# HOKKAIDO UNIVERSITY

| | |
|---|---|
| Title | Optimal Control for Absolutely Continuous Stochastic Processes and the Mass Transportation Problem |
| Author(s) | Mikami, Toshio |
| Citation | Electronic Communications in Probability, 7, 199-213 |
| Issue Date | 2002 |
| Doc URL | http://hdl.handle.net/2115/5927 |
| Type | article (author version) |
| Note(URL) | http://www.math.washington.edu/~ejpecp/ECP/viewarticle.php?id=1644 |
| File Information | ECP7.pdf |

Instructions for use

# Optimal control for absolutely continuous stochastic processes and the mass transportation problem

Toshio Mikami

*Department of Mathematics, Hokkaido University*

*Sapporo 060-0810, Japan*

mikami@math.sci.hokudai.ac.jp

November 14, 2002

### Abstract

We study the optimal control problem for $\mathbf{R}^d$-valued absolutely continuous stochastic processes with given marginal distributions at every time. When $d = 1$, we show the existence and the uniqueness of a minimizer which is a function of a time and an initial point. When $d > 1$, we show that a minimizer exists and that minimizers satisfy the same ordinary differential equation.

## 1 Introduction

Monge-Kantorovich problem (MKP for short) plays a crucial role in many fields and has been studied by many authors (see [2, 3, 7, 10, 12, 20] and the references therein).

1

Let $h : \mathbf{R}^d \mapsto [0, \infty)$ be convex, and $Q_0$ and $Q_1$ be Borel probability measures on $\mathbf{R}^d$, and put

$$\mu_h(Q_0, Q_1) := \inf E\Big[\int_0^1 h\Big(\frac{d\phi(t)}{dt}\Big)dt\Big], \tag{1}$$

where the infimum is taken over all absolutely continuous stochastic processes $\{\phi(t)\}_{0 \leq t \leq 1}$ for which $P\phi(t)^{-1} = Q_t (t = 0, 1)$. (In this paper we use the same notation $P$ for different probability measures, for the sake of simplicity, when it is not confusing.)

As a special case of MKPs, we introduce the following problem (see e.g. [2, 3] and also [18]).

Does there exist a minimizer $\{\phi^o(t)\}_{0 \leq t \leq 1}$, of (1.1), which is a function of $t$ and $\phi^o(0)$?

Suppose that there exist $p \in L^1([0, 1] \times \mathbf{R}^d : \mathbf{R}, dtdx)$ and $b(t, x) \in L^1([0, 1] \times \mathbf{R}^d : \mathbf{R}^d, p(t, x)dtdx)$ such that the following holds: for any $f \in C_o^\infty(\mathbf{R}^d)$ and any $t \in [0, 1]$,

$$\int_{\mathbf{R}^d} f(x)(p(t, x) - p(0, x))dx = \int_0^t ds \int_{\mathbf{R}^d} < \nabla f(x), b(s, x) > p(s, x)dx,$$

$$p(t, x) \geq 0 \quad dx - \text{a.e.}, \qquad \int_{\mathbf{R}^d} p(t, x)dx = 1. \tag{2}$$

Here $< \cdot, \cdot >$ denotes the inner product in $\mathbf{R}^d$ and $\nabla f(x) := (\partial f(x)/\partial x_i)_{i=1}^d$.

Put, for $n \geq 1$,

$$\mathbf{e}_n := \inf\Big\{E\Big[\int_0^1 h\Big(\frac{dY(t)}{dt}\Big)dt\Big] : \{Y(t)\}_{0 \leq t \leq 1} \in A_n\Big\}, \tag{3}$$

where $A_n$ is the set of all absolutely continuous stochastic processes $\{Y(t)\}_{0 \leq t \leq 1}$ for which $P(Y(t) \in dx) = p(t, x)dx$ for all $t = 0, 1/n, \cdots, 1$.

Then the minimizer of $\mathbf{e}_n$ can be constructed by those of

$$\mu_{\frac{h(n \cdot)}{n}}\Big(p\Big(\frac{k}{n}, x\Big)dx, p\Big(\frac{k+1}{n}, x\Big)dx\Big) \quad (k = 0, \cdots, n - 1)$$

(see (1.1) for notation). As $n \to \infty$, $\mathbf{e}_n$ formally converges to

$$\mathbf{e} := \inf\Big\{E\Big[\int_0^1 h\Big(\frac{dY(t)}{dt}\Big)dt\Big] : \{Y(t)\}_{0 \leq t \leq 1} \in A\Big\}, \tag{4}$$

2

where $A$ is the set of all absolutely continuous stochastic processes $\{Y(t)\}_{0 \le t \le 1}$ for which $P(Y(t) \in dx) = p(t, x)dx$ for all $t \in [0, 1]$.

In this sense, the minimizer of **e** can be considered as the continuum limit of those of $\mathbf{e}_n$ as $n \to \infty$.

In this paper, instead of $h(u)$, we would like to consider more general function $L(t, x; u) : [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto [0, \infty)$ which is convex in $u$, and study the minimizers of

$$\mathbf{e}^0 := \inf\{E[\int_0^1 L\Big(t, Y(t); \frac{dY(t)}{dt}\Big)dt] : \{Y(t)\}_{0 \le t \le 1} \in A\}. \tag{5}$$

**Remark 1** *It is easy to see that the set $A_n$ is not empty, but it is not trivial to show that the set $A$ is not empty if $b$ in (1.2) is not smooth. As a similar problem, that of the construction of a Markov diffusion process $\{X(t)\}_{0 \le t \le 1}$ such that $PX(t)^{-1}$ satisfies a given Fokker-Planck equation with nonsmooth coefficients is known and has been studied by many authors (see [4], [5], [15], [19] and the references therein).*

We would also like to point out that (1.1) and (1.5) can be formally considered as the zero-noise limits of h-path processes and variational processes, respectively, when $h = L = |u|^2$ (see [8] and [15], respectively).

More generally, we have the following.

Let $(\Omega, \mathbf{B}, P)$ be a probability space, and $\{\mathbf{B}_t\}_{t \ge 0}$ be a right continuous, increasing family of sub $\sigma$-fields of $\mathbf{B}$, and $X_o$ be a $\mathbf{R}^d$-valued, $\mathbf{B}_0$-adapted random variable such that $PX_o^{-1}(dx) = p(0, x)dx$, and $\{W(t)\}_{t \ge 0}$ denote a d-dimensional $(\mathbf{B}_t)$-Wiener process (see e.g. [11] or [13]).

For $\varepsilon > 0$ and a $\mathbf{R}^d$-valued $(\mathbf{B}_t)$-progressively measurable $\{u(t)\}_{0 \le t \le 1}$, put

$$X^{\varepsilon,u}(t) := X_o + \int_0^t u(s)ds + \varepsilon W(t) \quad (t \in [0, 1]). \tag{6}$$

Put also

$$\mathbf{e}^\varepsilon := \inf\{E[\int_0^1 L(t, X^{\varepsilon,u}(t); u(t))dt] : \{u(t)\}_{0 \le t \le 1} \in A^\varepsilon\} \quad (\varepsilon > 0), \tag{7}$$

where $A^\varepsilon := \{\{u(t)\}_{0 \le t \le 1} : P(X^{\varepsilon,u}(t) \in dx) = p(t, x)dx(0 \le t \le 1)\}$; and

3

$$\tilde{\mathbf{e}}^\varepsilon := \inf\{\int_0^1 \int_{\mathbf{R}^d} L(t,y;B(t,y))p(t,y)dtdy : B \in \tilde{A}^\varepsilon\} \quad (\varepsilon \geq 0), \qquad (8)$$

where $\tilde{A}^\varepsilon$ is the set of all $B(t,x) : [0,1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ for which the following holds: for any $f \in C_o^\infty(\mathbf{R}^d)$ and any $t \in [0,1]$,

$$\int_{\mathbf{R}^d} f(x)(p(t,x) - p(0,x))dx$$
$$= \int_0^t ds \int_{\mathbf{R}^d} \Big(\frac{\varepsilon^2 \triangle f(x)}{2} + < \nabla f(x), B(s,x) >\Big)p(s,x)dx.$$

Then we expect that the following holds:

$$\mathbf{e}^\varepsilon = \tilde{\mathbf{e}}^\varepsilon \to \mathbf{e}^0 = \tilde{\mathbf{e}}^0 \quad (\text{as } \varepsilon \to 0). \qquad (9)$$

In this paper we show that the set $A$ is not empty and (1.9) holds, and that a minimizer of $\mathbf{e}^0$ exists when the cost function $L(t,x;u)$ grows at least of order of $|u|^2$ as $u \to \infty$ (see Theorem 1 in section 2).

We also show that the minimizers satisfy the same ordinary differential equation (ODE for short) when $L$ is strictly convex in $u$ (see Theorem 2 in section 2). (In this paper we say that a function $\{\psi(t)\}_{0 \leq t \leq 1}$ satisfies an ODE if and only if it is absolutely continuous and $d\psi(t)/dt$ is a function of $t$ and $\psi(t)$, dt-a.e..)

When $d = 1$, we show the uniqueness of the minimizer of $\mathbf{e}^0$ (see Corollary 1 in section 2).

Since a stochastic process which satisfies an ODE is not always nonrandom, we would also like to know if the minimizer is a function of a time and an initial point. In fact, the following is known as Salisbury's problem (SP for short).

Is a continuous strong Markov process which is of bounded variation in time a function of an initial point and a time?

If $x(t)_{0 \leq t \leq 1}$ is a $\mathbf{R}$-valued strong Markov process, and if there exists a Borel measurable function $f$, on $\mathbf{R}$, such that $x(t) = x(0) + \int_0^t f(x(s))ds$ $(0 \leq t \leq 1)$, then SP has been solved positively by Çinlar and Jacod (see [6]). When $d > 1$, a counter example is known (see [21]).

4

When $d = 1$, we give a positive answer to SP for time-inhomogeneous stochastic processes (see Proposition 2 in section 4). This is a slight generalization of [6] where they made use of the result on time changes of Markov processes, in that the stochastic processes under consideration are time-inhomogeneous and need not be Markovian. In particular, we show, when $d = 1$, that $\{Y(t)\}_{0 \leq t \leq 1}, \in A$, which satisfies an ODE is unique and nonrandom. It will be used to show that the unique minimizer of $\mathbf{e}^0$ is a function of an initial point and of a time when $d = 1$ (see Corollary 1 and Theorem 3 in section 2).

**Remark 2** *When $d > 1$, $\{Y(t)\}_{0 \leq t \leq 1}, \in A$, which satisfies an ODE is not unique (see Proposition 1 in section 2).*

When $L(t, x; u) = |u|^2$ and $p(t, x)$ satisfies the Fokker-Planck equation with sufficiently smooth coefficients, the optimization problem (1.5) was considered in [16] where the minimizer exists uniquely and is a function of a time and an initial point, and where we used a different approach which depends on the form of $L(t, x; u) = |u|^2$.

Our main tool in the proof is the weak convergence method, the result on the construction of a Markov diffusion process from a family of marginal distributions, and the theory of Copulas.

In section 2 we state our main result. We first consider the case where a cost function $L(t, x; u)$ grows at least of order of $|u|^2$ as $u \to \infty$ and $d \geq 1$. Next we restrict our attention to the case where $L$ is a function of $u$ and $d = 1$. The proof is given in section 3. We discuss SP in section 4.

## 2  Main result

In this section we give our main result.

We state assumptions before we state the result when $d \geq 1$.

(H.0). $\tilde{\mathbf{e}}^0$ is finite (see (1.8) for notation).

(H.1). $L(t, x; u) : [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto [0, \infty)$ is convex in $u$, and as $h, \delta \downarrow 0$,

$$R(h, \delta) := \sup\left\{ \frac{L(t, x; u) - L(s, y; u)}{1 + L(s, y; u)} : |t - s| < h, |x - y| < \delta, u \in \mathbf{R}^d \right\} \downarrow 0.$$

(H.2). There exists $q \geq 2$ such that the following holds:

5

$$0 < \liminf_{|u| \to \infty} \frac{\inf\{L(t, x; u) : (t, x) \in [0, 1] \times \mathbf{R}^d\}}{|u|^q}, \tag{10}$$

$$\sup\left\{\frac{\sup_{z \in \partial_u L(t,x;u)} |z|}{(1 + |u|)^{q-1}} : (t, x, u) \in [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d\right\} \equiv C_{\nabla L} < \infty, \tag{11}$$

where $\partial_u L(t, x; u) := \{z \in \mathbf{R}^d : L(t, x; v) - L(t, x; u) \geq\ < z, v - u > \text{ for all } v \in \mathbf{R}^d\}$ ($t \in [0, 1]$, $x$, $u \in \mathbf{R}^d$).

(H.3). $p(t, \cdot)$ is absolutely continuous $dt$-a.e., and for $q$ in (H.2),

$$\int_0^1 \int_{\mathbf{R}^d} \left|\frac{\nabla_x p(t, x)}{p(t, x)}\right|^q p(t, x) dt dx < \infty. \tag{12}$$

**Remark 3** *If (H.0) does not hold, then $\mathbf{e}^0$ in (1.5) is infinite. (H.1) implies the continuity of $L(\cdot, \cdot; u)$ for each $u \in \mathbf{R}^d$. (H.2) holds if $L(t, x; u) = |u|^q$. We need (H.3) to make use of the result on the construction of a Markov diffusion process of which the marginal distribution at time $t$ is $p(t, x)dx$ ($0 \leq t \leq 1$). (2.3) holds if $b(t, x)$ in (1.2) is twice continuously differentiable with bounded derivatives up to the second order, and if $p(0, x)$ is absolutely continuous, and if the following holds:*

$$\int_{\mathbf{R}^d} \left|\frac{\nabla_x p(0, x)}{p(0, x)}\right|^q p(0, x) dx < \infty. \tag{13}$$

The following theorem implies the existence of a minimizer of $\mathbf{e}^0$ (see (1.5)-(1.8) for notations).

**Theorem 1** *Suppose that (H.0)-(H.3) hold. Then the sets $A^\varepsilon$ ($\varepsilon > 0$) and $A$ are not empty, and the following holds:*

$$\tilde{\mathbf{e}}^\varepsilon = \mathbf{e}^\varepsilon \to \mathbf{e}^0 = \tilde{\mathbf{e}}^0 \quad (\text{as } \varepsilon \to 0). \tag{14}$$

*In particular, for any $\{u^\varepsilon(t)\}_{0 \leq t \leq 1}, \in A^\varepsilon(\varepsilon > 0)$, for which*

$$\lim_{\varepsilon \to 0} E[\int_0^1 L(t, X^{\varepsilon, u^\varepsilon}(t); u^\varepsilon(t)) dt] = \mathbf{e}^0, \tag{15}$$

$\{\{X^{\varepsilon, u^\varepsilon}(t)\}_{0 \leq t \leq 1}\}_{\varepsilon > 0}$ *is tight in $C([0, 1] : \mathbf{R}^d)$, and any weak limit point of $\{X^{\varepsilon, u^\varepsilon}(t)\}_{0 \leq t \leq 1}$ as $\varepsilon \to 0$ is a minimizer of $\mathbf{e}^0$.*

The following theorem implies the uniqueness of the minimizer of $\tilde{\mathbf{e}}^0$ and that of the ODE which is satisfied by the minimizers of $\mathbf{e}^0$.

**Theorem 2** *Suppose that (H.0)-(H.3) hold. Then for any minimizer $\{X(t)\}_{0 \leq t \leq 1}$ of $\mathbf{e}^0$, $b^X(t,x) := E[dX(t)/dt|(t, X(t) = x)]$ is a minimizer of $\tilde{\mathbf{e}}^0$. Suppose in addition that $L$ is strictly convex in $u$. Then $\tilde{\mathbf{e}}^0$ has the unique minimizer $b^o(t,x)$ and the following holds: for any minimizer $\{X(t)\}_{0 \leq t \leq 1}$ of $\mathbf{e}^0$,*

$$X(t) = X(0) + \int_0^t b^o(s, X(s))ds \quad \text{for all } t \in [0,1], \quad a.s.. \tag{16}$$

**Remark 4** *By Theorems 1 and 2, if (H.0) with $L = |u|^2$ and (H.3) with $q = 2$ hold, then there exists a stochastic process $\{X(t)\}_{0 \leq t \leq 1}, \in A$, which satisfies an ODE.*

Since $b \in \tilde{A}^0$ is not always the gradient, in $x$, of a function, the following implies that the set $\tilde{A}^0$ does not always consist of only one point.

**Proposition 1** *Suppose that $L = |u|^2$, and that (H.0) and (H.3) with $q = 2$ hold, and that for any $M > 0$,*

$$ess.inf\{p(t,x) : t \in [0,1], |x| \leq M\} > 0. \tag{17}$$

*Then the unique minimizer of $\tilde{\mathbf{e}}^0$ can be written as $\nabla_x V(t,x)$, where $V(t, \cdot) \in H^1_{loc}(\mathbf{R}^d : \mathbf{R})$ $dt$-a.e..*

We next consider the one-dimensional case. Put

$$F_t(x) := \int_{(-\infty, x]} p(t,y)dy \quad (t \in [0,1], x \in \mathbf{R}),$$

$$F_t^{-1}(u) := \sup\{y \in \mathbf{R} : F_t(y) < u\} \quad (t \in [0,1], 0 < u < 1).$$

(H.3)'. $d = 1$, and $F_t(x)$ is differentiable and has the locally bounded first partial derivatives on $[0,1] \times \mathbf{R}$.

By Proposition 2 in section 4, we obtain the following.

**Corollary 1** *Suppose that (H.0)-(H.3) and (H.3)' hold, and that $L$ is strictly convex in $u$. Then the minimizer $\{X(t)\}_{0 \leq t \leq 1}$ of $\mathbf{e}^0$ is unique. Moreover, $\lim_{s \in \mathbf{Q} \cap [0,1], s \to t} F_s^{-1}(F_0(X(0)))$ exists and is equal to $X(t)$ for all $t \in [0,1]$ a.s..*

7

The theory of copulas allows us to treat a different set of assumptions by a different method (see (1.3)-(1.4) for notations).

(H.0)'. $\{\mathbf{e}_n\}_{n\geq 1}$ is bounded.

(H.1)'. $h : \mathbf{R} \mapsto [0, \infty)$ is even and convex.

(H.2)'. There exists $r > 1$ such that the following holds:

$$0 < \liminf_{|u|\to\infty} \frac{h(u)}{|u|^r}. \tag{18}$$

(H.3)". $d = 1$, and $p(t, x)$ is positive on $[0, 1] \times \mathbf{R}$.

**Theorem 3** *Suppose that (H.0)'-(H.2)' and (H.3)" hold. Then $\{F_t^{-1}(F_0(x))\}_{0\leq t\leq 1}$ on $(\mathbf{R}, \mathbf{B}(\mathbf{R}), p(0, x)dx)$ belongs to the set $A$ and is a minimizer of $\mathbf{e}$. Suppose in addition that (H.3)' holds. Then $\{F_t^{-1}(F_0(x))\}_{0\leq t\leq 1}$ is the unique minimizer, of $\mathbf{e}$, that satisfies an ODE.*

**Remark 5** *If $\{\mathbf{e}_n\}_{n\geq 1}$ is unbounded, then so is $\mathbf{e}$. By (H.1)', $\{\overline{X}(t) := F_t^{-1}(F_0(x))\}_{0\leq t\leq 1}$ satisfies the following (see e.g. [20, Chap. 3.1]): for any $t$ and $s \in [0, 1]$,*

$$\mu_h(p(s, x)dx, p(t, x)dx) = E_0[h(\overline{X}(t) - \overline{X}(s))] \tag{19}$$

*(see (1.1) for notation), where we put $P_0(dx) := p(0, x)dx$. Indeed,*

$$\overline{X}(t) = F_t^{-1}(F_s(\overline{X}(s))) \tag{20}$$

*since for a distribution $F$ on $\mathbf{R}$,*

$$F(F^{-1}(u)) = u \quad (0 < u < 1) \tag{21}$$

*(see e. g. [17]).*

# 3 Proof of the result

In this section we prove the result given in section 2.

Before we give the proof of Theorem 1, we state and prove three technical lemmas.

**Lemma 1** *Suppose that (H.2) holds. Then for any $\varepsilon > 0$, $\tilde{\mathbf{e}}^\varepsilon = \mathbf{e}^\varepsilon$.*

(Proof). For any $B^\varepsilon \in \tilde{A}^\varepsilon$ for which $\int_0^1 \int_{\mathbf{R}^d} L(t, x; B^\varepsilon(t, x)) p(t, x) dt dx$ is finite, there exists a Markov process $\{Z^\varepsilon(t)\}_{0 \le t \le 1}$ such that the following holds:

$$Z^\varepsilon(t) = X_o + \int_0^t B^\varepsilon(s, Z^\varepsilon(s)) ds + \varepsilon W(t), \qquad (22)$$

$$P(Z^\varepsilon(t) \in dx) = p(t, x) dx \quad (0 \le t \le 1), \qquad (23)$$

since $\int_0^1 \int_{\mathbf{R}^d} |B^\varepsilon(t, x)|^2 p(t, x) dt dx$ is finite by (H.2) (see [4] and [5]). This implies that $\{B^\varepsilon(t, Z^\varepsilon(t))\}_{0 \le t \le 1} \in A^\varepsilon$, and that the following holds:

$$\int_0^1 \int_{\mathbf{R}^d} L(t, x; B^\varepsilon(t, x)) p(t, x) dt dx = \int_0^1 E[L(t, Z^\varepsilon(t); B^\varepsilon(t, Z^\varepsilon(t)))] dt, \quad (24)$$

from which $\mathbf{e}^\varepsilon \le \tilde{\mathbf{e}}^\varepsilon$.

We next show that $\mathbf{e}^\varepsilon \ge \tilde{\mathbf{e}}^\varepsilon$.

For any $\{u^\varepsilon(t)\}_{0 \le t \le 1} \in A^\varepsilon$, $b^{\varepsilon, u^\varepsilon}(t, x) := E[u^\varepsilon(t) | (t, X^{\varepsilon, u^\varepsilon}(t) = x)] \in \tilde{A}^\varepsilon$.

Indeed, for any $f \in C_o^\infty(\mathbf{R}^d)$ and any $t \in [0, 1]$, by the Itô formula,

$$\int_{\mathbf{R}^d} f(x)(p(t, x) - p(0, x)) dx = E[f(X^{\varepsilon, u^\varepsilon}(t)) - f(X^{\varepsilon, u^\varepsilon}(0))] \qquad (25)$$

$$= \int_0^t E\left[\frac{\varepsilon^2}{2} \triangle f(X^{\varepsilon, u^\varepsilon}(s)) + < \nabla f(X^{\varepsilon, u^\varepsilon}(s)), u^\varepsilon(s) >\right] ds$$

$$= \int_0^t E\left[\frac{\varepsilon^2}{2} \triangle f(X^{\varepsilon, u^\varepsilon}(s)) + < \nabla f(X^{\varepsilon, u^\varepsilon}(s)), b^{\varepsilon, u^\varepsilon}(s, X^{\varepsilon, u^\varepsilon}(s)) >\right] ds$$

$$= \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{\varepsilon^2}{2} \triangle f(x) + < \nabla f(x), b^{\varepsilon, u^\varepsilon}(s, x) >\right) p(s, x) dx.$$

The following completes the proof: by Jensen's inequality,

$$\int_0^1 E[L(t, X^{\varepsilon, u^\varepsilon}(t); u^\varepsilon(t))] dt \qquad (26)$$

$$\ge \int_0^1 E[L(t, X^{\varepsilon, u^\varepsilon}(t); b^{\varepsilon, u^\varepsilon}(t, X^{\varepsilon, u^\varepsilon}(t)))] dt$$

$$= \int_0^1 \int_{\mathbf{R}^d} L(t, x; b^{\varepsilon, u^\varepsilon}(t, x)) p(t, x) dt dx.$$

Q. E. D.

The following lemma can be shown by the standard argument and the proof is omitted (see [13, p. 17, Theorem 4.2 and p. 33, Theorem 6.10]).

**Lemma 2** *For any $\{u^\varepsilon(t)\}_{0 \le t \le 1} \in A^\varepsilon$ $(\varepsilon > 0)$ for which $\{E[\int_0^1 |u^\varepsilon(t)|^2 dt]\}_{\varepsilon > 0}$ is bounded, $\{\{X^{\varepsilon, u^\varepsilon}(t)\}_{0 \le t \le 1}\}_{\varepsilon > 0}$ is tight in $C([0,1] : \mathbf{R}^d)$.*

**Lemma 3** *For any $\{u^{\varepsilon_n}(t)\}_{0 \le t \le 1} \in A^{\varepsilon_n}$ $(n \ge 1)$ $(\varepsilon_n \to 0$ as $n \to \infty)$ such that $\{E[\int_0^1 |u^{\varepsilon_n}(t)|^2 dt]\}_{n \ge 1}$ is bounded and that $\{X_n(t) := X^{\varepsilon_n, u^{\varepsilon_n}}(t)\}_{0 \le t \le 1}$ weakly converges as $n \to \infty$, the weak limit $\{X(t)\}_{0 \le t \le 1}$ in $C([0,1] : \mathbf{R}^d)$ is absolutely continuous.*

(Proof). We only have to show the following: for any $\delta > 0$ and any $m \ge 2$, $n \ge 1$ and any $s_{i,j}, t_{i,j} \in \mathbf{Q}$ for which $0 \le s_{i,j} \le t_{i,j} \le s_{i,j+1} \le t_{i,j+1} \le 1$ $(1 \le i \le n, 1 \le j \le m-1)$ and for which $\sum_{j=1}^m |t_{i,j} - s_{i,j}| \le \delta$ $(1 \le i \le n)$,

$$E[\max_{1 \le i \le n} (\sum_{j=1}^m |X(t_{i,j}) - X(s_{i,j})|)^2] \le \delta \liminf_{k \to \infty} E[\int_0^1 |u^{\varepsilon_k}(t)|^2 dt]. \qquad (27)$$

Indeed, by the monotone convergence theorem and by the continuity of $\{X(t)\}_{0 \le t \le 1}$, (3.6) implies that, for all $m \ge 2$,

$$E[\sup\{(\sum_{j=1}^m |X(t_j) - X(s_j)|)^2 : \sum_{j=1}^m |t_j - s_j| \le \delta \qquad (28)$$
$$, 0 \le s_j \le t_j \le s_{j+1} \le t_{j+1} \le 1 (1 \le j \le m-1)\}]$$
$$\le \delta \liminf_{k \to \infty} E[\int_0^1 |u^{\varepsilon_k}(t)|^2 dt].$$

The left hand side of (3.7) converges, as $m \to \infty$, to

$$E[\sup\{(\sum_{j=1}^m |X(t_j) - X(s_j)|)^2 : \sum_{j=1}^m |t_j - s_j| \le \delta, m \ge 2 \qquad (29)$$
$$, 0 \le s_j \le t_j \le s_{j+1} \le t_{j+1} \le 1 (1 \le j \le m-1)\}]$$

since the integrand on the left hand side of (3.7) is nondecreasing in $m$.
Hence by Fatou's lemma,

$$\lim_{\delta \to 0} (\sup\{(\sum_{j=1}^m |X(t_j) - X(s_j)|)^2 : \sum_{j=1}^m |t_j - s_j| \le \delta, m \ge 2 \qquad (30)$$
$$, 0 \le s_j \le t_j \le s_{j+1} \le t_{j+1} \le 1 (1 \le j \le m-1)\}) = 0 \quad \text{a.s.,}$$

since the integrand in (3.8) is nondecreasing in $\delta > 0$ and henceforth is convergent as $\delta \to 0$.

To complete the proof, we prove (3.6). By Jensen's inequality, for $i = 1, \cdots, n$ for which $\sum_{j=1}^{m} |t_{i,j} - s_{i,j}| > 0$,

$$
\begin{aligned}
&(\sum_{j=1}^{m} |X(t_{i,j}) - X(s_{i,j})|)^2 \\
&\leq \ (\sum_{j=1}^{m} |t_{i,j} - s_{i,j}|) \sum_{1 \leq j \leq m, s_{i,j} < t_{i,j}} \left| \frac{X(t_{i,j}) - X(s_{i,j})}{t_{i,j} - s_{i,j}} \right|^2 (t_{i,j} - s_{i,j}).
\end{aligned} \tag{31}
$$

Put $A_{mn} := \{t_{i,j}, s_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\}$ and $\{t_k\}_{1 \leq k \leq \#(A_{mn})} := A_{mn}$ so that $t_k < t_{k+1}$ for $k = 1, \cdots, \#(A_{mn}) - 1$, where $\#(A_{mn})$ denotes the cardinal number of the set $A_{mn}$. Then, by Jensen's inequality,

$$
\begin{aligned}
&\sum_{1 \leq j \leq m, s_{i,j} < t_{i,j}} \left| \frac{X(t_{i,j}) - X(s_{i,j})}{t_{i,j} - s_{i,j}} \right|^2 (t_{i,j} - s_{i,j}) \\
&\leq \ \sum_{1 \leq k \leq \#(A_{mn})-1} \left| \frac{X(t_k) - X(t_{k+1})}{t_{k+1} - t_k} \right|^2 (t_{k+1} - t_k).
\end{aligned} \tag{32}
$$

The following completes the proof: for any $k = 1, \cdots, \#(A_{mn}) - 1$,

$$
\begin{aligned}
E\left[ \left| \frac{X(t_k) - X(t_{k+1})}{t_{k+1} - t_k} \right|^2 \right] &\leq \ \liminf_{\ell \to \infty} E\left[ \left| \frac{X_\ell(t_k) - X_\ell(t_{k+1})}{t_{k+1} - t_k} \right|^2 \right] \\
&\leq \ \frac{1}{t_{k+1} - t_k} \liminf_{\ell \to \infty} E[\int_{t_k}^{t_{k+1}} |u^{\varepsilon_\ell}(t)|^2 dt].
\end{aligned} \tag{33}
$$

Q. E. D.

We prove Theorem 1 by Lemmas 1-3.
(Proof of Theorem 1). The proof of (2.5) is divided into the following:

$$
\limsup_{\varepsilon \to 0} \tilde{\mathbf{e}}^\varepsilon \ \leq \ \tilde{\mathbf{e}}^0, \tag{34}
$$

$$
\liminf_{\varepsilon \to 0} \mathbf{e}^\varepsilon \ \geq \ \mathbf{e}^0, \tag{35}
$$

11

since $\mathbf{e}^\varepsilon = \tilde{\mathbf{e}}^\varepsilon$ by Lemma 1, and since $\mathbf{e}^0 \geq \tilde{\mathbf{e}}^0$ in the same way as in the proof of the inequality $\mathbf{e}^\varepsilon \geq \tilde{\mathbf{e}}^\varepsilon$ (see (3.4)-(3.5)).

We first prove (3.13). For $B \in \tilde{A}^0$ for which $\int_0^1 \int_{\mathbf{R}^d} L(t, x; B(t, x))p(t, x)dtdx$ is finite and $\varepsilon > 0$, $B(t, x) + \varepsilon^2 \nabla p(t, x)/(2p(t, x)) \in \tilde{A}^\varepsilon$.

Indeed, for any $f \in C_o^\infty(\mathbf{R}^d)$ and any $t \in [0, 1]$,

$$\int_{\mathbf{R}^d} f(x)(p(t, x) - p(0, x))dx$$
$$= \int_0^t ds \int_{\mathbf{R}^d} < \nabla f(x), B(s, x) > p(s, x)dx$$
$$= \int_0^t ds \int_{\mathbf{R}^d} \left( \frac{\varepsilon^2}{2}\triangle f(x) + \left\langle \nabla f(x), B(s, x) + \frac{\varepsilon^2 \nabla p(s, x)}{2p(s, x)} \right\rangle \right)p(s, x)dx.$$

For any $t \in [0, 1]$, $x$, $u$, $v \in \mathbf{R}^d$, and $z \in \partial_u L(t, x; u + v)$, by (2.2),

$$L(t, x; u + v) \leq L(t, x; u) - < z, v > \qquad (36)$$
$$\leq L(t, x; u) + C_{\nabla L}(1 + |u + v|)^{q-1}|v|.$$

Putting $u = B(t, x)$ and $v = \varepsilon^2 \nabla p(t, x)/(2p(t, x))$ in (3.15), we have

$$\tilde{\mathbf{e}}^\varepsilon \leq \int_0^1 \int_{\mathbf{R}^d} L\left( t, x; B(t, x) + \frac{\varepsilon^2 \nabla p(t, x)}{2p(t, x)} \right)p(t, x)dtdx \qquad (37)$$
$$\leq \int_0^1 \int_{\mathbf{R}^d} C_{\nabla L}\left( 1 + \left| B(t, x) + \frac{\varepsilon^2 \nabla p(t, x)}{2p(t, x)} \right| \right)^{q-1} \left| \frac{\varepsilon^2 \nabla p(t, x)}{2p(t, x)} \right| p(t, x)dtdx$$
$$+ \int_0^1 \int_{\mathbf{R}^d} L(t, x; B(t, x))p(t, x)dtdx$$
$$\to \int_0^1 \int_{\mathbf{R}^d} L(t, x; B(t, x))p(t, x)dtdx \quad (\text{as } \varepsilon \to 0)$$

by (2.1) and (H.3), where we used the following in the last line of (3.16):

$$\frac{q-1}{q} + \frac{1}{q} = 1.$$

Next we prove (3.14). By Lemmas 2-3, we only have to show the following: for any $\{u^{\varepsilon_n}(t)\}_{0 \leq t \leq 1} \in A^{\varepsilon_n}$ $(n \geq 1)$ $(\varepsilon_n \to 0$ as $n \to \infty)$ for which

12

$\{X_n(t) := X^{\varepsilon_n, u^{\varepsilon_n}}(t)\}_{0 \le t \le 1}$ weakly converges, as $n \to \infty$, to a stochastic process $\{X(t)\}_{0 \le t \le 1}$, and for which $\{E[\int_0^1 L(t, X_n(t); u^{\varepsilon_n}(t))dt]\}_{n \ge 1}$ is bounded,

$$\liminf_{n \to \infty} E[\int_0^1 L(t, X_n(t); u^{\varepsilon_n}(t))dt] \ge E[\int_0^1 L\Big(t, X(t); \frac{dX(t)}{dt}\Big)dt]. \qquad (38)$$

We prove (3.17). For $\alpha \in (0,1)$ and $\delta > 0$,

$$E[\int_0^1 L(t, X_n(t); u^{\varepsilon_n}(t))dt] \qquad (39)$$
$$\ge \frac{1}{1 + R(\alpha, \delta)} E[\int_0^{1-\alpha} ds L\Big(s, X_n(s); \frac{1}{\alpha} \int_s^{s+\alpha} u^{\varepsilon_n}(t)dt\Big)$$
$$; \sup_{0 \le t, s \le 1, |t-s| < \alpha} |X_n(t) - X_n(s)| < \delta] - R(\alpha, \delta).$$

Indeed, if $\sup_{0 \le t, s \le 1, |t-s| < \alpha} |X_n(t) - X_n(s)| < \delta$, then for $s \in [0, 1-\alpha]$, by Jensen's inequality and (H.1),

$$L\Big(s, X_n(s); \frac{1}{\alpha} \int_s^{s+\alpha} u^{\varepsilon_n}(t)dt\Big) \le \frac{1}{\alpha} \int_s^{s+\alpha} L(s, X_n(s); u^{\varepsilon_n}(t))dt \quad (40)$$
$$\le R(\alpha, \delta) + \frac{1 + R(\alpha, \delta)}{\alpha} \int_s^{s+\alpha} L(t, X_n(t); u^{\varepsilon_n}(t))dt.$$

Hence putting $u = \int_s^{s+\alpha} u^{\varepsilon_n}(t)dt/\alpha$ and $v = (X_n(s + \alpha) - X_n(s) - \int_s^{s+\alpha} u^{\varepsilon_n}(t)dt)/\alpha$ in (3.15), we have, from (3.18),

$$E[\int_0^1 L(t, X_n(t); u^{\varepsilon_n}(t))dt] \qquad (41)$$
$$\ge \frac{1}{1 + R(\alpha, \delta)} E[\int_0^{1-\alpha} L\Big(s, X_n(s); \frac{X_n(s + \alpha) - X_n(s)}{\alpha}\Big)ds$$
$$; \sup_{0 \le t, s \le 1, |t-s| < \alpha} |X_n(t) - X_n(s)| < \delta]$$
$$- E[\int_0^{1-\alpha} C_{\nabla L}\Big(1 + \Big|\frac{X_n(s + \alpha) - X_n(s)}{\alpha}\Big|\Big)^{q-1}$$
$$\times \Big|\frac{\varepsilon_n}{\alpha}(W(s + \alpha) - W(s))\Big|ds] - R(\alpha, \delta).$$

13

Letting $n \to \infty$ and then $\alpha \to 0$ and $\delta \to 0$ in (3.20), we obtain (3.17).

Indeed, by Skorohod's theorem (see e.g. [13]), taking a new probability space, we can assume that $\{X_n(t)\}_{0 \le t \le 1}$ converges, as $n \to \infty$, to $\{X(t)\}_{0 \le t \le 1}$ in sup norm, a.s., and that the following holds: for any $\beta \in (0, \delta/3)$, by (H.1),

$$(1 + R(0, \beta))E[\int_0^{1-\alpha} L\left(s, X_n(s); \frac{X_n(s+\alpha) - X_n(s)}{\alpha}\right) ds$$
$$; \sup_{0 \le t, s \le 1, |t-s| < \alpha} |X_n(t) - X_n(s)| < \delta]$$
$$\ge E[\int_0^{1-\alpha} L\left(s, X(s); \frac{X_n(s+\alpha) - X_n(s)}{\alpha}\right) ds; \sup_{0 \le t \le 1} |X(t) - X_n(t)| < \beta$$
$$, \sup_{0 \le t, s \le 1, |t-s| < \alpha} |X(t) - X(s)| < \beta] - R(0, \beta).$$

The liminf of the right-hand side of this inequality as $n \to \infty$, and $\alpha \to 0$ and then $\beta \to 0$ is dominated by $E[\int_0^1 L(s, X(s); dX(s)/ds)ds]$ from below by Fatou's lemma. The second mean value on the right hand side of (3.20) can be shown to converge to zero as $n \to \infty$ in the same way as in (3.16) by (2.1).

(H.0) and (2.5) implies that the set $A$ and $A^\varepsilon$ ($\varepsilon > 0$) are not empty.

(2.5) and (3.17) completes the proof.

<div align="right">Q. E. D.</div>

(Proof of Theorem 2). $b^X(t, x)$ is a minimizer of $\tilde{\mathbf{e}}^0$ by (2.5) in the same way as in (3.4)-(3.5).

We prove the uniqueness of the minimizer of $\tilde{\mathbf{e}}^0$. Suppose that $b^o(t, x)$ is also a minimizer of $\tilde{\mathbf{e}}^0$. Then for any $\lambda \in (0, 1)$, $\lambda b^X + (1 - \lambda)b^o \in \tilde{A}^0$, and

$$\tilde{\mathbf{e}}^0 \le \int_0^1 \int_{\mathbf{R}^d} L(t, y; \lambda b^X(t, y) + (1 - \lambda)b^o(t, y))p(t, y)dtdy \qquad (42)$$
$$\le \lambda \int_0^1 \int_{\mathbf{R}^d} L(t, y; b^X(t, y))p(t, y)dtdy$$
$$+ (1 - \lambda) \int_0^1 \int_{\mathbf{R}^d} L(t, y; b^o(t, y))p(t, y)dtdy = \tilde{\mathbf{e}}^0.$$

By the strict convexity of $L$ in $u$,

$$b^X(t, x) = b^o(t, x), \quad p(t, x)dtdx - a.e.. \qquad (43)$$

<div align="center">14</div>

We prove (2.7). Since $L$ is strictly convex in $u$, the following holds:

$$\frac{dX(t)}{dt} = b^X(t, X(t)) \quad dtdP - a.e. \tag{44}$$

by (2.5) (see (3.5)). By (3.22),

$$E[\sup_{0 \le t \le 1} |X(t) - X(0) - \int_0^t b^o(s, X(s))ds|] \tag{45}$$

$$\le \int_0^1 E[|b^X(s, X(s)) - b^o(s, X(s))|]ds = 0.$$

Q. E. D.

(Proof of Proposition 1). From [15], $\tilde{\mathbf{e}}^\varepsilon = \mathbf{e}^\varepsilon$ for $\varepsilon > 0$, and the minimizer of $\tilde{\mathbf{e}}^\varepsilon$ can be written as $\nabla_x \Phi^\varepsilon(t, x)$, where $\Phi^\varepsilon(t, \cdot) \in H^1_{loc}(\mathbf{R}^d : \mathbf{R})$ dt-a.e.. Since $\{\nabla_x \Phi^\varepsilon\}_{0 < \varepsilon < 1}$ is strongly bounded in $L^2([0, 1] \times \mathbf{R}^d : \mathbf{R}^d, p(t, x)dtdx)$ by (2.5), it is weakly compact in $L^2([0, 1] \times \mathbf{R}^d : \mathbf{R}^d, p(t, x)dtdx)$ (see [9, p. 639]). We denote a weak limit point by $\Psi$. Then $\Psi$ is the unique minimizer of $\tilde{\mathbf{e}}^0$. Indeed, $\Psi \in \tilde{A}^0$, and by (2.5) and Fatou's lemma,

$$\tilde{\mathbf{e}}^0 = \lim_{\varepsilon \to 0} \int_0^1 \int_{\mathbf{R}^d} |\nabla_x \Phi^\varepsilon(t, y)|^2 p(t, y)dtdy \tag{46}$$

$$\ge \int_0^1 \int_{\mathbf{R}^d} |\Psi(t, y)|^2 p(t, y)dtdy \ge \tilde{\mathbf{e}}^0.$$

In particular, $\{\nabla_x \Phi^\varepsilon\}_{0 < \varepsilon < 1}$ converges, as $\varepsilon \to 0$, to $\Psi$, strongly in $L^2([0, 1] \times \mathbf{R}^d : \mathbf{R}^d, p(t, x)dtdx)$, which completes the proof in the same way as in [15, Proposition 3.1].

Q. E. D.

**Remark 6** *If $V(t, x)$ and $p(t, x)$ in Proposition 1 are sufficiently smooth, then*

$$\nabla_x \Phi^\varepsilon(t, x) = \nabla_x V(t, x) + \frac{\varepsilon^2 \nabla_x p(t, x)}{2p(t, x)}$$

*(see [16, section 1]).*

15

(Proof of Theorem 3). Put for $t \in [0, 1]$, $x \in \mathbf{R}$ and $n \geq 1$,

$$
\begin{aligned}
Y(t, x) &= F_t^{-1}(F_0(x)), && (47) \\
Y_n(t, x) &= Y\left(\frac{[nt]}{n}, x\right) && (48) \\
&\quad + n\left(t - \frac{[nt]}{n}\right)\left(Y\left(\frac{[nt]+1}{n}, x\right) - Y\left(\frac{[nt]}{n}, x\right)\right),
\end{aligned}
$$

where $[nt]$ denotes the integer part of $nt$.

Then by (H.3)", $Y(\cdot, x) \in C([0, 1] : \mathbf{R})$, $P_0(dx) := p(0, x)dx - a.s.$, and

$$
\lim_{n \to \infty} Y_n(t, x) = Y(t, x) \quad (0 \leq t \leq 1), \quad P_0 - a.s., \tag{49}
$$

and

$$
\mathbf{e}_n = E_0\left[\int_0^1 h\left(\frac{dY_n(t, x)}{dt}\right)dt\right] \quad (n \geq 1) \tag{50}
$$

(see Remark 5 in section 2 and [11, p. 35, Exam. 8.1]).

Hence in the same way as in the proof of Lemma 3, we can show that the following holds: for any $\delta > 0$

$$
\begin{aligned}
E_0[\sup\{(\sum_{j=1}^m |Y(t_j, x) - Y(s_j, x)|)^r : \sum_{j=1}^m |t_j - s_j| \leq \delta, m \geq 2 \quad (51) \\
, 0 \leq s_j \leq t_j \leq s_{j+1} \leq t_{j+1} \leq 1(1 \leq j \leq m - 1)\}] \\
\leq \quad \delta^{r-1} \liminf_{n \to \infty} E_0\left[\int_0^1 \left|\frac{dY_n(t, x)}{dt}\right|^r dt\right],
\end{aligned}
$$

which implies that $Y(\cdot, x)$ is absolutely continuous $P_0 - a.s.$, by (H.0)' and (H.2)'. In particular, $\{Y(t, x)\}_{0 \leq t \leq 1}$ on $(\mathbf{R}, \mathbf{B}(\mathbf{R}), P_0)$ belongs to the set $A$.

For $n \geq 1$ and $\alpha \in (0, 1)$, by Jensen's inequality and (H.1)',

$$
\begin{aligned}
\infty > \sup_{m \geq 1} \mathbf{e}_m \geq \mathbf{e}_n &\geq E_0\left[\int_0^{1-\alpha} ds\left(\frac{1}{\alpha}\int_s^{s+\alpha} h\left(\frac{dY_n(t, x)}{dt}\right)dt\right)\right] && (52) \\
&\geq E_0\left[\int_0^{1-\alpha} h\left(\frac{Y_n(s + \alpha, x) - Y_n(s, x)}{\alpha}\right)ds\right].
\end{aligned}
$$

16

Let $n \to \infty$ and then $\alpha \to 0$ in (3.31). Then the proof of the first part is over by Fatou's lemma since $\sup_{m \geq 1} \mathbf{e}_m \leq \mathbf{e}$.

The following together with Proposition 2 in section 4 completes the proof: by (2.12),

$$Y(t, x) = Y(0, x) + \int_0^t \frac{\partial F_s^{-1}(F_s(Y(s, x)))}{\partial s} \, ds \quad (0 \leq t \leq 1) \quad P_0 - a.s..$$

Q. E. D.

# 4  Appendix

In this section we solve SP positively for $\mathbf{R}$-valued, time-inhomogeneous stochastic processes.

**Proposition 2** *Suppose that (H.3)' holds, and that there exists $\{Y(t)\}_{0 \leq t \leq 1}$, $\in A$, which satisfies*

$$Y(t) = Y(0) + \int_0^t b^Y(s, Y(s)) ds \quad (0 \leq t \leq 1) \; a.s. \tag{53}$$

*for some $b^Y(t, x) \in L^1([0, 1] \times \mathbf{R} : \mathbf{R}, p(t, x) dt dx)$. Then the following holds:*

$$Y(t) = F_t^{-1}(F_0(Y(0))) \quad (t \in \mathbf{Q} \cap [0, 1]) \quad a.s.. \tag{54}$$

*In particular, $\lim_{s \in \mathbf{Q} \cap [0,1], s \to t} F_s^{-1}(F_0(Y(0)))$ exists and is equal to $Y(t)$ for all $t \in [0, 1]$ a.s..*

**Remark 7** *If $F_0$ is not continuous, then SP does not always have a positive answer. For example, put $Y(t) \equiv tY(\omega)$ for a $\mathbf{R}$-valued random variable $Y(\omega)$ on a probability space. Then $dY(t)/dt = Y(t)/t$ for $t > 0$. But, of course, $Y(t)$ is not a function of $t$ and $Y(0) \equiv 0$.*

(Proof of Proposition 2). It is easy to see that the following holds:

$$F_t(Y(t)) = F_0(Y(0)) \quad (t \in [0, 1]) \text{ a.s..} \tag{55}$$

Indeed,

$$\frac{\partial F_t(x)}{\partial t} = -b^Y(t, x)p(t, x), \quad dtdx - a.e.$$

since $b^Y(t, x) = b(t, x), \ p(t, x)dtdx - a.e.$, and henceforth by (H.3)'

$$E[\sup_{0 \leq t \leq 1} |F_t(Y(t)) - F_0(Y(0))|]$$
$$\leq \int_0^1 E[|\frac{\partial F_s(Y(s))}{\partial s} + p(s, Y(s))b^Y(s, Y(s))|]ds = 0.$$

Since $\{Y(t)\}_{0 \leq t \leq 1}$ is continuous, the proof is over by (4.3) and by the following:

$$P(F_t^{-1}(F_t(Y(t))) = Y(t)(t \in [0, 1] \cap \mathbf{Q})) = 1. \qquad (56)$$

We prove (4.4). For $(t, x) \in [0, 1] \times \mathbf{R}$ for which $F_t(x) \in (0, 1)$,

$$F_t^{-1}(F_t(x)) \leq x,$$

and for $t \in [0, 1]$, the set $\{x \in \mathbf{R} : F_t^{-1}(F_t(x)) < x, F_t(x) \in (0, 1)\}$ can be written as a union of at most countably many disjoint intervals of the form $(a, b]$ for which $P(a < Y(t) \leq b) = 0$, provided that it is not empty.

Indeed, if $F_t^{-1}(F_t(x)) < x$ and if $F_t(x) \in (0, 1)$, then

$$\{y \in \mathbf{R} : F_t^{-1}(F_t(y)) < y, F_t(y) = F_t(x)\}$$
$$= (F_t^{-1}(F_t(x)), \sup\{y \in \mathbf{R} : F_t(y) = F_t(x)\}].$$

<div align="right">Q. E. D.</div>

# References

[1] Breiman, L. (1992), *Probability*, SIAM, Philadelphia.

[2] Brenier, Y. and Benamou, J. D. (1999), *A numerical method for the optimal mass transport problem and related problems*, Contemporary Mathematics **226**, Amer. Math. Soc., Providence, 1-11.

[3] Brenier, Y. and Benamou, J. D. (2000), *A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem*, Numer. Math. **84**, 375-393.

[4] Carlen, E. A. (1984), *Conservative diffusions*, Commun. Math. Phys. **94**, 293-315.

[5] Carlen, E. A. (1986), *Existence and sample path properties of the diffusions in Nelson's stochastic machanics*, Lecture Notes in Mathematics **1158**, Springer, Berlin Heidelberg New York, 25-51.

[6] Çinlar, E. and Jacod, J. (1981), *Representation of semimartingale Markov processes in terms of Wiener processes and Poisson random measures*, Seminar on Stochastic Processes 1981, Birkhauser, Boston Basel Berlin, 159-242.

[7] Dall'Aglio, G. (1991), *Frèchet classes: the beginning*, Mathematics and its applications **67**, Kluwer Academic Publishers, Dordrecht Boston London, 1-12.

[8] Doob, J. L. (1990), *Stochastic processes*, John Wiley & Sons, Inc., New York.

[9] Evans, L. C. (1998), *Partial differential equations*, AMS, Providence.

[10] Evans, L. C. and Gangbo, W. (1999), *Differential equations methods for the Monge-Kantorovich mass transfer problem*, Mem. Amer. Math. Soc. **137**, No. 653.

[11] Fleming, W. H. and Soner, H. M. (1993), *Controlled Markov Processes and Viscosity Solutions*, Springer, Berlin Heidelberg New York.

[12] Gangbo, W and McCann, R. J. (1996), *The geometry of optimal transportation*, Acta Math. **177**, 113-161.

[13] Ikeda, N. and Watanabe, S. (1981), *Stochastic differential equations and diffusion processes*, North-Holland/Kodansha, Amsterdam New York Oxford Tokyo.

[14] Jordan, R., Kinderlehrer, D. and Otto, F. (1998), *The variational formulation of the Fokker-Planck equation*, SIAM J. Math. Anal. **29**, 1-17.

[15] Mikami, T. (1990), *Variational processes from the weak forward equation*, Commun. Math. Phys. **135**, 19-40.

[16] Mikami, T. (2000), *Dynamical systems in the variational formulation of the Fokker-Planck equation by the Wasserstein metric*, Appl. Math. Optim. **42**, 203-227.

[17] Nelsen, R. B. (1999), *An Introduction to Copulas*, Lecture Notes in Statistics **139**, Springer, Berlin Heidelberg New York.

[18] Otto, F. (2001), *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations **26**, 101-174.

[19] Quastel, J. and Varadhan, S. R. S. (1997), *Diffusion semigroups and diffusion processes corresponding to degenerate divergence form operators*, Comm. Pure Appl. Math. **50**, 667-706.

[20] Rachev, S. T. and Rüschendorf, L. (1998), *Mass transportation problems, Vol. I: Theory*, Springer, Berlin Heidelberg New York.

[21] Salisbury, T. S. (1986), *An increasing diffusion*, Seminar on Stochastic Processes 1984, Birkhauser, Boston Basel Berlin, 173-194.