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Singularities of ruled surfaces in $\mathbb{R}^3$

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Abstract

We study singularities of ruled surfaces in $\mathbb{R}^3$. The main result asserts that only cross-caps appear as singularities for generic ruled surfaces.

1. *Introduction*

The study of ruled surfaces in $\mathbb{R}^3$ is a classical subject in differential geometry. It has again been studied in some areas (i.e. Projective differential geometry [**16**], Computer-aided design [**7**, **18**], etc.). Generally, ruled surfaces have singularities. Recently there have appeared several articles concerning singularities of developable surfaces in $\mathbb{R}^3$ (cf. [**3**, **8**–**12**, **14**, **15**, **17**]). Developable surfaces are ruled surfaces which have vanishing Gauss curvature on the regular part. Another characterization is that developable surfaces are envelopes of one-parameter families of planes in $\mathbb{R}^3$, so that they have singularities of discriminants of such families. In these articles classifications of singularities of developable surfaces are given. Briefly speaking, the cuspidal edge, the cuspidal cross-cap or the swallowtail appear as singularities of developable surfaces in general (cf. Fig. 1).

On the other hand, the Gaussian curvature of the regular part of a ruled surface

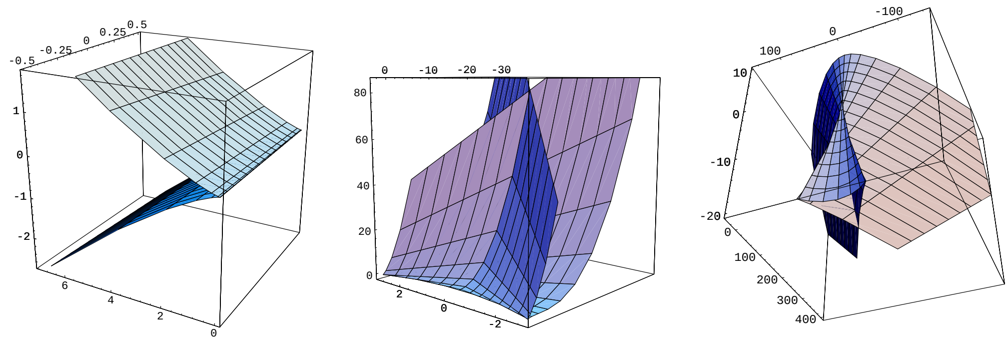

Fig. 1

is generally nonpositive. So the developable surface is a member of the special class of ruled surfaces. Therefore we have the natural question:

How are singularities of developable surfaces different from those of 'general' ruled surfaces?

In this paper we give a classification of singularities of general ruled surface. *A ruled surface* in $\mathbb{R}^3$ is (locally) the image of the map $F_{(\gamma,\delta)}\colon I \times J \to \mathbb{R}^3$ defined by $F_{(\gamma,\delta)}(t,u) = \gamma(t) + u\delta(t)$, where $\gamma\colon I \to \mathbb{R}^3$, $\delta\colon I \to S^2$ are smooth mappings and I, J are open intervals. We assume that I is bounded. We call γ *a base curve* and δ *a director curve*. The straightlines $u \mapsto \gamma(t) + u\delta(t)$ are called *rulings*.

In order to describe the main result in this paper we need some preparation. Let $f_i\colon (N_i, x_i) \to (P_i, y_i)$ $(i = 1, 2)$ be C^∞ map germs. We say that f_1, f_2 are $\mathscr{A}$-equivalent if there exist diffeomorphism germs $\phi\colon (N_1, x_1) \to (N_2, x_2)$ and $\psi\colon (P_1, y_1) \to (P_2, y_2)$ such that $\psi \circ f_1 = f_2 \circ \phi$. Let $C^\infty_{pr}(I, \mathbb{R}^3 \times S^2)$ be the space of smooth proper mappings $(\gamma, \delta)\colon I \to \mathbb{R}^3 \times S^2$ equipped with Whitney C^∞-topology, where I is an open interval. The following theorem is the main result in this paper which gives a 'generic' answer to the above question.

Theorem 1·1. *There exists an open dense subset $\mathscr{O} \subset C^\infty_{pr}(I, \mathbb{R}^3 \times S^2)$ such that the germ of the ruled surface $F_{(\gamma,\delta)}$ at any point (t_0, u_0) is an immersion germ or $\mathscr{A}$-equivalent to the cross-cap for any $(\gamma, \delta) \in \mathscr{O}$.*

Here, the cross-cap is the map germ defined by $(x_1, x_2) \mapsto (x_1^2, x_2, x_1x_2)$.

It is well known that any singular point for generic smooth mappings from a surface to $\mathbb{R}^3$ is the cross-cap (cf. [**1**, **5**, **13**, **19**]). The set of ruled surfaces is a very small subset in the space of all C^∞-mappings. The above theorem, however, asserts that the generic singularities of ruled surfaces are the same as those of C^∞-mappings. We remark that the cross-cap is realized as a singularity of a ruled surface as follows: consider curves $\gamma(t) = (t^2, 0, 0)$ and $\delta(t) = (0, 1/\sqrt{1+t^2}, t/\sqrt{1+t^2})$, then $F_{(\gamma,\delta)}(t,u)$ is the cross-cap (cf. Fig. 2 below) which corresponds to the normal form.

We can summarize the results of the above theorem as the following relations by referring to the previous results [**3**, **10**–**12**, **15**]:

$$\{\text{Singularities of generic developable surfaces}\} \neq \{\text{Singularities of generic ruled surfaces}\},$$

$$\{\text{Singularities of generic ruled surfaces}\} = \{\text{Singularities of generic } C^\infty\text{-mappings}\}.$$

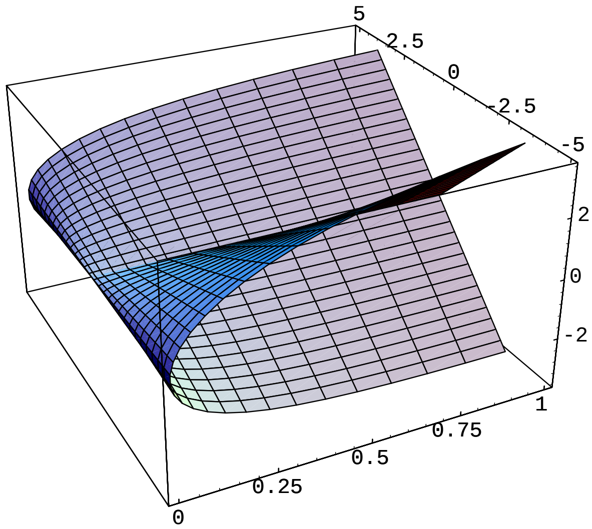

Fig. 2

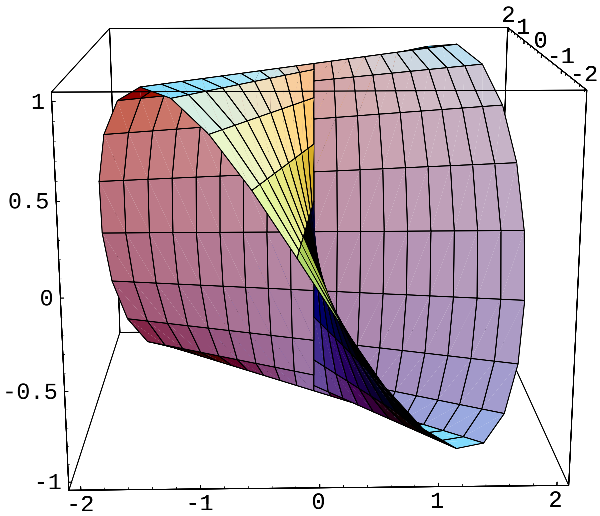

Fig. 3

One of the examples of ruled surfaces with cross-caps is *the Plücker conoid* which is given by $\gamma(\theta) = (0, 0, 2\cos\theta\sin\theta)$ and $\delta(\theta) = (\cos\theta, \sin\theta, 0)$ $(0 \leqslant \theta \leqslant 2\pi)$ (cf. Fig. 3).

We can also see a beautiful picture of the ruled surface at the home page of Banchoff [**2**].

In Section 2 we briefly review the classical theory of ruled surfaces. The idea of the proof of Theorem 1·1 is that we may locally regard the ruled surface as a one-dimensional unfolding of a map germ and apply the theory of unfoldings. In this case the parameter along rulings is considered to be the unfolding parameter. In Section 3 we prepare the general theory of unfoldings. The proof of Theorem 1·1 is given in Section 4.

This is the first paper of the authors' joint project entitled 'Geometry of ruled surfaces and line congruences'.

All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2. *Basic notions and a review of the classical theory*

We now present basic concepts and properties of ruled surfaces in $\mathbb{R}^3$. The classical theory has been given in [**6**]. However, ruled surfaces are not so popular now, so that we review the classical framework. For the ruled surface $F_{(\gamma,\delta)}$, if δ is a constant vector v, then the ruled surface $F_{(\gamma,v)}$ is a generalized cylinder. Therefore, the ruled surface $F_{(\gamma,\delta)}$ is said to be *noncylindrical* provided δ' never vanishes. Thus the rulings are always changing directions on a noncylindrical ruled surface. It is clear that the set $\mathcal{O}^1$ consisting of noncylindrical ruled surfaces is an open and dense subset in $C^\infty_{pr}(I, \mathbb{R}^3 \times S^2)$. Then we have the following lemma (cf. [**6**, lemmas **17·7**, **17·8**]).

Lemma 2·1. (1) *Let $F_{(\gamma,\delta)}(t,u)$ be a noncylindrical ruled surface. Then there exists a smooth curve $\sigma\colon I \to \mathbb{R}^3$ such that* Image $F_{(\gamma,\delta)}$ = Image $F_{(\sigma,\delta)}$ *and $\langle \sigma'(t), \delta'(t)\rangle = 0$, where $\langle , \rangle$ denotes the canonical inner product on $\mathbb{R}^3$. The curve $\sigma(t)$ is called the striction curve of $F_{(\gamma,\delta)}(t,u)$.*

(2) *The striction curve of a noncylindrical ruled surface $F_{(\gamma,\delta)}(t,u)$ does not depend on the choice of the base curve γ.*

We can specify the place where the singularities of the ruled surface are located.

Lemma 2·2. *Let $F_{(\sigma,\delta)}$ be a ruled surface with the striction curve σ. If $x_0 = F_{(\sigma,\delta)}(t_0,u_0)$ is a singular point of the ruled surface $F_{(\sigma,\delta)}$ then $u_0 = 0$ (i.e. $x_0 \in$* Image σ*). Moreover, if $\sigma'(t_0) \neq 0$, then the ruling through $\sigma(t_0)$ is tangent to σ at t_0.*

Proof. We can calculate the partial derivative of $F_{(\sigma,\delta)}$ as follows:

$$\frac{\partial F_{(\sigma,\delta)}}{\partial t}(t,u) = \sigma'(t) + u\delta'(t), \qquad \frac{\partial F_{(\sigma,\delta)}}{\partial u}(t,u) = \delta(t).$$

Therefore we have

$$\frac{\partial F_{(\sigma,\delta)}}{\partial t} \times \frac{\partial F_{(\sigma,\delta)}}{\partial u}(t,u) = \sigma'(t) \times \delta(t) + u\delta'(t) \times \delta(t),$$

where $\times$ denotes the vector product in $\mathbb{R}^3$.

Since $\|\delta(t)\| = \sqrt{\langle \delta(t), \delta(t)\rangle} \equiv 1$, we have $\langle \delta'(t), \delta(t)\rangle \equiv 0$. By the condition that $\langle \sigma'(t), \delta'(t)\rangle \equiv 0$ and the above, there exists a smooth function $\lambda(t)$ such that $\sigma'(t) \times \delta(t) = \lambda(t)\delta'(t)$. So we have

$$\begin{aligned}\left\| \frac{\partial F_{(\sigma,\delta)}}{\partial t} \times \frac{\partial F_{(\sigma,\delta)}}{\partial u}(t,u)\right\|^2 &= \|\lambda(t)\delta'(t) + u\delta'(t) \times \delta(t)\|^2 \\ &= \lambda(t)^2\|\delta'(t)\|^2 + 2\lambda(t)u\langle \delta'(t), \delta'(t) \times \delta(t)\rangle + u^2\|\delta'(t) \times \delta(t)\|^2 \\ &= (\lambda(t)^2 + u^2)\|\delta'(t)\|^2.\end{aligned}$$

Suppose that $x_0 = F_{(\sigma,\delta)}(t_0,u_0)$ is a singular point of the ruled surface $F_{(\sigma,\delta)}$, then

$$\left\| \frac{\partial F_{(\sigma,\delta)}}{\partial t} \times \frac{\partial F_{(\sigma,\delta)}}{\partial u}(t_0,u_0)\right\| = 0.$$

Since $F_{(\sigma,\delta)}$ is noncylindrical, this means that $u_0 = \lambda(t_0) = 0$.

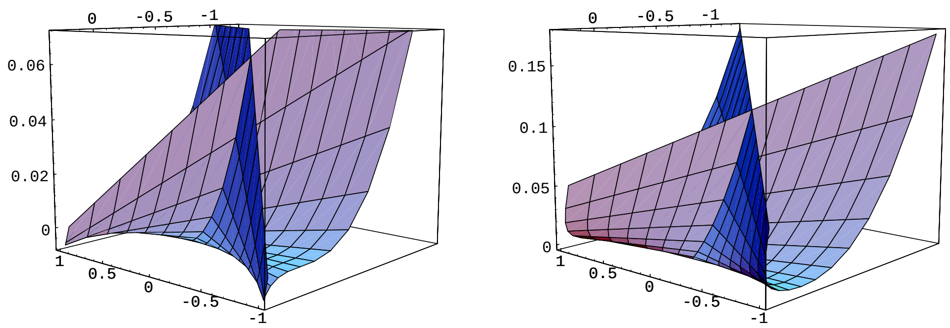

Fig. 4

By Lemma 2·2, the singularities of a ruled surface are located on the striction curve. If we consider the cross-cap $F_{(\gamma,\delta)}(t,u) = (t^2, u/\sqrt{1+t^2}, ut/\sqrt{1+t^2})$, then $\gamma'(t) = (2t,0,0)$ and $\delta'(t) = (0, -t/\sqrt{(1+t^2)^3}, 1/\sqrt{(1+t^2)^3})$. By definition, $\gamma(t)$ is the striction curve of $F_{(\gamma,\delta)}(t,u)$ and the singular point is $(0,0,0)$.

We also consider the following examples.

Example 2·3. Let $\gamma\colon I \to \mathbb{R}^3$ and $\delta\colon I \to S^2$ be curves given by $\gamma(t) = (0,0,f(t))$ and $\delta(t) = (\cos t, \sin t, 0)$. Then the ruled surface $F_{(\gamma,\delta)}(t,u) = (u\cos t, u\sin t, f(t))$ is called *a positive conoid*. We can easily calculate that singularities of $F_{(\gamma,\delta)}(t,u)$ is given by $u = 0,\ f'(t) = 0$ and the striction curve is $\gamma(t)$.

On the other hand, let $g\colon (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ be a smooth map germ. It has been known that the origin is the cross-cap if and only if there exists a local chart (x_1,x_2) around the origin such that the following conditions hold:

$$\frac{\partial g}{\partial x_1}(0) \neq 0,\ \frac{\partial g}{\partial x_2}(0) = 0 \quad \text{and} \quad \det\left(\frac{\partial g}{\partial x_1}(0), \frac{\partial^2 g}{\partial x_1 \partial x_2}(0), \frac{\partial^2 g}{\partial x_2^2}(0)\right) \neq 0.$$

By a direct calculation, $F_{(\gamma,\delta)}(t_0,0)$ is the cross-cap if and only if $f'(t_0) = 0$ and $f''(t_0) \neq 0$. The above condition means that t_0 is a Morse singular point of $f(t)$. Moreover, it is well-known that Morse functions are generic in the space of smooth functions. Therefore, this example confirms the assertion of the main theorem. One of the examples of positive conoids with cross-caps is the Plücker conoid which has been given in Section 1 (cf. Fig. 3).

Example 2·4. Consider the developable surface

$$F_{(\gamma,\delta)}(t,u) = (u, -2t^2 - \tfrac{3}{2}tu, t^4 + \tfrac{1}{2}t^3u)$$

with a cuspidal cross-cap. If we slightly perturb it into the ruled surface

$$F^{\varepsilon}_{(\gamma,\delta)}(t,u) = (u, -2t^2 - \tfrac{3}{2}tu, t^4 + \varepsilon t^2 + \tfrac{1}{2}t^3u),$$

we can easily show that the origin is a cross-cap. The situation is depicted in Fig. 4. The left picture is $F_{(\gamma,\delta)}(t,u)$ and the right one is $F^{0.2}_{(\gamma,\delta)}(t,u)$.

3. *Unfoldings*

For the proof of Theorem 1·1, we need to prepare and review the theory of one-dimensional unfoldings of map germs. The definition of *r-dimensional unfolding of*

$f_0\colon (\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ (originally due to Thom) is a germ $F\colon (\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R}^p\times\mathbb{R}^r,0)$ given by $F(x,u)=(f(x,u),u)$, where $f(x,u)$ is a germ of r dimensional parameterized families of germs with $f(x,0)=f_0(x)$. This definition depends on the coordinates of both spaces $(\mathbb{R}^n\times\mathbb{R}^r,0)$ and $(\mathbb{R}^p\times\mathbb{R}^r,0)$. We need the coordinate free definition of unfoldings [4]. Let $f\colon (N,x_0)\to(P,y_0)$ be a map-germ between manifolds. An *unfolding* of f is a triple (F,i,j) of map germs, where $i\colon (N,x_0)\to(N',x_0')$, $j\colon (P,y_0)\to(P',y_0')$ are immersions and j is transverse to F, such that $F\circ i=j\circ f$ and $(i,f)\colon N\to\{(x',y)\in N'\times P \mid F(x')=j(y)\}$ is a diffeomorphism germ. The *dimension* of (F,i,j) as an unfolding is $\dim N'-\dim N$. We can easily prove that the above two definitions are equivalent.

Lemma 3·1. *Let $F\colon (\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a map germ with the components of the form $F(t,u)=(F_1(t,u),F_2(t,u),F_3(t,u))$. Suppose that $(\partial F_3/\partial u)(0,0)\neq 0$. By the implicit function theorem, there exists a function germ $g\colon (\mathbb{R},0)\to(\mathbb{R},0)$ with $F_3^{-1}(0)=\{(t,g(t))\mid t\in(\mathbb{R},0)\}$. Let us consider immersion germs $i\colon (\mathbb{R},0)\to(\mathbb{R}^2,0)$ given by $i(t)=(t,g(t))$, $j\colon (\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ given by $j(y_1,y_2)=(y_1,y_2,0)$ and a map germ $f\colon (\mathbb{R},0)\to(\mathbb{R}^2,0)$ given by $f(t)=(F_1(t,g(t)),F_2(t,g(t)))$. Then the triple (F,i,j) is a one-dimensional unfolding of f.*

Proof. It is clear that $F\circ i=j\circ f$. Since $(\partial F_3/\partial u)(0,0)\neq 0$, F is transverse to j. We can easily show that

$$\{(t,u,y_1,y_2)\mid F(t,u)=j(y_1,y_2)\}=\{(t,g(t),F_1(t,g(t)),F_2(t,g(t))\mid t\in(\mathbb{R},0)\}.$$

Since $(i,f)\colon (\mathbb{R},0)\to(\mathbb{R}^2\times\mathbb{R}^2,0)$ is given by $(i,f)(t)=(t,g(t),F_1(t,g(t)),F_2(t,g(t)))$, it maps diffeomorphically onto the above set. This completes the proof.

Since the cross-cap is a stable singularity of map germs $(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$, we now discuss the stability of unfoldings. Let $\mathscr{E}_n$ be the *local ring* of function germs $(\mathbb{R}^n,0)\to\mathbb{R}$ and the unique maximal ideal is denoted by $\mathscr{M}_n$. For a map germ $f\colon (\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, we say that f is *infinitesimally $\mathscr{A}$-stable* if the following equality holds:

$$\mathscr{E}(n,p)=\left\langle \frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right\rangle_{\mathscr{E}_n}+f^*\mathscr{E}(p,p),$$

where $\mathscr{E}(n,p)$ denotes the $\mathscr{E}_n$-module of map germs $(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ and $f^*\colon \mathscr{E}(p,p)\to\mathscr{E}(n,p)$ is the *pull back map* given by $f^*(h)=h\circ f$. It is known that an infinitesimally $\mathscr{A}$-stable map germ $(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ is an immersion germ or the cross-cap [**1**, **5**, **13**, **19**].

For map germs $f,g\colon (\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, we say that they are $\mathscr{K}$-equivalent if there exists a diffeomorphism germ $\phi\colon (\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ such that $f^*(\mathscr{M}_p)\mathscr{E}_n=\phi^*\circ g^*(\mathscr{M}_p)\mathscr{E}_n$. The $\mathscr{K}$-equivalence is a equivalence relation among map germs. Let $J^k(n,p)$ be the *k-jet space* of map germ $(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$. For any $z=j^kf(0)\in J^k(n,p)$, we denote that

$$\mathscr{K}^k(z)=\{j^kg(0)\mid g \text{ is } \mathscr{K}\text{-equivalent to } f\}.$$

We call it *a $\mathscr{K}^k$-orbit* since it is the orbit of a certain Lie group action. For any map germ $f\colon (\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R}^p,0)$, we define a map germ $j_1^kf\colon (\mathbb{R}^n\times\mathbb{R}^r,0)\to J^k(n,p)$ by $j_1^kf(x_0,u_0)=j^kf_{u_0}(x_0)$, where $f_u(x)=f(x,u)$ and $j^kf_{u_0}(x_0)=j^k(f_{u_0}(x+x_0))(0)$. We have the following Lemma (cf. [**13**]).

LEMMA 3·2. *Under the same notation as the above,* $j_1^k f$ *is transverse to* $\mathscr{K}^k(j_1^k f_0(0))$ *if and only if*

$$\mathscr{E}(n,p)=\left\langle \frac{\partial f_0}{\partial x_1},\ldots,\frac{\partial f_0}{\partial x_n}\right\rangle_{\mathscr{E}_n}+f_0^*(\mathscr{M}_p)\mathscr{E}(n,p)+\left\langle \frac{\partial f}{\partial u_1}(x,0),\ldots,\frac{\partial f}{\partial u_r}(x,0),e_1,\ldots,e_p\right\rangle_{\mathbb{R}},$$

where $e_i=(0,\ldots,0,\overset{i}{1},0,\ldots,0)\in\mathbb{R}^p$.

The following lemma is folklore. However, we cannot find any context on where the proof is explicitly written. So we give the proof here.

LEMMA 3·3. *Let* $F\colon(\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R}^p\times\mathbb{R}^r,0)$ *be an unfolding of* f_0 *of the form* $F(x,u)=(f(x,u),u)$. *If* $j_1^k f$ *is transverse to* $\mathscr{K}^k(j^k f_0(0))$ *for sufficiently large* k, *then* F *is infinitesimally* $\mathscr{A}$*-stable.*

Proof. By Lemma 3·2 we may assume that

$$\mathscr{E}(n,p)=\left\langle \frac{\partial f_0}{\partial x_1},\ldots,\frac{\partial f_0}{\partial x_n}\right\rangle_{\mathscr{E}_n}+f_0^*(\mathscr{M}_p)\mathscr{E}(n,p)+\left\langle \frac{\partial f}{\partial u_1}(x,0),\ldots,\frac{\partial f}{\partial u_r}(x,0),e_1,\ldots,e_p\right\rangle_{\mathbb{R}}.$$

We can show that

$$\begin{aligned}\mathscr{E}(n+r,p)=&\left\langle \frac{\partial f_0}{\partial x_1},\ldots,\frac{\partial f_0}{\partial x_n}\right\rangle_{\mathscr{E}_{n+r}}+f_0^*(\mathscr{M}_p)\mathscr{E}(n+r,p)\\&+\left\langle \frac{\partial f}{\partial u_1}(x,0),\ldots,\frac{\partial f}{\partial u_r}(x,0),e_1,\ldots,e_p\right\rangle_{\mathscr{E}_r}+\mathscr{M}_r\mathscr{E}(n+r,p).\end{aligned}$$

We now apply the Malgrange preparation theorem (cf. [**13**, p. 141]) as follows: consider $M=\mathscr{E}(n+r,p)$ as an $\mathscr{E}_{n+r}$-module of finite type. Then we have the quotient $\mathscr{E}_r$-module $M_0=\mathscr{E}(n+r,p)/\mathscr{M}_r\mathscr{E}(n+r,p)$. We also consider an $\mathscr{E}_{n+r}$-submodule

$$N=\left\langle \frac{\partial f_0}{\partial x_1},\ldots,\frac{\partial f_0}{\partial x_n}\right\rangle_{\mathscr{E}_{n+r}}+f_0^*(\mathscr{M}_p)\mathscr{E}(n+r,p)$$

of M. By the previous equality and the Malgrange preparation theorem, we have

$$M=N+\left\langle \frac{\partial f}{\partial u_1}(x,0),\ldots,\frac{\partial f}{\partial u_r}(x,0),e_1,\ldots,e_p\right\rangle_{\mathscr{E}_r}.$$

This means that

$$\begin{aligned}\mathscr{E}(n+r,p)=&\left\langle \frac{\partial f_0}{\partial x_1},\ldots,\frac{\partial f_0}{\partial x_n}\right\rangle_{\mathscr{E}_{n+r}}+f_0^*(\mathscr{M}_p)\mathscr{E}(n+r,p)\\&+\left\langle \frac{\partial f}{\partial u_1}(x,0),\ldots,\frac{\partial f}{\partial u_r}(x,0),e_1,\ldots,e_p\right\rangle_{\mathscr{E}_r}.\end{aligned}$$

For any $\xi=(\xi_1,\xi_2)\in\mathscr{E}(n+r,p+r)=\mathscr{E}(n+r,p)\times\mathscr{E}(n+r,r)$, there exist $\lambda_i,\eta_i\in\mathscr{E}_{n+r}$and $\mu_i,\zeta_i\in\mathscr{E}_r$ such that

$$\xi_1=\sum_{i=1}^n\lambda_i\frac{\partial f_0}{\partial x_i}+\sum_{i=1}^p\eta_i f_{0,i}+\sum_{i=1}^r\mu_i\frac{\partial f_0}{\partial u_i}+\sum_{i=1}^p\zeta_i e_i.$$

Therefore, we have

$$(\xi_1, 0) = \sum_{i=1}^{n} \lambda_i \frac{\partial F}{\partial x_i} + \sum_{i=1}^{p} \eta_i (f_i, 0) + \sum_{i=1}^{r} \mu_i \left(\frac{\partial f}{\partial u_i}, 0 \right) + \sum_{i=1}^{p} \zeta_i (e_i, 0)$$
$$= \sum_{i=1}^{n} \lambda_i \frac{\partial F}{\partial x_i} + \sum_{i=1}^{r} \mu_i \frac{\partial F}{\partial u_i} + \sum_{i=1}^{p} \eta_i (f_i, 0) + \sum_{i=1}^{p} (\zeta_i - \mu_i)(e_i, 0).$$

Since $\zeta_i - \mu_i \in \mathscr{E}_r$, $\sum_{i=1}^{p} (\zeta_i - \mu_i)(e_i, 0) \in F^*\mathscr{E}(p+r, p+r)$. This means that

$$(\xi_1, 0) \in \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_r} \right\rangle_{\mathscr{E}_{n+r}} + F^*\mathscr{E}(p+r, p+r) + F^*(\mathscr{M}_{p+r})\mathscr{E}(n+r, p+r).$$

On the other hand, we have

$$(0, \xi_2) = \sum_{i=1}^{r} \xi_{2,i} \frac{\partial F}{\partial u_i} - \sum_{i=1}^{r} \xi_{2,i} \left(\frac{\partial f}{\partial u_i}, 0 \right).$$

By the same argument as those above, we have

$$(0, \xi_2) \in \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_r} \right\rangle_{\mathscr{E}_{n+r}} + F^*\mathscr{E}(p+r, p+r) + F^*(\mathscr{M}_{p+r})\mathscr{E}(n+r, p+r).$$

Hence, we have

$$\mathscr{E}(n+r, p+r) = \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_r} \right\rangle_{\mathscr{E}_{n+r}}$$
$$+ F^*\mathscr{E}(p+r, p+r) + F^*(\mathscr{M}_{p+r})\mathscr{E}(n+r, p+r).$$

Applying the Malgrange preparation theorem once again, we have

$$\mathscr{E}(n+r, p+r) = \left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_r} \right\rangle_{\mathscr{E}_{n+r}} + F^*\mathscr{E}(p+r, p+r).$$

4. *Generic classifications*

In this section we give the proof of Theorem 1·1. Since the infinitesimally $\mathscr{A}$-stable map germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is an immersion or the cross-cap, we now prove that the germ of the ruled surface $F_{(\gamma,\delta)}$ at any point is infinitesimally $\mathscr{A}$-stable for generic (γ, δ).

On the other hand, by the calculation of the proof of Lemma 2·2, the singular point of the ruled surface $F_{(\gamma,\delta)}$ is given by the condition that rank $(\gamma'(t) + u\delta'(t), \delta(t)) < 2$ and it is equivalent to the condition that two vectors $\gamma'(t) + u\delta'(t)$, $\delta(t)$ are parallel. Since $\delta(t) \neq 0$, rank $(\gamma'(t) + u\delta'(t), \delta(t)) \geqslant 1$.

We now regard the parameter u (i.e. the parameter along the ruling) of the ruled surface as the parameter of a one-dimensional unfolding. For any $(\gamma, \delta) \colon I \to \mathbb{R}^3 \times S^2$ with $\delta'(t) \neq 0$, we denote that $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $\delta(t) = (\delta_1(t), \delta_2(t), \delta_3(t))$, then we have the coordinate representation:

$$F_{(\gamma,\delta)}(t, u) = (\gamma_1(t) + u\delta_1(t), \gamma_2(t) + u\delta_2(t), \gamma_3(t) + u\delta_3(t)).$$

For any fixed $(t_0, u_0) \in I \times J$ with $\delta_3(t_0) \neq 0$, we define a non-empty open subset U_3 in I by

$$U_3 = \{t \in I \mid \delta_3(t) \neq 0\}.$$

We define a function $g_3(t)$ by $g_3(t) = -(\gamma_3(t) - y_0)/\delta_3(t)$ for any $t \in U_3$, where $y_0 = \gamma_3(t_0) + u_0\delta(t_0)$. Therefore, we have

$$F_{(\gamma,\delta)}(t,u) = \gamma(t) + g_3(t)\delta(t) + (u - g_3(t))\delta(t) = \gamma(T) + g_3(T)\delta(T) + U\delta(T)$$

for $T = t$, $U = u - g_3(t)$. We denote the above map as $\tilde{F}_{(\gamma,\delta)}(T,U)$. By Lemma 3·1, the map germ $\tilde{F}_{(\gamma,\delta)}(T,U)$ at $(t_0,0)$ is a one-dimensional unfolding of $\hat{\pi}_3 \circ \tilde{F}_{(\gamma,\delta)}(T,0) = (\gamma_1(T)+g_3(T)\delta_1(T), \gamma_2(T)+g_3(T)\delta_2(T))$, where $\hat{\pi}_3 \colon \mathbb{R}^3 \to \mathbb{R}^2$ is the canonical projection given by $\hat{\pi}_3(y_1,y_2,y_3) = (y_1,y_2)$. The following lemma is the basis for the proof of Theorem 1·1.

Lemma 4·1. *Let $W \subset J^k(1,2)$ be a submanifold. For any fixed map germ $\delta\colon I \to S^2$ with $\delta'(t) \neq 0$ and any fixed point $(t_0,u_0) \in I \times J$ with $\delta_3(t_0) \neq 0$, the set*

$${}_3T_{W,(t_0,u_0)} = \{\gamma \mid j_1^k\hat{\pi}_3 \circ \tilde{F}_{(\gamma,\delta)} \text{ is transverse to } W \text{ at } (t_0,u_0)\}$$

is a residual subset in $C^\infty(I,\mathbb{R}^3) \times \{\delta\}$.

Here, we consider that $C^\infty(I,\mathbb{R}^3 \times S^2) = C^\infty(I,\mathbb{R}^3) \times C^\infty(I,S^2)$ and relative topology on $C^\infty(I,\mathbb{R}^3) \times \{\delta\}$.

For the proof of Lemma 4·1, we need to apply the following fundamental transversality lemma of Thom (cf. [**5**, p. 53, lemma 4·6]).

Lemma 4·2. *Let X, B and Y be C^∞-manifolds. Let $j\colon B \to C^\infty(X,Y)$ be a mapping (not necessarily continuous) and define $\Phi\colon X \times B \to Y$ by $\Phi(x,b) = j(b)(x)$. Assume that Φ is smooth and transverse to the submanifold W of Y. Then the set $\{b \in B \mid j(b) \text{ is transverse to } W\}$ is dense in B.*

Proof of Lemma 4·1. Let $\{K_j\}_{j=1}^\infty$ be the countable set of open covering of W such that each closure $\bar{K}_j$ is compact. We define the following set

$${}_3T_{W,(t_0,u_0),K_j} = \left\{\gamma \mid \ j_1^k\hat{\pi}_3 \circ \tilde{F}_{(\gamma,\delta)} \text{ is transverse to } W \text{ with } j_1^k\hat{\pi}_3 \circ \tilde{F}_{(\gamma,\delta)}(t_0,u_0) \in \bar{K}_j\right\}.$$

We now prove that ${}_3T_{W,(t_0,u_0),K_j}$ is an open subset. For the purpose, we consider the following mapping

$$\hat{j}^k \colon C^\infty(U_3,\mathbb{R}^3) \to C^\infty(U_3 \times J, J^k(1,2))$$

defined by $\hat{j}^k(\gamma) = j^k\hat{\pi}_3 \circ \tilde{F}_{(\gamma,\delta)}$. It is clear that the mapping $\hat{j}^k$ is continuous. We also define a subset

$$O_{W,K_j} = \{g \in C^\infty(U_3 \times J, J^k(1,2)) \mid g \text{ is transverse to } W \text{ at } (t_0,u_0) \text{ with } g(t_0,u_0) \in K_j\},$$

then it is open (cf. [**5**]). Since the restriction map $\mathrm{res}_{U_3}\colon C^\infty(I,\mathbb{R}^3) \to C^\infty(U_3,\mathbb{R}^3)$ is continuous, ${}_3T_{W,(t_0,u_0),K_j} = (\mathrm{res}_{U_3})^{-1} \circ (\hat{j}^k)^{-1}(O_{W,K_j})$ is open. If we show that ${}_3T_{W,(t_0,u_0),K_j}$ is dense subset in $C^\infty(I,\mathbb{R}^3) \times \{\delta\}$, then $T_{W,(t_0,u_0)} = \bigcap_{j=1}^\infty T_{W,(t_0,u_0),K_j}$ is a residual subset.

Since res_{U_3} is surjective, it is enough to show that

$$T_{W,(t_0,u_0),K_j,U_3} = \left\{\gamma \in C^\infty(U_3,\mathbb{R}^3) \mid \ j_1^k\hat{\pi}_3 \circ \tilde{F}_{(\gamma,\delta)} \text{ is transverse to } W \text{ at } (t_0,u_0) \right.$$
$$\left. \text{with } j_1^k\hat{\pi}_3 \circ \tilde{F}_{(\gamma,\delta)}(t_0,u_0) \in \bar{K}_j\right\}$$

is a dense subset in $C^\infty(U_3,\mathbb{R}^3)$.

For any $\gamma \in C^\infty(U_3, \mathbb{R}^3)$ and $p = (p_1, p_2) \in P(1,2;k)$, we define a mapping $f_{(\gamma,p)}\colon U_3 \times J \to \mathbb{R}^2$ by

$$f_{(\gamma,p)}(t,u) = (\gamma_1(t) + p_1(t) + g_3(t)\delta_1(t) + u\delta_1(t), \gamma_2(t) + p_2(t) + g_3(t)\delta_2(t) + u\delta_2(t)),$$

where $P(1,2;k)$ denotes the space of pairs of polynomials (p_1, p_2) with degrees at most k without constant terms. We also define a mapping $\Phi\colon U_3 \times J \times P(1,2;k) \to J^k(1,2)$ by $\Phi(t,u,(p_1,p_2)) = j_1^k f_{(\gamma,p)}(t,u) = j^k f_{(\gamma,p),u}(t)$, where $f_{(\gamma,p),u}(t) = f_{(\gamma,p)}(t,u)$. We may consider that $P(1,2;k)$ is Euclidian space $\mathbb{R}^N$.

It is easy to show that Φ is a submersion, so that it is transverse to W. By Lemma 4·2,

$$\{p = (p_1,p_2) \in P(1,2;k) \mid \Phi_{(p_1,p_2)} \text{ is transverse to } W \text{ at } (t_0,u_0) \text{ with } \Phi_{(p_1,p_2)}(t_0,u_0) \in \bar{K}_j\}$$

is dense in $P(1,2,;k)$. Hence, we can find $(p_1,p_2)_1, (p_1,p_2)_2, (p_1,p_2)_3, \ldots$ in $P(1,2;k)$ converging to $(0,0)$ so that $\Phi_{(p_1,p_2)_i}$ is transverse to W on K_j. Since $\lim_{i\to\infty}(\gamma + ((p_1,p_2)_i, 0)) = \gamma$ in $C^\infty(U_3,\mathbb{R}^3)$, $T_{W,(t_0,u_0),K_j,U_3}$ is dense in $C^\infty(U_3,\mathbb{R}^3)$.

We remark that ${}_jT_{W,(t_0,u_0)}$ $(j = 1,2)$ can also be defined for $(t_0,u_0) \in I \times J$ with $\delta_j(t_0) \neq 0$ and the same assertion for ${}_jT_{W,(t_0,u_0)}$ as the above holds.

Proof of Theorem 1·1. Let $\mathscr{K}_i$ be the $\mathscr{K}$-orbit with codimension i in $J^k(1,2)$ for sufficiently large k. We also denote that $\Sigma(1,2) = \bigcap_{i\geqslant 3} \mathscr{K}_i \subset J^k(1,2)$. It has been known that $\Sigma(1,2)$ is a semi-algebraic subset in $J^k(1,2)$ with codimension greater than 2. Therefore we have the canonical stratification $\{\mathscr{S}_i\}_{i=1}^m$of $\Sigma(1,2)$ with $\operatorname{codim}\mathscr{S}_i > 2$. For any (t_0,u_0) with $\delta_3(t_0) \neq 0$, we denote that ${}_3T_{\Sigma(1,2),(t_0,u_0)} = \bigcap_{i=1}^m T_{\mathscr{S}_i,(t_0,u_0)}$. Since ${}_3T_{\mathscr{K}_i,(t_0,u_0)}$ and ${}_3T_{\Sigma(1,2),(t_0,u_0)}$ are residual subsets in $C^\infty_{pr}(I,\mathbb{R}^3 \times S^2)$, ${}_3\mathscr{O}_{(t_0,u_0)} = \bigcap_{i=1}^2 {}_3T_{\mathscr{K}_i,(t_0,u_0)} \bigcap T_{\Sigma(1,2),(t_0,u_0)}$ is also a residual subset in $C^\infty_{pr}(I,\mathbb{R}^3 \times S^2)$. By the remark after the proof of Lemma 4·1, ${}_j\mathscr{O}_{(t_0,u_0)}$ $(j = 1,2)$ are also residual subsets in $C^\infty_{pr}(I,\mathbb{R}^3 \times S^2)$ respectively. Therefore, for any fixed $(t_0,u_0) \in I \times J$, there exists a residual subset $\mathscr{O}_{(t_0,u_0)} \subset C^\infty_{pr}(I,\mathbb{R}^3 \times S^2)$ such that the map germ $F_{(\gamma,\delta)}$ at (t_0,u_0) is an infinitesimally $\mathscr{A}$-stable map germ for any $(\gamma,\delta) \in \mathscr{O}$ by Lemma 3·3. Since the infinitesimally $\mathscr{A}$-stable map germ $\mathbb{R}^2 \to \mathbb{R}^3$ is an immersion or the cross-cap and the singularities of $F_{(\gamma,\delta)}$ are located on the striction curve, there exists an open neighbourhood $U_{t_0} \subset I$ of t_0 such that $F_{(\gamma,\delta)}$ is an immersion on $U_{t_0} \times I - \{(t_0,u_0)\}$. Since $\bar{I}$ is compact, we can extend (γ,δ) slightly on an open interval $I_0 \supset \bar{I}$ and there exist finitely many U_{t_i} $(i = 1,\ldots,\ell)$ such open subsets as the above with $I = \bigcup_{i=1}^\ell U_{t_i}$. Then $\mathscr{O} = \bigcap_{i=1}^\ell \mathscr{O}_{(t_i,u_i)}$ is a residual subset of $C^\infty_{pr}(I,\mathbb{R}^3 \times S^2)$. It is clear that the germ $F_{(\gamma,\delta)}$ at any point $(t,u) \in I \times J$ is an immersion or the cross-cap for any $(\gamma,\delta) \in \mathscr{O}$. It is easy to show that the mapping $F_\sharp\colon C^\infty(I,\mathbb{R}^3 \times S^2) \to C^\infty(I \times J,\mathbb{R}^3)$ defined by $F_\sharp(\gamma,\delta) = F_{(\gamma,\delta)}$ is continuous. Since the cross-cap is the stable singularities of map germs $(\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$, the set $\mathscr{S} = \{f \in C^\infty(I \times J,\mathbb{R}^3) \mid f \text{ is an immersion or the cross-cap at any point } \in I \times J\}$ is an open subset. Therefore, $\mathscr{O} = F_\sharp^{-1}(\mathscr{S})$ is an open subset of $C^\infty_{pr}(I,\mathbb{R}^3 \times S^2)$. This completes the proof of Theorem 1·1.

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