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# A TWO DIMENSIONAL RANDOM CRYSTALLINE ALGORITHM FOR GAUSS CURVATURE FLOW 

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#### Abstract

We propose and study a random crystalline algorithm (a discrete approximation) of the Gauss curvature flow of smooth simple closed convex curves in $\mathbf{R}^{2}$ as a stepping stone to the full understanding of such a phenomenon as the wearing process of stones on beaches.


Keywords: random crystalline algorithm; Gauss curvature flow; closed curve
AMS 2000 Subject Classification: Primary 60D05
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## 1. Introduction.

The Gauss curvature flow of closed surfaces in $\mathbf{R}^{3}$ is a mathematical model of the wearing process of stones on beaches (see [3] and also [1], [6] and [11]).

We introduce the definition of the Gauss curvature flow of smooth closed convex hypersurfaces in $\mathbf{R}^{d+1}$. Let $\Gamma$ be a smooth closed convex hypersurface in $\mathbf{R}^{d+1}$ and $F: \mathbf{S}^{d} \mapsto \mathbf{R}^{d+1}$ be a parametric representation of $\Gamma$. Then a collection of $F(\cdot, t): \mathbf{S}^{d} \mapsto$ $\mathbf{R}^{d+1}$ of smooth closed convex hypersurfaces with parameter $t \in[0, T)$ for some $T>0$ is called Gauss curvature flow with initial state $\Gamma$ if the following holds:

$$
\begin{align*}
\frac{\partial F(s, t)}{\partial t} & =-K(s, t) n(s, t) \quad\left(s \in \mathbf{S}^{d}, 0<t<T\right)  \tag{1.1}\\
F(s, 0) & =F(s) \quad\left(s \in \mathbf{S}^{d}\right) \tag{1.2}
\end{align*}
$$

[^0]Figure 1

Figure 1: Motion of $F(\cdot, t)$ at $F(s, t)$ in $\mathbf{R}^{2}$
where $K(s, t)$ and $n(s, t)$ denote the Gauss curvature and the unit outward normal vector, respectively, at a point $F(s, t)$ on the hypersurface $\left\{F\left(s^{\prime}, t\right) \mid s^{\prime} \in \mathbf{S}^{d}\right\}$. In this paper we assume that the convex set with boundary $\left\{F(s, t) \mid s \in \mathbf{S}^{d}\right\}$ is non-increasing in $t$ (see Figure 1).

Suppose that $\Gamma$ is strictly convex. Then there exists the maximum $T^{*}$ of $T$ for which (1.1)-(1.2) has a unique smooth strictly convex solution and $\left\{F(s, t) \mid s \in \mathbf{S}^{d}\right\}$ converges to a point as $t \uparrow T^{*}$ (see [1], [6] and [11]).

In [8], H. Ishii proposed a discrete time approximation scheme for the Gauss curvature flow. We briefly introduce it. Suppose that we are given the strictly convex set $D$ with smooth boundary $\partial D$ in $\mathbf{R}^{d+1}$ at time $t=0$. Take $h>0$ and a function $V:[0, \infty) \mapsto[0, \infty)$. For every $s \in \mathbf{S}^{d}$, let $D_{s, h}$ denote the set which can be obtained by cutting off the volume $V(h)$ from the set $D$ in the direction $-s$ (see Figure 2). Put $\mathbf{D}_{0, h} \equiv D$ and $\mathbf{D}_{1, h} \equiv \cap_{s \in \mathbf{S}^{d}} D_{s, h}$. Define $\mathbf{D}_{n, h}$ inductively in $n$ until $n_{h} \equiv \max \{k \geq 1 \mid$ the volume of $\mathbf{D}_{k, h}$ is greater than $\left.V(h)\right\}+1$. Let $V(h) \rightarrow 0$ as $h \rightarrow 0$ in an appropriate rate. Then $\lim _{h \rightarrow 0} n_{h} h=T_{\text {max }}$, and the flow of $\partial \mathbf{D}_{[t / h], h}\left(0 \leq t \leq n_{h} h\right)$ converges to the Gauss curvature flow in Hausdorff metric uniformly in $t$ on every compact subset of $\left[0, T^{*}\right)$, where $[t / h]$ denotes the integer part of $[t / h]$. Notice that the time variable $t$ is discretized but the space variable $s$ is not in this approximation scheme.

Remark 1. Hausdorff metric of compact sets $A$ and $B \in \mathbf{R}^{d}$ is given by the following:


Figure 2: $D$ and $D_{s, h}$ in $\mathbf{R}^{2}$

$$
\begin{equation*}
d_{H}(A, B) \equiv \max \left(\max _{p \in A} \operatorname{dist}(p, B), \max _{q \in B} \operatorname{dist}(q, A)\right) \tag{1.3}
\end{equation*}
$$

A crystalline (or a polyhedral) approximation of the curvature flow of convex curves was studied by P. M. Girão and is useful in numerical analysis (see Theorem 1 given below, [4] and also [5] and the references therein). In [4], the space variable $s$ is discretized but the time variable $t$ is not. In case when the initial curve is not convex, the results of [4] have been generalized by K. Ishii and M. H. Soner (see [9] and the references therein for further information on this problem). The results of [4] have not been generalized to a class of closed convex hypersurfaces in $\mathbf{R}^{d+1}$ for $d \geq 2$. This is a well-known open problem.

Remark 2. Let $\Gamma$ be a smooth simple closed convex curve on $\mathbf{R}^{2}$. Fix a point $x_{0}$ on $\Gamma$. For any $x \in \Gamma$, let $s(x)$ be the length of the curve which connects $x_{0}$ and $x$ on $\Gamma$ clockwise. Then one can parametrize $x \in \Gamma$ by $s(x)$. Let $p_{1}(s(x))$ and $p_{2}(s(x))$ denote, respectively, the clockwise unit tangent vector and the unit outward normal vector at $x$ on $\Gamma$. Then the Gauss curvature $K(s(x))(\in \mathbf{R})$ at $x$ on $\Gamma$ satisfies the following:

$$
\begin{aligned}
& \frac{d p_{1}(s(x))}{d s}=-K(s(x)) p_{2}(s(x)) \\
& \frac{d p_{2}(s(x))}{d s}=K(s(x)) p_{1}(s(x))
\end{aligned}
$$



Figure 3: $\Gamma$ and $\Gamma_{6}$
We refer to [4] since it plays a crucial role in this paper. First of all we introduce one of the conventions in this paper. Every convex polygon with $n$ sides ( $n$-polygon for short) has outward normals $N_{n, i} \equiv(\cos (2 \pi i / n), \sin (2 \pi i / n))(i=0, \cdots, n-1)$. By the $i$ th side of the $n$-polygon we denote the side with the outward normal $N_{n, i}$.

Take a smooth simple closed convex curve $\Gamma$ on $\mathbf{R}^{2}$. For $n \geq 5$, let $\Gamma_{n}$ denote the $n$-polygon of which the $i$ th side is tangent to $\Gamma$ (see Figure 3). Let $\left\{\Gamma_{n}(t)\right\}_{0 \leq t<T_{n}^{*}}$ be the flow of $n$-polygons which can be defined as follows, where $T_{n}^{*}$ denotes the extinction time of $\Gamma_{n}(\cdot)$.

$$
\Gamma_{n}(0)=\Gamma_{n}
$$

and for $t \in\left[0, T_{n}^{*}\right)$, the inward normal velocity $V_{n, i}(t)$ of the $i$ th side of $\Gamma_{n}(t)$ is given by the following:

$$
\begin{equation*}
V_{n, i}(t)=2 \frac{\tan (\pi / n)}{\ell_{n, i}(t)} \tag{1.4}
\end{equation*}
$$

where $\ell_{n, i}(t)$ denotes the length of the $i$ th side of $\Gamma_{n}(t)$ (see Figure 4). It is known that there exists the Gauss curvature flow $\{\Gamma(t)\}_{0 \leq t<T^{*}}$ on $\mathbf{R}^{2}$, with $\Gamma(0)=\Gamma$, where $T^{*}$ denotes the extinction time of $\Gamma(t)$ (see [4]). Let $\Omega_{\ell, n}(t)$ and $\Omega(t)\left(\subset \mathbf{R}^{2}\right)$ be the closed convex sets such that $\partial \Omega_{\ell, n}(t)=\Gamma_{n}(t)$ and $\partial \Omega(t)=\Gamma(t)$, and such that $\Omega_{\ell, n}(t) \subset$ $\Omega_{\ell, n}(s)$ and $\Omega(t) \subset \Omega(s)$ if $0 \leq s \leq t$.

Then the following holds.


Figure 4: Motion of the $i$ th side of $\Gamma_{n}(t)$
Theorem 1. (see [4]). As $t \uparrow T^{*}, \Omega(t)$ converges in Hausdorff metric to a point or a segment. $\lim _{n \rightarrow \infty} T_{n}^{*}=T^{*}$, and for any $t \in\left[0, T^{*}\right)$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sup _{0 \leq s \leq t} d_{H}\left(\Omega_{\ell, n}(s), \Omega(s)\right)=0 \tag{1.5}
\end{equation*}
$$

Since the wearing process of stones on beaches is random, we would like to construct a stochastic model instead of a deterministic one such as Theorem 1.

In this paper we introduce the flow of random $n$-polygons with outward normals $N_{n, i}$ $(i=0, \cdots, n-1)$ and show that it converges in probability to the Gauss curvature flow of smooth simple closed convex curves on $\mathbf{R}^{2}$ as $n \rightarrow \infty$ in Hausdorff metric uniformly in $t$ on every compact subset of $\left[0, T^{*}\right)$ (see Theorem 2 in section 2).

In the proof we approximate the random $n$-polygon by $\Gamma_{n}(t)$ at time $t$ and use Theorem 1.

We use the word "Gauss" even for the curvature flow in $\mathbf{R}^{2}$ since a part of our idea that the volume is cut off from the stone is originally from the deterministic model of the Gauss curvature flow (see [8]).

In section 2 we introduce our random model and state our result which will be proved in section 4 . Technical lemmas will be stated and proved in section 3.

## 2. Main result.

We first introduce our random model.


Figure 5: The isogonal trapezoid with the height $h_{n}(x)$
Let $\{T(n)\}_{n \geq 1}$ be an increasing sequence of positive real numbers and put

$$
\begin{equation*}
\theta_{n}=\frac{2 \pi}{n} \tag{2.1}
\end{equation*}
$$

For $x>0$ and $n \geq 1$, put

$$
\begin{equation*}
h_{n}(x)=\frac{\tan \theta_{n}\left\{-x+\left(x^{2}+4\left(\cot \theta_{n}\right) \theta_{n} / T(n)\right)^{1 / 2}\right\}}{2} \tag{2.2}
\end{equation*}
$$

Remark 3. $h_{n}(x)$ is the height of the isogonal trapezoid, with the area $\theta_{n} / T(n)$, of which the lengths of upper and lower sides are $x$ and $x+2\left(\cot \theta_{n}\right) h_{n}(x)$ respectively (see Figure 5). In particular,

$$
\begin{equation*}
\left(x+\left(\cot \theta_{n}\right) h_{n}(x)\right) h_{n}(x)=\frac{\theta_{n}}{T(n)} \tag{2.3}
\end{equation*}
$$

For $n \geq 5$, we consider the Markov process $\left\{\left(X_{n, i}(t)\right)_{i=0}^{n-1}\right\}_{t \geq 0}$ on $\mathbf{R}^{n}$ such that $\left(X_{n, i}(0)\right)_{i=0}^{n-1}=\left(\ell_{n, i}(0)\right)_{i=0}^{n-1}$ (see (1.4)) and of which the generator is given by the following: for a bounded Borel measurable function $f: \mathbf{R}^{n} \mapsto \mathbf{R}$ and $x=\left(x_{i}\right)_{i=0}^{n-1} \in$ $\mathbf{R}^{n}$,

$$
\begin{align*}
L f(x)= & \frac{T(n) \tan \left(\theta_{n} / 2\right)}{\theta_{n} / 2} \sum_{i=0}^{n-1} I_{\left\{y \mid \min \left(y_{i-1}, y_{i+1}\right) \sin \theta_{n}>h_{n}\left(y_{i}\right)\right\}}(x)  \tag{2.4}\\
& \times\left[f\left(x+2\left(\cot \theta_{n}\right) h_{n}\left(x_{i}\right) \mathbf{e}_{n, i}-\frac{h_{n}\left(x_{i}\right)}{\sin \theta_{n}}\left(\mathbf{e}_{n, i-1}+\mathbf{e}_{n, i+1}\right)\right)-f(x)\right]
\end{align*}
$$

(see [2, Chap. 4, section 2]). Here $I_{A}(x)$ and $\left\{\mathbf{e}_{n, k}\right\}_{k=0}^{n-1}$ denote the indicator function of the set $A$ and the standard normal base in $\mathbf{R}^{n}$ respectively, and we put $\mathbf{e}_{n, n+k}=\mathbf{e}_{n, k}$ and $y_{n+k}=y_{k}(k=-1,0)$.

It is easy to see that one can construct the flow of random closed convex sets $\left\{\Omega_{X, n}(t)\right\}_{t \geq 0}$ in $\mathbf{R}^{2}$, surrounded by $n$-polygons, such that $\Omega_{X, n}(0)=\Omega_{\ell, n}(0)$, and that $\Omega_{X, n}(t) \subset \Omega_{X, n}(s)$ if $s \leq t$, and that the length of the $i$ th side of $\partial \Omega_{X, n}(t)$ is equal to $X_{n, i}(t)$.

We discuss the meaning of our model.
For $n \geq 5$, put

$$
\sigma_{n, i} \equiv\left\{\begin{aligned}
0 & \text { if } i=0 \\
\inf \left\{t>\sigma_{n, i-1}\left|\sum_{k=0}^{n-1}\right| X_{n, k}(t)-X_{n, k}(t-) \mid>0\right\} & \text { if } i \geq 1
\end{aligned}\right.
$$

where $X_{n, k}(t-) \equiv \lim _{s \uparrow t} X_{n, k}(s)$, and where we consider the right hand side as infinity if the set over which the infimum is taken is empty. Then

$$
P\left(\sigma_{n, i}<\sigma_{n, i+1} \quad \text { for all } i \text { for which } \sigma_{n, i}<\infty\right)=1
$$

Put

$$
A_{n} \equiv\left\{j \in\{0, \cdots, n-1\} \mid \min \left(X_{n, j-1}(0), X_{n, j+1}(0)\right) \sin \theta_{n}>h_{n}\left(X_{n, j}(0)\right)\right\}
$$

If the set $A_{n}$ is not empty, then $\sigma_{n, 1}$ is exponentially distributed with parameter $\left[\# A_{n} \cdot T(n) \tan \left(\theta_{n} / 2\right)\right] /\left(\theta_{n} / 2\right)$ (see [2, p. 163]), where we put $j=n+j$ for $j=-1$ and 0 , and where $\# A_{n}$ denotes the cardinal number of the set $A_{n}$. For any $k \in A_{n}$, the probability that the isogonal trapezoid with the area $\theta_{n} / T(n)$ is cut off from $\Omega_{X, n}(0)$ in the direction $-N_{n, k}=(-\cos (2 \pi k / n),-\sin (2 \pi k / n))$ at time $t=\sigma_{n, 1}$ is equal to $\left(\# A_{n}\right)^{-1}$ (see Figure 6).

If the set $A_{n}$ is empty, then $\sigma_{n, 1}=\infty$ and $X_{n, k}(0)=X_{n, k}(t)$ for all $k=0, \cdots, n-1$ and all $t \geq 0$ a.s..

The following also holds a.s.: $\left\{\Omega_{X, n}(t)\right\}$ continues to change the shape in a similar manner to above at times $t=\sigma_{n, i}$ which is finite; $\sigma_{n, i}$ is infinite if $i$ is greater than (the area of $\left.\Omega_{X, n}(0)\right) /\left(T(n)^{-1} \theta_{n}\right) ; \Omega_{X, n}(t)$ is an $n$-polygon for all $t \geq 0$.

The following is our main result.


Figure 6: The change of the $k$ th side of $\Omega_{X, n}(0)$

Theorem 2. Suppose that $\Gamma$ is a smooth simple closed convex curve on $\mathbf{R}^{2}$ and that the following holds:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} T(n) n^{-5}=\infty \tag{2.5}
\end{equation*}
$$

Then for any $t \in\left[0, T^{*}\right)$ and any $\eta>0$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} P\left(\sup _{0 \leq s \leq t} d_{H}\left(\Omega_{X, n}(s), \Omega(s)\right)<\eta\right)=1 . \tag{2.6}
\end{equation*}
$$

Remark 4. (2.5) implies that $\theta_{n} / T(n) \sim o\left(n^{-6}\right)$ (as $n \rightarrow \infty$ ), where $\theta_{n} / T(n)$ is the area of the isogonal trapezoid which is cut off from an $n$-polygon in our model.

Consider a convex stone which rotates randomly on a beach where waves are even. Our result suggests that the time evolution of the surface of such a stone can be considered as Gauss curvature flow.

## 3. Lemmas.

In this section we state and prove lemmas which will be used in the next section.
For $n \geq 1$ and $i=0, \cdots, n-1$, put

$$
\begin{align*}
D_{n, i}(t) & =\sum_{0<s \leq t} h_{n}\left(X_{n, i}(s-)\right) I_{\left(X_{n, i}(s-), \infty\right)}\left(X_{n, i}(s)\right) \quad(t \geq 0)  \tag{3.1}\\
d_{n, i}(t) & =\int_{0}^{t} \frac{2 \tan \left(\theta_{n} / 2\right)}{\ell_{n, i}(s)} d s \quad\left(0 \leq t<T_{n}^{*}\right) \tag{3.2}
\end{align*}
$$

Remark 5. $D_{n, i}(t)$ is the distance between the straight line which includes the $i$ th side of $\Omega_{X, n}(t)$ and that which includes the $i$ th side of $\Omega_{X, n}(0) . d_{n, i}(t)$ is also the distance between the straight line which includes the $i$ th side of $\Omega_{\ell, n}(t)$ and that which includes the $i$ th side of $\Omega_{\ell, n}(0)$.

Put the intersection point of the 0 th and the first sides of $\Omega_{\ell, n}(0)$ at the origin. Then the coordinate of the intersection point of the $i$ th and the $(i+1)$ th sides of $\Omega_{X, n}(t)$ and $\Omega_{\ell, n}(t)$ can be written as follows, respectively: for $t \geq 0$,

$$
\begin{array}{ll}
Y_{n, 0}(t)=\left(-D_{n, 0}(t), D_{n, 0}(t) \cot \theta_{n}-D_{n, 1}(t) / \sin \theta_{n}\right) & \text { if } i=0, \\
Y_{n, i}(t)=Y_{n, 0}(t)+\sum_{k=1}^{i} X_{n, k}(t)\left(-\sin \left(k \theta_{n}\right), \cos \left(k \theta_{n}\right)\right) \quad \text { if } i=1, \cdots, n-1,( \tag{3.4}
\end{array}
$$

and for $t \in\left[0, T_{n}^{*}\right)$

$$
\begin{align*}
& y_{n, 0}(t)=\left(-d_{n, 0}(t), d_{n, 0}(t) \cot \theta_{n}-d_{n, 1}(t) / \sin \theta_{n}\right) \quad \text { if } i=0,  \tag{3.5}\\
& y_{n, i}(t)=y_{n, 0}(t)+\sum_{k=1}^{i} \ell_{n, k}(t)\left(-\sin \left(k \theta_{n}\right), \cos \left(k \theta_{n}\right)\right) \quad \text { if } i=1, \cdots, n-1 . \tag{3.6}
\end{align*}
$$

Remark 6. $X_{n, i}(t)=\left|Y_{n, i}(t)-Y_{n, i-1}(t)\right|$ for $t \geq 0$ and $\ell_{n, i}(t)=\left|y_{n, i}(t)-y_{n, i-1}(t)\right|$ for $t \in\left[0, T_{n}^{*}\right)$, where we put $\left(Y_{n, i}(t), y_{n, i}(t)\right)=\left(Y_{n, n+i}(t), y_{n, n+i}(t)\right)$ for $i=-1,0$.

The time evolution of $\left\{y_{n, i}(t)\right\}_{0 \leq t<T_{n}^{*}}(n \geq 5, i=0, \cdots, n-1)$ can be given by the following.

Lemma 1. For $n \geq 5, i=0, \cdots, n-1$, and $s \in\left(0, T_{n}^{*}\right)$,

$$
\begin{equation*}
\frac{d y_{n, i}(s)}{d s}=\frac{\left(\sin \left(i \theta_{n}\right),-\cos \left(i \theta_{n}\right)\right)}{\ell_{n, i+1}(s) \cos ^{2}\left(\theta_{n} / 2\right)}-\frac{\left(\sin \left((i+1) \theta_{n}\right),-\cos \left((i+1) \theta_{n}\right)\right)}{\ell_{n, i}(s) \cos ^{2}\left(\theta_{n} / 2\right)} \tag{3.7}
\end{equation*}
$$

where we put $\ell_{n, n}(s)=\ell_{n, 0}(s)$.

Proof. It is known that $\left\{\ell_{n, i}(t)\right\}_{i=0}^{n-1}$ satisfies the following (see [4]):

$$
\begin{equation*}
\frac{d \ell_{n, i}(t)}{d t}=\left(\frac{2 \cos \theta_{n}}{\ell_{n, i}(t)}-\frac{1}{\ell_{n, i+1}(t)}-\frac{1}{\ell_{n, i-1}(t)}\right) \frac{1}{\cos ^{2}\left(\theta_{n} / 2\right)} \tag{3.8}
\end{equation*}
$$

where we put $\ell_{n, n+k}(t)=\ell_{n, k}(t)(k=-1,0)$.
(3.7) can be proved inductively in $i$, by (3.2), (3.5)-(3.6) and by the following:

$$
\begin{align*}
\sin \left((i-1) \theta_{n}\right)+\sin \left((i+1) \theta_{n}\right) & =2 \cos \theta_{n} \sin \left(i \theta_{n}\right)  \tag{3.9}\\
\cos \left((i-1) \theta_{n}\right)+\cos \left((i+1) \theta_{n}\right) & =2 \cos \theta_{n} \cos \left(i \theta_{n}\right) \tag{3.10}
\end{align*}
$$

Before we state and prove the following lemma, we give some notation. Put for $\delta \in\left(0, T_{n}^{*}\right)$,

$$
\begin{align*}
C_{n}(\delta) & =n \min \left\{\ell_{n, k}(s) \mid 0 \leq k \leq n-1,0 \leq s \leq T_{n}^{*}-\delta\right\}  \tag{3.11}\\
\tau_{n, \delta} & =\inf \left\{t>0 \mid C_{n}(\delta) /(2 n) \geq \min \left\{X_{n, k}(t) ; 0 \leq k \leq n-1\right\}\right\} \tag{3.12}
\end{align*}
$$

For any $f \in C_{o}^{2}\left(\mathbf{R}^{2 n} ; \mathbf{R}\right)$ and $y=\left(y_{i}\right)_{i=0}^{2 n-1} \in \mathbf{R}^{2 n}$, put

$$
\begin{gathered}
\tilde{L} f(y)=\frac{T(n) \tan \left(\theta_{n} / 2\right)}{\theta_{n} / 2} \sum_{i=0}^{n-1}\left\{f \left(y+\frac{h_{n}\left(\left\{\left|y_{2 i}-y_{2(i-1)}\right|^{2}+\left|y_{2 i+1}-y_{2 i-1}\right|^{2}\right\}^{1 / 2}\right)}{\sin \theta_{n}}\right.\right. \\
\times\left(\left[\sin \left((i-1) \theta_{n}\right)\right] \mathbf{e}_{2 n, 2(i-1)}-\left[\cos \left((i-1) \theta_{n}\right)\right] \mathbf{e}_{2 n, 2 i-1}\right. \\
\left.\left.\left.\quad-\left[\sin \left((i+1) \theta_{n}\right)\right] \mathbf{e}_{2 n, 2 i}+\left[\cos \left((i+1) \theta_{n}\right)\right] \mathbf{e}_{2 n, 2 i+1}\right)\right)-f(y)\right\}
\end{gathered}
$$

(see (2.4) for the convention of the notation).
Remark 7. For $\left(y_{2 i}, y_{2 i+1}\right) \in \mathbf{R}^{2}(i=0, \cdots, n-1)$, put

$$
x_{i} \equiv\left\{\left(y_{2 i}-y_{2(i-1)}\right)^{2}+\left(y_{2 i+1}-y_{2 i-1}\right)^{2}\right\}^{1 / 2}
$$

where we put $\left(y_{2 i}, y_{2 i+1}\right)=\left(y_{2(n+i)}, y_{2(n+i)+1}\right)$ for $i=-1,0$. If

$$
\min \left(x_{i+1}, x_{i-1}\right) \sin \theta_{n}>h_{n}\left(x_{i}\right)
$$

$$
\left(y_{2 i}, y_{2 i+1}\right)-\left(y_{2(i-1)}, y_{2 i-1}\right)=x_{i}\left(-\sin \left(i \theta_{n}\right), \cos \left(i \theta_{n}\right)\right)
$$

for all $i=0, \cdots, n-1$, then for any $g \in C_{o}^{2}\left(\mathbf{R}^{n} ; \mathbf{R}\right)$,

$$
\tilde{L} g\left(\left(\left\{\left(y_{2 i}-y_{2(i-1)}\right)^{2}+\left(y_{2 i+1}-y_{2 i-1}\right)^{2}\right\}^{1 / 2}\right)_{i=0}^{n-1}\right)=L g\left(\left(x_{i}\right)_{i=0}^{n-1}\right)
$$

Put also $\mathbf{Y}_{n}(t)=\left(Y_{n, k}(t)\right)_{k=0}^{n-1}$. Then the time evolution of $\left\{\mathbf{Y}_{n}(t)\right\}_{0 \leq t}$ for sufficiently large $n$ can be given by the following.

Lemma 2. Suppose that (2.5) holds. Then for any $\delta \in\left(0, T^{*}\right)$, there exists $n_{1} \in \mathbf{N}$ such that for any $n \geq n_{1}$ and any $f \in C_{o}^{2}\left(\mathbf{R}^{2 n} ; \mathbf{R}\right), \delta$ is less than $T_{n}^{*}$ and

$$
f\left(\mathbf{Y}_{n}\left(\min \left(t, \tau_{n, \delta}\right)\right)\right)=f\left(\mathbf{Y}_{n}(0)\right)+\int_{0}^{\min \left(t, \tau_{n, \delta}\right)} \tilde{L} f\left(\mathbf{Y}_{n}(s)\right) d s+M^{\left[f\left(\mathbf{Y}_{n}\right)\right]}\left(\min \left(t, \tau_{n, \delta}\right)\right)
$$

for $t \geq 0$, P-a.s., where $M^{\left[f\left(\mathbf{Y}_{n}\right)\right]}(t)$ denotes a purely discontinuous martingale part of $f\left(\mathbf{Y}_{n}(t)\right)$.

Proof. Take $n_{0} \in \mathbf{N}$ such that $\delta<T_{n}^{*}$ for any $n \geq n_{0}$, which is possible from Theorem 1. First we show that there exists $n_{1} \geq n_{0}$ such that for any $n \geq n_{1}$ and $k=0, \cdots, n-1$

$$
\begin{equation*}
\min \left(X_{n, k-1}(t), X_{n, k+1}(t)\right) \sin \theta_{n}>h_{n}\left(X_{n, k}(t)\right) \quad \text { for } t \in\left[0, \tau_{n, \delta}\right) \text { P-a.s. } \tag{3.13}
\end{equation*}
$$

where we put $X_{n, i}(t)=X_{n, n+i}(t)$ for $i=-1,0$.
By (14) of [4], $\left\{C_{k}(\delta)^{-1}\right\}_{k=n_{0}}^{\infty}$ defined in (3.11) is bounded. Therefore there exists $n_{1} \geq n_{0}$ such that for any $n \geq n_{1}$

$$
\frac{C_{n}(\delta)}{2 n} \sin \theta_{n}>\frac{2 n}{C_{n}(\delta)} \frac{\theta_{n}}{T(n)}
$$

by (2.5). Hence, for $k, i=0, \cdots, n-1$ and $t \in\left[0, \tau_{n, \delta}\right)$

$$
X_{n, k}(t) \sin \theta_{n}>\frac{C_{n}(\delta)}{2 n} \sin \theta_{n}>\frac{2 n}{C_{n}(\delta)} \frac{\theta_{n}}{T(n)}>\frac{1}{X_{n, i}(t)} \frac{\theta_{n}}{T(n)}>h_{n}\left(X_{n, i}(t)\right) \quad \text { a.s. }
$$

by (2.3), which implies (3.13).
By (3.3)-(3.4), we have the following:

$$
\begin{aligned}
\mathbf{Y}_{n}(s)=-x_{1} & \sum_{k=0}^{n-1} \mathbf{e}_{2 n, 2 k}+\left(x_{1} \cot \theta_{n}-x_{3} / \sin \theta_{n}\right) \sum_{k=0}^{n-1} \mathbf{e}_{2 n, 2 k+1} \\
& +\sum_{i=1}^{n-1} x_{2 i}\left[-\left(\sin \left(i \theta_{n}\right)\right) \sum_{k=i}^{n-1} \mathbf{e}_{2 n, 2 k}+\left(\cos \left(i \theta_{n}\right)\right) \sum_{k=i}^{n-1} \mathbf{e}_{2 n, 2 k+1}\right]
\end{aligned}
$$

with $x_{2 i}=X_{n, i}(s)$ and $x_{2 i+1}=D_{n, i}(s)(i=0, \cdots, n-1)$. Therefore, from (2.4), (3.1), (3.9)-(3.10) and (3.13), by the Itô formula (see [9]), the proof is over (see Remark 5).

The following lemma plays a crucial role when we approximate $\Omega_{\ell, n}$ by $\Omega_{X, n}$.

Lemma 3. Suppose that (2.5) holds. Then for any $\delta \in\left(0, T^{*}\right)$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{2} E\left[\sup _{0 \leq t \leq \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)} \sum_{i=0}^{n-1}\left|Y_{n, i}(t)-y_{n, i}(t)\right|^{2}\right]=0 \tag{3.14}
\end{equation*}
$$

Proof. For $n_{1} \in \mathbf{N}$ in Lemma 2, there exists a positive constant $C$ such that the following which will be proved later holds: for any $n \geq n_{1}$ and $t \in\left[0, T_{n}^{*}-\delta\right]$,

$$
\begin{align*}
& E\left[\sup _{0 \leq s \leq \min \left(t, \tau_{n, \delta}\right)} \sum_{k=0}^{n-1}\left|y_{n, k}(s)-Y_{n, k}(s)\right|^{2}\right]  \tag{3.15}\\
\leq & C T(n)^{-1} n^{3}+C \int_{0}^{t} E\left[\sup _{0 \leq u \leq \min \left(s, \tau_{n, \delta}\right)} \sum_{k=0}^{n-1}\left|y_{n, k}(u)-Y_{n, k}(u)\right|^{2}\right] d s .
\end{align*}
$$

This implies (3.14), by Gronwall's inequality, from (2.5).
We prove (3.15) to complete the proof. For any $n \geq n_{1}$, by Lemmas 1 and 2, the following holds: for $t \in\left[0, \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)\right]$,

$$
\begin{aligned}
& \sum_{k=0}^{n-1}\left|y_{n, k}(t)-Y_{n, k}(t)\right|^{2} \\
& =\frac{2}{\cos ^{2}\left(\theta_{n} / 2\right)} \sum_{k=0}^{n-1} \int_{0}^{t}<y_{n, k}(s)-Y_{n, k}(s), \\
& \\
& \quad\left(\frac{1}{\ell_{n, k+1}(s)}-\frac{1}{X_{n, k+1}(s)}\right)\left(\sin \left(k \theta_{n}\right),-\cos \left(k \theta_{n}\right)\right) \\
& \quad+\left(\frac{1}{\ell_{n, k}(s)}-\frac{1}{X_{n, k}(s)}\right)\left(-\sin \left((k+1) \theta_{n}\right), \cos \left((k+1) \theta_{n}\right)\right)>d s \\
& \quad+\frac{2}{\cos ^{2}\left(\theta_{n} / 2\right)} \sum_{k=0}^{n-1} \int_{0}^{t}<y_{n, k}(t)-Y_{n, k}(s), \\
& \\
& \quad\left[-\frac{T(n)}{\theta_{n}} h_{n}\left(X_{n, k+1}(s)\right)+\frac{1}{X_{n, k+1}(s)}\right]\left(\sin \left(k \theta_{n}\right),-\cos \left(k \theta_{n}\right)\right) \\
& \quad+\left[\frac{T(n)}{\theta_{n}} h_{n}\left(X_{n, k}(s)\right)-\frac{1}{X_{n, k}(s)}\right]\left(\sin \left((k+1) \theta_{n}\right),-\cos \left((k+1) \theta_{n}\right)\right)>d s \\
& \quad+
\end{aligned}
$$

where $M(t)$ denotes a purely discontinuous martingale part of $\sum_{k=0}^{n-1}\left|y_{n, k}(t)-Y_{n, k}(t)\right|^{2}$.

Since $\left\{C_{k}^{-1}(\delta)\right\}_{k \geq n_{1}}$ is bounded by (14) of [4], we only have to show the following (3.16)-(3.19) to complete the proof: for $t \in\left[0, \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)\right]$,

$$
\begin{align*}
& \frac{2}{\cos ^{2}\left(\theta_{n} / 2\right)} \sum_{k=0}^{n-1} \int_{0}^{t}<y_{n, k}(s)-Y_{n, k}(s),  \tag{3.16}\\
& \quad\left(\frac{1}{\ell_{n, k+1}(s)}-\frac{1}{X_{n, k+1}(s)}\right)\left(\sin \left(k \theta_{n}\right),-\cos \left(k \theta_{n}\right)\right) \\
& \quad+\left(\frac{1}{\ell_{n, k}(s)}-\frac{1}{X_{n, k}(s)}\right)\left(-\sin \left((k+1) \theta_{n}\right), \cos \left((k+1) \theta_{n}\right)\right)>d s \\
& \leq \frac{4 n^{2} \sin ^{2} \theta_{n}}{C_{n}(\delta)^{2} \cos \theta_{n} \cos ^{2}\left(\theta_{n} / 2\right)} \int_{0}^{t} \sup _{0 \leq u \leq \min \left(s, \tau_{n, \delta}\right)} \sum_{k=0}^{n-1}\left|y_{n, k}(u)-Y_{n, k}(u)\right|^{2} d s,
\end{align*}
$$

$$
\begin{align*}
& \frac{2}{\cos ^{2}\left(\theta_{n} / 2\right)} \sum_{k=0}^{n-1} \int_{0}^{t}<y_{n, k}(t)-Y_{n, k}(s),  \tag{3.17}\\
& \quad\left[-\frac{T(n)}{\theta_{n}} h_{n}\left(X_{n, k+1}(s)\right)+\frac{1}{X_{n, k+1}(s)}\right]\left(\sin \left(k \theta_{n}\right),-\cos \left(k \theta_{n}\right)\right) \\
& \quad+\left[\frac{T(n)}{\theta_{n}} h_{n}\left(X_{n, k}(s)\right)-\frac{1}{X_{n, k}(s)}\right]\left(\sin \left((k+1) \theta_{n}\right),-\cos \left((k+1) \theta_{n}\right)\right)>d s \\
& \leq 2 \int_{0}^{t} \sup _{0 \leq u \leq \min \left(s, \tau_{n, \delta}\right)} \sum_{k=0}^{n-1}\left|y_{n, k}(u)-Y_{n, k}(u)\right|^{2} d s+2 n t\left|\frac{8 n^{3} \theta_{n}}{T(n) C_{n}(\delta)^{3} \sin \theta_{n}}\right|^{2},
\end{align*}
$$

$$
\begin{equation*}
2 \frac{T(n) \tan \left(\theta_{n} / 2\right)}{\theta_{n} / 2} \sum_{k=0}^{n-1} \int_{0}^{t}\left|\frac{h_{n}\left(X_{n, k}(s)\right)}{\sin \theta_{n}}\right|^{2} d s \leq \frac{8 n^{3} t \theta_{n}}{T(n) C_{n}(\delta)^{2} \cos ^{2}\left(\theta_{n} / 2\right) \sin \theta_{n}}, \tag{3.18}
\end{equation*}
$$

and for $t \in\left[0, T_{n}^{*}-\delta\right]$,

$$
\begin{align*}
&\left\{E\left[\sup _{0 \leq s \leq \min \left(t, \tau_{n, \delta}\right)}|M(s)|^{2}\right]\right\}^{1 / 2}  \tag{3.19}\\
& \leq \frac{32 n^{2} \theta_{n}}{T(n) C_{n}(\delta)^{2} \sin \theta_{n} \cos ^{2}\left(\theta_{n} / 2\right)}+\frac{6 n^{3} t \theta_{n}^{2}}{\left(T(n) C_{n}(\delta) \sin \theta_{n}\right)^{2}} \\
&+3 \int_{0}^{t} E\left[\sup _{0 \leq u \leq \min \left(s, \tau_{n, \delta}\right)} \sum_{k=0}^{n-1}\left|y_{n, k}(u)-Y_{n, k}(u)\right|^{2}\right] d s .
\end{align*}
$$

We first prove (3.16). By (3.4) and (3.6), for $k=0, \cdots, n-1$ and $s \in\left[0, T_{n}^{*}\right)$,

$$
\begin{aligned}
& y_{n, k+1}(s)-Y_{n, k+1}(s) \\
= & y_{n, k}(s)-Y_{n, k}(s)+\left(\ell_{n, k+1}(s)-X_{n, k+1}(s)\right)\left(-\sin \left((k+1) \theta_{n}\right), \cos \left((k+1) \theta_{n}\right)\right) .
\end{aligned}
$$

Hence for $s \in\left[0, T_{n}^{*}\right)$,

$$
\begin{aligned}
\sum_{k=0}^{n-1}< & y_{n, k}(s)-Y_{n, k}(s),\left(\frac{1}{\ell_{n, k+1}(s)}-\frac{1}{X_{n, k+1}(s)}\right)\left(\sin \left(k \theta_{n}\right),-\cos \left(k \theta_{n}\right)\right) \\
& +\left(\frac{1}{\ell_{n, k}(s)}-\frac{1}{X_{n, k}(s)}\right)\left(-\sin \left((k+1) \theta_{n}\right), \cos \left((k+1) \theta_{n}\right)\right)> \\
=\sum_{k=0}^{n-1} & <y_{n, k}(s)-Y_{n, k}(s),-2\left(\sin \theta_{n}\right)\left(\cos \left((k+1) \theta_{n}\right), \sin \left((k+1) \theta_{n}\right)\right)> \\
& \times\left(\frac{1}{\ell_{n, k+1}(s)}-\frac{1}{X_{n, k+1}(s)}\right) \\
& +\sum_{k=0}^{n-1}\left(\cos \theta_{n}\right)\left(\ell_{n, k}(s)-X_{n, k}(s)\right)\left(\frac{1}{\ell_{n, k}(s)}-\frac{1}{X_{n, k}(s)}\right)
\end{aligned}
$$

This together with (3.11)-(3.12) and the following implies (3.16) : for $s \in\left[0, T_{n}^{*}\right)$,

$$
\begin{aligned}
& \quad<y_{n, k}(s)-Y_{n, k}(s),-2\left(\sin \theta_{n}\right)\left(\cos \left((k+1) \theta_{n}\right), \sin \left((k+1) \theta_{n}\right)\right)> \\
& \times\left(\frac{1}{\ell_{n, k+1}(s)}-\frac{1}{X_{n, k+1}(s)}\right) \\
& \leq \frac{\left|y_{n, k}(s)-Y_{n, k}(s)\right|^{2} \sin ^{2} \theta_{n}}{\ell_{n, k+1}(s) X_{n, k+1}(s) \cos \theta_{n}}+\frac{\left(\ell_{n, k+1}(s)-X_{n, k+1}(s)\right)^{2} \cos \theta_{n}}{\ell_{n, k+1}(s) X_{n, k+1}(s)}
\end{aligned}
$$

(3.17) can be proved by (3.12) and by the following: for $x>0$,

$$
\frac{1}{x}-\frac{T(n)}{\theta_{n}} h_{n}(x)=\frac{\theta_{n} \cot \theta_{n}}{T(n) x^{3}}\left(\frac{2}{1+\left(1+4 x^{-2} T(n)^{-1} \theta_{n} \cot \theta_{n}\right)^{1 / 2}}\right)^{2}
$$

since $\cos \theta_{n}<\cos ^{2}\left(\theta_{n} / 2\right)$.
(3.18) is true, since $h_{n}(x)<\theta_{n}(T(n) x)^{-1}$ by (2.3).

Finally we prove (3.19). For $t \in\left[0, T_{n}^{*}-\delta\right]$,

$$
\begin{aligned}
& E\left[\sup _{0 \leq s \leq \min \left(t, \tau_{n, \delta}\right)}|M(s)|^{2}\right] \\
& \leq \quad 4 E\left[\left|M\left(\min \left(t, \tau_{n, \delta}\right)\right)\right|^{2}\right] \quad \text { (by Doob's inequality) } \\
&= 4 \frac{T(n) \tan \left(\theta_{n} / 2\right)}{\theta_{n} / 2} \sum_{i=0}^{n-1} E\left[\int _ { 0 } ^ { \operatorname { m i n } ( t , \tau _ { n , \delta } ) } \left(-2 \frac{h_{n}\left(X_{n, i}(s)\right)}{\sin \theta_{n}}\right.\right. \\
& \quad \times\left[<y_{n, i-1}(s)-Y_{n, i-1}(s),\left(\sin \left((i-1) \theta_{n}\right),-\cos \left((i-1) \theta_{n}\right)\right)>\right. \\
&\left.\quad+<y_{n, i}(s)-Y_{n, i}(s),\left(-\sin \left((i+1) \theta_{n}\right), \cos \left((i+1) \theta_{n}\right)\right)>\right] \\
&\left.\left.\quad+2\left|\frac{h_{n}\left(X_{n, i}(s)\right)}{\sin \theta_{n}}\right|^{2}\right)^{2} d s\right] .
\end{aligned}
$$

For $s \in\left[0, \min \left(t, \tau_{n, \delta}\right)\right)$ and $i=0, \cdots, n-1$, by (3.12),

$$
\begin{aligned}
& \left(-2 \frac{h_{n}\left(X_{n, i}(s)\right)}{\sin \theta_{n}}\left[<y_{n, i-1}(s)-Y_{n, i-1}(s),\left(\sin \left((i-1) \theta_{n}\right),-\cos \left((i-1) \theta_{n}\right)\right)>\right.\right. \\
& \left.\left.\quad+\quad<y_{n, i}(s)-Y_{n, i}(s),\left(-\sin \left((i+1) \theta_{n}\right), \cos \left((i+1) \theta_{n}\right)\right)>\right]+2\left|\frac{h_{n}\left(X_{n, i}(s)\right)}{\sin \theta_{n}}\right|^{2}\right)^{2} \\
& \leq 4\left|\frac{2 n \theta_{n}}{T(n) C_{n}(\delta) \sin \theta_{n}}\right|^{2}\left(\left|y_{n, i-1}(s)-Y_{n, i-1}(s)\right|+\left|y_{n, i}(s)-Y_{n, i}(s)\right|\right. \\
& \left.\quad+\left|\frac{2 n \theta_{n}}{T(n) C_{n}(\delta) \sin \theta_{n}}\right|\right)^{2}
\end{aligned}
$$

since $h_{n}(x)<\theta_{n}(T(n) x)^{-1}$ by (2.3).
Use the inequality $(x y)^{1 / 2} \leq(x+y) / 2(x, y>0)$ for

$$
x=4 \frac{T(n) \tan \left(\theta_{n} / 2\right)}{\theta_{n} / 2} \times 4\left|\frac{2 n \theta_{n}}{T(n) C_{n}(\delta) \sin \theta_{n}}\right|^{2}
$$

$y=\sum_{i=0}^{n-1} E\left[\int_{0}^{\min \left(t, \tau_{n, \delta}\right)}\left(\left|y_{n, i-1}(s)-Y_{n, i-1}(s)\right|+\left|y_{n, i}(s)-Y_{n, i}(s)\right|+\left|\frac{2 n \theta_{n}}{T(n) C_{n}(\delta) \sin \theta_{n}}\right|\right)^{2} d s\right]$.
Use also the inequality $(x+y+z)^{2} \leq 3\left(x^{2}+y^{2}+z^{2}\right)$ for $x=\left|y_{n, i-1}(s)-Y_{n, i-1}(s)\right|$, $y=\left|y_{n, i}(s)-Y_{n, i}(s)\right|$ and $z=\left|\left(2 n \theta_{n}\right) /\left(T(n) C_{n}(\delta) \sin \theta_{n}\right)\right|$. Then we obtain (3.19).

## 4. Proof of Main Result.

In this section we prove Theorem 2 by making use of lemmas given in section 3 .

Proof of Theorem 2. For any $t \in\left(0, T^{*}\right)$ and any $\eta>0$, take $n_{2} \in \mathbf{N}$ such that for any $n \geq n_{2}$

$$
\begin{aligned}
t & <T_{n}^{*}-\left(T^{*}-t\right) / 2 \\
\sup _{0 \leq s \leq t} d_{H}\left(\Omega_{\ell, n}(s), \Omega(s)\right) & <\eta / 2
\end{aligned}
$$

which is possible Theorem 1. Put $\delta=\left(T^{*}-t\right) / 2$. Then for any $n \geq n_{2}$,

$$
\begin{align*}
& P\left(\sup _{0 \leq s \leq t} d_{H}\left(\Omega_{X, n}(s), \Omega(s)\right) \geq \eta\right)  \tag{4.1}\\
\leq & P\left(\sup _{0 \leq s \leq t} d_{H}\left(\Omega_{X, n}(s), \Omega_{\ell, n}(s)\right) \geq \eta / 2\right) \\
\leq & P\left(\sup _{0 \leq s \leq T_{n}^{*}-\delta} d_{H}\left(\Omega_{X, n}(s), \Omega_{\ell, n}(s)\right) \geq \eta / 2\right) \\
\leq & P\left(\tau_{n, \delta}<T_{n}^{*}-\delta\right)+P\left(\sup _{0 \leq s \leq \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)} d_{H}\left(\Omega_{X, n}(s), \Omega_{\ell, n}(s)\right) \geq \eta / 2\right) .
\end{align*}
$$

The first probability on the last part of (4.1) can be shown to converge to zero as $n \rightarrow \infty$ as follows: by Chebychev's inequality,

$$
\begin{aligned}
& P\left(\tau_{n, \delta}<T_{n}^{*}-\delta\right) \\
\leq & P\left(\sup _{0 \leq s \leq \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)} \max _{k=0}^{n-1}\left|X_{n, k}(s)-\ell_{n, k}(s)\right| \geq C_{n}(\delta) /(2 n)\right) \\
\leq & P\left(\sup _{0 \leq s \leq \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)}^{n-1} \max _{k=0}\left|Y_{n, k}(s)-y_{n, k}(s)\right| \geq C_{n}(\delta) /(4 n)\right) \\
\leq & \left(4 n C_{n}(\delta)^{-1}\right)^{2} E\left[\sup _{0 \leq s \leq \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)} \sum_{k=0}^{n-1}\left|Y_{n, k}(s)-y_{n, k}(s)\right|^{2}\right] \\
\rightarrow & 0, \text { as } n \rightarrow \infty \text { by Lemma } 3,
\end{aligned}
$$

since

$$
\left|X_{n, k}(s)-\ell_{n, k}(s)\right| \leq\left|Y_{n, k}(s)-y_{n, k}(s)\right|+\left|Y_{n, k-1}(s)-y_{n, k-1}(s)\right|
$$

by (3.20) and since $\lim \sup _{k \rightarrow \infty} C_{k}(\delta)^{-1}$ is finite by (14) of [4].
The second probability on the last part of (4.1) can be shown to converge to zero as $n \rightarrow \infty$ as follows: by Chebychev's inequality,

$$
\begin{aligned}
& P\left(\sup _{0 \leq s \leq \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)} d_{H}\left(\Omega_{X, n}(s), \Omega_{\ell, n}(s)\right) \geq \eta / 2\right) \\
\leq & P\left(\sup _{0 \leq s \leq \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)}^{n-1} \max _{k=0}^{n=0}\left|Y_{n, k}(s)-y_{n, k}(s)\right| \geq \eta / 2\right) \\
\leq & (\eta / 2)^{-2} E\left[\sup _{0 \leq s \leq \min \left(T_{n}^{*}-\delta, \tau_{n, \delta}\right)} \sum_{k=0}^{n-1}\left|Y_{n, k}(s)-y_{n, k}(s)\right|^{2}\right] \\
\rightarrow & 0, \text { as } n \rightarrow \infty \text { by Lemma } 3,
\end{aligned}
$$

since

$$
d_{H}\left(\Omega_{X, n}(s), \Omega_{\ell, n}(s)\right) \leq \prod_{k=0}^{n-1}\left|Y_{n, k}(s)-y_{n, k}(s)\right| .
$$

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