Statistical hypersurfaces in the space of Hessian curvature zero II

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Abstract

We construct cylindrical statistical immersions between spaces of Hessian curvature zero.

The words "statistical submanifold" can be found in the paper [5] in 1989, which was written by Vos in the context of statistical inference or information geometry. Although the history of this geometry is not so short, it is hard to find classical differential geometric approaches for the study of statistical submanifolds. In this paper, we would like to continue to try it after [2], and give some of basic examples of statistical submanifolds apart from applications for statistics. In other words, we will study immersions between statistical manifolds preserving statistical structures, which are called *statistical immersions*, in particular, called *statistical hypersurfaces* if the codimension equals one. We take a space N^n in Definition 1.1, which can be considered as a basic model of a statistical manifold of dimension n. The space N^n has been known as a Hessian manifold of constant Hessian curvature zero. In [2], a statistical hypersurface of a Hessian manifold of constant Hessian curvature negative into the space N^{n+1} is uniquely determined. Besides, there exist no statistical hypersurfaces of a Hessian manifold of constant Hessian curvature positive into the space N^{n+1}. On the other hand, we have plenty of statistical hypersurfaces of N^n into N^{n+1}.

In this paper, we determine statistical diffeomorphisms of N^n onto itself, and statistical hypersurfaces of N^n into N^{n+1} with vanishing statistical second fundamental form (Propositions 2.1 and 2.2). Moreover, we explicitly construct and determine statistical immersions of a domain of N^2 into N^3 of cylinder type (Theorem 3.1).

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1 Preliminaries

A triple (M, ∇, g) of a manifold, an affine connection and a Riemannian metric on it is called a *statistical manifold* if the connection ∇ is of torsion free and the Codazzi equation $(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z)$ holds for any vector fields $X, Y, Z \in \Gamma(TM)$. Furthermore, a statistical manifold (M, ∇, g) is called a *Hessian manifold* if the connection ∇ is flat. In this case, we can naturally define the Kähler structure on the tangent bundle of M. A Hessian manifold (M, ∇, g) is said to be of *constant Hessian curvature* c $(\in \mathbb{R})$ if the holomorphic sectional curvature of the corresponding Kähler metric is constant $-c$, which expresses that

$$(\nabla_X K)(Y,Z) = -\frac{c}{2}\{g(X,Y)Z + g(X,Z)Y\}, \quad X, Y, Z \in \Gamma(TM),$$

where K is the $(1,2)$ tensor field given by

$$K(X,Y) := \nabla_X Y - \nabla^g_X Y,$$

and ∇^g is the Levi-Civita connection of g (See [4]). We should remark that

$$C(X,Y,Z) := (\nabla_X g)(Y,Z) = -2g(K(X,Y),Z)$$

for any vector fields $X, Y, Z \in \Gamma(TM)$.
Among the Hessian manifolds of constant Hessian curvature zero, a non-trivial and basic model is given as follows:

Definition 1.1. Set $N^n := ((\mathbb{R}^+)^n, \nabla^{N^n}, g_{N^n})$ as

$$(\mathbb{R}^+)^n := \{y = {}^t(y^1, \dots, y^n) \in \mathbb{R}^n \mid y^1 > 0, \cdots, y^n > 0\},$$
$$\nabla^{N^n}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = -\delta_{ij}(y^j)^{-1}\frac{\partial}{\partial y^j}, \qquad g_{N^n} := g_{E^n}|_{(\mathbb{R}^+)^n} = \sum_{j=1}^n (dy^j)^2,$$

where g_{E^n} is the Euclidean metric on $\mathbb{R}^n$. Then N^n is a Hessian manifold of constant Hessian curvature zero, and besides ∇^{N^n} is complete. We call N^n the *space of Hessian curvature zero* of dimension n in this paper. □

A characterization of the space N^n is given in [3]. We calculate that

$$C_{N^n} = -2\sum_{j=1}^n (y^j)^{-1} dy^j \otimes dy^j \otimes dy^j.$$

Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical manifold and $f : M \to \widetilde{M}$ an immersion of a manifold M. Define g and ∇ on M by

$$g = f^*\widetilde{g}, \quad g(\nabla_X Y, Z) = \widetilde{g}(\widetilde{\nabla}_X f_* Y, f_* Z), \quad X, Y, Z \in \Gamma(TM).$$

Then (M, ∇, g) is a statistical manifold, and (∇, g) is called the statistical structure *induced by f from* $(\widetilde{\nabla}, \widetilde{g})$. An immersion between statistical manifolds is called a *statistical immersion* if the induced structure coincides with the original one. Let (M, ∇, g) and $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be statistical manifolds with C and $\widetilde{C}$, respectively. An isometric immersion

$f : M \to \widetilde{M}$ is a statistical immersion if and only if $f^*\widetilde{C} = C$. A statistical immersion of codimension one is called a *statistical hypersurface* as well.
Let $f : (M, \nabla, g) \to (\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical hypersurface with a unit normal vector field n. Define $h, II \in \Gamma(TM^{(0,2)})$ by

$$\begin{aligned} \widetilde{\nabla}_X f_* Y &= f_* \nabla_X Y + h(X,Y)n, \\ \nabla^{\widetilde{g}}_X f_* Y &= f_* \nabla^g_X Y + II(X,Y)n, \quad X, Y \in \Gamma(TM), \end{aligned}$$

where II is the Riemannian second fundamental form of f, and h should be called the *statistical second fundamental form* of f. Let $\widetilde{\nabla}^*$ be the dual connection of $\widetilde{\nabla}$ with respect to $\widetilde{g}$. By definition,

$$W\widetilde{g}(U,V) = \widetilde{g}(\widetilde{\nabla}_W U, V) + \widetilde{g}(U, \widetilde{\nabla}^*_W V)$$

holds for $U, V, W \in \Gamma(T\widetilde{M})$. We can define another second fundamental form h^* by

$$\widetilde{\nabla}^*_X f_* Y = f_* \nabla^*_X Y + h^*(X,Y)n.$$

It should remark that ∇^* is the dual connection of ∇ with respect to g, and that

$$II = \frac{1}{2}(h + h^*). \tag{1.1}$$

We should remark that if $h = II = 0$, then (∇, g) is a Hessian structure of Hessian curvature zero.

2 Statistical Hyperplanes

Let us determine a statistical diffeomorphism $\varphi : N^n \to N^n$ and a statistical hypersurface $f : N^n \to N^{n+1}$ with vanishing statistical second fundamental form in a global setting.

Proposition 2.1. *A statistical diffeomorphism between N^n is given as $\varphi : (\mathbb{R}^+)^n \ni (y^i) \mapsto (y^{\sigma(i)}) \in (\mathbb{R}^+)^n$ for some permutation σ on $\{1, \ldots, n\}$.*

The proof will be omitted because it follows from the argument similar to Step 2 of the proof for Proposition 2.2. Let $M_{n+1,n}(\mathbb{R})$ denote the set of all the $(n+1) \times n$ real matrices.

Proposition 2.2. *Let $f : N^n \to N^{n+1}$ be a statistical hypersurface. If the statistical second fundamental form h vanishes, then so does the Riemannian one, and moreover, f is a hyperplane expressed as*

$$f^\alpha((y^i)) = e^{b^\alpha}(y^1)^{l^\alpha_1} \cdots (y^n)^{l^\alpha_n},$$

for some $L = (l^\alpha_j) \in M_{n+1,n}(\mathbb{R})$ and $b = (b^\alpha) \in \mathbb{R}^{n+1}$ such that (1) $l^\alpha_j = 0$ or 1 for $\alpha = 1, \ldots, n+1$, $j = 1, \ldots, n$, (2) rank $L = n$, *(3) $l^\alpha_1 + \cdots + l^\alpha_n \leq 1$.*

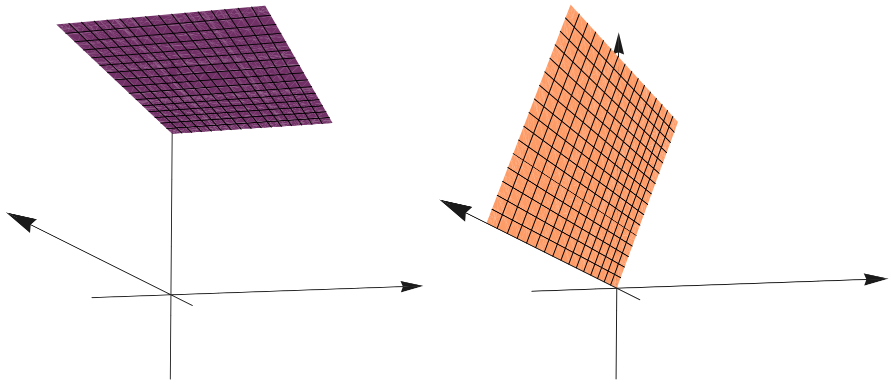

Figure 1: Examples of f in Proposition 2.2

Proof. [Step 1] We claim the following in general: Let $f : (M, \nabla, g) \to (\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical hypersurface. If (∇, g) and $(\widetilde{\nabla}, \widetilde{g})$ are Hessian structures of Hessian curvature zero, and if the statistical second fundamental form h vanishes, then so does the Riemannian second fundamental form II.
Let us prove the claim. According to [1], the tangential component of $(\widetilde{\nabla}_X \widetilde{K})(f_*Y, f_*Z)$ is given as

$$(\nabla_X K)(Y, Z) - b(Y, Z)A^*X + h(X, Y)B^*Z + h(X, Z)B^*Y$$

for any vector fields $X, Y, Z \in \Gamma(TM)$, where

$$\begin{aligned} &b = h - II, \quad B^* = A^* - S, \\ &\widetilde{\nabla}_X n = -f_*A^*X + \tau^*(X)n, \quad \nabla^{\widetilde{g}}_X n = -f_*SX. \end{aligned}$$

By the assumption, we get

$$0 = -b(Y, Z)A^*X = II(Y, Z)A^*X.$$

If A^* vanishes, so does h^*, since $g(A^*X, Y) = h^*(X, Y)$. The formula (1.1) implies that the Riemannian second fundamental form vanishes.
[Step 2] In the Euclidean differential geometry, $II = 0$ implies f is a part of a hyperplane, and we may assume that

$$f^\alpha((y^i)) = \sum a^\alpha_j y^j + b^\alpha, \qquad \text{for some } A = (a^\alpha_j) \in M_{n+1,n}(\mathbb{R}),\ b = (b^\alpha) \in \mathbb{R}^{n+1}.$$

The indexes run as follows: $i, j, k, l = 1, \ldots, n$ and $\alpha, \beta, \gamma = 1, \ldots, n+1$. Since f is isometric and $f((\mathbb{R}^+)^n) \subset (\mathbb{R}^+)^{n+1}$, we have

$$\sum a^\alpha_i a^\alpha_j = \delta_{ij}, \qquad a^\alpha_i \geq 0, \quad b^\alpha \geq 0. \tag{2.1}$$

The direct calculation implies that the condition $f^*C_{N^{n+1}} = C_{N^n}$ is equivalent to

$$(y^i)^{-1}\delta_{ij}\delta_{ik} = \sum a^\gamma_i a^\gamma_j a^\gamma_k \left(\sum a^\gamma_l y^l + b^\gamma\right)^{-1}. \tag{2.2}$$

Since $\operatorname{rank} A = n$, for any $i \in \{1, \ldots, n\}$ there exists $\gamma \in \{1, \ldots, n+1\}$ such that $a^\gamma_i \neq 0$. Let $\gamma_1(i), \ldots, \gamma_p(i)$ be the indexes such that $a^{\gamma_1(i)}_i \cdots a^{\gamma_p(i)}_i \neq 0$. Using (2.1), we write (2.2) with $i = j \neq k$ as $0 = (a_i^\gamma)^2 a_k^\gamma (\sum a_l^\gamma y^l + b^\gamma)^{-1}$ for each γ, and then $0 = a_k^{\gamma_q(i)}(\sum a_l^{\gamma_q(i)} y^l + b^{\gamma_q(i)})^{-1}$, which implies

$$a_k^{\gamma_1(i)} = \cdots = a_k^{\gamma_p(i)} = 0 \quad \text{for } k \neq i \in \{1, \ldots, n\}, \tag{2.3}$$

and

$$(a_i^{\gamma_1(i)})^2 + \cdots + (a_i^{\gamma_p(i)})^2 = 1. \tag{2.4}$$

Using (2.1) again, from (2.2) with $i = j = k$ we get that for each i,

$$\begin{aligned} 1 &= \sum_\gamma y^i (a_i^\gamma)^3 (\sum_l a_l^\gamma y^l + b^\gamma)^{-1} \\ &= \sum_q y^i (a_i^{\gamma_q(i)})^3 (a_i^{\gamma_q(i)} y^i + b^{\gamma_q(i)})^{-1} \\ &= \sum_q (a_i^{\gamma_q(i)})^3 (a_i^{\gamma_q(i)} + \frac{b^{\gamma_q(i)}}{y^i})^{-1} \\ &\leq \sum_q (a_i^{\gamma_q(i)})^2 = 1, \end{aligned}$$

which implies

$$b^{\gamma_1(i)} = \cdots = b^{\gamma_p(i)} = 0 \quad \text{for } i \in \{1, \ldots, n\}. \tag{2.5}$$

The conditions (2.3), (2.4) and (2.5) can be rewrite as in the statement in our proposition. □

3 Statistical Cylinders

As non-trivial simple examples, we construct cylinders preserving statistical structures by solving a system of ordinary differential equations.

Theorem 3.1. *Let f be a statistical immersion of a domain $\mathbb{R}^+ \times I$ of N^2 into N^3 of the form $f(s,t) := {}^t(s, u(t), v(t)) \in (\mathbb{R}^+)^3$. Then $u(t)$ and $v(t)$ are given by*

$$\begin{cases} u(t) = \left(at^{2/3} + b\right)^{3/2}, \\ v(t) = (1 - a^3)^{-1} \left\{(1 - a^3)t^{2/3} - a^2 b\right\}^{3/2} \end{cases} \tag{3.1}$$

for constants $a \in (-\infty, 1)$ and $b \in \mathbb{R}$.

Proof. Since we calculate

$$\begin{aligned} f^* g_{N^3} &= ds^2 + \left(u'(t)^2 + v'(t)^2\right) dt^2, \\ f^* C_{N^3} &= -2\left\{s^{-1} ds^3 + \left(u(t)^{-1} u'(t)^3 + v(t)^{-1} v'(t)^3\right) dt^3\right\}, \end{aligned}$$

we should determine functions u, v on an interval $I \subset (0, \infty)$ satisfying

$$\begin{aligned} &u'(t)^2 + v'(t)^2 = 1, \\ &u(t)^{-1} u'(t)^3 + v(t)^{-1} v'(t)^3 = t^{-1}, \\ &u(t) > 0, \quad v(t) > 0. \end{aligned}$$

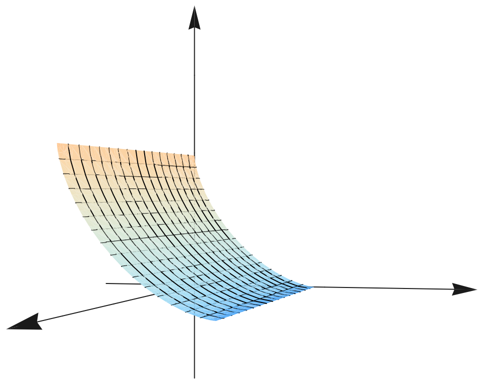

Figure 2: f for $a = -2$ and $b = 3/4$ in Theorem 3.1

Setting

$$p(T) := u(T^{3/2})^{2/3} \quad \text{and} \quad q(T) := v(T^{3/2})^{2/3}, \tag{3.2}$$

we get that they are equivalent to the following:

$$p(T)p'(T)^2 + q(T)q'(T)^2 = T, \tag{3.3}$$
$$p'(T)^3 + q'(T)^3 = 1, \tag{3.4}$$
$$p(T) > 0, \quad q(T) > 0. \tag{3.5}$$

Differentiating (3.4) and (3.3), we have $p'^2p'' + q'^2q'' = 0$ and $pp'p'' + qq'q'' = 0$, that is,

$$\begin{bmatrix} p(T) & q(T) \\ p'(T) & q'(T) \end{bmatrix} \begin{bmatrix} p'(T)p''(T) \\ q'(T)q''(T) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that the following (3.6) or (3.7) occurs:

$$p'(T)p''(T) = q'(T)q''(T) = 0, \tag{3.6}$$
$$p(T)q'(T) - q(T)p'(T) = 0. \tag{3.7}$$

In the case (3.7), there exists a positive constant $\tilde{a}$ such that

$$p(T) = (1 + \tilde{a}^3)^{-1/3}T, \quad q(T) = \tilde{a}(1 + \tilde{a}^3)^{-1/3}T, \tag{3.8}$$

which are derived from $q = \tilde{a}p$ with (3.3) and (3.4).
In the case (3.6), there exist constants a, b, c, d such that

$$a^3 + c^3 = 1, \quad a^2b + c^2d = 0,$$
$$p(T) = aT + b, \quad q(T) = cT + d,$$

in which the first two equations are derived from (3.3) and (3.4). Accordingly, we have

$$\begin{array}{lll} p(T) = aT + b, & q(T) = (1 - a^3)^{1/3}T - a^2(1 - a^3)^{-2/3}b, & \text{if } a < 1, \\ p(T) = T, & q(T) = d, & \text{if } a = 1, \\ p(T) = aT + b, & q(T) = -(a^3 - 1)^{1/3}T - a^2(a^3 - 1)^{-2/3}b, & \text{if } a > 1. \end{array} \tag{3.9}$$

The case (3.8) is considered as the case when $b = 0$ and $a = (1+\tilde{a}^3)^{-1/3}$ in $(3.9)_1$. Besides, $(3.9)_2$ is considered as the case when $a = 0$ and $b = d$ in $(3.9)_1$, and $(3.9)_3$ for constants

$\widetilde{a}(> 1), \widetilde{b}$ is considered as the case when $a = -(\widetilde{a}^3 - 1)^{1/3}$ and $b = -\widetilde{a}^2(\widetilde{a}^3 - 1)^{-2/3}\widetilde{b}$ in $(3.9)_1$. To sum up, we can say that functions p and q satisfying (3.3), (3.4) and (3.5) are given by $(3.9)_1$. Using (3.2), we obtain (3.1). □

Under the setting in Theorem 3.1, if $b = 0$, the interval I of the domain can be the whole $\mathbb{R}^+$ and then f is a plane given in Proposition 2.2.

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