projection of $A_n(F_k)$ as in the proof of Proposition 2.1. And we define an automorphism α of A_n by the limit of $\mathrm{Ad}(U_1 U_2 \cdots U_k)$. Then we can see that α has the property that for any $N \in \mathbb{N}$ there is a sequence $(e_i^{(k)})$ of N-cycles in A_n such that $\max_i \|\alpha(e_i^{(k)}) - e_{i+1}^{(k)}\| \to 0$, $N[e_0^{(k)}](x) \to 1$ uniformly in x on every compact subset of $\Lambda_n \setminus \mathrm{Ex}(\Lambda_n)$, and $\|[e_i^{(k)}, x]\| \to 0$ for $x \in A_n$ (see, e.g., [**2**]). However, then the ideal J_k of A_n generated by $e_i^{(k)}$ must satisfy $\bigcap_k J_k = \{0\}$.

Our purpose is to prove, by induction, that the properties defined above follow universally. The following lemmas give the basic step for this induction, which are more or less presented in [**19**].

LEMMA 4.3. *The* 2*-shift has the approximate Rohlin property.*

Proof. By using the CAR algebra formulation we show in [**19**] that for any $N \in \mathbb{N}$ there is a sequence of 2^N-cycles in A_2 satisfying the required properties with $M = N$ and $C = 2^N$. □

LEMMA 4.4. *The* 1*-shift has the stable one-cocycle property.*

Proof. Let $m \in \mathbb{N}$ and $k \in \mathbb{Z}_+^{1+m}$ such that $k_0 = 0$ and $k_j > 0$ for $j > 0$ and define

$$B = A_{1+m} \Big/ \sum_{i=1}^{m} I(k_i e_i).$$

We have to show that the shift σ defined on B has the one-cocycle property.

Let φ_0 be the unique character of B. What we have to prove is that for any $u \in \mathcal{U}(B)$ with $\varphi_0(u) = 1$, there is a sequence (v_n) in $\mathcal{U}(B)$ such that $\|u - v_n\sigma(v_n)^*\| \to 0$; and two other versions.

By the following Lemma 4.5 and Lemmas 3.2 and 3.3 we find a finite decreasing sequence $(J_i)_{i=0}^N$ of ideals of A_{1+m} such that $J_0 = A_{1+m}$, $J_1 = \sum_{i=1}^m I(e_i)$, $J_N \subset \sum_{i=1}^m I(k_i e_i)$, $J_{N-1} \not\subset \sum_{j=1}^m I(k_j e_j)$, $J_{i-1}/J_i \cong \mathcal{K}$ for $i > 1$, and the automorphism, denoted by σ, induced on J_{i-1}/J_i by the shift σ has the one-cocycle property. (In Lemma 4.5, since $J_N = \mathrm{Ker}(\rho_N) \cap J_{N-1} = \sum_{j=1}^m I(k_j e_j) \cap J_{N-1}$, it follows that $J_{N-1} \not\subset \sum_{j=1}^m I(k_j e_j)$.)

Let $u_1 \in \mathcal{U}(A_{1+m})$ be such that $Q(u_1) = u$, where Q is the quotient map of A_{1+m} onto B. Since $\varphi_0(Q(u_1)) = 1$, we have that $u_1 \in J_1 + 1$.

From the short exact sequence

$$0 \to J_2 \to J_1 \to J_1/J_2 \to 0,$$

and the one-cocycle property for the shift on J_1/J_2 applied to $u_1 + J_2$, proved in Lemma 3.3, we find unitaries $u_2 \in J_2 + 1$ and $v_1 \in J_1 + 1$ such that

$$\|v_1^* u_1 \sigma(v_1) - u_2\| < N^{-1}\epsilon.$$

Repeating this process, we have unitaries $u_i \in J_i + 1$ for $i \le N$ and unitaries $v_i \in J_i + 1$ for $i < N$ such that

$$\|v_i^* u_i \sigma(v_i) - u_{i+1}\| < N^{-1}\epsilon$$

for $i < N$. Then it follows that

$$\|v_1 v_2 \cdots v_{N-1} u_N \sigma(v_1 v_2 \cdots v_{N-1}) - u_1\| < \epsilon.$$

Thus, for $v = Q(v_1 v_2 \cdots v_{N-1}) \in B$ we get that $\|v\sigma(v)^* - u\| < \epsilon$. This concludes the proof of the first property of Definition 4.1.

We turn to the second property of Definition 4.1. Let G be a finite subset of $\mathbb{Z}$. To get v_{N-1} from $A_{1+m}(\mathbb{Z} \setminus G) \cap J_{N-1} + 1$ in the above arguments based on Lemma 3.3, u_{N-1} must be from $A_{1+m}(\mathbb{Z} \setminus F)$ for some finite subset $F_1 \subset \mathbb{Z}$ with $F_1 \supset G$. Here we used the fact that requiring $v_{N-1} \in A_{1+m}(\mathbb{Z} \setminus G) \cap J_{N-1} + 1$ modulo J_N is as strong as requiring that $v_{N-1} p \in \mathbb{C} p$ for some $p \in J_{N-1}$ which is a projection modulo J_N since $J_{N-1}/J_N \cong \mathcal{K}$. For that to be true, we must have that $v_{N-2}, u_{N-2} \in A_{1+m}(\mathbb{Z} \setminus F_1)$, which in turn requires that $u_{N-2} \in A_{1+m}(\mathbb{Z} \setminus F_2)$ for some finite subset $F_2 \supset F_1$. In this way we can conclude that there is a finite subset F_{N-1} of $\mathbb{Z}$ such that if u_1 is a unitary from $A_{1+m}(\mathbb{Z} \setminus F_{N-1})$, then all $v_1, \ldots, v_{N-1}$ will be chosen from $A_{1+m}(\mathbb{Z} \setminus G)$ in the above arguments; then $v = Q(v_1 \cdots v_{N-1}) \in B(\mathbb{Z} \setminus G)$ is the desired unitary. This concludes the proof in this case.

Finally, we come to the last property of Definition 4.1. Let $\mathcal{G}$ be a finite subset of B; by lifting each element of $\mathcal{G}$ to A_{1+m} we regard $\mathcal{G}$ as a subset of A_{1+m}. We choose a finite subset G of $\mathbb{Z}$ such that if v is a unitary in $A_{1+m}(\mathbb{Z} \setminus G)$ then $\|[v, x]\| < 1/2N$, $x \in \mathcal{G}$. To get $v_{N-1} \in J_{N-1} + 1$ for u_{N-1} such that $v_{N-1} \in A_{1+m}(\mathbb{Z} \setminus G)$, we must require that $\|[u_{N-1}, x] + J_N\| < 1, x \in \mathcal{F}_1$ for some finite subset of $\mathcal{F}_1$ of A_{1+m}. For that we have to require that $v_{N-2} \in A_{1+m}(\mathbb{Z} \setminus G_1)$ and $\|[u_{N-2}, x] + J_N\| < 1/3N$, $x \in \mathcal{F}_1$ for some finite subset G_1 of $\mathbb{Z}$ with $G_1 \supset G$. Again for this to be true, we have to require that $\|[u_{N-2}, x] + J_N\| < 1/N$, $x \in \mathcal{F}_2$ for some finite subset $\mathcal{F}_2$ of A_{1+m}. In this way we obtain a finite subset $\mathcal{F}_{N-1}$ of A_{1+m} such that if $u_1 \in J_1 + 1$ satisfies that $\|[u_1, x] + J_N\| < 1/N$ for $x \in \mathcal{F}_{N-1}$, then all $v_1, \ldots, v_{N-1}$ can be taken from $A_{1+m}(\mathbb{Z} \setminus G)$. Then $v = Q(v_1 \cdots v_{N-1})$ would satisfy the required condition. (Thus, since $B = A_{1+m}/\sum_{j=1}^m I(k_j e_j)$ is of type I, properties (2) and (3) are equivalent for this case.) $\square$

LEMMA 4.5. *There exists a finite sequence $(\rho_i)_{i=1}^N$ of irreducible representations of A_{1+m} of the form $\pi_{(\infty,\ell)}$, $\ell \in (\mathbb{Z}_+)^m$ and a decreasing sequence $(J_i)_{i=0}^N$ of ideals of A_{1+m} such that $J_0 = A_{1+m}$, $\rho(J_{i-1}) = \mathcal{K}$ or $\mathbb{C}$, $\mathrm{Ker}(\rho_i | J_{i-1}) = J_i$, and*

$$\mathrm{Ker}(\rho_i) \supset \sum_{j=1}^m I(k_j e_j),$$

$$J_N \subset \sum_{j=1}^m I(k_j e_j).$$

Proof. Let $\rho_1 = \pi_{(\infty,0,\ldots,0)}$ and $J_1 = \mathrm{Ker}\,\rho_1$. Then it follows that $J_1 = \sum_{j=1}^m I(e_j) \supset \sum_{j=1}^m I(k_j e_j)$. We prove the assertion by induction as follows.

Suppose that we are given an ideal J of A_{1+m} such that

$$J = \sum_{w \in \mathcal{C}} I(w),$$

where $\mathcal{C}$ is a subset of $\mathbb{Z}_+^{1+m}$ such that $w \le k$ and $|w| = \sum_{j=0}^m w_j \ge 1$ for $w \in \mathcal{C}$. We may suppose that no pairs in $\mathcal{C}$ are comparable. (If $w \ge w'$ in the sense that $w_j \ge w'_j$, then $I(w) \subset I(w')$; so we may remove w from $\mathcal{C}$.)

Let

$$r = \min\{|w| \mid w \in \mathcal{C}, w_j < k_j \text{ for all } j = 1, \ldots, m\}.$$

For each $w \in \{w \in \mathcal{C} \mid |w| = r\}$, we define $\hat{w} \in (\mathbb{Z}_+ \cup \{\infty\})^{1+m}$ by $\hat{w}_0 = \infty$ and $\hat{w}_j = w_j$ for $j > 0$ and consider $\pi_{\hat{w}}$. For $z \in \mathcal{C} \setminus \{w\}$, we have that $I(z) \subset \operatorname{Ker} \pi_{\hat{w}}$. (Either $z_j \geq k_j > w_j$ for some $j = 1, 2, \ldots, m$ or $z_j > w_j$ for some $j > 0$ as $|z| \geq |w|$.) Then $\pi_{\hat{w}}(J) = \mathcal{K}$ and

$$\operatorname{Ker}(\pi_{\hat{w}}|J) = \sum_{z \in \mathcal{C} \setminus \{w\}} I(z) + \sum_{j=1}^{m} I(w + e_j).$$

Note also that $\operatorname{Ker} \pi_{\hat{w}} \supset \sum_{j=1}^{m} I((w_j + 1)e_j) \supset \sum_{j=1}^{m} I(k_j e_j)$.

We now set J to be $\operatorname{Ker}(\pi_{\hat{w}}|J)$ and $\mathcal{C}$ to be an appropriate subset of $\mathcal{C} \setminus \{w\} \cup \{w + e_j \mid j = 1, \ldots, m\}$. Repeating this process for a finite number of times we can increase $r = \min\{|w| \mid w \in \mathcal{C}, w_j < k_j \text{ for all } j > 0\}$. Eventually we reach the situation where for any $w \in \mathcal{C}$ there is $j > 0$ such that $w_j = k_j$. Hence, we get $J = \sum_{w \in \mathcal{C}} I(w) \subset \sum_{j=1}^{m} I(k_j e_j)$. This completes the proof. □

The following result combined with Lemma 4.3 shows that the n-shift has the approximate Rohlin property for all $n \geq 2$.

LEMMA 4.6. *Let $n = 2, 3, \ldots$. If the n-shift has the approximate Rohlin property, then the $n + 1$-shift has also the approximate Rohlin property.*

Proof. Let $K, L, M \in \mathbb{N}$ be such that $M \gg L$ and let $Z_{M,K} = \{mM \mid m = 0, 1, \ldots, K-1\}$ and $Z_{M,K,L} = \bigcup\{Z_{M,K} + m \mid m = 0, 1, \ldots, L-1\}$. We define an embedding ι of M_n into $\bigotimes_{\{0\}} M_{n+1} \subset \bigotimes_{\mathbb{Z}} M_{n+1}$ by $\iota(e_{ij}) = e_{ij}^{(0)}$, where $(e_{ij}^{(0)})$ are the canonical matrix units for M_{n+1} at $0 \in \mathbb{Z}$ as a C^*-subalgebra of $\bigotimes_{\mathbb{Z}} M_{n+1}$; in particular, $1 - \iota_0(1) = e_{nn}^{(0)}$. We then define a homomorphism ψ_0 of M_n into $\bigotimes_{Z_{M,K}} M_{n+1} \subset \bigotimes_{\mathbb{Z}} M_{n+1}$ by

$$\begin{aligned}\psi_0(x) = \iota_0(x) + e_{nn}^{0}\sigma^{M}(\iota_0(x)) + e_{nn}^{(0)}e_{nn}^{(L)}\sigma^{2M}(\iota_0(x)) + \cdots \\ + e_{nn}^{(0)} \cdots e_{nn}^{((K-2)L)}\sigma^{(K-1)M}(\iota_0(x)).\end{aligned}$$

We set $\psi_k = \sigma^k \psi_0$ for $k = 1, 2, \ldots, M-1$; note that the ranges of $\psi_k, k = 0, 1, \ldots, M-1$ mutually commute and that $L \ll M$. We then define a homomorphism ψ of $\bigotimes_0^{L-1} M_n$ into $\bigotimes_{Z_{M,K,L}} M_{n+1} \subset \bigotimes_{\mathbb{Z}} M_{n+1}$ by $\psi = \psi_0 \otimes \psi_1 \otimes \cdots \otimes \psi_{L-1}$. Note that $\psi_L(1)\psi\sigma = \psi_0(1)\sigma\psi$ on $\bigotimes_0^{L-2} M_n$ and that the range of $\psi|\big(\bigotimes_0^{L-1} M_n\big)^{\gamma}$ is contained in $A_{n+1} = \big(\bigotimes_{\mathbb{Z}} M_{n+1}\big)^{\gamma}$, where γ is the *gauge action* of $\mathbb{T}^{n-1}$ (or $\mathbb{T}^n$ for the latter case). We thus regard ψ as a homomorphism of $B_L = \big(\bigotimes_0^{L-1} M_n\big)^{\gamma}$ into A_{n+1}.

We recall that there is a tracial state τ_λ on A_{n+1} to each

$$\lambda \in \Lambda_{n+1} = \left\{\lambda \in [0, 1]^n \;\middle|\; \sum_{i=1}^{n} \lambda_i \leq 1\right\}.$$

Note that $\tau_\lambda\psi_0(1) = 1 - \lambda_n^K$ and $\tau_\lambda\psi(1) = (1 - \lambda_n^K)^L$. For a projection $e \in B_L$, we have that $\tau_\lambda\psi(e) = 0$ for $\lambda_n = 1$ and

$$\tau_\lambda\psi(e) = (1 - \lambda_n^K)^L \tau_\mu(e)$$

for $\lambda_n < 1$, where

$$\mu = \left(\frac{\lambda_1}{1 - \lambda_n}, \ldots, \frac{\lambda_{n-1}}{1 - \lambda_n}\right) \in \Lambda_n.$$

Hence, if $[e] = g \in \mathbb{Z}[y_1, \ldots, y_{n-1}]$ in $K_0(A_n)$ under the embedding $B_L \subset A_n$, then $f = [\psi(e)]$ in $K_0(A_{n+1})$ is given by

$$f(x_1, \ldots, x_n) = (1 - x_n^K)^L g(x_1/(1 - x_n), \ldots, x_{n-1}/(1 - x_n)),$$

which must be a polynomial in $x_1, \ldots, x_n$, since g is at most of order L. If g is strictly positive on $\Lambda_n \setminus \mathrm{Ex}(\Lambda_n)$, then f is strictly positive on $\Lambda_{n+1} \setminus \bigcup_{i=0}^{n-1}[i, n]$, where $[i, n]$ is the edges between $x_i = 1$ and $x_n = 1$ for $i = 0, 1, \ldots, n-1$. We can make f arbitrarily close to 1 uniformly on a compact subset of $\Lambda_{n+1} \setminus \bigcup_{i=0}^{n-1}[i, n]$ by making g close to 1 on a compact subset of $\Lambda_n \setminus \mathrm{Ex}(\Lambda_n)$. If $g(y_1, \ldots, y_{n-1}) \geq C(y_0 \cdots y_{n-1})^S$ for some $S \in \mathbb{N}$ and $C > 0$, where $y_0 = 1 - y_1 - \cdots - y_{n-1}$, then $f(x) \geq C(1 - x_n^K)^L (1 - x_n)^{-nS} (x_0 \cdots x_{n-1})^S$ for $x \in \Lambda_{n+1}$ because

$$1 - \frac{x_1}{1 - x_n} - \frac{x_2}{1 - x_n} - \cdots - \frac{x_{n-1}}{1 - x_n} = \frac{x_0}{1 - x_n}.$$

Note that if K is sufficiently large, then $(1 - x_n^K)^L \approx 1$ uniformly for x_n in $[0, 1 - \delta]$ for $\delta > 0$. Thus, for such a choice of K, we have that

$$f(x) \geq C(x_0 x_1 \cdots x_{n-1})^S$$

for $x \in \Lambda_{n+1}$ with $x_n \in [0, 1 - \delta]$.

Since $\psi_L(1)\psi\sigma = \psi_0(1)\sigma\psi$ on $\left(\bigotimes_0^{L-2} M_n\right)^\gamma \otimes 1 \subset B_L$, ψ does not really intertwine the shifts σ even approximately. We have to modify ψ as follows.

By using the lemma below and taking a sufficiently large M, we define an embedding $\phi : B_L \to A_{n+1}$ by $\phi(x) = \psi(x)p$, where p is a projection which resides outside $Z_{M,K,L}$ and satisfies that $\|\sigma(\phi(1)) - \phi(1)\| < \epsilon$ for a prescribed $\epsilon > 0$ and $\tau_\lambda(\phi(1)) \geq (1 - \lambda_n^K)^{L+L_\epsilon}$ with some $L_\epsilon \in \mathbb{N}$. Since $\|\sigma(\psi(1)p) - \psi(1)p\| < \epsilon$ and $\sigma(\psi(1)p) = \psi_L(1)\sigma(\psi(1)p)$, it follows that $\|\psi_L(1)\psi(1)p - \psi(1)p\| < 2\epsilon$. If $e \in B_L \subset A_n$ is a projection such that $\sigma(e) \in B_L$, then it follows that $\|\sigma\phi(e) - \phi\sigma(e)\| < 3\epsilon$ because $\sigma\phi(e) = \sigma(\psi(e))\sigma(\psi(1)p) \approx \sigma\psi(e)\psi(1)p = \sigma\psi(e)\psi_0(1)\psi(1)p = \psi\sigma(e)\psi_L(1)\psi(1)p \approx \psi\sigma(e)p = \phi\sigma(e)$. Since this modification of ψ introduces only the factor $(1 - \lambda_n^K)^{L_\epsilon}$ in the estimate of $\tau_\lambda(\phi(e))$ for a projection $e \in B_L$, we have a similar estimate for $f = [\phi(e)]$ as in the previous paragraph.

Let $N \in \mathbb{N}$. Then there are $N_1, C, S \in \mathbb{N}$ with $N_1 \geq N$ satisfying the following: we have an N_1-cycle (e_i) in A_n such that $\|\sigma(e_i) - e_{i+1}\| \approx 0$, $N_1[e_i](x_1, \ldots, x_{n-1})$ is close to 1 uniformly on a compact subset of $\Lambda_n \setminus \mathrm{Ex}(\Lambda_n)$, and

$$[e_i](x_1, \ldots, x_{n-1}) \geq C(x_0 x_1 \cdots x_{n-1})^S$$

on Λ_n, and

$$[e_i](x_1, \ldots, x_{n-1}) \geq c \sum_{i \neq j} (x_i x_j)^S$$

on Λ_n for some $c > 0$ (which may depend on the choice of e_i).

We may assume that $e_i \in \bigotimes_{m=0}^{L-1} M_n$ for some L and we embed e_i into A_{n+1} by ϕ which depends on the choice of K, M, p. Thus, for any $\epsilon > 0$ we get an N_1-cycle (E_i) in A_{n+1} such that $\|\sigma(E_i) - E_{i+1}\| < \epsilon$,

$$N_1[E_i](x_1, \ldots, x_n) > 1 - \epsilon$$

for $x \in \Lambda_{n+1}$ such that $\min_{i \neq n} \operatorname{dist}(x, [i, n]) > \epsilon$, where the distance dist on Λ_{n+1} is defined by $\operatorname{dist}(\lambda, \nu) = \max\{|\lambda_i - \mu_i|; i = 0, 1, \ldots, n\}$, and

$$[E_i](x_1, \ldots, x_n) \geq C(x_0 x_1 \cdots x_n)^S$$

on the subset of Λ_{n+1} consisting of x with $x_n < 1 - \epsilon$, and

$$[E_i](x_1, \ldots, x_n) \geq c \sum_{i \neq j \leq n-1} (x_i x_j)^S$$

on the above subset for some $c > 0$.

In this way we embed a Rohlin cycle in A_n into A_{n+1}. Note that there are $n + 1$ types of embedding by assigning the role played by the nth coordinate in the above to the other coordinates. We combine the Rohlin cycles so obtained to get the desired one. The proof is continued after the following lemmas.

LEMMA 4.7. *For any $\epsilon > 0$ there exists an $L_\epsilon \in \mathbb{N}$ such that if $M > L + 2L_\epsilon$, there is a projection p in $\left(\bigotimes_{Z'} M_{n+1}\right)^\gamma$, where $Z' = \bigcup\{Z_{M,K} + m \mid m = -1, -2, \ldots, -L_\epsilon;\ m = L, L+1, \ldots, L + L_\epsilon - 1\}$ is disjoint from $Z_{M,K,L} = \bigcup\{Z_{M,K} + m \mid m = 0, 1, \ldots, L-1\}$ such that $f = \psi(1)p$ is a projection satisfying $\|\sigma(f) - f\| < \epsilon$ and $\tau_\lambda(f) \geq (1 - \lambda_n^K)^{L+2L_\epsilon}$ for $\lambda \in \Lambda_{n+1}$.*

Proof. Let $e = \psi_0(1)$ as a projection in $\left(\bigotimes_{Z_{M,K}} M_{n+1}\right)^\gamma$. Then $\psi(1)$ looks like $e \otimes e \otimes \cdots \otimes e$ (in $\left(\bigotimes_{Z_{M,K,L}} M_{n+1}\right)^\gamma = \left(\bigotimes_0^{L-1} \bigotimes_{Z_{M,K}} M_{n+1}\right)^\gamma$), where e repeats L times. If $Z'' = Z' \cup Z_{M,K,L}$, then $\bigotimes_{Z''} M_{n+1}$ is the tensor product of $\bigotimes_{Z_{M,K}} M_{n+1}$ indexed by m between $-L_\epsilon$ and $L + L_\epsilon - 1$ inclusive, where σ shifts each factor to right and $\psi(1)$ resides at the tensor product between 0 and $L - 1$. Hence, the problem reduces to finding an almost σ invariant projection in $\left(\bigotimes_{-L_\epsilon}^{L+l_\epsilon-2} M_{n+1}\right)^\gamma$ dominates $e \otimes e \otimes \cdots \otimes e$ ($L + 2L_\epsilon$ times) and is dominated by $\psi(1) = e \otimes e \otimes \cdots \otimes e$ (L times). This follows from the following lemma. □

LEMMA 4.8. *Let $m \in \mathbb{N}$ and let $A = \left(\bigotimes_{\mathbb{Z}} M_m\right)^\gamma$, where γ is a restriction of the (infinite tensor product type) action β of $\mathcal{U}_m = \mathcal{U}(M_m)$ to a compact abelian subgroup of $\mathcal{U}_m$ and let σ be the shift automorphism of A to the right. Let e be a projection in $(M_m)^\gamma$. For $K \in \mathbb{N}$, let $e_K = \bigotimes_{-K}^{K} e$. For $L, N \in \mathbb{N}$ there is a projection $p \in \left(\bigotimes_{-N-2L}^{N+2L} M_m\right)^\gamma \subset A$ such that $e_{N+L} \leq p \leq e_N$ and $\|\sigma(p) - p\|$ is of the order of $L^{-1/2}$. Moreover, p can be chosen to satisfy that if φ is a state on M_m and $\tau = \bigotimes_{\mathbb{Z}} \varphi$ is a tracial state on A, then*

$$\tau(p) = \varphi(e)^{2N+2L+1}(1 + L(1 - \varphi(e))).$$

Proof. Let $f = e_{N+L}$ and let $f_k = \sigma^k(f)(1 - \sigma^{k-1}(f))$ and $f_{-k} = \sigma^{-k}(f)(1 - \sigma^{-k+1}(f))$ for $k = 1, 2, \ldots, L$, which are all projections. If $1 \leq k < \ell$, then

$$f_k f_\ell \leq \sigma^k(f)\sigma^\ell(f)(1 - \sigma^{\ell-1}(f)) = 0$$

because $\sigma^k(f)\sigma^{\ell-1}(f) \geq \sigma^k(f)\sigma^\ell(f)$. If $k, \ell \geq 1$, then

$$f_{-k} f_\ell \leq \sigma^{-k}(f)(1 - \sigma^{-k+1}(f))\sigma^\ell(f) = 0$$

because $\sigma^{-k}(f)\sigma^{\ell}(f) \leq \sigma^{k}(f)\sigma^{-k+1}(f)$. Thus, one can conclude that the projections f_k, $0 < |k| \leq L$ are mutually orthogonal and orthogonal to f. Note also that they are mutually equivalent and that σ shifts the sequence of projections

$$f_{-L}, f_{-L+1}, \ldots, f_{-1}$$

to the right, the last one f_{-1} to $f(1-\sigma(f))$, which is a subprojection of f, and a subprojection, $f(1-\sigma^{-1}(f))$, of f to f_1, and shifts the sequence

$$f_1, f_2, \ldots, f_L$$

to the right except for the last f_L. Let $v \in A$ be such that $v^*v = f_{-L}$ and $vv^* = f_1$ and define

$$p = f + \sum_{k=1}^{L}\left(\frac{k}{L}f_{-L+k-1} + \frac{L-k}{L}f_k + \frac{\sqrt{k(L-k)}}{L}\sigma^{k-1}(v+v^*)\right).$$

Since all f_k, f_{-k}, f are dominated by e_N and $p \geq f = e_{N+L}$, it follows that $e_{N+L} \leq p \leq e_N$. We can also estimate $\|\sigma(p)-p\|$ as required (see [**15**] for details).

If τ is a tracial state on A as in the statement, then $\tau(f) = \varphi(e)^{2N+2L+1}$ and $\tau(f_k) = \tau(f)(1-\varphi(e))$. Since $\tau(v) = 0$, we get that $\tau(p) = \tau(f) + \tau(f)(1-\varphi(e))L = \tau(f)(1+L(1-\varphi(e)))$. This concludes the proof. □

LEMMA 4.9. *For each $i = 0, 1, \ldots, n$ and $\epsilon > 0$ there exists a projection $f_i \in A_{n+1}$ such that $\|\sigma(f_i) - f_i\| < \epsilon$ and $\tau_\lambda(f_i)$ depends only on $\lambda_i \in [0,1]$ and is a decreasing function in λ_i and for any $\lambda \in \Lambda_{n+1}$,*

$$\tau_\lambda(f_i) < \epsilon \quad \text{if } \lambda_i > \epsilon,$$
$$\tau_\lambda(f_i) = 1 \quad \text{if } \lambda_i = 0,$$

from which there exists a $\delta > 0$ such that if $\lambda_i < \delta$ then $\tau_\lambda(f_i) > 1-\epsilon$. Moreover f_i can be chosen from $\big(\bigotimes_Z M_{n+1}\big)^\gamma$ for some finite subset Z of $\mathbb{Z}$ and have the following property. If $\lambda \in \Lambda_{n+1}$ satisfies that $\mathrm{dist}(\lambda, \mathrm{Ex}(\Lambda_{n+1})) \geq \epsilon + 2\delta$ *and* $\mathrm{dist}(\lambda, [i,j]) < \delta$ *for some $i \neq j$, then $\tau_\lambda(f_i) < \epsilon$, $\tau_\lambda(f_j) < \epsilon$, and $\tau_\lambda(f_k) > 1-\epsilon$ for all $k \neq i, j$.*

Proof. Let $E_i^{(m)} = \sum_{j\neq i} e_{jj}^{(m)} = 1 - e_{ii}^{(m)}$. It is shown by the previous lemma that for large $L, N \in \mathbb{N}$ there is a projection $f_i \in \big(\bigotimes_{m=-2L-N}^{2L+N} M_{n+1}\big)^\gamma$ between $\bigotimes_{m=-L-N}^{L+N} E_i^{(m)}$ and $\bigotimes_{m=-N}^{N} E_i^{(m)}$ such that $\|\sigma(f_i) - f_i\|$ is of order $L^{-1/2}$. By using the explicit formula for f_i given there, we have that

$$\tau_\lambda(f_i) = (1-\lambda_i)^{2N+2L+1}(1+L\lambda_i)$$

for $\lambda \in \Lambda_{n+1}$. Then if $\lambda_i > \epsilon$, then $\tau_\lambda(f) \leq \tau_\lambda\big(\bigotimes_{m=-N}^{N} E_i^{(m)}\big) < (1-\epsilon)^{2N+1}$ and if $\lambda_i = 0$ then $\tau_\lambda(f_i) = 1$. By choosing L, N sufficiently large, this concludes the proof of the first part.

To show the second part suppose, in contrast, that $\mathrm{dist}(\lambda, \mathrm{Ex}(\Lambda_{n+1})) \geq \epsilon + 2\delta$, $\mathrm{dist}(\lambda, [i,j]) < \delta$, and $\tau_\lambda(f_i) \geq \epsilon$; from the last condition it follows that $\lambda_i \leq \epsilon$. Since $\min_{0\leq t\leq 1}\max\{|\lambda_i - t|, |\lambda_j - (1-t)|\} < \delta$, there is a $t \in [0,1]$ such that $t - \lambda_i < \delta$ and $1-t-\lambda_j < \delta$, which implies that $\lambda_j > 1-\lambda_i-2\delta \geq 1-\epsilon-2\delta$. Since this contradicts that $\mathrm{dist}(\lambda, \mathrm{Ex}(\Lambda_{n+1})) \geq \epsilon + 2\delta$, we get that $\tau_\lambda(f_i) < \epsilon$. The same argument yields that $\tau_\lambda(f_j) < \epsilon$. Since $\lambda_k < \delta$ for $k \neq i, j$, it follows that $\tau_\lambda(f_k) > 1-\epsilon$ for such a k. □

Continuation of the proof of Lemma 4.6. By the previous lemma for any small $\epsilon > 0$ we choose projections $f_0, f_1, \dots, f_{n-1}$ in A_{n+1} such that $\tau_\lambda(f_i) < \epsilon$ if $\lambda_i > \epsilon$, $\tau_\lambda(f_i) = 1$ if $\lambda_i = 0$, and $\|\sigma(f_i) - f_i\| \approx 0$ for all i. More explicitly we use the *same formula* (as a function of $E_i^{(m)} = 1 - e_{ii}^{(m)}$) for constructing f_i; hence, we have that $\tau_\lambda(f_i) = \tau_\mu(f_j)$ if $\lambda_i = \mu_j$. Furthermore, we assume that all the f_i reside at different places (by applying powers of σ), i.e. there are finite subsets $Z_0, \dots, Z_{n-1}$ of $\mathbb{Z}$ which are mutually disjoint such that $f_i \in \left(\bigotimes_{Z_i} M_{n+1}\right)^\gamma$; in particular, the f_i commute with each other and $[f_i f_j] = [f_i][f_j]$ in $\mathbb{Z}[x_1, \dots, x_n]$ for $i \neq j$.

We suppose that $\epsilon < (n+2)^{-1}$ and note that $\delta > 0$ is chosen so that if $\lambda_k < \delta$ then $\tau_\lambda(f_k) > 1 - \epsilon$; obviously $\delta < \epsilon$. Let $N, N_1, C, S \in \mathbb{N}$ be as in the proof before the interruption. For each $k = 0, 1, \dots, n$ let (E_i^k) be an N_1-cycle in A_{n+1} such that $\|\sigma(E_i^k) - E_{i+1}^k\| \approx 0$,

$$N_1[E_0^k](x_1, \dots, x_n) > 1 - \epsilon$$

if $\min_{i \neq k} \operatorname{dist}(x, [i, k]) \geq \delta$, and

$$[E_0^k] \geq C(x_0 x_1 \cdots x_n)^S, \quad [E_0^k] \geq c \sum_{i \neq j; i, j \neq k} (x_i x_j)^S$$

for some $c > 0$, if $x_k \leq 1 - n\delta$. We suppose that all (E_i^k) and f_k reside at disjoint subsets of $\mathbb{Z}$ for various k; in particular, we have assumed that $f_k E_i^k$ is a projection and that $\tau_\lambda(f_k E_i^k) = \tau_\lambda(f_k)\tau_\lambda(E_i^k)$. Now we define an N_1-cycle:

$$E_i = \bigvee_{k=0}^{n} (f_k E_i^k) + \left(1 - \bigvee_{k=0}^{n} f_k\right) E_i^0.$$

Here we have use the fact that all $f_k E_i^k$ commute with each other. We show that (E_i) is the required N_1-cycle in A_{n+1}.

Since $\|\sigma(f_k E_i^k) - f_k E_{i+1}^k\| \approx 0$ and $\|\sigma(F) - F\| \approx 0$ where $F = \bigvee_{k=0}^n f_k = f_0 + f_1(1 - f_1) + f_2(1 - f_0 - f_1(1 - f_0)) + \cdots$, we obtain that $\|\sigma(E_i) - E_{i+1}\| \approx 0$.

Let $\lambda \in \Lambda_{n+1}$ with $\lambda_0 = 1 - \lambda_1 - \cdots - \lambda_n$. We take $k \in \{0, 1, \dots, n\}$ such that $\lambda_k = \min\{\lambda_i \mid i = 0, \dots, n\}$. Then we have that $\tau_\lambda(f_k) = \max\{\tau_\lambda(f_i) \mid i = 0, \dots, n\}$ since $\lambda_i \mapsto \tau_\lambda(f_i)$ is decreasing and $\tau_\lambda(f_i) = \tau_\lambda(f_j)$ if $\lambda_i = \lambda_j$. If $\tau_\lambda(f_k) \geq 1/(n+2) > \epsilon$, then we have that $\lambda_k \leq \epsilon$.

If $\tau_\lambda(f_i) < 1/(n+2) < 1 - \epsilon$ for all i, then $\lambda_i \geq \delta$ and $\tau_\lambda(F) \leq \sum_{i=0}^n \tau_\lambda(f_i) < (n+1)/(n+2) < 1 - 1/(n+2)$. Note that for any i,

$$\lambda_i = 1 - \sum_{j \neq i} \lambda_j < 1 - n\delta.$$

Thus, we have two cases for $\lambda \in \Lambda_{n+1}$: (I) the smallest of all λ_i is given by λ_k satisfying $\lambda_k \leq \epsilon < 1 - n\delta$ and $\tau_\lambda(f_k) \geq 1/(n+2)$; and (II) $\tau_\lambda(1 - F) > 1/(n+2)$, $\lambda_i \geq \delta$ for all i, and $\lambda_0 \leq 1 - n\delta$.

In the first case

$$\tau_\lambda(E_0) \geq \tau_\lambda(f_k)\tau_\lambda(E_0^k) \geq \frac{C}{n+2}(\lambda_0 \lambda_1 \cdots \lambda_n)^S.$$

In the second case

$$\tau_\lambda(E_0) \geq \tau_\lambda(1-F)\tau_\lambda(E_0^0) \geq \frac{C}{n+2}(\lambda_0\lambda_1\cdots\lambda_n)^S.$$

In either case we have one of the required estimates from below.

To prove the other estimate, we have, for the first case,

$$\tau_\lambda(E_0) \geq \frac{c}{n+2}\sum_{i\neq j;\, i,j\neq k}(\lambda_i\lambda_j)^S,$$

where $\lambda_k < 1-n\delta$ is used. Since λ_k is the smallest in λ_i, we have that for each $i \neq k$

$$(\lambda_i\lambda_k)^S \leq 1/(n-1)\sum_{j\neq i,k}(\lambda_i\lambda_j)^S \leq \sum_{j\neq i,k}(\lambda_i\lambda_j)^S,$$

which implies that

$$\begin{aligned}\sum_{i\neq j}(\lambda_i\lambda_j)^S &= \sum_{i\neq j;i,j\neq k}(\lambda_i\lambda_j)^S + \sum_{i\neq k}(\lambda_i\lambda_k)^S\\ &\leq \sum_{i\neq j;i,j\neq k}(\lambda_i\lambda_j)^S + \sum_{i\neq k}\sum_{j\neq i,k}(\lambda_i\lambda_j)^S\\ &= 3\sum_{i\neq j;i,j\neq k}(\lambda_i\lambda_j)^S.\end{aligned}$$

Hence, we have that

$$\tau_\lambda(E_0) \geq \frac{c}{3(n+2)}\sum_{i\neq j}(\lambda_i\lambda_j)^S.$$

We have, for the second case, that

$$\tau_\lambda(E_0) \geq \frac{c}{n+2}\sum_{i\neq j;i,j>0}(\lambda_i\lambda_j)^S.$$

Since $(\lambda_0\lambda_i)^S \leq \lambda_i^S \leq (n-1)^{-1}\delta^{-S}\sum_{j\neq i,0}(\lambda_i\lambda_j)^S \leq \delta^{-S}\sum_{j\neq i,0}(\lambda_i\lambda_j)^S$, we have that

$$\sum_{i\neq j;i,j>0}(\lambda_i\lambda_j)^S \geq \frac{1}{1+2\delta^{-S}}\sum_{i\neq j}(\lambda_i\lambda_j)^S.$$

Hence, in this case we get that

$$\tau_\lambda(E_0) \geq \frac{c}{(n+2)(1+2\delta^{-S})}\sum_{i\neq j}(\lambda_i\lambda_j)^S.$$

Combining these cases we conclude that there is a $c > 0$ such that

$$[E_0](x_1,\ldots,x_n) \geq c\sum_{i\neq j}(x_ix_j)^S$$

on Λ_{n+1}.

Let $\lambda \in \Lambda_{n+1}$ such that $\mathrm{dist}(\lambda, \mathrm{Ex}(\Lambda_{n+1})) \geq 3\epsilon(>\epsilon+2\delta)$.

If $\mathrm{dist}(\lambda, [i, j]) < \delta$ for some edge $[i, j]$, then by the previous lemma, $\tau_\lambda(f_i) < \epsilon$, $\tau_\lambda(f_j) < \epsilon$, and $\tau_\lambda(f_k) > 1 - \epsilon$ for $k \neq i, j$. Hence, for $k \neq i, j$, we have that

$$N_1 \tau_\lambda(E_0) \geq N_1 \tau_\lambda(f_k) \tau_\lambda(E_0^k) > (1-\epsilon)^2,$$

where we have used the fact that $\mathrm{dist}(\lambda, [k, \ell]) \geq \delta$ for any ℓ.

If $\mathrm{dist}(\lambda, [i, j]) \geq \delta$ for any edge $[i, j]$, then $\tau_\lambda(f_k) \leq 1 - \epsilon$ and $\tau_\lambda(E_0^k) > 1 - \epsilon$ for all k. Since $\tau_\lambda\big(1 - \bigvee_{k=0}^n (f_k E_0^k)\big) = \prod_{k=0}^n \tau_\lambda(1 - f_k E_0^k) \leq \prod_{k=0}^n (1 - \tau_\lambda(f_k)(1-\epsilon))$, we have that

$$N_1 \tau(E_0) \geq 1 - \prod_{k=0}^n (1 - \tau_\lambda(f_k)(1-\epsilon)) + \prod_{k=0}^n (1 - \tau_\lambda(f_k)) \cdot (1-\epsilon).$$

If ϵ is sufficiently small, then the right-hand side is approximately equal to $1 - \epsilon \prod_{k=0}^n (1 - \tau_\lambda(f_k))\big(1 + \sum_{k=0}^n \tau_\lambda(f_k)/(1 - \tau_\lambda(f_k))\big)$, which is bigger than $1 - (n+2)\epsilon$.

This shows that $N_1[E_0](x_1, \ldots, x_n)$ is close to 1 uniformly on the compact subset of Λ_{n+1} consisting of points distant at least 3ϵ from the vertices. $\square$

The following lemma follows from the proof of the above lemma.

LEMMA 4.10. *Let $n, m \in \mathbb{N}$ with $n \geq 2$. Suppose that the n-shift has the approximate Rohlin property. Then for any $N \in \mathbb{N}$ there are $N_1, S \in \mathbb{N}$ and $C > 0$ as in Definition 4.2 such that there is a sequence $(e_i^{(k)})$ of N_1-cycles in A_{n+m} with $E_k = \sum_{i=0}^{N_1-1} e_i^{(k)}$ and a decreasing sequence (δ_k) of positive numbers with* $\lim_k \delta_k = 0$ *satisfying*

$$\max_i \|\sigma(e_i^{(k)}) - E_{i+1}^{(k)}\| \to 0, \quad k \to \infty,$$
$$[E_k](x) \geq C(x_0 x_1 \cdots x_{n-1})^S, \quad x \in \Lambda_{n+m}^{(\delta_k)},$$
$$[E_k](x) \geq c_k \sum_{i \neq j;\, i,j<n} (x_i x_j)^S, \quad x \in \Lambda_{n+m}^{(\delta_k)}$$

for some $c_k > 0$, and $[E_k](x)$ converges to 1 *uniformly on every compact subset of $\Lambda_{n+m}^{(0)} \setminus \mathrm{Ex}(\Lambda_{n+m})$, where*

$$\Lambda_{n+m}^{(\delta)} = \left\{ x \in \Lambda_{n+m} \;\middle|\; \sum_{i=0}^{m-1} x_{n+i} < 1 - \delta \right\}$$

for $\delta \geq 0$.

Proof. Let $K, L, M \in \mathbb{N}$ be such that $M \gg L$ and let $Z_{M,K} = \{mM \mid m = 0, 1, \ldots, K-1\}$ and $Z_{M,K,L} = \bigcup\{Z_{M,K} + m \mid m = 0, 1, \ldots, L\}$. We define a (non-unital) embedding ι_0 of M_n into $\bigotimes_{\mathbb{Z}} M_{n+m}$ by $\iota_0(e_{ij}) = e_{ij}^{(0)}$, where $e_{ij}^{(0)}$ are matrix units of M_{n+m} at $0 \in \mathbb{Z}$. In particular, $f_0 = 1 - \iota_0(1) = \sum_{i=n}^{n+m-1} e_{ii}^{(0)}$. We then define a homomorphism ψ_0 into $\bigotimes_{Z_{M,K}} M_{n+m}$ by

$$\psi_0(x) = \iota_0(x) + f_0 \cdot \iota_M(x) + f_0 f_M \cdot \iota_{2M}(x) + \cdots + f_0 \cdots f_{(K-2)M} \cdot \iota_{(K-1)M}(x),$$

where $\iota_M(x) = \sigma^M \iota_0(x)$, $f_M = \sigma^M(f_0)$, etc. By setting $\psi_k = \sigma^k \psi_0$, we define a homomorphism ψ of $\bigotimes_0^{L-1} M_n$ into $\bigotimes_{Z_{M,K,L}} M_{n+m} \subset \bigotimes_{\mathbb{Z}} M_{n+m}$ by $\psi_0 \otimes \psi_1 \otimes \cdots \otimes \psi_{L-1}$.

Then we proceed just as in the proof of Lemma 4.6. In particular, we should note that for a projection $e \in B_L = \left(\bigotimes_0^{L-1} M_n\right)^\gamma$ and $\lambda \in \Lambda_{n+m}$ with $c \equiv \sum_{i=0}^{m-1} \lambda_{n+i} < 1$,

$$\tau_\lambda \psi(e) = (1 - c^K)^L \tau_\mu(e),$$

where $\mu = (1-c)^{-1}(\lambda_1, \ldots, \lambda_{n-1}) \in \Lambda_n$. □

LEMMA 4.11. *Let $n, m \in \mathbb{Z}_+$ and let $S, M \in \mathbb{N}$ with $M > S$. Let e be a non-zero projection in A_{n+m} such that the i-vanguard $V^i([e])$ for $i = 0, 1, \ldots, n-1$ is confined in the S-deep face generated by the jth vertices for $j = 0, \ldots, n-1$, i.e. if $v \in V^i([e])$ for $i = 0, 1, \ldots, n-1$, then $v_j \le S$ for $j < n$ (and $j \neq i$) and $v_j = 0$ for $j \ge n$. Let f be a projection in $I\left(\sum_{i=0}^{n-1} Me_i\right)$ such that $[e] - [f]$ is strictly positive on $\Lambda_{n+m}^{(\delta)} = \{x \in \Lambda_{n+m} \mid x_n + \cdots + x_{n+m-1} < \delta\}$ for some $\delta > 0$ except for the n vertices of Λ_{n+m-1} inside. Then for any $k = (k_0, k_1, \ldots, k_{m-1}) \in \mathbb{N}^m$ it follows that $[Q(e)] \ge [Q(f)]$, where Q is the quotient map of A_{n+m} onto*

$$B = A_{n+m} \Big/ \sum_{i=0}^{m-1} I(k_i e_{n+i}).$$

Proof. We show that there are $C_{n+i} \in \mathbb{N}$ for $i = 0, 1, \ldots, m-1$ such that

$$[e] + \sum_{i=0}^{m-1} C_{n+i}(x_0 x_1 \cdots x_{n-1})^M x_{n+i}^{k_i} \ge [f]$$

in $\mathbb{Z}[x_1, \ldots, x_{n+m-1}]$. Since the order ideal corresponding to $\sum_{i=0}^{m-1} I(k_i e_{n+i})$ is

$$\sum_{i=0}^{m-1} x_{n+i}^{k_i} \mathbb{Z}[x_1, \ldots, x_{n+m-1}],$$

this will give the result.

Since $[f] = (x_0 \cdots x_{n-1})^M r$ with some $r \in \mathbb{Z}[x_1, \ldots, x_{n+m-1}]$, let D be the maximum of r as a function on $\Lambda_{n+m} \setminus \Lambda_{n+m}^{(\delta)}$. We choose $C_i \in \mathbb{N}$ sufficiently large so that

$$q(x_1, \ldots, x_{n+m-1}) = \sum_{i=0}^{m-1} C_{n+i} x_{n+i}^{k_i} > D$$

on $\Lambda_{n+m} \setminus \Lambda_{n+m}^{(\delta)}$. Note that the i-vanguard $V^i(q - r)$ of $q - r$ with $i \ge n$ consists of one point 0. If $q_1 = (x_0 \cdots x_{n-1})^M q$, then it follows that $[e] + q_1 - [f]$ is strictly positive on Λ_{n+m-1} except for the vertices. (If it vanishes at some point, excluding the endpoints, on the edge $[i, j]$ with $i < n$, $j \ge n$, it vanishes on the whole $[i, j]$, which contradicts the assumption that it is strictly positive near the vertex $x_i = 1$.) We have to check the positivity condition on the vanguards of $[e] + q_1 - [f]$ for each $i = 0, 1, \ldots, n+m-1$.

Let $L \in \mathbb{N}$ be such that $[e]$ and $q_1 - [f]$ can be expressed in terms of x^v, $v \in V_L$:

$$[e] = \sum_{v \in V_L} b_v x^v, \quad q_1 - [f] = \sum_{v \in V_L} d_w x^w,$$

where we suppose that $b_v \ge 0$ for all $v \in V_L$.

Let $v \in V_L^i([e] + q_1 - [f])$ with $i = 0, \ldots, n-1$. If $v_j > S$ for some $j \neq i$ in $\{0, 1, \ldots, n-1\}$, then v does not belong to the i vanguard $V_L^i([e])$ of $[e]$ by the assumption. Hence, $v_j \geq M$ for all $j \neq i$ in $\{0, \ldots, n-1\}$ (otherwise $d_w = 0$ for any w in front of v). Hence, there must be a $v' \in V_L^i([e])$ such that v' is in front of v (since $v'_j \leq S < M \leq v_j$ for $j \neq i$ in $\{0, 1, \ldots, n-1\}$ and $v_j = 0$ for $j \geq n$). As $b_{v'} + d_{v'} = b_{v'} > 0$, v cannot belong to $V_L^i([e] + q_1 - [f])$. This shows that if $v \in V_L^i([e] + q_1 - [f])$, then $v_j \leq S$ for all $j \neq i$ in $\{0, \ldots, n-1\}$; so its coefficient is positive because it is non-zero and has contributions only from $[e]$; v must belong to $V_L^i([e])$.

Next let $v \in V_L^i([e] + q_1 - [f])$ with $i = n, \ldots, n+m-1$. If $v_j < M$ for some $j = 0, \ldots, n-1$, then $d_v = 0$ and hence $b_v + d_v = b_v$ must be positive. If $v_j \geq M$ for all $j = 0, 1, \ldots, n-1$, then v must be $\sum_{j=0}^{n-1} Me_j + (L - nM)e_i$, where $d_v > 0$ (which is the value taken by $q - r$ on $x_i = 1$); hence $b_v + d_v \geq d_v > 0$.

Thus, we conclude the proof that the coefficients of $[e] + q_1 - [f]$ takes positive values on the vanguards. Hence, $[e] + q_1 - [f] \geq 0$ as claimed. □

The following lemma, together with Lemmas 4.3, 4.4 and 4.6, completes the induction, thus implying the main theorem of this note. We closely follow the arguments given in the proof of Theorem 2.8 of [**19**], where we must apologize since there was some confusion in details.

LEMMA 4.12. *Let $n \in \mathbb{N}$ with $n \geq 2$. If the n-shift has the approximate Rohlin property and $(n-1)$-shift has the stable one-cocycle property, then the n-shift has the stable one-cocycle property.*

Proof. Let $m \in \mathbb{Z}_+$ and $k = (k_0, k_1, \ldots, k_{m-1}) \in \mathbb{N}^m$. We show that the shift σ on the quotient

$$B = A_{n+m} \Big/ \sum_{i=0}^{m-1} I(k_i e_{n+i})$$

has the one-cocycle property.

Let $u \in \mathcal{U}(B)$ be such that $\varphi_i(u) = 1$ for $i = 0, 1, \ldots, n-1$, where φ_i is the character on B induced from the character on A_n denoted by the same symbol through

$$A_n \cong A_{n+m} \Big/ \sum_{i=0}^{m-1} I(e_{n+i}).$$

We have to show that for any $\epsilon > 0$ there is a $v \in \mathcal{U}(B)$ such that $\|u - v\sigma(v)^*\| < \epsilon$ and also the two other versions. First we just concentrate on approximating u by $v\sigma(v)^*$.

We denote by Q the quotient map of A_{n+m} onto B. Let $M \in \mathbb{N}$. From the short exact sequence

$$0 \to Q(I(Me_0)) \to B \to B/Q(I(Me_0)) \to 0$$

and the assumption that the shift has the one-cocycle property for $B/Q(I(Me_0)) \cong A_{n+m}/\big(I(Me_0) + \sum_{i=0}^{m-1} I(k_i e_{n+i})\big)$, there is a $v \in \mathcal{U}(B)$ such that

$$\|u - v\sigma(v^*) + Q(I(Me_0))\| \approx 0,$$

where we have used that $\varphi_i(Q(u)) = 1$ for $i = 1, \ldots, n-1$. Thus, by taking a unitary in $Q(I(Me_0)) + 1$ close to $v^*u\sigma(v)$ instead of u, we may suppose that $u \in Q(I(Me_0)) + 1$.

Similarly, from the short exact sequence

$$0 \to Q(I(Me_0 + Me_1)) \to Q(I(Me_0)) \to Q(I(Me_0))/Q(I(Me_0 + Me_1)) \to 0,$$

where the quotient is isomorphic to $I(Me_0)/(I(Me_0 + Me_1) + \sum_{i=0}^{m-1} I(Me_0 + k_i e_{n+i}))$ which is an ideal of $A_{n+m}/(I(Me_1) + \sum_{i=0}^{m-1} I(k_i e_{n+i}))$, there is a unitary $v \in Q(I(Me_0)) + 1$ such that

$$\|u - v\sigma(v^*) + Q(I(Me_0 + Me_1))\| \approx 0.$$

Thus, we may assume that $u \in Q(I(Me_0 + Me_1)) + 1$. Repeating this process, we reach the following conclusion. For any unitary $u \in \mathcal{U}(B)$, $M \in \mathbb{N}$, and $\epsilon > 0$ we find a unitary $u' \in Q(J_M) + 1$ and a unitary $v \in B$ such that $\|v^* u\sigma(v) - u'\| < \epsilon$, where $J_M = I\big(\sum_{i=0}^{n-1} Me_i\big)$. We may further suppose that there is a projection $f \in Q(J_M)$ such that $\sigma(f) \approx f$ and $u' = u'f + 1 - f$.

We now have to approximate u' by $v\sigma(v)^*$ for some $v \in \mathcal{U}(B)$ by assuming that M is sufficiently large.

We proceed as follows. By mapping a Rohlin cycle for the shift on A_n into A_{n+m} (by Lemma 4.10) and then into the quotient B, we try to approximate u' by $v\sigma(v)^*$. The Rohlin cycle, so-embedded, must commute with u' and some translates of u' under σ. If the sum E of projections in the Rohlin cycle dominated f, the *support* of u', then we would be finished by using the now-standard arguments based on the Rohlin cycles. However, E is not the identity and will never dominate f, because to assure the commutativity we have to make the Rohlin cycle reside outside of where u' resides. Thus, the problem is how to deal with the left-over part $u'f(1-E)$ or $u'' = u'f(1-E) + 1 - f(1-E)$.

To approximate u'' by $v\sigma(v)^*$ we use another similar method. We construct Rohlin cycles for σ and for $\mathrm{Ad}\, u'' \circ \sigma$ such that they are of the same type and the sum of projections covers $f(1-E)$. For example, let (e_i) and (e_i') be N-cycles such that $F = \sum_{i=0}^{N-1} e_i = \sum_{i=0}^{N-1} e_i' \geq f(1-E)$, $\sigma(e_i) \approx e_{i+1}$, and $\mathrm{Ad}\, u''\sigma(e_i') \approx e_{i+1}'$; then take a partial isometry w such that $w^*w = e_0$ and $ww^* = e_0'$, and the desired unitary v can be obtained by modifying

$$\sum_{i=0}^{N-1} (u''\sigma)^i(v) + 1 - F.$$

(As a matter of fact we need two Rohlin cycles for each of σ and $\mathrm{Ad}\, u''\sigma$ to make F large enough to cover $f(1-E)$, and we apply this method to u' directly.)

We now denote by u the unitary in $Q(J_M) + 1$ for some $M \in \mathbb{N}$ which we have to approximate by a unitary of the form $v\sigma(v)^*$.

Let $\epsilon > 0$ and $\delta > 0$; we specify δ for the given ϵ later.

First we choose $N_1 \in \mathbb{N}$ such that $2\pi N_1^{-1} < \epsilon$; we construct a set of Rohlin cycles of length N_1, $N_1 + 1$, and of longer length for each of σ and $\mathrm{Ad}\, u'\sigma$ to approximate u' by a unitary of the form $v\sigma(v)^*$ within the error of order ϵ.

We then choose $N_2 \in \mathbb{N}$ such that if $U \in M_{N_2+1}$ is a unitary with eigenvalues $\exp(2\pi ik/N_2)$; $k = 0, 1, \ldots, N_2$, then there is a N_1-cycle (e_i) and a $N_1 + 1$ cycle (e_i')

in M_{N_2+1} such that $\sum_{i=0}^{N_1-1} e_i + \sum_{i=0}^{N_1} e'_i = 1$ and

$$\|\mathrm{Ad}\, U(e_i) - e_{i+1}\| < \delta, \quad \|\mathrm{Ad}\, U(e'_i) - e'_{i+1}\| < \delta.$$

By making N_2 larger if necessary, we obtain a sequence $(e_{2i}^{(k)})$ of N_2-cycles in A_{n+m} with $E_{2,k} = \sum_i e_{2,i}^{(k)}$ such that

$$\max_i \|\sigma(e_{2,i}^{(k)}) - e_{2,i+1}^{(k)}\| < \delta_k,$$
$$[E_{2,k}](x) \geq C(x_0x_1\cdots x_{n-1})^{M_2}, \quad x \in \Lambda_{n+m}^{(\delta_k)},$$
$$[E_{2,k}](x) \geq c_k \sum_{i\neq j; i,j<n} (x_ix_j)^{M_2}, \quad x \in \Lambda_{n+m}^{(\delta_k)},$$

and $[E_{2,k}](x)$ converges to 1 uniformly on every compact subset of $\Lambda_{n+m}^{(0)} \setminus \mathrm{Ex}(\Lambda_{n+m})$, where $C > 0$ is a constant independent of k as well as $M_2 \in \mathbb{N}$, $c_k > 0$ is a constant, (δ_k) is a sequence of positive numbers decreasing to 0, and $\Lambda_{n+m}^{(\delta)} = \left\{x \in \Lambda_{n+m} \mid \sum_{i=0}^{m-1} x_{n+i} < \delta\right\}$.

Let $\delta' > 0$ be sufficiently small and let $N_3 \in \mathbb{N}$ be such that $2N_3^{-1/2} < \delta'$; δ' will be chosen for δ and N_2. By taking N'-cycles for σ for $N' \geq N_2N_3$, we obtain a sequence $(e_{3i}^{(k)})$ of N_3 projections in A_{n+m} with $E_{3,k} = \sum_i e_{3,i}^{(k)}$ such that

$$\max_i \|\sigma^{N_2}(e_{3,i}^{(k)}) - e_{3,i+1}^{(k)}\| < \delta_k, \quad i = 0, \ldots, N_3 - 2,$$
$$[E_{3,k}](x) \geq C'(x_0x_1\cdots x_{n-1})^{M_3}, \quad x \in \Lambda_{n+m}^{(\delta_k)},$$
$$[E_{3,k}](x) \geq c'_k \sum_{i\neq j; i,j<n} (x_ix_j)^{M_3}, \quad x \in \Lambda_{n+m}^{(\delta_k)},$$

and $[E_{3,k}](x)$ becomes larger than $1 - 1/N_3$ uniformly on every compact subset of $\Lambda_{n+m}^{(0)} \setminus \mathrm{Ex}(\Lambda_{n+m})$, where $C' > 0$ is a constant independent of k as well as $M_3 \in \mathbb{N}$, and $c'_k > 0$ is a constant. By taking a smaller one, we set $C' = C$ and $c_k = c'_k$.

We set $M = M_2 + M_3 + 1$. We now assume that the unitary $u \in B$ belongs to $Q(J_M)+1$. We also assume that there is a projection $f \in J_M$ such that $\|\sigma(f) - f\| < N_2^{-1}\delta'$ and $u = uQ(f) + 1 - Q(f)$ and f is *local* in the sense that f belongs to $A_{n+m}(Z_1) = \left(\bigotimes_{Z_1} M_{n+m}\right)^{\gamma}$ for some finite subset Z_1. Further we may assume that $(e_{2,i}^{(k)})_i$ and $(e_{3,i}^{(k)})_i$ are local, and they as well as f reside at disjoint subsets.

Then we have that $[f(1 - E_{2,k})] = [f][1 - E_{2,k}]$, $[f] \in (x_0x_1\cdots x_{n-1})^M\mathbb{Z}[x_1, \ldots, x_{n+m-1}]$,

$$[f(1 - E_{2,k})](x) \leq D(x_0x_1\cdots x_{n-1})^M, \quad x \in \Lambda_{n+m}$$

for some $D > 0$ and $[f(1 - E_{2,k})](x)$ converges to 0 uniformly on every compact subset of $\Lambda_{n+m}^{(0)} \setminus \mathrm{Ex}(\Lambda_{n+m})$ and hence of $\Lambda_{n+m}^{(0)}$. On the other hand, we have that

$$[e_{20}^{(k)}e_{30}^{(k)}] \geq C^2(x_0x_1\cdots x_{n-1})^{M-1}, \quad x \in \Lambda_{n+m}^{(\delta_k)},$$
$$[e_{20}^{(k)}e_{30}^{(k)}] \geq c_k^2 \sum_{i\neq j; i,j<n} (x_ix_j)^{M-1}, \quad x \in \Lambda_{n+m}^{(\delta_k)},$$

and $[e_{20}^{(k)}e_{30}^{(k)}](x)$ becomes larger than $(N_2N_3)^{-1}(1 - 1/N_3)$ uniformly on every compact subset of $\Lambda_{n+m}^{(0)} \setminus \mathrm{Ex}(\Lambda_{n+m})$.

If $e_{2,i}^{(k)}$ is constructed from a projection $e_i' \in A_n$ with $[e_i'] = g \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ through the mapping ϕ into A_{n+m} (see the proof of Lemma 4.6), then $[e_{2,i}^{(k)}]$ is given by

$$(1 - X^K)^{L'}(1 + L'X) \cdot (1 - X^K)^L g(x_1(1 - X)^{-1}, \ldots, x_{n-1}(1 - X)^{-1}),$$

where L' depends only on the order of $\|\sigma\phi(e_i') - \phi\sigma(e_i')\|$ and

$$X = \sum_{i=n}^{n+m-1} x_i.$$

Thus, if $g(x) = g(x_1, \ldots, x_{n-1}) = \sum_{v \in V_L} c_v x^v$, $[e_{20}^{(k)}]$ is given as

$$(1 + Xp(X))G_L(x_1, \ldots, x_{n+m-1}),$$

where $p(X)$ is a polynomial in $X = x_n + \cdots + x_{n+m-1}$ and

$$G_L(x_1, \ldots, x_{n+m-1}) = \sum_{v \in V_L} (1 - x_1 - \cdots - x_{n+m-1})^{v_0} x_1^{v_1} \cdots x_{n-1}^{v_{n-1}}.$$

Hence, for each $i = 0, \ldots, n - 1$, the i-vanguard $V^i([e_{20}^{(k)}])$ of $[e_{20}^{(k)}]$ is the same as that of G_L. More precisely $V^i([e_{20}^{(k)}])$ is confined in the M_2-deep face generated by the ith vertices for $i = 0, \ldots, n - 1$, i.e. in the set of $v : \{0, 1, \ldots, n + m - 1\} \setminus \{i\} \to \mathbb{Z}_+$ with $v_j \le M_2$ for $j < n$ (and $j \ne i$) and $v_j = 0$ for $j \ge n$.

Since

$$\left\{ x \in \Lambda_{n+m} \;\middle|\; \sum_{i=0}^{m-1} x_{n+i} < \frac{1}{2},\, D(x_0 \cdots x_{n-1})^M < C^2(x_0 \cdots x_{n-1})^{M-1} \right\}$$

contains a small neighborhood of each vertex $x_i = 1$ (except for the vertex) for $i = 0, 1, \ldots, n - 1$, we have that for all sufficiently large k,

$$[f(1 - E_{2,k})](x) < [e_{2,0}^{(k)} e_{3,0}^{(k)}](x)$$

is strict on $\{x \in \Lambda_{n+m} \mid \sum_{i=0}^{m-1} x_{n+i} < \frac{1}{2}\}$ except for the n vertices $x_i = 1$ for $i = 0, \ldots, n - 1$. Note also the i-vanguards of $[e_{2,0}^{(k)} e_{3,0}^{(k)}]$ with $i = 0, \ldots, n - 1$ are confined in the $M_2 + M_3$ deep face generated by the ith vertices for $i = 0, \ldots, n - 1$. Hence, we can conclude by Lemma 4.11 that

$$[Q(f(1 - E_{2,k}))] \le [Q(e_{2,0}^{(k)} e_{3,0}^{(k)})]$$

for all sufficiently large k.

The rest of the arguments proceed as in the proof of Theorem 2.8 of [**19**].

From now on we will work in B so that we denote $Q(f)$, $Q(e_{2,i}^{(k)})$, etc., simply by f, $e_{2,i}^{(k)}$ etc. There should be no confusion.

We take a sufficiently large k. First we find a partial isometry onto $f(1 - E_{2,k}) \in B$ from a subprojection of $e_{2,0}^{(k)} \in B$, which is almost σ^{N_2}-invariant, by using $(e_{3,i}^{(k)})$ in B as follows.

Let b be a partial isometry in $B = A_{n+m} / \sum_{i=0}^{m-1} I(k_i e_{n+i})$ such that

$$bb^* = f(1 - E_{2,k}), \quad b^*b \le e_{2,0}^{(k)} e_{3,0}^{(k)}.$$

Note that we have assumed that f, $(e_{2,i}^{(k)})$ and $(e_{3,i}^{(k)})$ reside at disjoint subsets of $\mathbb{Z}$, i.e. there are finite subsets Z_1, Z_2, Z_3 of $\mathbb{Z}$, which are mutually disjoint, such that $f \in B(Z_1) = Q(A_{n+m}(Z_1)) = Q\big(\bigotimes_{Z_1} M_{n+m}\big)^{\gamma}$, $e_{2,i}^{(k)} \in B(Z_2)$, and $e_{3,i}^{(k)} \in B(Z_3)$. (Here we do not mean that Z_1, Z_2, Z_3 are independent of k; we pick up one k eventually.) We may suppose that there are $x, y, z \in \mathcal{U}(B)$ such that $x \in B(Z_1)$, $y \in B(Z_2)$, $z \in B(Z_3)$, $\|x-1\|$ is of order $\|\sigma(f) - f\| < N_2^{-1}\delta'$, $\|y-1\| \approx 0$, $\|z-1\| \approx 0$, $\operatorname{Ad} x\sigma(f) = f$, $\operatorname{Ad} y\sigma(e_{2,i}^{(k)}) = e_{2,i+1}^{(k)}$ and $\operatorname{Ad} z(\operatorname{Ad} y\sigma)^{N_2}(e_{3,i}^{(k)}) = \operatorname{Ad} z\sigma^{N_2}(e_{3,i}^{(k)}) = e_{3,i+1}^{(k)}$ for $i < N_3 - 1$. (If the original Z_1, Z_2, Z_3 are not only disjoint, but mutually far away by N_2, this is certainly possible.) Note that the projection $f \in Q(J_M)$ is chosen so that $u = uf + 1 - f$. We define

$$d = N_3^{-1/2} \sum_{i=0}^{N_3-1} (\operatorname{Ad} z(\operatorname{Ad}(xy)\sigma)^{N_2})^i(b).$$

Then it follows that d is indeed a partial isometry such that $dd^* = f(1 - E_{2,k})$, $d^*d \le e_{2,0}^{(k)}$ and $\|\operatorname{Ad} z(\operatorname{Ad}(xy)\sigma)^{N_2}(d) - d\| < 2N_3^{-1/2} < \delta'$. By assuming that k is sufficiently large, i.e. $\|y-1\|$ and $\|z-1\|$ are sufficiently small, and by the assumption that $\|x-1\| \lesssim N_2^{-1}\delta'$, we can assume that the unitary $w \in B$ defined through $(\operatorname{Ad}(xy)\sigma)^{N_2} = \operatorname{Ad} w\sigma^{N_2}$ satisfies $\|w - 1\| \lesssim \delta'$. Since $\operatorname{Ad} w\sigma^{N_2}(e_{2,0}^{(k)}) = e_{2,0}^{(k)}$ and $\|\operatorname{Ad} w\sigma^{N_2}(d^*d) - d^*d\| \lesssim 2\delta'$, there is a unitary $\zeta \in B$ such that $\zeta = \zeta e_{2,0}^{(k)} + 1 - e_{2,0}^{(k)}$ and $\|\zeta - 1\| \lesssim \delta'$ such that $\operatorname{Ad}\zeta \operatorname{Ad} w\sigma^{N_2}(d^*d) = d^*d$. In any case we have that $\|\sigma^{N_2}(d) - d\|$ is at most of order of δ'.

By using the partial isometry d, we define D to be the C^*-algebra D generated by $(\operatorname{Ad}(\zeta xy)\,\sigma)^k(d)$, $k = 0, 1, \ldots, N_2 - 1$. Then D is isomorphic to M_{N_2+1} and its identity is left invariant under $\operatorname{Ad}(\zeta xy)\sigma$, since $\operatorname{Ad}(\zeta xy)\sigma(dd^*) = dd^* = f(1 - E_{2,k})$ and $(\operatorname{Ad}(\zeta xy)\sigma)^{N_2}(d^*d) = d^*d$. Since $\|\zeta\sigma'(\zeta)(\sigma')^2(\zeta)\cdots(\sigma')^{N_2-1}(\zeta) - 1\|$ is of order of δ', where $\sigma' = \operatorname{Ad}(xy)\sigma$, we have that $\|(\operatorname{Ad}(\zeta xy)\sigma)^{N_2}(d) - d\|$ is of order of δ'. Note that if $(\operatorname{Ad}(\zeta xy)\sigma)^{N_2}(d) = d$ were true, then $\operatorname{Ad}(\zeta xy)\sigma$ would leave D invariant and be implemented by a unitary U with eigenvalues $\exp(2\pi k N_2^{-1})$, $k = 0, 1, \ldots, N_2$. Since $\|\zeta xy - 1\|$ is of the order of δ', it follows that $\|(\sigma - \operatorname{Ad} U)|D\| < \delta$ with some unitary U as above for a suitable choice of δ' (which can depend on δ and N_2). Then by the choice of N_2 we obtain a N_1-cycle $(f_{1,i})$ and a $(N_1 + 1)$-cycle $(f_{2,i})$ in D such that $\sum_i f_{1,i} + \sum_i f_{2,i} = 1_D$ and

$$\|\sigma(f_{1,i}) - f_{1,i+1}\| < 2\delta, \quad \|\sigma(f_{2,i}) - f_{2,i+1}\| < 2\delta.$$

Let $f_{3,i} = e_{2,i}^{(k)} - (\operatorname{Ad}(\zeta xy)\sigma)^i(d^*d)$ for $i = 0, 1, \ldots, N_2 - 1$. Then it follows that

$$\operatorname{Ad}(\zeta xy)\sigma(f_{3,i}) = f_{3,i+1};$$

hence $(f_{3,i})$ forms a N_2-cycle in B such that

$$\|\sigma(f_{3,i}) - f_{3,i+1}\| < \delta + \delta' < 2\delta.$$

Thus, we have obtained the three cycles $(f_{j,i})$, $j = 1, 2, 3$, which in particular satisfy that

$$\sum_{i=0}^{N_1-1} f_{1,i} + \sum_{i=0}^{N_1} f_{2,i} + \sum_{i=0}^{N_2-1} f_{3,i} = f \vee E_{2,k}.$$

We apply the same argument to $\operatorname{Ad} u\sigma$ instead of σ. Note that $\operatorname{Ad}(xyu)\sigma(e_{2,i}^{(k)}) = e_{2,i+1}^{(k)}$, $\operatorname{Ad}(xyu)\sigma(f(1-E_{2,k})) = f(1-E_{2,k})$ and $\operatorname{Ad} z(\operatorname{Ad}(xyu)\sigma)^{N_2}(e_{3,i}^{(k)}) = \operatorname{Ad} z\sigma^{N_2}(e_{3,i}^{(k)}) = e_{3,i+1}^{(k)}$. With the same b as above, we define a partial isometry d' onto $f(1-E_{2,k})$ from a subprojection of $e_{2,0}^{(k)}$ by

$$d' = N_3^{-1/2} \sum_{i=0}^{N_3-1} (\operatorname{Ad} z(\operatorname{Ad}(xyu)\sigma)^{N_2})^i(b),$$

which satisfies that $\|\operatorname{Ad} z(\operatorname{Ad}(xyu)\sigma)^{N_2}(d') - d'\| < 2N_3^{-1} < \delta'$, where we should note that $\|\operatorname{Ad} z(\operatorname{Ad}(xyu)\sigma)^{N_2} - \sigma^{N_2}\| \lesssim \delta'$. Let $\zeta' \in B$ be a unitary such that $\zeta' = \zeta' e_{2,0}^{(k)} + 1 - e_{2,0}^{(k)}$ and $\operatorname{Ad}\zeta'(\operatorname{Ad}(xyu)\sigma)^{N_2}((d')^*d') = (d')^*d'$ and $\|\zeta' - 1\|$ is of order of δ'. Let D' denote the C^*-algebra generated by $(\operatorname{Ad}(\zeta' xyu)\sigma)^i(d')$, $i = 0, 1, \ldots, N_2-1$. Note that $[1_D] = (N_2+1)[f(1-E_{2,k})] = [1_{D'}]$ in $K_0(B)$. Hence, by using the same formula used to define $f_{1,i}$ and $f_{2,i}$ we obtain a N_1-cycle $(f'_{1,i})$ and a (N_1+1)-cycle $(f'_{2,i})$ in D' such that $[f'_{1,i}] = [f_{1,i}]$, $[f'_{2,i}] = [f_{2,i}]$, $\sum_i f'_{1,i} + \sum_i f'_{2,i} = 1_{D'}$, and

$$\|\operatorname{Ad} u\sigma(f'_{1,i}) - f'_{1,i+1}\| < 2\delta, \quad \|\operatorname{Ad} u\sigma(f'_{2,i}) - f'_{2,i+1}\| < 2\delta.$$

Let $f'_{3,i} = e_{2,i}^{(k)} - (\operatorname{Ad}(\zeta' xyu)\sigma)^i((d')^*d')$, whence $\operatorname{Ad}(\zeta' xyu)\sigma(f'_{3,i}) = f'_{3,i+1}$. Then $(f_{3,i})$ forms a N_2-cycle such that

$$\|\sigma(f_{3,i}) - f_{3,i+1}\| < 2\delta.$$

Hence, we get the three cycles $(f'_{j,i})$, $j = 1, 2, 3$, which in particular satisfy that $[f'_{j,i}] = [f_{j,i}]$ and that the sum of all the projections $f'_{j,i}$ is $f \vee E_{2,k}$.

After having these cycles with appropriate permutation property for σ and $\operatorname{Ad} u\sigma$, we proceed exactly as in [**19**]. We choose partial isometries $b_1, b_2, b_3 \in B$ such that

$$b_j b_j^* = e'_{j,0}, \quad b_j^* b_j = e_{j,0}.$$

Then the unitary v which satisfies that $u\sigma(v) \approx v$ is obtained by modifying

$$v_1 = \sum_{i=0}^{N_1-1} (L_u\sigma)^i(b_1) + \sum_{i=0}^{N_1} (L_u\sigma)^i(b_2) + \sum_{i=0}^{N_2-1} (L_u\sigma)^i(b_3),$$

where L_u denotes the left multiplication of u. The necessary modifications are done as follows. First we choose $Y, Y' \in \mathcal{U}(B)$ such that $\|Y - 1\| \approx 0$, $\|Y' - 1\| \approx 0$, $\operatorname{Ad} Y\sigma(f_{j,i}) = f_{j,i+1}$ and $\operatorname{Ad}(Y'u)\sigma(f'_{j,i}) = f'_{j,i+1}$. Such Y (respectively Y') can be obtained by modifying ζxy (respectively $\zeta' xy$). Second we choose sequences $(z_{1i})_{i=0}^{N_1-1}$ in $\mathcal{U}(f_{1,0}Bf_{1,0})$, $(z_{2i})_{i=0}^{N_1}$ in $\mathcal{U}(f_{2,0}Bf_{2,0})$ and $(z_{3i})_{i=0}^{N_2-1}$ in $f_{3,0}Bf_{3,0}$ such that $z_{1,N_1-1} = f_{1,0}$, $z_{2,N_1} = f_{2,0}$, $z_{3,N_2-1} = f_{3,0}$,

$$\begin{aligned} z_{1,0} &= b_1^*(L_{Y'u}R_{Y^*}\sigma)^{N_1}(b_1), \\ z_{2,0} &= b_2^*(L_{Y'u}R_{Y^*}\sigma)^{N_1+1}(b_2^*), \\ z_{3,0} &= b_3^*(L_{Y'u}R_{Y^*}\sigma)^{N_2}(b_3), \end{aligned}$$

and

$$\|z_{j,i} - z_{j,i+1}\| < 2\pi N_1^{-1} < \epsilon,$$

where L_a (respectively R_a) denotes the left (respectively right) multiplication of a. We then define a unitary v by

$$v = \sum_{i=0}^{N_1-1} (L_{Y'u} R_{Y^*}\sigma)^i (b_1 z_{1i}) + \sum_{i=0}^{N_1} (L_{Y'u} R_{Y^*}\sigma)^i (b_2 z_{2i}) + \sum_{i=0}^{N_2-1} (L_{Y'u} R_{Y^*}\sigma)^i (b_3 z_{3i}) + 1 - f \vee E_{2,k}.$$

Then it follows that v is almost invariant under $L_{Y'u} R_{Y^*}\sigma$ (up to the order of $\epsilon > 2\pi/N_1$). Since $Y \approx 1$ and $Y' \approx 1$, it follows that $u\sigma(v) \approx v$ or $u \approx v\sigma(v)^*$.

We now turn to the second property of the one-cocycle property. In the above proof we should note that the unitary v essentially resides on $Z_1 \cup Z_2 \cup Z_3$ (or belong to $B(Z_1 \cup Z_2 \cup Z_3)$). We have a total control of Z_2 (where $(e^{(k)}_{2,i})$ resides) and Z_3 (where $(e^{(k)}_{3,i})$ resides); i.e. by applying σ many times we can make Z_2, Z_3 as far away as we want. Hence, we only have to control Z_1, where f resides.

This amounts to showing that for any finite subset F of $\mathbb{Z}$, there is a finite subset G of $\mathbb{Z}$ such that for any $u \in B(\mathbb{Z} \setminus G)$, there are unitaries $u' \in Q(J_M) + 1$ and $v \in B(Z_1')$ such that $\|v^* u\sigma(v) - u'\| < \epsilon$, where Z_1' must be a finite subset of $\mathbb{Z}$ disjoint from F. If we assume that $u \in B(Z_1'')$ with a finite subset Z_1'' disjoint from G, then u' essentially belong to $B(Z_1)$, where $Z_1 = Z_1' \cup Z_1''$ is disjoint from F. (We then apply the above arguments to u' instead of u.)

To get such u' we apply the second condition of the one-cocycle property for the $(n-1)$-shift a finite number of times, as discussed in the beginning of this proof. Hence, it is indeed possible.

We now turn to the third property of the one-cocycle property. In the above proof we have chosen four partial isometries b, b_1, b_2, b_3, which we now have to choose more carefully, i.e. to make them almost commute with a prescribed finite subset of B requiring some commutativity condition on the unitary u and the projection f, and the cycles $e^{(k)}_{j,i}$.

To find a partial isometry b onto $Q(1 - E_{2,k})$ from a subprojection of $Q(e^{(k)}_{2,0} e^{(k)}_{3,0})$ we have derived the condition

$$[Q(f(1 - E_{2,k}))] \le [Q(e^{(k)}_{2,0} e^{(k)}_{3,0})].$$

Since we have to impose the condition that b should commute with a prescribed finite-dimensional C^*-subalgebra B_1 of B, we have to assume that u, f, $e^{(k)}_{2,i}$, and $e^{(k)}_{3,i}$ almost commute with B_1 and moreover that

$$[Q(f(1 - E_{2,k}))P] \le [Q(e^{(k)}_{2,0} e^{(k)}_{3,0})P]$$

for all minimal central projections P in B_1.

The first part of the condition is easy. The near commutativity of $e^{(k)}_{2,i}$ and $e^{(k)}_{3,i}$ with B_1 can be assumed just by shifting them by using σ. The near commutativity of f can also be assumed by making f large (in the ideal retaining the condition $\sigma(f) \approx f$). The near commutativity of u follows from the induction assumption, where we use the stable one-cocycle property for the $(n-1)$-shift a finite number of times. Hence, we concentrate on the second part.

By lifting B_1 to a finite-dimensional C^*-subalgebra of A_{n+m}, we regard B_1 as a C^*-subalgebra of $A_{n+m}(Z_0)$ for some finite subset Z_0 of $\mathbb{Z}$. We can assume that $u, f \in B_1'$ and $Z_2 \cup Z_3$ is disjoint from Z_0. From the latter it follows that

$$[e_{2,0}^{(k)} e_{3,0}^{(k)} p] = [e_{2,0}^{(k)} e_{3,0}^{(k)}][p] \geq C^2 (x_0 \cdots x_{n-1})^{M_2+M_3} [p]$$

for a minimal central projection p in B_1.

Let $M_0 = |Z_0|$ and let p be a minimal central projection in B_1. Let $p = \sum_{v \in V'} p_v$, where p_v is a non-zero projection such that $[p_v] = c_v x^v$ with $c_v \in \mathbb{N}$ and V' is a subset of V_{M_0}. Suppose that $f \in J_M$ for some $M \in \mathbb{N}$ with $M > M_0$. Since $f p_v$ belongs to the ideal $J_M \cap I(v) = I\big(\sum_{i=0}^{n-1} M e_i + \sum_{i=n}^{n+m-1} v_i e_i\big)$, one can estimate that

$$[f(1 - E_{2,k})p](x) \leq [fp](x) \leq (x_0 x_1 \cdots x_{n-1})^M \sum_{v \in V'} D_v x^v[n]$$

on Λ_{n+m} for some $D_v \geq 0$, where $x^v[n] = x_n^{v_n} \cdots x_{n+m-1}^{v_{n+m-1}}$.

Set $M = M_0 + M_2 + M_3 + 1$. Since

$$[e_{2,0}^{(k)} e_{3,0}^{(k)} p](x) \geq C^2 (x_0 \cdots x_{n-1})^{M_0+M_2+M_3} \sum_{v \in V'} c_v x^v[n]$$

on Λ_{n+m}, it follows that

$$\{x \in \Lambda_{n+m} \mid [f(1 - E_{2,k})p](x) < [e_{2,0}^{(k)} e_{3,0}^{(k)} p](x)\}$$

includes a neighborhood of the vertex $x_i = 1$, excluding the vertex itself, for $i = 0, \ldots, n-1$. Hence, for a sufficiently large k we have that for all minimal central projections p in B_1,

$$[f(1 - E_{2,k})p](x) < [e_{2,0}^{(k)} e_{3,0}^{(k)} p](x)$$

on $\Lambda_{n+m}^{(1/2)}$.

As in the proof of Lemma 4.11 we choose $C_{n+i} \in \mathbb{N}$ such that

$$\sum_{i=0}^{m-1} C_{n+i} x_{n+i}^{k_{n+i}} - (x_0 \cdots x_{n-1})^{-M} [f(1 - E_{2,k})p]$$

is strictly positive on $\Lambda_{n+m} \setminus \Lambda_{n+m}^{(1/2)}$. Hence,

$$[e_{2,0}^{(k)} e_{3,0}^{(k)} p] + q - [f(1 - E_{2,k})p]$$

is strictly positive on Λ_{n+m} except for the vertices $x_0 = 1, \ldots, x_{n-1} = 1$, where $q = (x_0 \cdots x_{n-1})^M \sum_{i=0}^{m-1} C_{n+i} x_{n+i}^{k_{n+i}}$. We express each term as linear combinations of x^v, $v \in V_L$ for a sufficiently large $L \in \mathbb{N}$. We have to show that the coefficient is positive on each point of the i-vanguard V_L^i of $[e_{2,0}^{(k)} e_{3,0}^{(k)} p] + q - [f(1 - E_{2,k})p]$ for $i = 0, \ldots, n-1$.

Let $v \in V_L^i$. If $v_j < M$ for some $j \neq i$ in $\{0, 1, \ldots, n-1\}$, the contribution from $[f(1 - E_{2,k})p]$ must be zero; hence the coefficient is positive.

Suppose that $v_j \geq M$ for all $j \neq i$ in $\{0, \ldots, n-1\}$. If $v_\ell \geq k_\ell$ for some $\ell \geq n$, then $w = \sum_{j \neq i; j < n} M e_j + (L - (n-1)M - k_\ell) e_i + k_\ell e_\ell$ is in front of v and the contribution to the coefficient of x^w from $q - [f(1 - E_{2,k})p]$, and hence from $[e_{2,0}^{(k)} e_{3,0}^{(k)} p] + q - [f(1 - E_{2,k})p]$, must be positive, which entails that $w = v$ with positive coefficient of x^v. So we now have to consider the case $v_\ell < k_\ell$ for all $\ell \geq n$. In this case the contribution to the coefficient of x^v from q is zero. If there is no $w \in V'$ such that $D_w \neq 0$ and $w_\ell \leq v_\ell$ for all $\ell \geq n$, then the contribution from $[f(1 - E_{2,k})p]$ is also zero; thus the coefficient of x^v must be positive because it is only contributed from $[e_{2,0}^{(k)} e_{3,0}^{(k)} p]$. Hence, we are left with the case where there must be $w \in V'$ such that $D_w \neq 0$ and $w_\ell \leq v_\ell$ for all $\ell \geq n$. Then there must be $v' \in V_L$ such that $v'_j < M$ for $j \neq i$ in $\{0, 1, \ldots, n-1\}$ and $v'_\ell = w_\ell$ for all $\ell \geq n$ such that the contribution to the coefficient of $x^{v'}$ from $[e_{2,0}^{(k)} e_{3,0}^{(k)} p]$ is non-zero. Since the contribution to it from $[f(1 - E_{2,k})p]$ must be zero, this implies that v cannot be in V_L^i. Thus, we have shown that the coefficients are all positive on the vanguards V_L^i.

Hence, if C_{n+i} are sufficiently large as well as k, we can conclude that $[e_{2,0}^{(k)} e_{3,0}^{(k)} p] + q - [f(1 - E_{2,k})p]$ is positive for all minimal central projections $p \in B_1$. Thus, it follows that

$$[Q(e_{2,0}^{(k)} e_{3,0}^{(k)} p)] - [Q(f(1 - E_{2,k})p)]$$

is positive on the quotient B for all minimal central projection $p \in B_1$. Hence, we can choose a partial isometry b in the commutant of B_1 such that

$$bb^* = Q(f(1 - E_{2,k})), \quad b^*b \leq Q(e_{2,0}^{(k)} e_{3,0}^{(k)}).$$

We then define partial isometries d, d' onto $Q(f(1 - E_{2,k}))$ and construct C^*-subalgebras D, D' of B (which are isomorphic to M_{N_2+1}) as before.

We now work in the quotient $B = Q(A_{n+m})$; so we omit the symbol Q. Define

$$V = \sum_{i=0}^{N_2-1} (\mathrm{Ad}(\zeta' xyu)\sigma)^i((d')^*)(\mathrm{Ad}(\zeta xy)\sigma)^i(d) + f(1 - E_{2,k}).$$

Then V is a partial isometry, which we can assume almost commute with a prescribed finite subset of B by imposing such a condition on u, d, d'. Note that the map $\mathrm{Ad}\, V$: $x \mapsto VxV^*$ defines an isomorphism of D onto D_1 and satisfies that $\mathrm{Ad}(\zeta' xyu)\sigma \circ \mathrm{Ad}\, V \approx \mathrm{Ad}\, V \circ \mathrm{Ad}(\zeta xy)\sigma$ on D, where $\mathrm{Ad}(\zeta xy)\sigma$ (respectively $\mathrm{Ad}(\zeta' xyu)\sigma$) leaves D (respectively D') almost invariant. Thus, if we choose $f_{j,i}$, $j = 1, 2$, then we may define $f'_{j,i} = Vf_{j,i}V^*$, which implies that we may set $b_1 = Vf_{1,0}$ and $b_2 = Vf_{2,0}$, which almost commute with a prescribed finite subset.

Since $V^*V = 1_D$, the N_2-cycle $(f_{3,i})$ is defined by

$$f_{3,i} = e_{2,i}^{(k)}(1 - V^*V)$$

and similarly

$$f'_{3,i} = e_{2,i}^{(k)}(1 - VV^*).$$

Since we can assume that V as well as $e_{2,i}^{(k)}$ almost commutes with elements of a prescribed finite-dimensional C^*-subalgebra B_1 of B, we have that $[f_{3,i}p] = [f'_{3,i}p]$ for all minimal central projection p in B_1. Thus, we can find a partial isometry b_3 which almost commutes with B_1 and $b_3^*b_3 = f_{3,0}$ and $b_3b_3^* = f'_{3,0}$.

Having chosen b, b_1, b_2, and b_3 as above, the third property also follows as the first property. $\square$

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