Invariant dynamical systems embedded in the N-vortex problem on a sphere with pole vortices

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Abstract

We are concerned with the system of N vortex points on a sphere with two fixed vortex points at the poles. This article gives a reduction method of the system to invariant dynamical systems when all the vortex points have the same strength. It is carried out by considering the invariant property of the system with respect to the shift and pole reversal transformations, for which the polygonal ring configuration of the N vortex points at the line of latitude, called "N-ring", remains unchanged. We prove that there exists a $2p$-dimensional invariant dynamical system reduced by the p-shift transformation for arbitrary factor p of N. The p-shift invariant system is equivalent to the p-vortex points system generated by the averaged Hamiltonian with the modified pole vortices. It is also shown that the system can be reduced by the pole reversal transformation when the pole vortices are identical. Since the reduced dynamical systems are defined in the linear space spanned by the eigenvectors given in the linear stability analysis for the N-ring, we obtain the inclusion relation among the invariant reduced dynamical systems. This allows us to decompose the system of a large number of vortex points into a collection of invariant reduced subsystems.

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1 Introduction

We consider the motion of the inviscid and incompressible flow on a sphere. Specifically, we focus on the motion of the vortex points, in which the vorticity concentrates discretely. Since the strength of the vortex point, which is the circulation around the point, is conserved according to Kelvin's theorem, the vortex point behaves like a material point, and it is advected by the velocity field that the other vortex points induce.

Now, let (Θ_m, Ψ_m) denote the position of the mth vortex point in the spherical coordinates. The equations of the N-vortex points with the identical strength $\Gamma^{(N)} = 2\pi/N$ on the sphere are given by

$$\dot{\Theta}_m = -\frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N}\frac{\sin\Theta_j \sin(\Psi_m-\Psi_j)}{1-\cos\gamma_{mj}} \equiv F_m, \tag{1}$$

$$\begin{aligned}\dot{\Psi}_m &= -\frac{\Gamma^{(N)}}{4\pi\sin\Theta_m}\sum_{j\neq m}^{N}\frac{\cos\Theta_m\sin\Theta_j\cos(\Psi_m-\Psi_j)-\sin\Theta_m\cos\Theta_j}{1-\cos\gamma_{mj}} \\ &\quad +\frac{\Gamma_1}{4\pi}\frac{1}{1-\cos\Theta_m}-\frac{\Gamma_2}{4\pi}\frac{1}{1+\cos\Theta_m} \equiv G_m, \qquad m=1,2,\cdots,N, \end{aligned}\tag{2}$$

in which γ_{mj} represents the central angle between the mth and the jth vortex points, and

$$\cos\gamma_{mj} = \cos\Theta_m\cos\Theta_j + \sin\Theta_m\sin\Theta_j\cos(\Psi_m-\Psi_j).$$

The last two forcing terms in the equation (2) represent the flow fields induced by the pole vortices. The strengths of the north and the south pole vortices are denoted by Γ_1 and Γ_2 respectively. They are formally introduced in order to incorporate an effect of rotation of the sphere locally.

The equations (1) and (2) define the dynamical system in the $2N$-dimensional phase space $\mathbb{P}_N \equiv [0,\pi]^N \times (\mathbb{R}/2\pi\mathbb{Z})^N$, which we rewrite in the following vector form:

$$\frac{d\vec{x}}{dt} = \mathbb{F}(\vec{x}),$$

where the map $\mathbb{F} : \mathbb{P}_N \to \mathbb{R}^{2N}$ gives the vector field for the position $\vec{x} \in \mathbb{P}_N$,

$$\mathbb{F} : (\Theta_1,\cdots,\Theta_N,\Psi_1,\cdots,\Psi_N) \mapsto (F_1,\cdots,F_N,G_1,\cdots,G_N).$$

We call the dynamical system the "N-vortex system" or the "N-vortex problem" with the identical strength. This is the Hamiltonian dynamical system[7, 12], whose Hamiltonian is represented by

$$\begin{aligned} H &= -\frac{\left(\Gamma^{(N)}\right)^2}{8\pi}\sum_{m=1}^{N}\sum_{j\neq m}^{N}\log(1-\cos\gamma_{mj}) \\ &\quad -\frac{\Gamma_1\Gamma^{(N)}}{4\pi}\sum_{m=1}^{N}\log(1-\cos\Theta_m) - \frac{\Gamma_2\Gamma^{(N)}}{4\pi}\sum_{m=1}^{N}\log(1+\cos\Theta_m). \end{aligned}\tag{3}$$

The solution of the equations (1) and (2) exists globally in time, since the self-similar collapse of the vortex points never occurs when the strengths of the vortex points are the same[5]. We also note that the system has the invariant quantity $\sum_{m=1}^{N} \cos \Theta_m$ due to the invariance of the Hamiltonian with respect to the rotation around the pole.

The N-vortex system attracts many researchers as a nonlinear Hamiltonian dynamical system[12]. For instance, the system of the three vortex points is integrable in the absence of the pole vortices and its motion was studied well[5, 6, 15]. Many relative fixed configurations of the N vortex points were systematically found[10, 11, 13]. Relative periodic orbits were also determined by the invariance of the Hamiltonian under the action of groups[9, 19]. In the meantime, when the N vortex points are spaced equally along the line of latitude, the polygonal ring configuration is called "N-ring". The motion of the N-ring has been investigated in particular, since the ring configuration of the vortex structure is often observed in the numerical research of the atmospheric phenomena[4, 14, 16]; The linear and nonlinear stability analysis of the N-ring with and without the pole vortices were given[1, 2, 3, 8, 17]. The unstable motion of the perturbed N-ring was investigated[17, 18].

On the other hand, the N-vortex problem appears when the Euler equations are solved by the vortex method; Discretizing the vorticity region at the initial time with a cluster of vortex points, we investigate the evolution of the vortex points as an approximated solution of the Euler equations. In order to attain accurate approximation, the number of the discretizing vortex points must be very large. However, mathematical analysis of the vortex-points system gets more difficult in general as the number of the vortex points increases. Therefore we sometimes reduce the N-vortex system to low-dimensional systems by assuming a certain symmetry, and then study them as embedded subsystems. For instance, in the papers [17] and [18], the N-vortex system was successfully reduced to the integrable two-dimensional systems, with which the existence of the periodic, the heteroclinic and the homoclinic orbits and their stability were investigated. Thus the reduced systems help us understand the dynamics of the large number of the N-vortex points.

In the article, we give a reduction method of the N-vortex system to invariant dynamical systems. In §2, the linear stability analysis of the N-ring[17, 18] is reviewed. It is required to characterize the invariant systems in the following sections. In §3, we show that it is possible to reduced the system by considering the invariant property for a shift transformation. The reduced dynamical system exists for every factor p of N, and it is equivalent to the p-vortex system generated by the averaged Hamiltonian on the sphere when the strengths of the pole vortices are modified suitably. In §4, we reduce the N-vortex system by the invariance with respect to a pole reversal transformation when the strength of the north pole vortex is equivalent to that of the south pole vortex. We conclude and discuss the results in the last section.

2 Preliminary results

We give a brief summary of the linear stability analysis for the N-ring [17, 18], which is expressed by

$$\Theta_m = \theta_0, \qquad \Psi_m = \frac{2\pi m}{N}, \qquad m = 1, 2, \cdots, N. \tag{4}$$

The N-ring is a relative equilibrium for the equations (1) and (2) rotating with the constant velocity $V_0(N)$ in the longitudinal direction,

$$V_0(N) = \frac{\Gamma_1 - \Gamma_2}{4\pi \sin^2\theta_0} + \frac{(\Gamma_1 + \Gamma_2 + 2\pi)\cos\theta_0}{4\pi \sin^2\theta_0} - \frac{1}{2N}\frac{\cos\theta_0}{\sin^2\theta_0}.$$

When we add small perturbations to the equilibrium,

$$\Theta_m(t) = \theta_0 + \epsilon\theta_m(t), \qquad \Psi_m(t) = \frac{2\pi m}{N} + V_0(N)t + \epsilon\varphi_m(t), \qquad |\epsilon| \ll 1, \tag{5}$$

we obtain the linearized equations of $O(\epsilon)$ for the perturbations:

$$\dot{\theta}_m = \frac{1}{2N\sin\theta_0}\sum_{j\neq m}^{N}\frac{\varphi_m - \varphi_j}{1 - \cos\frac{2\pi}{N}(m-j)}, \tag{6}$$

$$\dot{\varphi}_m = \frac{1}{2N\sin^3\theta_0}\sum_{j\neq m}^{N}\frac{\theta_m - \theta_j}{1 - \cos\frac{2\pi}{N}(m-j)} + B_N\theta_m. \tag{7}$$

The parameter B_N is denoted by

$$B_N = \frac{1+\cos^2\theta_0}{2N\sin^3\theta_0} - \frac{\kappa_1(1+\cos^2\theta_0)}{2\sin^3\theta_0} - \frac{\kappa_2\cos\theta_0}{2\sin^3\theta_0}, \tag{8}$$

in which κ_1 and κ_2 are defined by

$$\kappa_1 = \frac{\Gamma_1 + \Gamma_2 + 2\pi}{2\pi}, \qquad \kappa_2 = \frac{\Gamma_1 - \Gamma_2}{\pi}.$$

Then, we have obtained the eigenvalues and their corresponding eigenvectors for the linearized equations (6) and (7).

Theorem 1. *For $m = 0, 1, \cdots, N-1$, the eigenvalues $\lambda_m^{\pm}$ are represented by*

$$\lambda_m^{\pm} = \pm\sqrt{\xi_m\eta_m}, \tag{9}$$

in which

$$\xi_m = \frac{m(N-m)}{2N\sin\theta_0}, \qquad \eta_m = \frac{m(N-m)}{2N\sin^3\theta_0} + B_N. \tag{10}$$

It follows from (9) and (10) that we have $\lambda_m^\pm = \lambda_{N-m}^\pm$. The symmetry indicates that when the number of the vortex points is even, say $N = 2M$, we have the zero eigenvalues $\lambda_0^\pm = 0$, the double eigenvalues $\lambda_m^\pm$ for $m = 1, \cdots, M-1$, and the simple eigenvalues $\lambda_M^\pm$. On the other hand, for the odd case, i.e. $N = 2M+1$, the eigenvalues $\lambda_0^\pm$ are zero, and all the other eigenvalues $\lambda_m^\pm$ are double for $m = 1, \cdots, M$.

The following theorem provides us with the explicit representation of the eigenvectors corresponding to the eigenvalues.

Theorem 2. *Let the vectors $\vec{\phi}_m^\pm$ and $\vec{\psi}_m^\pm$ be defined by*

$$\begin{aligned} \vec{\psi}_m^\pm &= {}^t\Big(\sqrt{\xi_m}, \sqrt{\xi_m}\cos\frac{2\pi}{N}m, \cdots, \sqrt{\xi_m}\cos\frac{2\pi}{N}(N-1)m, \\ &\qquad \pm\sqrt{\eta_m}, \pm\sqrt{\eta_m}\cos\frac{2\pi}{N}m, \cdots, \pm\sqrt{\eta_m}\cos\frac{2\pi}{N}(N-1)m\Big), \quad (11) \\ \vec{\phi}_m^\pm &= {}^t\Big(0, \sqrt{\xi_m}\sin\frac{2\pi}{N}m, \cdots, \sqrt{\xi_m}\sin\frac{2\pi}{N}(N-1)m, \\ &\qquad 0, \pm\sqrt{\eta_m}\sin\frac{2\pi}{N}m, \cdots, \pm\sqrt{\eta_m}\sin\frac{2\pi}{N}(N-1)m\Big), \quad (12) \end{aligned}$$

for $m = 1, \cdots, M = [N/2]$, in which $[x]$ denotes the largest integer less than or equals x. Then, the vectors $\vec{\phi}_m^\pm$ and $\vec{\psi}_m^\pm$ are the eigenvectors corresponding to the eigenvalues $\lambda_m^\pm$ for $m = 1, \cdots, M-1$. When $N = 2M$, the vectors $\vec{\psi}_M^\pm$ are the eigenvectors for the simple $\lambda_M^\pm$, whereas $\vec{\phi}_M^\pm$ and $\vec{\psi}_M^\pm$ are the eigenvectors corresponding to the double $\lambda_M^\pm$ for $N = 2M+1$. Furthermore, the eigenvectors are linearly independent.

Since the multiplicity of the eigenvalues $\lambda_M^\pm$ for the even case is different from that for the odd case, so are the corresponding eigenvectors. In the present article, we use these eigenvectors to characterize invariant dynamical systems embedded in the real phase space $\mathbb{P}_N$. However, while the eigenvectors (11) and (12) are real vectors for $\eta_m > 0$, they have the pure imaginary component $\sqrt{\eta_m}$ when η_m is negative. Then noting that $\vec{\phi}_m^-$ is the complex conjugate of $\vec{\phi}_m^+$ for the negative case, we newly define the real eigenvectors by $(\vec{\phi}_m^+ + \vec{\phi}_m^-)/2$ and $(\vec{\phi}_m^+ - \vec{\phi}_m^-)/2\mathrm{i}$. In the same way, we redefine the real eigenvectors with respect to $\vec{\psi}_m^\pm$. Hence, regardless of the sign of η_m, we can construct the linearly independent real eigenvectors. In what follows, for the sake of convenience, the redefined real eigenvectors are denoted by $\vec{\phi}_m^\pm$ and $\vec{\psi}_m^\pm$ for $\eta_m < 0$. On the other hand, since the total number of the non-zero eigenvectors given in Theorem 2 is $2N-2$, we need two more linearly independent vectors, which are given as follows [18].

Lemma 3. *Let $\vec{\zeta}^\pm$ be defined by*

$$\vec{\zeta}^\pm = \frac{1}{\sqrt{2N}}\,{}^t(1, 1, \cdots, 1, \pm1, \pm1, \cdots, \pm1). \quad (13)$$

Then, they satisfy $(\vec{\psi}_m^{\pm}, \vec{\zeta}^{\pm}) = 0$, $(\vec{\phi}_m^{\pm}, \vec{\zeta}^{\pm}) = 0$ *for all* m.

3 Reduction by the shift invariance

We define the p-shift transformation for the vortex points. For a given point $(\Theta_1, \cdots, \Theta_N, \Psi_1, \cdots, \Psi_N) \in \mathbb{P}_N$, the angular rotation by the degree $\frac{2\pi}{N}p$, $r_p : \mathbb{P}_N \to \mathbb{P}_N$, and the circular p-shift of variables $s_p : \mathbb{P}_N \to \mathbb{P}_N$ are given by

$$r_p, s_p : (\Theta_1, \cdots, \Theta_N, \Psi_1, \cdots, \Psi_N) \to (\Theta'_1, \cdots, \Theta'_N, \Psi'_1, \cdots, \Psi'_N),$$

in which

$$r_p : \Theta'_m = \Theta_m, \qquad \Psi'_m = \Psi_m + \frac{2\pi}{N}p \mod 2\pi, \qquad \text{for} \ \ m = 1, \ldots, N,$$

and

$$\begin{array}{llll} s_p : & \Theta'_m = \Theta_{N-p+m}, & \Psi'_m = \Psi_{N-p+m}, & \text{for} \ \ m = 1, \ldots, p, \\ & \Theta'_m = \Theta_{m-p}, & \Psi'_m = \Psi_{m-p}, & \text{for} \ \ m = p+1, \ldots, N. \end{array}$$

The p-shift transformation σ_p is defined by $\sigma_p = r_p \circ s_p$, which is specified by

$$\begin{array}{llll} \sigma_p : & \Theta'_m = \Theta_{N-p+m}, & \Psi'_m = \Psi_{N-p+m} + \frac{2\pi p}{N}, & \text{for} \ \ m = 1, \ldots, p, \\ & \Theta'_m = \Theta_{m-p}, & \Psi'_m = \Psi_{m-p} + \frac{2\pi p}{N}, & \text{for} \ \ m = p+1, \ldots, N. \end{array} \tag{14}$$

On the other hand, we introduce the circular p-shift map for the vector field,

$$\Sigma_p : (F_1, \cdots, F_N, G_1, \cdots, G_N) \mapsto (F'_1, \cdots, F'_N, G'_1, \cdots, G'_N),$$

in which

$$\begin{array}{llll} \Sigma_p : & F'_m = F_{N-p+m}, & G'_m = G_{N-p+m}, & \text{for} \ \ m = 1, \ldots, p, \\ & F'_m = F_{m-p}, & G'_m = G_{m-p}, & \text{for} \ \ m = p+1, \ldots, N. \end{array} \tag{15}$$

Then we have the following lemma.

Lemma 4. *For* $\vec{x} \in \mathbb{P}_N$, $\Sigma_p \mathbb{F}(\vec{x}) = \mathbb{F}(\sigma_p \vec{x})$.

Proof: For the sake of convenience, we use the following notations.

$$\begin{aligned} f(\Theta_m, \Theta_j, \Psi_m - \Psi_j) &= \frac{\sin\Theta_j \sin(\Psi_m - \Psi_j)}{1 - \cos\gamma_{mj}}, \\ g(\Theta_m, \Theta_j, \Psi_m - \Psi_j) &= \frac{\cos\Theta_m \sin\Theta_j \cos(\Psi_m - \Psi_j) - \sin\Theta_m \cos\Theta_j}{1 - \cos\gamma_{mj}}. \end{aligned}$$

For $m = 1, \ldots, p$, it follows from (14) that

$$\begin{aligned}
&\sum_{j\neq m}^{N} f(\Theta'_m, \Theta'_j, \Psi'_m - \Psi'_j) = \sum_{j\neq m}^{p} f(\Theta'_m, \Theta'_j, \Psi'_m - \Psi'_j) + \sum_{j=p+1}^{N} f(\Theta'_m, \Theta'_j, \Psi'_m - \Psi'_j) \\
= &\sum_{j\neq m}^{p} f(\Theta_{N-p+m}, \Theta_{N-p+j}, \Psi_{N-p+m} - \Psi_{N-p+j}) + \sum_{j=p+1}^{N} f(\Theta_{N-p+m}, \Theta_{j-p}, \Psi_{N-p+m} - \Psi_{j-p}) \\
= &\sum_{j'\neq N-p+m}^{N} f(\Theta_{N-p+m}, \Theta_{j'}, \Psi_{N-p+m} - \Psi_{j'}) + \sum_{j'=1}^{N-p} f(\Theta_{N-p+m}, \Theta_{j'}, \Psi_{N-p+m} - \Psi_{j'}) \\
= &\sum_{j'\neq N-p+m}^{N} f(\Theta_{N-p+m}, \Theta_{j'}, \Psi_{N-p+m} - \Psi_{j'}).
\end{aligned}$$

In the third equality, we change the summation variable $j' = N - p + j$ in the first summation, and $j' = j - p$ in the second summation. Hence, we have

$$\begin{aligned}
F_m(\sigma_p \vec{x}) &= -\frac{\Gamma^{(N)}}{4\pi} \sum_{j\neq m}^{N} f(\Theta'_m, \Theta'_j, \Psi'_m - \Psi'_j) \\
&= -\frac{\Gamma^{(N)}}{4\pi} \sum_{j\neq N-p+m}^{N} f(\Theta_{N-p+m}, \Theta_j, \Psi_{N-p+m} - \Psi_j) = F_{N-p+m}(\vec{x}).
\end{aligned}$$

Regarding G_m for $m = 1, \ldots, p$, since

$$\begin{aligned}
&\sum_{j\neq m}^{N} g(\Theta'_m, \Theta'_j, \Psi'_m - \Psi'_j) = \sum_{j\neq m}^{p} g(\Theta'_m, \Theta'_j, \Psi'_m - \Psi'_j) + \sum_{j=p+1}^{N} g(\Theta'_m, \Theta'_j, \Psi'_m - \Psi'_j) \\
= &\sum_{j\neq m}^{p} g(\Theta_{N-p+m}, \Theta_{N-p+j}, \Psi_{N-p+m} - \Psi_{N-p+j}) + \sum_{j=p+1}^{N} g(\Theta_{N-p+m}, \Theta_{j-p}, \Psi_{N-p+m} - \Psi_{j-p}) \\
= &\sum_{j'\neq N-p+m}^{N} g(\Theta_{N-p+m}, \Theta_{j'}, \Psi_{N-p+m} - \Psi_{j'}) + \sum_{j'=1}^{N-p} g(\Theta_{N-p+m}, \Theta_{j'}, \Psi_{N-p+m} - \Psi_{j'}) \\
= &\sum_{j'\neq N-p+m}^{N} g(\Theta_{N-p+m}, \Theta_{j'}, \Psi_{N-p+m} - \Psi_{j'}),
\end{aligned}$$

we obtain

$$\begin{aligned}
G_m(\sigma_p\vec{x}) &= -\frac{\Gamma^{(N)}}{4\pi\sin\Theta'_m}\sum_{j\neq m}^{N} g(\Theta'_m,\Theta'_j,\Psi'_m-\Psi'_j) \\
&\quad +\frac{\Gamma_1}{4\pi}\frac{1}{1-\cos\Theta'_m}-\frac{\Gamma_2}{4\pi}\frac{1}{1+\cos\Theta'_m} \\
&= -\frac{\Gamma^{(N)}}{4\pi\sin\Theta_{N-p+m}}\sum_{j\neq N-p+m}^{N} g(\Theta_{N-p+m},\Theta_j,\Psi_{N-p+m}-\Psi_j) \\
&\quad -\frac{\Gamma_1}{4\pi}\frac{1}{1-\cos\Theta_{N-p+m}}-\frac{\Gamma_2}{4\pi}\frac{1}{1+\cos\Theta_{N-p+m}} = G_{N-p+m}(\vec{x}).
\end{aligned}$$

In the similar way, we show the relations $F_m(\sigma_p\vec{x}) = F_{m-p}(\vec{x})$ and $G_m(\sigma_p\vec{x}) = G_{m-p}(\vec{x})$ for $m = p+1,\dots,N$. $\square$

The next proposition claims that the σ_p invariance persists for all the time when it holds at the initial moment.

Proposition 5. *Let $N = pq$, $(p,q \in \mathbb{N})$. If the solution $\vec{x}$ of (1) and (2) is σ_p invariant at the initial time, i.e. $\sigma_p\vec{x}(0) = \vec{x}(0)$, then $\sigma_p\vec{x}(t) = \vec{x}(t)$ for $t \geq 0$.*

Proof. Since $N = pq$ and $\sigma_p\vec{x}(0) = \vec{x}(0)$, we have $\sigma_p^k\vec{x}(0) = \vec{x}(0)$ for $k = 0,\dots,q-1$, which is equivalently expressed by

$$\Theta_{kp+m}(0) = \Theta_m(0), \qquad \Psi_{kp+m}(0) = \Psi_m(0) + \frac{2\pi}{N}kp = \Psi_m(0) + \frac{2\pi}{q}k, \tag{16}$$

for $k = 0,\dots,q-1$ and $p = 1,\dots,m$.

On the other hand, when $\vec{x}$ is σ_p invariant, it follows from Lemma 4 that $\Sigma_p\mathbb{F}(\vec{x}) = \mathbb{F}(\sigma_p\vec{x}) = \mathbb{F}(\vec{x})$. In other words, $F_{N-p+m} = F_m$, $G_{N-p+m} = G_m$ for $m = 1,\dots,p$ and $F_{m-p} = F_m$, $G_{m-p} = G_m$ for $m = p+1,\dots,N$. Hence, due to $N = pq$, we have

$$\dot{\Theta}_{kp+m} - \dot{\Theta}_m = F_{kp+m} - F_m = 0, \qquad \dot{\Psi}_{kp+m} - \dot{\Psi}_m = G_{kp+m} - G_m = 0,$$

for $k = 0,\dots,q-1$ and $m = 1,\dots p$. Hence, if $\vec{x}$ satisfies the initial condition (16), then we obtain

$$\Theta_{kp+m}(t) = \Theta_m(t), \qquad \Psi_{kp+m}(t) = \Psi_m(t) + \frac{2\pi}{q}k, \tag{17}$$

for $k = 0,\dots,q-1$ and $m = 1,\dots,p$, which indicates that $\vec{x}(t)$ is invariant with respect to σ_p for all the time. $\square$

The relation (17) indicates that the motion of the vortex points $(\Theta_{kp+m},\Psi_{kp+m})$ for $k = 1,\dots,q-1$ is automatically determined by that of the vortex point

(Θ_m, Ψ_m). In other words, the N vortex points are divided into the q clusters of the p vortex points. Therefore, we expect that the σ_p invariant N-vortex system defines the $2p$-dimensional dynamical system embedded in $\mathbb{P}_N$. However, Proposition 5 just claims that if there exists a σ_p invariant point, then the evolution starting from the point remains σ_p invariant. Thus, we need to show that there really exists the $2p$-dimensional σ_p invariant subspace of $\mathbb{P}_N$, in which the reduced system is defined. Here, we use the linearly independent real vectors $\vec{\psi}^{\pm}_m$, $\vec{\phi}^{\pm}_m$ and $\vec{\zeta}^{\pm}$ in order to characterize the σ_p invariant subspace. We need the following lemma for the purpose.

Lemma 6. *Let $N = pq$, $(p,\, q \in \mathbb{N})$. The vectors $\vec{\psi}^{\pm}_{kq}$ and $\vec{\phi}^{\pm}_{kq}$ for $p \le kq \le M$, and $\vec{\zeta}^{\pm}$ are invariant with respect to s_p. Moreover, the number of the s_p invariant eigenvectors is $2p$.*

Proof. First we prove the s_p invariance of the eigenvectors. It is obvious that the vectors $\vec{\zeta}^{\pm}$ is s_p invariant. The eigenvectors $\vec{\psi}^{\pm}_{kq}$ and $\vec{\phi}_{kq}$ are also s_p invariant, since the eigenvectors are expressed by (11) and (12), and

$$\cos\frac{2\pi}{N}(k+j)mq = \cos\frac{2\pi}{N}jmq, \qquad \sin\frac{2\pi}{N}(kp+j)mq = \sin\frac{2\pi}{N}jmq,$$

hold for m, k and $j \in \mathbb{Z}$ due to $N = pq$.

Next we show the number of the s_p invariant eigenvectors is $2p$ by considering the following three cases separately; First, when $N = 2M$ and $p = 2p'$, then q is a factor of M, i.e. $p'q = M$. Hence, the eigenvectors $\vec{\psi}^{\pm}_M$ are s_p invariant. Thus, the $2p$ eigenvectors $\vec{\psi}^{\pm}_{kq}$ for $k = 1, \ldots, p'$, $\vec{\phi}^{\pm}_{kq}$ for $k = 1, \ldots, p'-1$ and $\vec{\zeta}^{\pm}$ are s_p invariant. Second, when $N = 2M$ and $p = 2p'+1$, since $\vec{\psi}^{\pm}_M$ are no longer s_p invariant, the number of the s_p invariant eigenvectors is

$$4\left[\frac{M}{q}\right] + 2 = 4\left[p' + \frac{1}{2}\right] + 2 = 4p' + 2 = 2p.$$

Finally, when $N = 2M+1$, since p must be odd, namely $p = 2p'+1$. So the number of the s_p invariant eigenvectors is

$$4\left[\frac{M}{q}\right] + 2 = 4\left[p' + \frac{1}{2} - \frac{1}{2q}\right] + 2 = 4p' + 2 = 2p. \qquad \square$$

It follows from Lemma 6 that the σ_p invariant subspace of $\mathbb{P}_N$ is represented by the linear combination of the s_p invariant eigenvectors.

Proposition 7. *Let $N = pq$, $(p,\, q \in \mathbb{N})$. Then, the σ_p invariant subspace of $\mathbb{P}_N$, say $\mathbb{P}_N(\sigma_p)$, is the set of*

$$\vec{x} = \vec{x}_0 + \sum_k \left(a^+_k \vec{\psi}^+_{kq} + a^-_k \vec{\psi}^-_{kq} + b^+_k \vec{\phi}^+_{kq} + b^-_k \vec{\phi}^-_{kq}\right) + c^+\vec{\zeta}^+ + c^-\vec{\zeta}^-,$$

in which $\vec{x}_0 = \left(\theta_0, \theta_0, \cdots, \theta_0, 0, \frac{2\pi}{N}, \cdots, \frac{2\pi}{N}(N-1)\right)$ *and* $a_k^\pm$, $b_k^\pm$, $c^\pm$ *are real coefficients.*

Proof. We note that the transformation σ_p is expressed by the following form,

$$\sigma_p \vec{x} = \left(0, \cdots, 0, \frac{2\pi}{N}p, \cdots, \frac{2\pi}{N}p\right) + s_p \vec{x}.$$

Since $\sigma_p \vec{x}_0 = \vec{x}_0$ and Lemma 6, we have

$$\begin{aligned} \sigma_p \vec{x} &= \left(0, \cdots, 0, \frac{2\pi}{N}p, \cdots, \frac{2\pi}{N}p\right) + s_p \vec{x}_0 \\ &\quad + s_p \left(\sum_k \left(a_k^+ \vec{\psi}_{kq}^+ + a_k^- \vec{\psi}_{kq}^- + b_k^+ \vec{\phi}_{kq}^+ + b_k^- \vec{\phi}_{kq}^- \right) + c^+ \vec{\zeta}^+ + c^- \vec{\zeta}^- \right) \\ &= \vec{x}_0 + \sum_k \left(a_k^+ \vec{\psi}_{kq}^+ + a_k^- \vec{\psi}_{kq}^- + b_k^+ \vec{\phi}_{kq}^+ + b_k^- \vec{\phi}_{kq}^- \right) + c^+ \vec{\zeta}^+ + c^- \vec{\zeta}^- = \vec{x}. \qquad \Box \end{aligned}$$

The subspace is defined for the N-ring at arbitrary latitude θ_0. This is why the reduced system is always available to describe the unstable motion of the perturbed N-ring successfully[17]. Proposition 7 also yields the inclusion relation between the two invariant subspaces, $\mathbb{P}_N(\sigma_p)$ and $\mathbb{P}_N(\sigma_q)$.

Corollary 8. *Suppose that integers* p *and* q *are the factors of* N*, and* p *also divides* q*. Then, the* σ_p *invariant subspace is included by the* σ_q *invariant subspace, namely* $\mathbb{P}_N(\sigma_p) \subset \mathbb{P}_N(\sigma_q)$*.*

Proof. From the assumptions, there exist integers m, p' and q' such that $q = mp$ and $N = pp' = qq'$. For the arbitrary s_p invariant $\vec{\psi}_{kp'}^\pm$ and $\vec{\phi}_{kq'}^\pm$, they are also s_q invariant eigenvectors, since $kp' = k\frac{N}{p} = k\frac{qq'}{p} = kmq'$. Consequently, we have $\mathbb{P}_N(\sigma_p) \subset \mathbb{P}_N(\sigma_q)$ $\Box$.

In the following theorem, we characterize the σ_p invariant dynamical system.

Theorem 9. *Let* $N = pq$, $(p, q \in \mathbb{N})$*. The* σ_p *invariant dynamical system is equivalent to the system of the* p*-vortex points generated by the averaged Hamiltonian,*

$$\begin{aligned} H &= -\frac{\left(\Gamma^{(p)}\right)^2}{8\pi} \sum_{m=1}^{p} \sum_{j \neq m}^{p} \frac{1}{q} \sum_{l=0}^{q-1} h\left(\Theta_m, \Theta_j, \Psi_m - \Psi_j - \frac{2\pi}{q} l\right) \\ &\quad - \frac{\Gamma_1' \Gamma^{(p)}}{4\pi} \sum_{m=1}^{p} \log(1 - \cos\Theta_m) - \frac{\Gamma_2' \Gamma^{(p)}}{4\pi} \sum_{m=1}^{p} \log(1 + \cos\Theta_m), \end{aligned} \tag{18}$$

in which $\Gamma^{(p)} = 2\pi/p$, Γ_1' *and* Γ_2' *are the modified strengths of the pole vortices given as follows,*

$$\Gamma_1' = \Gamma_1 + \frac{1}{2}\Gamma^{(p)}\left(1 - \frac{1}{q}\right), \qquad \Gamma_2' = \Gamma_2 + \frac{1}{2}\Gamma^{(p)}\left(1 - \frac{1}{q}\right). \tag{19}$$

Proof. Let h be defined by $h(\Theta, \Theta', \Psi) = \log(1 - \cos\Theta\cos\Theta' - \sin\Theta\sin\Theta'\cos\Psi)$ for the convenience. Since the dynamical system is invariant with respect to σ_p, the N vortex points satisfy the relation (17). Then the summations in the Hamiltonian (3) are taken separately with respect to the q clusters of the p vortex points. Accordingly, the Hamiltonian is rewritten by

$$\begin{aligned} H &= -\frac{\left(\Gamma^{(N)}\right)^2}{8\pi}\sum_{k=0}^{q-1}\sum_{m=1}^{p}\left(\sum_{l\neq k}^{q-1} h(\Theta_{kp+m}, \Theta_{lp+m}, \Psi_{kp+m} - \Psi_{lp+m})\right. \\ &\quad \left. + \sum_{l=0}^{q-1}\sum_{j\neq m}^{p} h(\Theta_{kp+m}, \Theta_{lp+j}, \Psi_{kp+m} - \Psi_{lp+j})\right) \\ &\quad -\frac{\Gamma_1\Gamma^{(N)}}{4\pi}\sum_{k=0}^{q-1}\sum_{m=1}^{p}\log(1 - \cos\Theta_{kp+m}) - \frac{\Gamma_2\Gamma^{(N)}}{4\pi}\sum_{k=0}^{q-1}\sum_{m=1}^{p}\log(1 + \cos\Theta_{kp+m}) \\ &= -\frac{\left(\Gamma^{(N)}\right)^2}{8\pi}\sum_{k=0}^{q-1}\sum_{m=1}^{p}\left(\sum_{l\neq k}^{q-1} h(\Theta_m, \Theta_m, 2\pi(k-l)/q)\right. \\ &\quad \left. + \sum_{l=0}^{q-1}\sum_{j\neq m}^{p} h(\Theta_m, \Theta_j, \Psi_m - \Psi_j - 2\pi(k-l)/q)\right) \\ &\quad -\frac{\Gamma_1\Gamma^{(N)}}{4\pi}\sum_{k=0}^{q-1}\sum_{m=1}^{p}\log(1 - \cos\Theta_m) - \frac{\Gamma_2\Gamma^{(N)}}{4\pi}\sum_{k=0}^{q-1}\sum_{m=1}^{p}\log(1 + \cos\Theta_m). \end{aligned}$$

By using $h(\Theta_m, \Theta_m, 2\pi(k-l)/q) = \log(1 - \cos^2\Theta_m) + \log(1 - \cos(2\pi(k-l)/q)$ in the first term and summing up with respect to k, we have

$$\begin{aligned} H &= -\frac{\left(\Gamma^{(p)}\right)^2}{4\pi}(1 - 1/q)\sum_{m=1}^{p}\log(1 - \cos^2\Theta_m) - \frac{\left(\Gamma^{(N)}\right)^2}{8\pi}p\sum_{k=0}^{q-1}\sum_{l\neq k}^{q-1}\log(1 - \cos(2\pi(k-l)/q)) \\ &\quad -\frac{\left(\Gamma^{(p)}\right)^2}{8\pi}\sum_{m=1}^{p}\sum_{j\neq m}^{p}\frac{1}{q}\sum_{l=0}^{q-1} h\left(\Theta_m, \Theta_j, \Psi_m - \Psi_j - \frac{2\pi}{q}l\right) \\ &\quad -\frac{\Gamma_1\Gamma^{(p)}}{4\pi}\sum_{m=1}^{p}\log(1 - \cos\Theta_m) - \frac{\Gamma_2\Gamma^{(p)}}{4\pi}\sum_{m=1}^{p}\log(1 + \cos\Theta_m). \end{aligned}$$

Since the second term is constant, eliminating it from the Hamiltonian, we have (18). The Hamiltonian (18) is $2\pi/q$ periodic in the Ψ direction. In the first term of the Hamiltonian, the function h is averaged over the q period, so we call it the averaged Hamiltonian of period q. Thus the σ_p invariant dynamical system is equivalent to the p-vortex system generated by the averaged Hamiltonian with the modified pole vortices. $\square$

Because of the $2\pi/q$-periodicity of the averaged Hamiltonian, the p-vortex system is defined in the restricted region $[0,\pi]^p \times [0, 2\pi/q]^p$ and extended in the whole space $\mathbb{P}_N$ by shifting the restricted space q times in the Ψ direction. For example, when N is even, the 2-vortex system with the averaged Hamiltonian is embedded in the N-vortex system. Since the 2-vortex system is integrable due to the invariant quantity $\cos\Theta_1 + \cos\Theta_2 = \text{Const.}$, it is sufficient to observe the contour plot of the averaged Hamiltonian. Figure 1 shows the contour plots of the averaged Hamiltonian reduced by the σ_2 invariance for $N = 6$ and $N = 10$ with various strengths of the pole vortices. They are plotted in the phase space $(\Phi = \Psi_1 - \Psi_2, \Theta_1)$. The invariant quantity is $\cos\Theta_1 + \cos\Theta_2 = 2\cos\theta_0$ with $\theta_0 = \frac{\pi}{3}$. According to Proposition 7, each of the subspaces $\mathbb{P}_6(\sigma_2)$ and $\mathbb{P}_{10}(\sigma_2)$ contains the 6-ring and the 10-ring at the latitude $\theta_0 = \frac{\pi}{3}$ respectively. The structure of the contour plots is $2\pi/3$-periodic for $N = 6$ and $2\pi/5$-periodic for $N = 10$ in the Φ direction. The reduced dynamical system has already been investigated to describe the unstable motion of the even vortex points by numerical means[17]. Finally, we remark that the σ_2 invariant dynamical system is in fact the same as the one reduced by the invariance of the Hamiltonian under the action of the dihedral group, which was used to find relative periodic orbits[19].

4 Reduction by the pole reversal invariance

In this section, we reduce the N-vortex system by the invariant property in terms of the pole reversal transformation, which is defined differently for the odd vortex points and the even vortex points.

When the number of the vortex points is odd, $N = 2M+1$, we define the pole reversal transformation around the point vortex (Θ_1, Ψ_1), say $\pi_o : \mathbb{P}_N \to \mathbb{P}_N$, by the following three steps.

(1) We rotate the system in the longitudinal direction by the degree $-\Psi_1$ so that the vortex point (Θ_1, Ψ_1) is in the xz-plane:

$$(\Theta_1, \cdots, \Theta_N, \Psi_1, \cdots, \Psi_N) \mapsto (\Theta_1, \Theta_2, \cdots, \Theta_N, 0, \cdots, \Psi_N - \Psi_1).$$

(2) Then we rotate the system around the x-axis by the degree π. The operation interchanges the north and the south poles:

$$\begin{aligned}&(\Theta_1, \Theta_2, \cdots, \Theta_N, 0, \Psi_2 - \Psi_1, \cdots, \Psi_N - \Psi_1)\\ &\qquad \mapsto (\pi - \Theta_1, \pi - \Theta_N, \cdots, \pi - \Theta_2, 0, \Psi_1 - \Psi_N, \cdots, \Psi_1 - \Psi_2).\end{aligned}$$

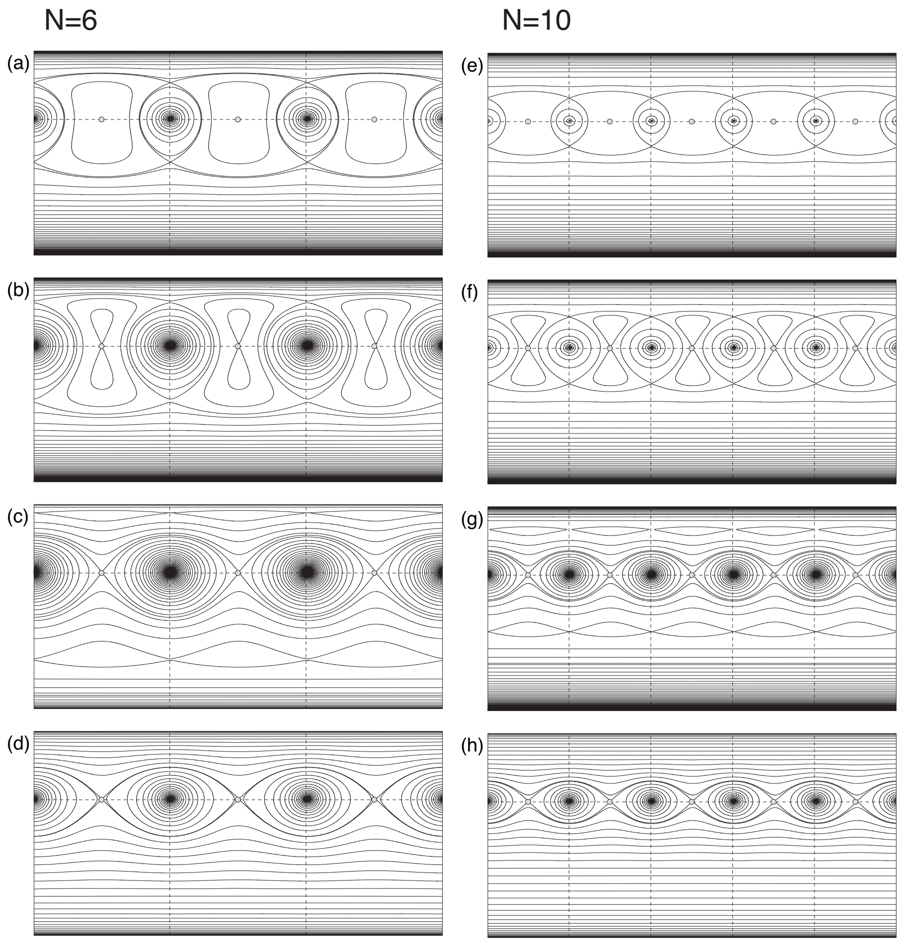

Figure 1: Contour plots of the averaged Hamiltonian (18) reduced by the σ_2 shift invariance for $N = 6$ (a-d) and $N = 10$ (e-h) when the pole vortices are identical, i.e. $\Gamma_1 = \Gamma_2$. The strength of the pole vortex is (a) 0.6π, (b) 0.2π, (c) -0.2π, (d) -0.6π, (e) 1.5π, (f) 0.6π, (g) 0.0, and (h) -0.6π respectively. They are plotted in the region $(\Phi = \Psi_1 - \Psi_2, \Theta_1) \in [0, 2\pi] \times [0, \pi]$.

(3) Finally we rotate back the system in the angular direction by the degree Ψ_1:

$$\begin{aligned}&(\pi-\Theta_1,\pi-\Theta_N,\cdots,\pi-\Theta_2,0,\Psi_1-\Psi_N,\cdots,\Psi_1-\Psi_2)\\ &\qquad\mapsto(\pi-\Theta_1,\pi-\Theta_N,\cdots,\pi-\Theta_2,\Psi_1,2\Psi_1-\Psi_N,\cdots,2\Psi_1-\Psi_2).\end{aligned}$$

Consequently, the transformation π_o is specified by

$$\pi_o:(\Theta_1,\cdots,\Theta_N,\Psi_1,\cdots,\Psi_N)\to(\Theta_1',\cdots,\Theta_N',\Psi_1',\cdots,\Psi_N'),$$

in which

$$\begin{array}{lll}\Theta_1'=\pi-\Theta_1, & \Psi_1'=\Psi_1, & \\ \Theta_m'=\pi-\Theta_{N-m+2}, & \Psi_m'=2\Psi_1-\Psi_{N-m+2}, & \text{for } m\neq 1.\end{array} \tag{20}$$

The transformation for the vector field by the pole reversal is given by

$$\Pi_o:(F_1,\cdots,F_N,G_1,\cdots,G_N)\to(F_1',\cdots,F_N',G_1',\cdots,G_N'),$$

where

$$\begin{array}{lll}F_1'=-F_1, & G_1'=-G_1, & \\ F_m'=-F_{N-m+2}, & G_m'=-G_{N-m+2}, & \text{for } m\neq 1.\end{array}$$

When the strengths of the pole vortices are the same, we have the following lemma.

Lemma 10. *Let $N=2M+1$. If $\Gamma_1=\Gamma_2$, then $\Pi_o\mathbb{F}(\vec{x})=\mathbb{F}(\pi_o\vec{x})$ for $\vec{x}\in\mathbb{P}_N$.*

Proof. It follows from (20) that we obtain

$$\begin{aligned}F_1(\pi_o\vec{x}) &= -\frac{\Gamma^{(N)}}{4\pi}\sum_{j=2}^{N}f(\pi-\Theta_1,\pi-\Theta_{N-j+2},\Psi_{N-j+2}-\Psi_1)\\ &= \frac{\Gamma^{(N)}}{4\pi}\sum_{j=2}^{N}f(\Theta_1,\Theta_{N-j+2},\Psi_1-\Psi_{N-j+2})=-F_1(\vec{x}),\\ G_1(\pi_o\vec{x}) &= -\frac{\Gamma^{(N)}}{4\pi}\sum_{j=2}^{N}g(\pi-\Theta_1,\pi-\Theta_{N-j+2},\Psi_{N-j+2}-\Psi_1)+\frac{\Gamma_1}{2\pi}\frac{\cos(\pi-\Theta_1)}{\sin(\pi-\Theta_1)}\\ &= \frac{\Gamma^{(N)}}{4\pi}\sum_{j=2}^{N}g(\Theta_1,\Theta_{N-j+2},\Psi_1-\Psi_{N-j+2})-\frac{\Gamma_1}{2\pi}\frac{\cos\Theta_1}{\sin\Theta_1}=-G_1(\vec{x}).\end{aligned}$$

Similarly, for $m \neq 1$, we have

$$
\begin{aligned}
F_m(\pi_o\vec{x}) &= -\frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N} f(\pi-\Theta_{N-m+2},\pi-\Theta_{N-j+2},\Psi_{N-j+2}-\Psi_{N-m+2}) \\
&= \frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N} f(\Theta_{N-m+2},\Theta_{N-j+2},\Psi_{N-m+2}-\Psi_{N-j+2}) \\
&= \frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq N-m+2}^{N} f(\Theta_{N-m+2},\Theta_{j},\Psi_{N-m+2}-\Psi_{j}) = -F_{N-m+2}(\vec{x}), \\
G_m(\pi_o\vec{x}) &= -\frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N} g(\pi-\Theta_{N-m+2},\pi-\Theta_{N-j+2},\Psi_{N-j+2}-\Psi_{N-m+2}) \\
&\quad +\frac{\Gamma_1}{2\pi}\frac{\cos(\pi-\Theta_{N-m+2})}{\sin(\pi-\Theta_{N-m+2})} \\
&= \frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N} g(\Theta_{N-m+2},\Theta_{N-j+2},\Psi_{N-m+2}-\Psi_{N-j+2}) - \frac{\Gamma_1}{2\pi}\frac{\cos\Theta_{N-m+2}}{\sin\Theta_{N-m+2}} \\
&= -G_{N-m+2}(\vec{x}). \qquad \square
\end{aligned}
$$

It is easy to show that if the initial data $\vec{x}(0)$ is invariant with respect to π_o, then the invariant property holds for all the time.

Lemma 11. *Let $N = 2M+1$ and $\Gamma_1 = \Gamma_2$. If $\pi_o\vec{x}(0) = \vec{x}(0)$, then $\pi_o\vec{x}(t) = \vec{x}(t)$ for $t \geq 0$.*

Proof. The π_o-invariance of the system yields

$$
\Theta_1 = \frac{\pi}{2}, \quad \Psi_1 = 0, \quad \Theta_m + \Theta_{N-m+2} = \pi, \quad \Psi_m + \Psi_{N-m+2} = 0, \quad \text{for } \; m \neq 1. \tag{21}
$$

On the other hand, it follows from Lemma 10 that $\Pi_o\mathbb{F}(\vec{x}) = \mathbb{F}(\pi_o\vec{x}) = \mathbb{F}(\vec{x})$, that is to say, $\dot{\Theta}_1 = F_1 = 0$, $\dot{\Psi}_1 = G_1 = 0$, and

$$
\dot{\Theta}_m + \dot{\Theta}_{N-m+2} = F_m + F_{N-m+2} = 0, \qquad \dot{\Psi}_m + \dot{\Psi}_{N-m+2} = G_m + G_{N-m+2} = 0, \tag{22}
$$

for $m \neq 1$. Hence, the π_o-invariant relation (21) holds for all the time if it is satisfied at the initial time. $\square$

Lemma 11 indicates that the N-vortex system can be reduced to the $2M$-dimensional invariant dynamical system as long as the π_o-invariant subspace exists. The following lemma guarantees the existence of the $2M$-dimensional π_o invariant subspace.

Lemma 12. *Let $N = 2M+1$, $\Gamma_1 = \Gamma_2$ and $\vec{x}_0$ is represented by*

$$
\vec{x}_0 = \left(\frac{\pi}{2}, \cdots, \frac{\pi}{2}, 0, \frac{2\pi}{N}, \cdots, \frac{2\pi}{N}M, -\frac{2\pi}{N}M, \cdots, -\frac{2\pi}{N}\right).
$$

Then, the $2M$-dimensional set of

$$\vec{x} = \vec{x}_0 + \sum_{k=1}^{M} \left(b_k^+ \vec{\phi}_k^+ + b_k^- \vec{\phi}_k^- \right), \qquad b_k^\pm \in \mathbb{R}, \tag{23}$$

is invariant with respect to the transformation π_o, which is denoted by $\mathbb{P}_N(\pi_o)$.

Proof. It is easy to see the eigenvectors $\vec{\phi}_k^\pm$ are invariant with respect to the following transformation,

$$\pi_o' : (\Theta_1, \Theta_2, \cdots, \Theta_N, \Psi_1, \Psi_2, \cdots, \Psi_N) \to -(\Theta_1, \Theta_N, \cdots, \Theta_2, \Psi_1, \Psi_N, \cdots, \Psi_2).$$

Since $\Psi_1 = 0$ in (23), we can rewrite π_o by $\pi_o \vec{x} = (\pi, \cdots, \pi, 0, \cdots, 0) + \pi_o' \vec{x}$. Hence, since $\pi_o \vec{x}_0 = \vec{x}_0$, we have

$$\begin{aligned} \pi_o \vec{x} &= (\pi, \cdots, \pi, 0, \cdots, 0) + \pi_o' \vec{x}_0 + \pi_o' \sum_{k=1}^{M} \left(b_k^+ \vec{\phi}_k^+ + b_k^- \vec{\phi}_k^- \right) \\ &= (\pi, \cdots, \pi, 0, \cdots, 0) + \pi_o' \vec{x}_0 + \sum_{k=1}^{M} \left(b_k^+ \vec{\phi}_k^+ + b_k^- \vec{\phi}_k^- \right) \\ &= \vec{x}_0 + \sum_{k=1}^{M} \left(b_k^+ \vec{\phi}_k^+ + b_k^- \vec{\phi}_k^- \right) = \vec{x}. \qquad \square \end{aligned}$$

We implement the similar reduction for the even vortex points, $N = 2M$. The pole reversal transformation $\pi_e : \mathbb{P}_N \to \mathbb{P}_N$ and the accompanied transformation for the vector field $\Pi_e : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ are defined by

$$\pi_e : (\Theta_1, \cdots, \Theta_N, \Psi_1, \cdots, \Psi_N) \to (\pi - \Theta_N', \cdots, \pi - \Theta_1', -\Psi_N', \cdots, -\Psi_1'), \tag{24}$$

and

$$\Pi_e : (F_1, \cdots, F_N, G_1, \cdots, G_N) \to (-F_N', \cdots, -F_1', -G_N', \cdots, -G_1').$$

Then, we have the following lemma for the even case.

Lemma 13. *Let $N = 2M$. If $\Gamma_1 = \Gamma_2$, then $\Pi_e \mathbb{F}(\vec{x}) = \mathbb{F}(\pi_e \vec{x})$ for $\vec{x} \in \mathbb{P}_N$.*

Proof.

$$
\begin{aligned}
F_m(\pi_e\vec{x}) &= -\frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N} f(\pi-\Theta_{N-m+1},\pi-\Theta_{N-j+1},\Psi_{N-j+1}-\Psi_{N-m+1}) \\
&= \frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N} f(\Theta_{N-m+1},\Theta_{N-j+1},\Psi_{N-m+1}-\Psi_{N-j+1}) \\
&= -F_{N-m+1}(\vec{x}), \\
G_m(\pi_e\vec{x}) &= -\frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N} g(\pi-\Theta_{N-m+1},\pi-\Theta_{N-j+1},\Psi_{N-j+1}-\Psi_{N-m+1}) \\
&\quad +\frac{\Gamma_1}{2\pi}\frac{\cos(\pi-\Theta_{N-m+1})}{\sin(\pi-\Theta_{N-m+1})} \\
&= \frac{\Gamma^{(N)}}{4\pi}\sum_{j\neq m}^{N} g(\Theta_{N-m+1},\Theta_{N-j+1},\Psi_{N-m+1}-\Psi_{N-j+1}) - \frac{\Gamma_1}{2\pi}\frac{\cos\Theta_{N-m+1}}{\sin\Theta_{N-m+1}} \\
&= -G_{N-m+1}(\vec{x}). \qquad \square
\end{aligned}
$$

Lemma 14. *Let $N=2M$ and $\Gamma_1=\Gamma_2$. If $\pi_e\vec{x}(0)=\vec{x}(0)$, then $\pi_e\vec{x}(t)=\vec{x}(t)$ for $t\geq 0$.*

Proof. The initial condition $\pi_e\vec{x}(0)=\vec{x}(0)$ yields $\Theta_m(0)+\Theta_{N-m+1}(0)=\pi$ and $\Psi_m(0)+\Psi_{N-m+1}(0)=0$. On the other hand, if $\pi_e\vec{x}=\vec{x}$ holds, then $\Pi_e\mathbb{F}(\vec{x})=\mathbb{F}(\pi_e\vec{x})=\mathbb{F}(\vec{x})$, i.e.

$$
\dot{\Theta}_m+\dot{\Theta}_{N-m+1}=F_m+F_{N-m+1}=0, \qquad \dot{\Psi}_m+\dot{\Psi}_{N-m+1}=G_m+G_{N-m+1}=0.
$$

Hence, we have

$$
\Theta_m+\Theta_{N-m+1}=\pi, \qquad \Psi_m+\Psi_{N-m+1}=0, \tag{25}
$$

for $t\geq 0$. $\square$

We give the existence of the $2M$-dimensional π_e-invariant subspace. The proof is now easy.

Lemma 15. *Let $N=2M$, $\Gamma_1=\Gamma_2$ and $\vec{x}_0$ is denoted by*

$$
\vec{x}_0=\left(\frac{\pi}{2},\cdots,\frac{\pi}{2},0,\cdots,\frac{2\pi}{N}(N-1)\right).
$$

Then, the $2M$-dimensional set of

$$
\vec{x}=\vec{x}_0+\sum_{k=1}^{M}\left(a_k^+\vec{\psi}_k^+ + a_k^-\vec{\psi}_k^-\right), \qquad a_k^\pm\in\mathbb{R}, \tag{26}
$$

is invariant with respect to the transformation π_e, which is denoted by $\mathbb{P}_N(\pi_e)$.

Lemma 12 and Lemma 15 indicate that the N-vortex system can be reduced to the $2M$-dimensional invariant dynamical system defined in the subspace of $\mathbb{P}_N$ containing the N-ring at the equator. The reduced system due to the pole reversal invariance exists only when the north and the south pole vortices are identical. Note that the invariant quantity $\sum_{m=1}^{N} \cos\Theta_m = 0$ is automatically satisfied for the reduced system because of (22) and (25). Thus we conclude the above results as follows.

Theorem 16. *Suppose that the strengths of the pole vortices are equivalent. Then, there exists the $2M$-dimensional invariant dynamical system reduced by the pole reversal transformation π_e or π_o that contains the N-ring at the equator.*

Lemma 12 shows that for $N = 3$, the π_o invariant space $\mathbb{P}_3(\pi_o)$ is spanned by the eigenvectors $\vec{\phi}_1^{\pm}$, and thus it is the two-dimensional integrable dynamical system. It has already been studied in order to describe the complex recurrent motion of the unstable perturbed 3-ring at the equator[18].

Finally, applying the reductions due to the shift and the pole reversal transformations simultaneously, we further reduce the system.

Corollary 17. *We assume that the strengths of the pole vortices are the same, and odd N has the factor p. Then, if $\sigma_p\pi_o\vec{x}(0) = \vec{x}(0)$, we have $\sigma_p\pi_o\vec{x}(t) = \vec{x}(t)$ for $t \geq 0$. The $\sigma_p\pi_o$-invariant subspace of $\mathbb{P}_N$ is the set of*

$$\vec{x} = \vec{x}_0 + \sum_k \left(b_k^+ \vec{\phi}_{kq}^+ + b_k^- \vec{\phi}_{kq}^- \right), \qquad b_k^{\pm} \in \mathbb{R},$$

in which

$$\vec{x}_0 = \left(\frac{\pi}{2}, \cdots, \frac{\pi}{2}, 0, \frac{2\pi}{N}, \cdots, \frac{2\pi}{N}M, -\frac{2\pi}{N}M, \cdots, -\frac{2\pi}{N} \right).$$

Proof. It follows from Lemma 4 and Lemma 10 that we have $\Sigma_p\Pi_o\mathbb{F}(\vec{x}) = \mathbb{F}(\sigma_p\pi_o\vec{x})$. The $\sigma_p\pi_o$-invariance of the solution is derived from the fact with the similar arguments as in Proposition 5 and Lemma 11. The linear representation of the invariant space is easily obtained, since the eigenvectors $\phi_{kq}^{\pm}$ and $\vec{x}_0$ are invariant with respect to $\sigma_p\pi_o$. $\square$

We have the similar result for the even case. The proof is now easy.

Corollary 18. *We assume that the strengths of the pole vortices are the same, and even N has the factor p. Then, if $\sigma_p\pi_e\vec{x}(0) = \vec{x}(0)$, we have $\sigma_p\pi_e\vec{x}(t) = \vec{x}(t)$ for $t \geq 0$. The $\sigma_p\pi_e$-invariant subspace of $\mathbb{P}_N$ is the set of*

$$\vec{x} = \vec{x}_0 + \sum_k \left(a_k^+ \vec{\psi}_{kq}^+ + a_k^- \vec{\psi}_{kq}^- \right), \qquad a_k^{\pm} \in \mathbb{R},$$

in which

$$\vec{x}_0 = \left(\frac{\pi}{2}, \cdots, \frac{\pi}{2}, 0, \cdots, \frac{2\pi}{N}(N-1) \right).$$

The $\sigma_p\pi_o$-invariant and $\sigma_p\pi_e$-invariant spaces are denoted by $\mathbb{P}_N(\sigma_p\pi_o)$ and $\mathbb{P}_N(\sigma_p\pi_e)$ respectively. Using these corollaries, we give all the invariant systems and their corresponding linear bases of the invariant subspaces from $N = 6$ to 12 in Table 1. We also show the inclusion relations between the invariant dynamical systems in Figure 2.

5 Conclusion and discussion

We have obtained the reduced invariant dynamical systems embedded in the system of identical N-vortex points on the sphere with pole vortices by using the p-shift transformation and the pole reversal transformation. We have shown that for any factor p of N, there exists the $2p$-dimensional invariant dynamical system reduced by the p-shift invariance, and it is equivalent to the p-vortex points system on the sphere with the modified pole vortices, which is generated by the averaged Hamiltonian (18). We also give the existence of the dynamical system reduced by the pole reversal transformation when both of the pole vortices are equivalent. The reduced dynamical systems play an important role in an understanding of the dynamics of N vortex points as embedded elements in the dynamical system.

The reduction of the dynamical system due to the group invariance of the Hamiltonian has been proposed in order to determine the relative periodic orbits for the vortex-point system [9, 19]. The present reduction method is based on the similar idea of theirs in the sense that we focus on the invariant property of the relative fixed configuration under the shift and the pole reversal transformations. However, while the purpose of their method is to reduce the N vortex system to the two-dimensional dynamical system, the present method gives us the collection of the invariant dynamical systems embedded in the N-vortex system and their inclusion relation, which allows us to decompose the N-vortex system into the systems of a small number of vortex points. Furthermore, since the invariant subspace is spanned by the eigenvectors obtained in the linear stability analysis of the N-ring, it is possible to connect the stability of the eigenvalues corresponding to the eigenvectors with that of the invariant systems as we have done in the paper[17].

The p-shift and the pole reversal transformations used in §3 and §4 are introduced so that the N-ring becomes a fixed point for them. It suggests that it is possible to obtain different invariant dynamical systems embedded in the N-vortex system when we apply the similar reduction by introducing the transformations that make the other relative equilibria of the N-vortex points unchanged.

N	Invariant Systems	Linear Basis
6	$\mathbb{P}_6(\sigma_2)$ $\mathbb{P}_6(\sigma_3)$ $\mathbb{P}_6(\pi_e)$ $\mathbb{P}_6(\sigma_2\pi_e)$ $\mathbb{P}_6(\sigma_3\pi_e)$	$\vec{\psi}_3^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_2^{\pm}, \vec{\phi}_2^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_1^{\pm}, \vec{\psi}_2^{\pm}, \vec{\psi}_3^{\pm}$ $\vec{\psi}_3^{\pm}$ $\vec{\psi}_2^{\pm}$
7	$\mathbb{P}_7(\pi_o)$	$\vec{\phi}_1^{\pm}, \vec{\phi}_2^{\pm}, \vec{\phi}_3^{\pm}$
8	$\mathbb{P}_8(\sigma_4)$ $\mathbb{P}_8(\sigma_2)$ $\mathbb{P}_8(\pi_e)$ $\mathbb{P}_8(\sigma_4\pi_e)$ $\mathbb{P}_8(\sigma_2\pi_e)$	$\vec{\psi}_2^{\pm}, \vec{\phi}_2^{\pm}, \vec{\psi}_4^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_4^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_1^{\pm}, \vec{\psi}_2^{\pm}, \vec{\psi}_3^{\pm}, \vec{\psi}_4^{\pm}$ $\vec{\psi}_2^{\pm}, \vec{\psi}_4^{\pm}$ $\vec{\psi}_4^{\pm}$
9	$\mathbb{P}_9(\sigma_3)$ $\mathbb{P}_9(\pi_o)$ $\mathbb{P}_9(\sigma_3\pi_o)$	$\vec{\psi}_3^{\pm}, \vec{\phi}_3^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\phi}_1^{\pm}, \vec{\phi}_2^{\pm}, \vec{\phi}_3^{\pm}, \vec{\phi}_4^{\pm}$ $\vec{\phi}_3^{\pm}$
10	$\mathbb{P}_{10}(\sigma_5)$ $\mathbb{P}_{10}(\sigma_2)$ $\mathbb{P}_{10}(\pi_e)$ $\mathbb{P}_{10}(\sigma_5\pi_e)$ $\mathbb{P}_{10}(\sigma_2\pi_e)$	$\vec{\psi}_2^{\pm}, \vec{\phi}_2^{\pm}, \vec{\psi}_4^{\pm}, \vec{\phi}_4^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_5^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_1^{\pm}, \vec{\psi}_2^{\pm}, \vec{\psi}_3^{\pm}, \vec{\psi}_4^{\pm}, \vec{\psi}_5^{\pm}$ $\vec{\psi}_2^{\pm}, \vec{\psi}_4^{\pm}$ $\vec{\psi}_5^{\pm}$
11	$\mathbb{P}_{11}(\pi_o)$	$\vec{\phi}_1^{\pm}, \vec{\phi}_2^{\pm}, \vec{\phi}_3^{\pm}, \vec{\phi}_4^{\pm}, \vec{\phi}_5^{\pm}$
12	$\mathbb{P}_{12}(\sigma_6)$ $\mathbb{P}_{12}(\sigma_4)$ $\mathbb{P}_{12}(\sigma_3)$ $\mathbb{P}_{12}(\sigma_2)$ $\mathbb{P}_{12}(\pi_e)$ $\mathbb{P}_{12}(\sigma_6\pi_e)$ $\mathbb{P}_{12}(\sigma_4\pi_e)$ $\mathbb{P}_{12}(\sigma_3\pi_e)$ $\mathbb{P}_{12}(\sigma_2\pi_e)$	$\vec{\psi}_2^{\pm}, \vec{\phi}_2^{\pm}, \vec{\psi}_4^{\pm}, \vec{\phi}_4^{\pm}, \vec{\psi}_6^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_3^{\pm}, \vec{\phi}_3^{\pm}, \vec{\psi}_6^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_4^{\pm}, \vec{\phi}_4^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_6^{\pm}, \vec{\zeta}^{\pm}$ $\vec{\psi}_1^{\pm}, \vec{\psi}_2^{\pm}, \vec{\psi}_3^{\pm}, \vec{\psi}_4^{\pm}, \vec{\psi}_5^{\pm}, \vec{\psi}_6^{\pm}$ $\vec{\psi}_2^{\pm}, \vec{\psi}_4^{\pm}, \vec{\psi}_6^{\pm}$ $\vec{\psi}_3^{\pm}, \vec{\psi}_6^{\pm}$ $\vec{\psi}_4^{\pm}$ $\vec{\psi}_6^{\pm}$

Table 1: Invariant dynamical systems reduced by the shift and the pole reversal transformations from $N = 6$ to 12 when the strengths of the pole vortices are identical.

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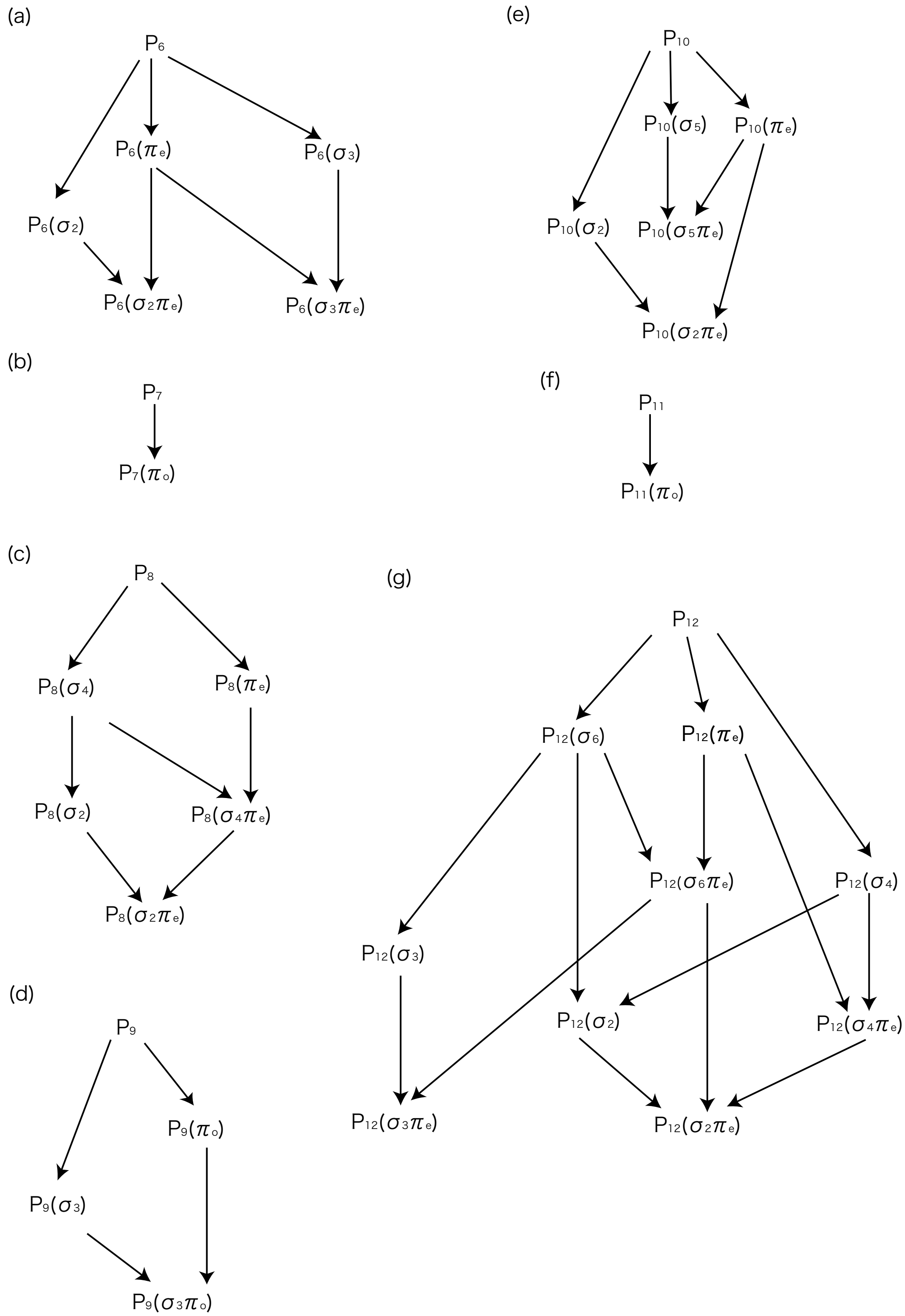

Figure 2: Inclusion relations between the invariant dynamical systems for (a) $N = 6$, (b) $N = 7$, (c) $N = 8$, (d) $N = 9$, (e) $N = 10$, (f) $N = 11$ and (g) $N = 12$. The relation $A \to B$ symbolizes $B \subset A$.