A numerical approach toward the p-adic Beilinson conjecture for elliptic curves over $\mathbb{Q}$

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with Appendix B by François Brunault

Abstract

Restricting ourselves to elliptic curves over $\mathbb{Q}$, we reformulate the p-adic Beilinson conjecture due to Perrin-Riou, which is customized to our computational approach. We then develop a new algorithm for numerical verifications of the p-adic Beilinson conjecture, which is based on the theory of rigid cohomology and F-isocrystals.

1 Introduction

Since Dirichlet proved his famous class number formula in the 19th century, the regulator has been one of the most important subjects in algebraic number theory. In the 20th century, several generalizations of Dirichlet's formula were proposed in the context of higher K-theory by Lichtenbaum, Borel, etc. In particular, Beilinson's conjecture on special values of motivic L-functions ([Bei], [Sch]) is an important milestone, and this mysterious conjecture is still attracting lots of researchers in arithmetic geometry.

The p-adic analogue of the Beilinson conjecture was formulated by Perrin-Riou [PR]; we shall call it the *p-adic Beilinson conjecture*. Roughly speaking, it asserts a certain formula on p-adic regulators and the special values of p-adic L-functions, which is conceptionally described as

$$\frac{p\text{-adic } L\text{-value}}{p\text{-adic regulator}} = \frac{L\text{-value}}{\text{Beilinson regulator}}.$$

However, the above involves a serious issue. At the present, there is no general definition or even candidate of the p-adic counterparts of motivic L-functions. Perrin-Riou needed to formulate the statement together with the existence of p-adic L-functions simultaneously, and that leads to a very complicated statement.

On the other hand, restricting ourselves to the case of elliptic curves over $\mathbb{Q}$, the p-adic L-functions are defined by the following people.

- Mazur and Swinnerton-Dyer [MS]
- Vishik [Vi], Amice and Vélu [AV]
- Pollack and Stevens [PS2], Bellaïche [B] (critical slope p-adic L-functions).

Thus one can re-formulate the p-adic Beilinson conjecture in terms of the above p-adic L-functions (cf. [PR, 4.2]). However there still remains a problem. In general, the regulators are only determined up to multiplication by $\mathbb{Q}^\times$, in other words there is no canonical manner to remove the ambiguity of $\mathbb{Q}^\times$. The p-adic Beilinson conjecture is stated by fixing bases of several cohomology groups, though the choice does not matter by replacing the normalization of her p-adic L-functions if necessary. On the other hand, the normalization of the p-adic L-functions by Mazur and Swinnerton-Dyer et al is fixed and it is a natural one. Therefore, we need to choose a suitable basis of cohomology groups according to this, and it is not so obvious to do this.

In this paper we write down the p-adic Beilinson conjecture for elliptic curves over $\mathbb{Q}$ in terms of the p-adic L-functions by Mazur and Swinnerton-Dyer et al. To be precise, let E be an elliptic curve E over $\mathbb{Q}$, and γ a root of the characteristic polynomial $X^2 - a_p(E)X + p$ of the Frobenius. Then, for an element $\xi \in H^2_{\mathscr{M}}(E, \mathbb{Q}(n+2))$ of the motivic cohomology group, we introduce $R_\infty(h^1(E)(-n), \xi)$ the *Beilinson regulator without ambiguity of* $\mathbb{Q}^\times$, and $R_{p,\gamma}(h^1(E)(-n), \xi)$ the *p-adic regulator without ambiguity of* $\mathbb{Q}^\times$. Our formulation of the p-adic Beilinson conjecture is described as follows (Conjecture 3.3):

$$\frac{L_{p,\gamma}(E, \omega^{-n-1}, -n)}{R_{p,\gamma}(h^1(E)(-n), \xi)} = \pm \frac{L'(E, -n)}{R_\infty(h^1(E)(-n), \xi)}, \quad n \geq 0 \tag{1.1}$$

where ω is the Teichmüller character. We also give a formulation for elliptic modular forms of weight $k \geq 2$ with rational Fourier coefficients (Conjecture 3.1). Our statement is more accessible than the original one, especially with regard to numerical verifications. The latter half of this paper is devoted to giving numerical verifications of our statement in the case $n = 0$:

$$\frac{L_{p,\gamma}(E, \omega^{-1}, 0)}{R_{p,\gamma}(h^1(E), \xi)} = \pm \frac{L'(E, 0)}{R_\infty(h^1(E), \xi)}. \tag{1.2}$$

Theorem 1.1 *Let $f : X \to S \subset \mathbb{P}^1_{\mathbb{Q}}$ be an elliptic fibration over a Zariski open set S which has at least one split multiplicative fiber. Let $X_a = f^{-1}(a)$ be the fiber at a $\mathbb{Q}$-rational point $a \in S(\mathbb{Q})$. Let $\xi \in H^2_{\mathscr{M}}(X, \mathbb{Q}(2))$. Then, under some mild assumption on p, there is an algorithm for computing the p-adic regulator*

$$R_{p,\gamma}(h^1(X_a), \xi|_{X_a}) \tag{1.3}$$

modulo p^n for each $n \geq 1$.

In Theorem 1.1, the central role is played by the theory of *rigid cohomology* and *F-isocrystals*. The precise statement of the algorithm requires lots of notation, which shall be given in §5. We here give an outline and idea of our algorithm.

Let $y^2 = 4x^3 - g_2 x - g_3$ be the Weierstrass equation of f with $g_2, g_3 \in \mathscr{O}(S)$. Let $\omega = dx/y$ and $\eta = x dx/y$. We take an inhomogeneous coordinate λ of $\mathbb{P}^1_{\mathbb{Q}}$ such that f has a split multiplicative reduction at $\lambda = 0$. Let σ be the p-th Frobenius on $\mathbb{Z}_p((\lambda))^\wedge$ given by

$\sigma(\lambda)=a^{1-p}\lambda^p$ where $\mathbb{Z}_p((\lambda))^\wedge$ denotes the p-adic completion of $\mathbb{Z}_p[[\lambda]][\lambda^{-1}]$. According to the main result of [AM], one can associate an element

$$\varepsilon_1^{(\sigma)}(\lambda)\omega+\varepsilon_2^{(\sigma)}(\lambda)\eta\in H^1_{\mathrm{dR}}(X/S)\otimes_{\mathscr{O}(S)}\mathscr{O}(S)^\dagger$$

to $\xi\in H^2_{\mathscr{M}}(X,\mathbb{Q}(2))$ where $\mathscr{O}(S)^\dagger$ denotes the weak completion of $\mathscr{O}(S)$, and its reduction modulo $\lambda-a$ provides the syntomic regulator

$$\mathrm{reg}_{\mathrm{syn}}(\xi|_{X_a})=\varepsilon_1^{(\sigma)}(a)\omega+\varepsilon_2^{(\sigma)}(a)\eta\in H^1_{\mathrm{dR}}(X_a/\mathbb{Q}_p).$$

The p-adic regulator $R_{p,\gamma}(h^1(X_a),\xi|_{X_a})$ is obtained from the syntomic regulator. Since $\varepsilon_i^{(\sigma)}(\lambda)$ are overconvergent functions, their reduction are rational functions,

$$\varepsilon_i^{(\sigma)}(\lambda)\equiv\frac{P_n(\lambda)}{Q_n(\lambda)}\mod p^n$$

where P_n,Q_n are polynomials. The algorithm is done by exploring the explicit descriptions of P_n,Q_n. To do this, we employ the deformation method by Lauder [L]. We take another basis $\widehat{\omega},\widehat{\eta}\in H^1_{\mathrm{dR}}(X/S)\otimes_{\mathscr{O}(S)}\mathbb{Z}_p((\lambda))^\wedge$ as in §4.3 (see also Proposition 7.3), and let

$$\varepsilon_1^{(\sigma)}(\lambda)\omega+\varepsilon_2^{(\sigma)}(\lambda)\eta=E_1^{(\sigma)}(\lambda)\widehat{\omega}+E_2^{(\sigma)}(\lambda)\widehat{\eta}.$$

See (5.5) for the transformation formula between $\{\varepsilon_i^{(\sigma)}(\lambda)\}$ and $\{E_i^{(\sigma)}(\lambda)\}$. Note that $E_i^{(\sigma)}(\lambda)$ are no longer overconvergent. The first step (§5.1 **Step 1**) is to compute the series expansions of $E_i^{(\sigma)}(\lambda)$ except the constant term of $E_2^{(\sigma)}(\lambda)$. Thanks to the property of the basis $\{\widehat{\omega},\widehat{\eta}\}$, there are certain simple differential equations for $E_i^{(\sigma)}(\lambda)$ (see (5.3), (5.4)), which are determined from the several invariants of the elliptic fibration f (e.g. j-invariant) and

$$\mathrm{dlog}(\xi)=g_\xi(\lambda)\frac{d\lambda}{\lambda}\wedge\omega.$$

One can compute the series expansions of $E_i^{(\sigma)}(\lambda)$ from the differential equations. The technical key in our algorithm is Kedlaya-Tuitman [KT, Theorem 2.1], which provides an explicit bound N_n such that

$$\deg P_n<N_n,\quad \deg Q_n<N_n.$$

This estimate allows us to determine the constant term of $E_2^{(\sigma)}(\lambda)$ modulo p^n uniquely. See §5.1 **Step 2** (also Remark 5.1) for the detail. This completes the computations of the series expansions of $\varepsilon_i^{(\sigma)}(\lambda)$ modulo p^n until the degree N_n. The final step (§5.1 **Step 3**) is to get the full descriptions of P_n,Q_n. Firstly, one can take $Q_n(\lambda)$ to be a product $\prod_i(\lambda-a_i)^{n_i}$ where $\lambda=a_i$ are points of $\mathbb{A}^1\setminus S$, and one has explicit bounds of n_i's by Kedlaya-Tuitman. Once we have $Q_n(\lambda)$, it follows from the fact deg $P_n<N_n$ that we have

$$P_n(\lambda)\equiv\varepsilon_i^{(\sigma)}(\lambda)\cdot Q_n(\lambda)\mod\lambda^{N_n},$$

which completes the computation of P_n, Q_n. We thus have

$$\varepsilon_i^{(\sigma)}(a) \equiv \frac{P_n(a)}{Q_n(a)} \mod p^n$$

where the right hand side denotes the evaluation of the rational function.

In the paper [A2], the first author introduces a new kind of p-adic hypergeometric functions, which we denote by $\mathscr{F}_{\underline{a}}^{(\sigma)}(\lambda)$. For some examples of elliptic curves, the syntomic regulators can be described in terms of $\mathscr{F}_{\underline{a}}^{(\sigma)}(\lambda)$ ([A2, §5]). This provides an alternative algorithm for computing $R_{p,\alpha}(h^1(X_a), \xi|_{X_a})$ when $\mathrm{ord}_p(\alpha) = 0$.

We mainly focus on the numerical verifications of (1.2), while there is an example which is proven.

Theorem 1.2 (Theorem 6.2) *Let E be the elliptic curve defined by an equation $y^2 = x(1-x)(1+3x)$ over $\mathbb{Q}$. Let*

$$\xi = \left\{ \frac{y+1-x}{y-1+x}, \frac{4x^2}{(1-x)^2} \right\} \in K_2(E).$$

Let p be a prime at which E has a good reduction. Let γ be a root of the eigenpolynomial of E modulo p such that $\mathrm{ord}_p(\gamma) < 1$. Then (1.2) holds.

The proof uses a formula of Rogers-Zudilin for computing the right hand side of (1.2), and Brunault's formula for computing the left hand side, together with a comparison between the symbol ξ and the Beilinson-Kato element (Appendix B).

There have been several papers concerning the p-adic Beilinson conjecture, Bannai-Kings [BK], Kings-Loeffler-Zerbes [KLZ], Niklas [N], Brunault [Br1], Bertolini-Darmon [BD] and probably more. A noteworthy point is that our approach is entirely different from theirs. For example, many papers rely on Kato's seminal paper [Ka3] where the Beilinson-Kato elements are introduced. Ours is less powerful than the method using [Ka3] while ours allows an approach to new examples such as the Beilinson-Kato elements [Ka3] vanish in the p-adic cohomology groups $H^1_f(G_{\mathbb{Q}}, H^1_{\text{ét}}(\overline{E}, \mathbb{Q}_p(2)))$ (Remark 6.1). A number of such numerical verifications convince us of our reformulation of the p-adic Beilinson conjecture.

The paper is organized as follows. §2 is a review of p-adic L-functions for elliptic modular forms, which includes the recent development of critical slope L-functions. We give a precise statement of the p-adic Beilinson conjecture for elliptic curves in §3 (Conjectures 3.1, 3.3). In §4, we provide the algorithm for computing the p-adic regulators (1.3), which is summed up in §5.1. With use of this algorithm, we give numerical verifications of the p-adic Beilinson conjecture for explicit examples of elliptic curves in §6 where we use SAGE, Pari/GP or MAGMA for computing the special values of (p-adic) L-functions. The values of critical slope p-adic L-functions were kindly provided by Professor Robert Pollack. In §7 Appendix A, we sum up the standard results on the Gauss-Manin connection for elliptic fibrations, which are used especially in §4. Most results in Appendix A are probably known to experts. However since the authors do not find a suitable literature, we give a comprehensive exposition for the convenience of the reader. Finally we compare the symbol in Theorem 1.2 with the Beilinson-Kato element in §8 Appendix B.

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2 p-adic L-functions for elliptic modular forms

Here we recall some basic facts on the cyclotomic p-adic L-functions for elliptic modular forms. Let N be a positive integer greater than 3. Fix a prime $p > 2$ not dividing N and fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let K be a subfield of $\overline{\mathbb{Q}}$. Let $S_{k+2}(\Gamma_1(N), K)$ be the space of cuspforms of weight $k+2$ for $\Gamma_1(N)$ whose Fourier coefficients belong to K. Let f be a normalized eigenform of weight $k+2$ for $\Gamma_0(N)$ with character χ_f. We denote the n-th Fourier coefficient of f by $a_n(f)$. Let α and β be the two roots of $X^2 - a_p(f)X + \chi_f(p)p^{k+1}$ with $\mathrm{ord}_p(\alpha) \leq \mathrm{ord}_p(\beta)$. Then $f_\alpha(z) = f(z) - \beta f(pz)$ and $f_\beta(z) = f(z) - \alpha f(pz)$ are normalized cuspforms for $\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)$ of weight $k+2$. Moreover they are also eigenforms for U_p-operator and satisfy $U_p f_\alpha(z) = \alpha f_\alpha(z)$ and $U_p f_\beta(z) = \beta f_\beta(z)$. The modular forms f_α and f_β are called p-refinements (or p-stabilizations) of f. If $\mathrm{ord}_p(\beta) = k+1$, the p-refinement f_β is called a cuspform of critical slope. If f is non-ordinary, then α and β satisfy $\mathrm{ord}_p(\alpha) \leq \mathrm{ord}_p(\beta) < k+1$. In this case, we have p-adic L-functions $L_{p,\alpha}(f, \chi, s)$ and $L_{p,\beta}(f, \chi, s)$ constructed by Vishik [Vi] and Amice-Vélu [AV] for any Dirichlet characters[1] χ. If f is ordinary, α satisfies $\mathrm{ord}_p(\alpha) = 0$. Therefore we also have p-adic L-function $L_{p,\alpha}(f, \chi, s)$. For the critical slope case, Pollack-Stevens [PS1, PS2] and Bellaïche [B] constructed p-adic L-functions $L_{p,\beta}(f, \chi, s)$ under certain assumptions. In this section, we recall the constructions briefly.

2.1 Preliminaries

Let $\mathcal{H} = \mathbb{Z}[T_\ell\,(\ell \nmid N),\, U_q\,(q|N)]$ be the Hecke algebra. For a commutative ring R, let $\mathscr{P}_k(R)$ denote the space of homogeneous polynomials of degree k in X and Y with R-coefficients. Let $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$. Then Δ_0 is a left $\mathrm{GL}_2(\mathbb{Q})$-module by the linear fractional transformation.

Let $Y_1(N)$ be the open modular curve over $\mathbb{Q}$ with level $\Gamma_1(N)$-structure and $\pi : \mathscr{E} \to Y_1(N)$ the universal elliptic curve. The line bundle $\underline{\omega} = \pi_* \Omega^1_{\mathscr{E}/Y_1(N)}$ is naturally extended to the modular curve $X_1(N)$. Then we have an identification

$$S_{k+2}(\Gamma_1(N), K) = H^0(X_1(N)_K, \underline{\omega}^k \otimes \Omega^1_{X_1(N)}).$$

[1] In this paper, we only consider the case that χ is a Dirichlet character of p-power conductor.

For the local system $\mathscr{L}_k^{\mathrm{B}} = \mathrm{Sym}^k R^1\pi_*\mathbb{Z}$, we also have a canonical identification

$$H_c^1(Y_1(N)(\mathbb{C}), \mathscr{L}_k^{\mathrm{B}}) \cong \mathrm{Hom}_{\Gamma_1(N)}(\Delta_0, \mathscr{P}_k(\mathbb{Z})).$$

For simplicity, we denote $H_c^1(\Gamma_1(N), \mathscr{P}_k(R)) = H_c^1(Y_1(N)(\mathbb{C}), \mathscr{L}_k^{\mathrm{B}}) \otimes R$ for a commutative ring R. The element $\iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ defines an involution. If R is a $\mathbb{Z}[1/2]$-module, then we have a decomposition

$$H_c^1(\Gamma_1(N), \mathscr{P}_k(R)) = H_c^1(\Gamma_1(N), \mathscr{P}_k(R))^+ \oplus H_c^1(\Gamma_1(N), \mathscr{P}_k(R))^-,$$

where $H_c^1(\Gamma_1(N), \mathscr{P}_k(R))^\pm = \{x \in H_c^1(\Gamma_1(N), \mathscr{P}_k(R)) | \iota(x) = \pm x\}$. It is known that the action of ι coincides with the action of the complex conjugation (see [Ka3, Proof of Lemma 7.19]). For a cuspform $g \in S_{k+2}(\Gamma_1(N), \mathbb{C})$, we define a cohomology class $\xi_g \in H_c^1(\Gamma_1(N), \mathscr{P}_k(\mathbb{C})) \cong \mathrm{Hom}_{\Gamma_1(N)}(\Delta_0, \mathscr{P}_k(\mathbb{C}))$ by

$$\xi_g(\{x\} - \{y\}) = 2\pi i \int_x^y g(z)(zX + Y)^k dz.$$

Let K be a subfield of $\overline{\mathbb{Q}}_p$ and $\phi : \mathcal{H} \to K$ be an algebra homomorphism and V a finite dimensional vector space over K with an action of $\mathcal{H}$. For $\alpha \in K$ and $T \in \mathcal{H}$, let $V[\alpha, T]$ be the eigenspace for T acting on V with the eigenvalue α and we put

$$V[\phi] = \bigcap_{T \in \mathcal{H}} V[(\phi(T), T)].$$

Similarly, let $V_{(\alpha,T)}$ be the generalized eigenspace for T acting on V with the eigenvalue α and define the ϕ-isotypic subspace $V_{(\phi)}$ of V by

$$V_{(\phi)} = \bigcap_{T \in \mathcal{H}} V_{(\phi(T),T)}.$$

For a normalized eigenform $f(z)$ in $S_{k+2}(\Gamma_1(N), \overline{\mathbb{Q}})$, we put $L = \mathbb{Q}(\{a_n(f)\}_{n=1}^\infty)$. It is well-known that L is a finite extension of $\mathbb{Q}$. Then we can define the homomorphism $\phi_f : \mathcal{H} \to L$ by $\phi_f(T_\ell) = a_\ell(f)$ for $\ell \nmid N$ and $\phi_f(U_q) = a_q(f)$ for $q | N$. For simplicity, we denote $V[f] = V[\phi_f]$ and $V_{(f)} = V_{(\phi_f)}$.

2.2 Result of Amice-Vélu and Vishik

Let f be a normalized newform in $S_{k+2}(\Gamma_1(N), \overline{\mathbb{Q}})$ and denote the complex conjugate cuspform $\sum_{n=1}^\infty \overline{a_n(f)} q^n$ of f by f^*. Let v be the valuation induced by the embedding $L \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\mathscr{O}_{L,v}$ the discrete valuation ring defined by v. We define

$$H_c^1(\Gamma_1(N), \mathscr{P}_k(\mathscr{O}_{L,v}))^\pm[f] = H_c^1(\Gamma_1(N), \mathscr{P}_k(L))^\pm[f] \cap H_c^1(\Gamma_1(N), \mathscr{P}_k(\mathscr{O}_{L,v})).$$

Then it is known that $H_c^1(\Gamma_1(N), \mathscr{P}_k(\mathscr{O}_{L,v}))^\pm[f]$ is a free $\mathscr{O}_{L,v}$-module of rank one. We fix a generator $u^\pm$ in $H_c^1(\Gamma_1(N), \mathscr{P}_k(\mathscr{O}_{L,v}))^\pm[f^*]$. For $0 \neq \eta \in S_{k+2}(\Gamma_1(N), L)[f^*]$ and $u = (u^+, u^-)$, we define a complex number $\Omega_f^\pm = \Omega_f^\pm(\eta, u)$ by

$$\xi_\eta^\pm = \Omega_f^\pm \cdot u^\pm,$$

where $\xi_\eta^\pm$ is the projection to $H_c^1(\Gamma_1(N), \mathscr{P}_k(\mathbb{C}))^\pm[f^*]$ of ξ_η.

Remark 2.1 *1. By multiplicity one, η is a nonzero multiple of f by an element in $L^\times$.*

2. The generator $u^\pm$ is determined only up to an element in $\mathscr{O}_{L,v}^\times$. Since

$$H_c^1(\Gamma_1(N), \mathscr{P}_k(\mathscr{O}_L))^\pm[f^*] = H_c^1(\Gamma_1(N), \mathscr{P}_k(L))^\pm[f] \cap H_c^1(\Gamma_1(N), \mathscr{P}_k(\mathscr{O}_L))$$

is not a free $\mathscr{O}_L$-module in general, we do not have a canonical choice of $u^\pm$ to define the periods. Therefore, our p-adic L-functions will depend on the choice of $u^\pm$.

Let f be a normalized eigenform of weight $k+2$ for $\Gamma_0(N)$ with character χ_f. Let α be a root of $X^2 - a_p(f)X + \chi_f(p)p^{k+1}$. For $a, m \in \mathbb{Q}$ and $P \in \mathscr{P}_k(\mathbb{C})$, we put

$$\lambda_{f,\alpha}^\pm(a,m)(P) = \frac{\pi i}{\Omega_f^\pm}\left\{\int_\infty^{a/m} f_\alpha(z)P(-mz+a,1)dz \pm \int_\infty^{-a/m} f_\alpha(z)P(mz+a,1)dz\right\}.$$

Theorem 2.2 *(Vishik [Vi], Amice-Vélu [AV]) Let f be a normalized eigenform of weight $k+2$ for $\Gamma_0(N)$ with character χ_f. Let α be a root of $X^2 - a_p(f)X + \chi_f(p)p^{k+1}$ satisfying $\mathrm{ord}_p(\alpha) < k+1$. Then there exists a unique p-adic distribution $\mu_{f,\alpha}^\pm$ on $\mathbb{Z}_p^\times$ satisfying the following properties:*

1. $\displaystyle\int_{a+p^\nu\mathbb{Z}_p} x^j d\mu_{f,\alpha}^\pm = \frac{1}{\alpha^\nu}\lambda_{f,\alpha}^\pm(a,p^\nu)(X^jY^{k-j})$ *for $\nu > 0$ and $0 \le j \le k$.*

2. *For $0 \le i < \mathrm{ord}_p(\alpha)$, $\displaystyle\sup_{a\in\mathbb{Z}_p^\times}\left|\int_{a+p^\nu\mathbb{Z}_p}(x-a)^i d\mu_{f,\alpha}^\pm\right|_p = o(p^{\nu(\mathrm{ord}_p(\alpha)-i)})$ when ν tends to infinity (i.e. $\mu_{f,\alpha}^\pm$ is $\mathrm{ord}_p(\alpha)$-admissible).*

For $x \in \mathbb{Z}_p^\times$, we write $x = \omega(x)\langle x\rangle$ with $\omega(x) \in \mu_{p-1}(\mathbb{Z}_p)$ and $\langle x\rangle \in 1 + p\mathbb{Z}_p$. Let $\chi : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}^\times$ be a character of conductor p^n. Now we set $\mu_{f,\alpha} = \mu_{f,\alpha}^+ + \mu_{f,\alpha}^-$ and define the p-adic L-function $L_{p,\alpha}(f,\chi,s) = L_{p,\alpha,\eta,u}(f,\chi,s)$ by

$$L_{p,\alpha}(f,\chi,s) = \int_{\mathbb{Z}_p^\times}\chi(x)\langle x\rangle^{s-1}d\mu_{f,\alpha}$$

for $s \in \mathbb{Z}_p$. Then we have the following interpolation property.

Theorem 2.3 *(Vishik [Vi], Amice-Vélu [AV]) Assume that the root α satisfies $\mathrm{ord}_p(\alpha) < k+1$. For a primitive character $\chi : (\mathbb{Z}/p^\nu\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ and $1 \le j \le k+1$, we have*

$$L_{p,\alpha}(f,\omega^{j-1}\chi,j) = \begin{cases} (1-\chi_f(p)p^{k+1-j}\alpha^{-1})(1-p^{j-1}\alpha^{-1})\dfrac{(j-1)!}{(-2\pi i)^{j-1}}\dfrac{L(f,j)}{\Omega_f^{\epsilon(\mathbf{1},j)}} & \text{if } \chi = \mathbf{1}, \\ \left(\dfrac{p^{j\nu}}{\alpha^\nu}\right)\cdot\dfrac{(j-1)!}{(-2\pi i)^{j-1}\tau(\overline{\chi})}\cdot\dfrac{L(f,\overline{\chi},j)}{\Omega_f^{\epsilon(\chi,j)}} & \text{if } \chi \ne \mathbf{1}, \end{cases}$$

where $\epsilon(\chi,j) = (-1)^{(j-1)}\cdot\chi(-1)$ and τ is the Gauss sum defined by

$$\tau(\overline{\chi}) = \sum_{a\in(\mathbb{Z}/p^\nu\mathbb{Z})^\times}\overline{\chi}(a)e^{2\pi i a/p^\nu}.$$

2.3 Critical slope p-adic L-functions

If $\mathrm{ord}_p(\beta) = k + 1$, it is said that β has a critical slope. Here we briefly recall the construction of critical slope p-adic L-functions following Pollack-Stevens [PS1, PS2]. Let p be a prime and N a positive integer satisfying $(p, N) = 1$. We denote $\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)$ and $S_0(p) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \,\middle|\, p \nmid a, p \mid c, \det g \neq 0 \right\}$. For a right $\mathbb{Z}_p[S_0(p)]$-module M, we denote the locally constant sheaf associated to M on the open modular curve $Y_{\Gamma_0}(\mathbb{C}) = \mathbb{H}/\Gamma_0$ by $\widetilde{M}$, where $\mathbb{H}$ is the upper half plane. For simplicity, we assume that the order of the torsion elements of Γ_0 are prime to p or M is a $\mathbb{Z}_p$-module with $p > 3$. Recall that $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ is the degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$. For $\varphi \in \mathrm{Hom}(\Delta_0, M)$ and $g \in S_0(p)$, we put $(\varphi|g)(D) = \varphi(gD)|g$ for $D \in \Delta_0$. Define Hecke operators T_ℓ ($\ell \nmid N$) and U_q ($q|N$) by $\varphi|T_\ell = \varphi| \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{\ell-1} \varphi| \begin{pmatrix} 1 & a \\ 0 & \ell \end{pmatrix}$ and $\varphi|U_q = \sum_{a=0}^{q-1} \varphi| \begin{pmatrix} 1 & a \\ 0 & q \end{pmatrix}$. We denote $H_c^1(\Gamma_0, M) = H_c^1(Y_{\Gamma_0}(\mathbb{C}), \widetilde{M})$ as before. Then Ash-Stevens [AS, Proposition 4.6] proved that there exists a canonical Hecke equivariant isomorphism $H_c^1(\Gamma_0, M) \cong \mathrm{Hom}_{\Gamma_0}(\Delta_0, M)$, where $\mathrm{Hom}_{\Gamma_0}(\Delta_0, M) = \{\varphi \in \mathrm{Hom}(\Delta_0, M) \mid \varphi|g = \varphi \text{ for all } g \in \Gamma_0\}$ is the Γ_0-invariant homomorphisms. From now on, we will identify these modules by the above isomorphism. If M is a Banach module and Γ_0 acts on M as unitary operators, then $H_c^1(\Gamma_0, M)$ is also a Banach space with the norm defined by $||\Psi|| = \sup_{D \in \Delta_0} ||\Psi(D)||$ for $\Psi \in H_c^1(\Gamma_0, M)$.

For $r \in |\mathbb{C}_p^\times|_p$, we denote

$$B[\mathbb{Z}_p, r] = \{z \in \mathbb{C}_p \,|\, \text{there exists } a \in \mathbb{Z}_p \text{ such that } |z - a|_p \leq r\}.$$

Then $B[\mathbb{Z}_p, r]$ is the $\mathbb{C}_p$-valued points of a $\mathbb{Q}_p$-affinoid variety. Let $\mathbb{A}[r]$ be the $\mathbb{Q}_p$-Banach algebra of $\mathbb{Q}_p$-affinoid functions on $B[\mathbb{Z}_p, r]$ with the norm $|| \cdot ||_r$ on $\mathbb{A}[r]$ defined by $||f||_r = \sup_{z \in B[\mathbb{Z}_p, r]} |f(z)|_p$. If $r > r'$, the restriction map $\mathbb{A}[r] \to \mathbb{A}[r']$ is injective, completely continuous and has dense image. Then we define the topological spaces $\mathscr{A}(\mathbb{Z}_p) = \varinjlim_{s>0} \mathbb{A}[s]$ with the inductive limit topology. Let $\mathscr{D}[r]$ (resp. $\mathscr{D}(\mathbb{Z}_p)$) be the space of continuous $\mathbb{Q}_p$-linear functionals on $\mathscr{A}[r]$ (resp. $\mathscr{A}(\mathbb{Z}_p)$) endowed with the strong topology. Fix a positive integer k and denote $\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \,\middle|\, p \nmid a, p \mid c, ad - bc \neq 0 \right\}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$ and $f \in \mathbb{A}[r]$, we set

$$(g \cdot_k f)(z) = (a + cz)^k \cdot f\left(\frac{b + dz}{a + cz}\right).$$

For $g \in \Sigma_0(p)$ and $\mu \in \mathbb{D}[r]$, we also set

$$(\mu|_k g)(f) = \mu(g \cdot_k f).$$

This action induces the action of $\Sigma_0(p)$ on $\mathscr{D}(\mathbb{Z}_p)$. To emphasize the role of k in this action, we write this module as $\mathscr{D}_k(\mathbb{Z}_p)$.

Let $\mathscr{P}_k = \mathscr{P}_k(\mathbb{Q}_p)$ be the space of homogeneous polynomials with $\mathbb{Q}_p$-coefficients of degree k in X and Y. We endow the space $\mathscr{P}_k$ with the structure of a right $\mathrm{GL}_2(\mathbb{Q}_p)$-module by

$$(P|g)(x) = P(dX - cY, -bX + aY),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $P \in \mathscr{P}_k$. Then we have a $\Sigma_0(p)$-equivariant map

$$\rho_k : \mathscr{D}_k \to \mathscr{P}_k$$

given by $\mu \mapsto \displaystyle\int (Y - zX)^k d\mu(z)$ and it induces the specialization map

$$\rho_k^* : H_c^1(\Gamma_0, \mathscr{D}_k) \to H_c^1(\Gamma_0, \mathscr{P}_k).$$

Theorem 2.4 *([PS2, Theorem 6.7]) Let f_β be a critical slope eigenform in $S_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$ (i.e. f_β is the critical p-refinement of a p-ordinary eigenform f in $S_{k+2}(\Gamma_1(N), \overline{\mathbb{Q}}_p)$). Assume that the order of torsion elements in Γ_0 are prime to p. Then the map induced by the specialization map*

$$\left(H_c^1(\Gamma_0, \mathscr{D}_k) \otimes \overline{\mathbb{Q}}_p\right)_{(f_\beta)} \xrightarrow{\sim} \left(H_c^1(\Gamma_0, \mathscr{P}_k) \otimes \overline{\mathbb{Q}}_p\right)_{(f_\beta)}$$

is an isomorphism if and only if f_β is non-θ-critical (i.e. f_β does not belong to the image of the θ-operator $\theta^{k+1} : S_{-k}^\dagger(\Gamma_0, \overline{\mathbb{Q}}_p) \to S_{k+2}^\dagger(\Gamma_0, \overline{\mathbb{Q}}_p)$ which acts on the q-expansion by $\left(q\dfrac{d}{dq}\right)^{k+1}$, where $S_r^\dagger(\Gamma_0, \overline{\mathbb{Q}}_p)$ is the space of overconvergent cuspforms of weight r).

For a normalized eigenform $f_\beta(z)$ in $S_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$, we put $K = \mathbb{Q}_p(\{a_n(f_\beta)\}_{n=1}^\infty)$. Then it is known that $(H_c^1(\Gamma_0, \mathscr{P}_k)^\pm \otimes K)_{(f_\beta)}$ are one dimensional vector spaces over K. We choose $\varphi_{f_\beta}^\pm = \varphi_{f_\beta,\eta,u}^\pm$ in $(H_c^1(\Gamma_0, \mathscr{P}_k)^\pm \otimes K)_{(f_\beta)}$ as

$$\varphi_{f_\beta}^\pm(\{x\} - \{y\}) = \frac{\pi i}{\Omega_f^\pm}\left(\int_x^y f_\beta(z)(zX + Y)^k dz \pm \int_{-x}^{-y} f_\beta(z)(zX - Y)^k dz\right)$$

and put $\varphi_{f_\beta} = \varphi_{f_\beta}^+ + \varphi_{f_\beta}^-$. Recall that $\Omega_f^\pm = \Omega_f^\pm(\eta, u)$ is the period determined by η and $u = (u^+, u^-)$.

Assume that f_β is a critical slope eigenform in $S_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$. Furthermore we suppose that f_β is non-θ-critical. Then by Theorem 2.4, there exists a unique element Φ_{f_β} such that the image of Φ_{f_β} under the specialization map $H_c^1(\Gamma_0, \mathscr{D}_k)_{(f_\beta)} \xrightarrow{\sim} H_c^1(\Gamma_0, \mathscr{P}_k)_{(f_\beta)}$ is equal to φ_{f_β}. Since we have the identification $H_c^1(\Gamma_0, \mathscr{D}_k) = \mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathscr{D}_k)$, $\mu_{f_\beta} = \Phi_{f_\beta}(\{\infty\} - \{0\})|_{\mathbb{Z}_p^\times}$ defines an element in $\mathscr{D}(\mathbb{Z}_p^\times)$. Therefore we can define the critical slope p-adic L-function $L_{p,\beta}(f, \chi, s) = L_{p,\beta,\eta,u}(f, \chi, s)$ by

$$L_{p,\beta}(f, \chi, s) = \int_{\mathbb{Z}_p^\times} \chi(x)\langle x\rangle^{s-1} d\mu_{f_\beta}$$

for each finite order character χ of $\mathbb{Z}_p^\times$. Then the critical slope p-adic L-function $L_{p,\beta}(f, \chi, s)$ satisfies the same interpolation property of Theorem 2.3. However the p-adic distribution μ_{f_β} is not characterized by the interpolation property in the critical slope case.

3 p-adic Beilinson conjecture for non-critical values of p-adic L-functions

In this section, we give a reformulation of the p-adic Beilinson conjecture for the special values of the cyclotomic p-adic L-functions in the case of cuspforms with rational Fourier coefficients. For the case of elliptic curves over $\mathbb{Q}$, the conjecture can be simplified. Since we will focus on the case of elliptic curves over $\mathbb{Q}$ later, we also explain the formulation of the conjecture for the special case. First, we begin with the review of Beilinson regulators and the syntomic regulators.

For a smooth scheme X over a commutative ring A, let $\Omega^\bullet_{X/A}$ denote the algebraic de Rham complex, and $H^*_{\mathrm{dR}}(X/A) := \mathbb{H}^*_{\mathrm{zar}}(X, \Omega^\bullet_{X/A})$ the algebraic de Rham cohomology where $\mathbb{H}^*_{\mathrm{zar}}(X, -)$ denotes the hypercohomology of the Zariski site of X.

3.1 Deligne-Beilinson cohomology and real regulators

Let X be a smooth quasi-projective variety over $\mathbb{C}$, and $\overline{X}$ a smooth completion such that $D := \overline{X} \setminus X$ is a normal crossing divisor (abbreviated NCD). Let $j : X \hookrightarrow \overline{X}$ be the open immersion. Let $\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D)$ denote the analytic de Rham complex with log poles along D, which is quasi-isomorphic to the de Rham complex $j_*\Omega^\bullet_{X^{\mathrm{an}}}$. Let $\mathrm{Fil}^r\,\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D) := \Omega^{\bullet\geq r}_{\overline{X}^{\mathrm{an}}}(\log D)$ be the stupid filtration.

For an integer $r \geq 0$, the *Deligne-Beilinson complex* is

$$\mathbb{Z}(r)_{\mathscr{D},X,\overline{X}} := \mathrm{Cone}[Rj_*\mathbb{Z}_X(r) \oplus \mathrm{Fil}^r\,\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D) \xrightarrow{\epsilon - i} j_*\Omega^\bullet_{X^{\mathrm{an}}}]$$

the cone of complexes of sheaves on the analytic site $\overline{X}^{\mathrm{an}}$ where $\mathbb{Z}_X(r) := (2\pi i)^r\mathbb{Z}_X$ and $\epsilon : \mathbb{Z}_X(r) \to \Omega^\bullet_{X^{\mathrm{an}}}$ is the natural map, and $i : \mathrm{Fil}^r\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D) \to \Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D) \to j_*\Omega^\bullet_{X^{\mathrm{an}}}$ is the composition. The *Deligne-Beilinson cohomology* is defined to be the cohomology

$$H^i_{\mathscr{D}}(X, \mathbb{Z}(r)) := \mathbb{H}^i(\overline{X}^{an}, \mathbb{Z}(r)_{\mathscr{D},X,\overline{X}}).$$

This depends only on X, and functorial with respect to X ([EV, Lemma 2.8]). The Deligne-Beilinson cohomology $H^i_{\mathscr{D}}(X, A(r))$ with coefficients in a ring $A \subset \mathbb{R}$ is defined by replacing $\mathbb{Z}$ with A in the above. The distinguished triangle

$$\mathrm{Cone}[\mathrm{Fil}^r\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D) \xrightarrow{\epsilon} j_*\Omega^\bullet_{X^{\mathrm{an}}}] \longrightarrow \mathbb{Z}(r)_{\mathscr{D},X,\overline{X}} \longrightarrow Rj_*\mathbb{Z}_X(r)$$

gives rise to an exact sequence

$$\cdots \to H^{i-1}_{\mathrm{dR}}(X/\mathbb{C})/\mathrm{Fil}^r \to H^i_{\mathscr{D}}(X, \mathbb{Z}(r)) \longrightarrow H^i_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z}(r)) \to H^i_{\mathrm{dR}}(X/\mathbb{C})/\mathrm{Fil}^r \to \cdots \tag{3.1}$$

where $\mathrm{Fil}^\bullet = \mathrm{Fil}^\bullet H_{\mathrm{dR}}(X/\mathbb{C})$ denotes Deligne's Hodge filtration and H^i_{B} is the Betti cohomology. Here we note that the analytic de Rham cohomology $\mathbb{H}^i(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$ agrees with the algebraic de Rham cohomology $H^i_{\mathrm{dR}}(X/\mathbb{C})$.

Let $X_{\mathbb{R}}$ be a smooth variety over $\mathbb{R}$, and $\overline{X}_{\mathbb{R}}$ a smooth compactification such that $D_{\mathbb{R}} := \overline{X}_{\mathbb{R}} \setminus X_{\mathbb{R}}$ is a NCD. Let $\overline{X} = \overline{X}_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ etc. and denote by $\overline{X}(\mathbb{C})$ the complex points.

Then complex conjugation $z \to \bar{z}$ induces the anti-holomorphic map $F_\infty : \overline{X}(\mathbb{C}) \to \overline{X}(\mathbb{C})$, which is called the *infinite Frobenius*. Let $\omega \mapsto \overline{\omega}$ denote complex conjugation on smooth differential forms on $X(\mathbb{C})$. The map

$$\omega \longmapsto \overline{F_\infty^*(\omega)}, \quad \omega \in \Omega^i_{\overline{X}^{\mathrm{an}}}(\log D)$$

induces maps

$$F_\infty^{-1}\mathbb{Z}_X(r) \longrightarrow \mathbb{Z}_X(r), \quad F_\infty^{-1}\mathbb{Z}(r)_{\mathscr{D},X,\overline{X}} \longrightarrow \mathbb{Z}(r)_{\mathscr{D},X,\overline{X}}$$

of sheaves on the smooth manifold $\overline{X}(\mathbb{C})$. We thus have an action on the cohomology $H^i_{\mathscr{D}}(X, \mathbb{Z}(r))$, $H^i_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z}(r))$ and so on, which we write by the same notation F_∞. As is easily seen, this is an involution (i.e. $F_\infty^2 = \mathrm{id}$). We write H^+ to be the F_∞-fixed part, and H^- the anti-fixed part. We have

$$H^i_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z}(r))^+ = H^i_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z}(s))^{(-1)^{r-s}} \otimes \mathbb{Z}(r-s).$$

Define the *real Deligne-Beilinson cohomology* $H^i_{\mathscr{D}}(X_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(r))$ to be the F_∞-fixed part ([Sch, §2])

$$H^i_{\mathscr{D}}(X_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(r)) := H^i_{\mathscr{D}}(X, \mathbb{R}(r))^+.$$

By the theory of the universal Chern classes, the *Beilinson regulator map* (the higher Chern class map)

$$\mathrm{reg}^{i,r}_{\mathscr{D}} : K_{2r-i}(X) \longrightarrow H^i_{\mathscr{D}}(X, \mathbb{Z}(r)) \tag{3.2}$$

is defined ([Sch, §4, p.29–30]). It induces the real regulator map

$$\mathrm{reg}^{i,r}_{\mathscr{D}} : K_{2r-i}(X_{\mathbb{R}}) \longrightarrow H^i_{\mathscr{D}}(X_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(r)) \tag{3.3}$$

to the real Deligne-Beilinson cohomology group.

3.2 Syntomic cohomology and syntomic regulators

Let W be the Witt ring of a perfect field k of characteristic $p > 0$, and put $K := \mathrm{Frac}\, W$. Let X be a smooth projective W-scheme. We write $X_k := X \times_W k$ and $X_K := X \times_W K$. Then the *syntomic cohomology* of Fontaine and Messing

$$H^i_{\mathrm{syn}}(X, \mathbb{Z}_p(r)), \quad r \geq 0$$

is defined ([FM], [Ka1]). There is an exact sequence

$$\cdots \to \mathrm{Fil}^r H^{i-1}_{\mathrm{crys}}(X_k/W) \xrightarrow{1-\frac{\Phi}{p^r}} H^{i-1}_{\mathrm{crys}}(X_k/W) \to H^i_{\mathrm{syn}}(X, \mathbb{Z}_p(r)) \to \mathrm{Fil}^r H^i_{\mathrm{crys}}(X_k/W) \to \cdots \tag{3.4}$$

which is a counterpart of (3.1), where Φ is the p-th Frobenius on the crystalline cohomology $H^i_{\mathrm{crys}}(X_k/W)$, and $\mathrm{Fil}^\bullet$ is the Hodge filtration via the comparison $H^i_{\mathrm{crys}}(X_k/W) \cong$

$H^i_{\mathrm{dR}}(X/W)$. If we drop the assumption that X is proper over W, the construction of the syntomic cohomology in [FM] or [Ka1] does not give a "correct cohomology theory", as the crystalline cohomology groups may not have finite rank. For a non-proper variety, we need the *rigid syntomic cohomology* $H^i_{\text{rig-syn}}(X, \mathbb{Q}_p(r))$ by Besser [Be1] (see also [NN, 1B]), which enjoys the expected properties.

Let X be a smooth W-scheme. Then the *syntomic regulator map*

$$\mathrm{reg}^{i,r}_{\text{rig-syn}} : K_{2r-i}(X) \longrightarrow H^i_{\text{rig-syn}}(X, \mathbb{Q}_p(r)) \tag{3.5}$$

is defined ([Be1, Theorem 7.5], [NN, Theorem A]). This is compatible with the étale regulator map

$$\mathrm{reg}^{i,r}_{\text{ét}} : K_{2r-i}(X_K) \longrightarrow H^i_{\text{ét}}(X_K, \mathbb{Q}_p(r)) \tag{3.6}$$

via the syntomic period map (loc.cit. Theorem A (7))

$$c : H^i_{\text{rig-syn}}(X, \mathbb{Q}_p(r)) \longrightarrow H^i_{\text{ét}}(X_K, \mathbb{Q}_p(r)).$$

If X is projective over W, the rigid syntomic cohomology agrees with the syntomic cohomology of Fontaine-Messing. In this case we write the subscript "syn" instead of "rig-syn".

Let X be a smooth projective W-scheme. The composition ρ in the following diagram

$$\begin{array}{ccccc}
H^{i-1}_{\text{crys}}(X_k/W)/(1-p^{-r}\Phi)\mathrm{Fil}^r & \xrightarrow{(3.4)} & H^i_{\text{syn}}(X, \mathbb{Q}_p(r)) & \xrightarrow{c} & H^i_{\text{ét}}(X, \mathbb{Q}_p(r)) \\
\uparrow {\scriptstyle 1-p^{-r}\Phi} & & & \nearrow {\scriptstyle \rho} & \\
H^{i-1}_{\text{crys}}(X_k/W)/\mathrm{Fil}^r & & & &
\end{array}$$

induces a map

$$H^{i-1}_{\text{crys}}(X_k/W)/\mathrm{Fil}^r \longrightarrow H^1(G_K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))).$$

This agrees with the Bloch-Kato exponential map ([Be1, Proposition 9.11], [NN, Proposition 1.1]).

3.3 p-adic Beilinson conjecture for modular forms

In this section, we assume that the Fourier coefficients of f belong to $\mathbb{Q}$ for simplicity. For a projective smooth variety over $\mathbb{Q}$, let $H^i_{\mathscr{M}}(X, \mathbb{Q}(j)) = K_{2j-i}(X)^{(j)}_{\mathbb{Q}}$ be the motivic cohomology group and let $H^i_{\mathscr{M}}(X, \mathbb{Q}(j))_{\mathbb{Z}}$ denote the integral part defined by Scholl [Sc2]. Recall that $\pi : \mathscr{E} \to X_1(N)$ is the universal generalized elliptic curve over the modular curve. Let $\mathscr{E}^k = \mathscr{E} \times_{X_1(N)} \cdots \times_{X_1(N)} \mathscr{E}$ be the k-fold fiber product of $\mathscr{E}$ over the modular curve $X_1(N)$ and W_k the canonical desingularization of $\mathscr{E}^k$(See [Sc1] or an appendix in [BDP] by Brian Conrad for details). W_k is a smooth projective variety over $\mathbb{Q}$ of dimension $k+1$. Fix a prime p not dividing N. Let $\mathcal{W}_k$ be the integral model over $\mathbb{Z}_p$. For $a \in (\mathbb{Z}/$

$N\mathbb{Z})^k$, let σ_a denote the automorphism on $\mathscr{E}^k$ obtained from the translation by the sections of order N, which naturally extends to W_k. Then we define the projector $\varepsilon_k^{(1)}$ by

$$\varepsilon_k^{(1)} = \frac{1}{N^k} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^k} \sigma_a.$$

Let $\mathfrak{S}_k$ be the symmetric group on k letters. Multiplication by -1 on $\mathscr{E}$ and the natural permutation action of $\mathfrak{S}_k$ on $\mathscr{E}^k$ give rise to an action of the semi-direct product $\{\pm 1\}^k \rtimes \mathfrak{S}_k$ on $\mathscr{E}^k$. This action also extends to W_k. Let $\mu : \{\pm 1\}^k \rtimes \mathfrak{S}_k \to \{\pm 1\}$ be the character which is multiplication on $\{\pm 1\}^k$ and the sign character on $\mathfrak{S}_k$. Then

$$\varepsilon_k^{(2)} = \frac{1}{2^k \cdot k!} \sum_{\sigma \in \{\pm 1\}^k \rtimes \mathfrak{S}_k} \mu(\sigma)\sigma \in \mathbb{Q}[\mathrm{Aut}(W_k/X_1(N))]$$

defines a projector. The projectors $\varepsilon_k^{(1)}$ and $\varepsilon_k^{(2)}$ commute, and therefore the composition $\varepsilon_k = \varepsilon_k^{(1)}\varepsilon_k^{(2)}$ defines a projector. Then the de Rham realization of the Chow motive $M = (W_k, \varepsilon_k)$ is canonically isomorphic to the parabolic de Rham cohomology:

$$H^*_{\mathrm{dR}}(M/\mathbb{Q}) = \varepsilon_k H^*_{\mathrm{dR}}(W_k/\mathbb{Q}) = \varepsilon_k H^{k+1}_{\mathrm{dR}}(W_k/\mathbb{Q}) \cong H^1_{\mathrm{par}}(X_1(N), \mathrm{Sym}^k \mathscr{L}, \nabla),$$

where $\mathscr{L} = \mathbb{R}^1\pi_*(0 \to \mathscr{O}_{\mathscr{E}} \to \Omega^1_{\mathscr{E}/X_1(N)} \to 0)$ is the relative de Rham cohomology sheaf on $X_1(N)$. The space of cuspforms $S_{k+2}(\Gamma_1(N), \mathbb{Q}) = H^0(X_1(N), \underline{\omega}^k \otimes \Omega^1_{X_1(N)})$ is naturally identified with a subspace of the parabolic de Rham cohomology $H^1_{\mathrm{par}}(X_1(N), \mathrm{Sym}^k \mathscr{L}, \nabla)$. Moreover, we have $\mathrm{Fil}^0\, \varepsilon_k H^*_{\mathrm{dR}}(W_k/\mathbb{Q}) = H^1_{\mathrm{par}}(X_1(N), \mathrm{Sym}^k \mathscr{L}, \nabla)$,

$$\mathrm{Fil}^1\, \varepsilon_k H^*_{\mathrm{dR}}(W_k/\mathbb{Q}) = \cdots = \mathrm{Fil}^{k+1}\, \varepsilon_k H^*_{\mathrm{dR}}(W_k/\mathbb{Q}) = H^0(X_1(N), \underline{\omega}^k \otimes \Omega^1_{X_1(N)})$$

and $\mathrm{Fil}^{k+2}\, \varepsilon_k H^*_{\mathrm{dR}}(W_k/\mathbb{Q}) = 0$, where Fil^i denotes the i-th step in the Hodge filtration on $\varepsilon_k H^*_{\mathrm{dR}}(W_k/\mathbb{Q})$. For the Betti realization, we have a canonical identification

$$H^*_{\mathrm{B}}(M, \mathbb{Q}) = \varepsilon_k H^*_{\mathrm{B}}(W_k(\mathbb{C}), \mathbb{Q}) = \varepsilon_k H^{k+1}_{\mathrm{B}}(W_k(\mathbb{C}), \mathbb{Q}) \cong H^1_{\mathrm{par}}(X_1(N)(\mathbb{C}), \mathscr{L}^{\mathrm{B}}_{k,\mathbb{Q}}),$$

where $\mathscr{L}^{\mathrm{B}}_{k,\mathbb{Q}} = \mathscr{L}^{\mathrm{B}}_k \otimes_{\mathbb{Z}} \mathbb{Q}$ (recall that $\mathscr{L}^{\mathrm{B}}_k = \mathrm{Sym}^k R^1\pi_*\mathbb{Z}$ as in 2.1) and

$$H^1_{\mathrm{par}}(X_1(N)(\mathbb{C}), \mathscr{L}^{\mathrm{B}}_{k,\mathbb{Q}}) = \mathrm{Im}\left[H^1_c(X_1(N)(\mathbb{C}), \mathscr{L}^{\mathrm{B}}_{k,\mathbb{Q}}) \to H^1(X_1(N)(\mathbb{C}), \mathscr{L}^{\mathrm{B}}_{k,\mathbb{Q}})\right]$$

is the parabolic Betti cohomology.

Let f be a normalized newform in $S_{k+2}(\Gamma_1(N), \mathbb{Q})$ (note that $f = f^*$ in this case). Then the element $u^\pm$ is determined only up to sign. Also we fix a non-zero element η in $S_{k+2}(\Gamma_1(N), \mathbb{Q})[f] = H^0(X_1(N), \underline{\omega}^k \otimes \Omega_{X_1(N)})[f]$. As we mentioned above, η can be identified with an element in $\varepsilon_k H^{k+1}_{\mathrm{dR}}(W_k/\mathbb{Q})[f]$ and $u^\pm$ can be viewed as an element in

$$H^*_{\mathrm{B}}(M, \mathbb{Q})[f]^\pm = \varepsilon_k H^*_{\mathrm{B}}(W_k(\mathbb{C}), \mathbb{Q})^\pm[f] = \varepsilon_k H^{k+1}_{\mathrm{B}}(W_k(\mathbb{C}), \mathbb{Q})^\pm[f].$$

Let γ be an eigenvalue of the Frobenius Φ on

$$(\varepsilon_k H^{k+1}_{\mathrm{dR}}(W_k/\mathbb{Q}_p)\otimes\overline{\mathbb{Q}}_p)[f]\cong(\varepsilon_k H^{k+1}_{\mathrm{crys}}(\mathcal{W}_{k,\mathbb{F}_p}/\mathbb{Z}_p)\otimes\overline{\mathbb{Q}}_p)[f]$$

and choose a non-zero eigenvector v_γ in $(\varepsilon_k H^{k+1}_{\mathrm{dR}}(W_k/\mathbb{Q}_p)\otimes\mathbb{Q}_p(\gamma))[f]$ with eigenvalue γ. Assume that the polynomial $X^2-a_p(f)X+p^{k+1}$ has distinct roots[2]. Under this assumption, the subspace of $(\varepsilon_k H^{k+1}_{\mathrm{dR}}(W_k/\mathbb{Q}_p)\otimes\mathbb{Q}_p(\gamma))[f]$ where Φ acts by the multiplication by γ is one dimensional. Therefore v_γ is uniquely determined up to a non-zero scalar. Let

$$\mathrm{reg}_{\mathrm{syn}}:\varepsilon_k H^{k+2}_{\mathscr{M}}(W_k,\mathbb{Q}(n+k+2))_{\mathbb{Z}}\to\varepsilon_k H^{k+2}_{\mathrm{syn}}(W_k,\mathbb{Q}_p(n+k+2))\cong\varepsilon_k H^{k+1}_{\mathrm{dR}}(W_k/\mathbb{Q}_p)$$

be the syntomic regulator map. Let

$$\mathrm{pr}_{\mathrm{dR},f}:\varepsilon_k H^{k+1}_{\mathrm{dR}}(W_k/\mathbb{Q}_p)\to\varepsilon_k H^{k+1}_{\mathrm{dR}}(W_k/\mathbb{Q}_p)[f]$$

denote the projection to the f-isotypic subspace. Assume that η and v_γ are linearly independent[3] over $\overline{\mathbb{Q}}_p$. For an element $\xi\in\varepsilon_k H^{k+2}_{\mathscr{M}}(W_k,\mathbb{Q}(n+k+2))_{\mathbb{Z}}$, we set

$$R_{p,\gamma,\eta}(M(f)(-n),\xi)=\Gamma^*(-n)(1-p^{k+1+n}\gamma^{-1})\frac{\mathrm{Tr}(\mathrm{pr}_{\mathrm{dR},f}\circ\mathrm{reg}_{\mathrm{syn}}(\xi)\cup v_\gamma)}{\mathrm{Tr}(\eta\cup v_\gamma)}\in\mathbb{Q}_p(\gamma),$$

where

$$\Gamma^*(-n)=\begin{cases}(-n-1)! & \text{if } n\le -1,\\ \dfrac{(-1)^n}{n!} & \text{if } n\ge 0.\end{cases}$$

Note that $R_{p,\gamma,\eta}(M(f)(-n),\xi)$ is independent of the choice of v_γ, since v_γ is unique up to a non-zero scalar. On the other hand, we have the Beilinson regulator map

$$\begin{aligned}\mathrm{reg}_{\mathscr{D}}:\varepsilon_k H^{k+2}_{\mathscr{M}}(W_k,\mathbb{Q}(n+k+2))_{\mathbb{Z}}&\to\varepsilon_k H^{k+2}_{\mathscr{D}}(W_{k,\mathbb{R}}/\mathbb{R},\mathbb{R}(n+k+2))\\&\cong\varepsilon_k H^{k+1}_{\mathrm{B}}(W_k(\mathbb{C}),\mathbb{R}(n+k+1))^+\\&\cong\varepsilon_k H^{k+1}_{\mathrm{B}}(W_k(\mathbb{C}),\mathbb{R})^{(-1)^{n+k+1}}\otimes\mathbb{R}(n+k+1).\end{aligned}$$

We denote the projection to the f-isotypic subspace by

$$\mathrm{pr}_{\mathrm{B},f}:\varepsilon_k H^{k+1}_{\mathrm{B}}(W_k(\mathbb{C}),\mathbb{R})^{(-1)^m}\otimes\mathbb{R}(m)\to\varepsilon_k H^{k+1}_{\mathrm{B}}(W_k(\mathbb{C}),\mathbb{R})^{(-1)^m}[f]\otimes\mathbb{R}(m),$$

where $m=n+k+1$. Now we define

$$R_{\infty,u}(M(f)(-n),\xi)=\frac{1}{(2\pi i)^{n+k+1}}\mathrm{Tr}(\mathrm{pr}_{\mathrm{B},f}\circ\mathrm{reg}_{\mathscr{D}}(\xi)\cup u^{(-1)^{n+k+1}})\in\mathbb{R}.$$

By the Beilinson conjecture, it is expected that the ratio $\dfrac{L'(f,-n)}{R_{\infty,u}(M(f)(-n),\xi)}$ is a rational number.

[2]This is conjectured in general and known for $k=0$.

[3]We exclude the case that η and v_γ are linearly dependent, since $R_{p,\gamma,\eta}(M(f)(-n),\xi)$ is not defined. However, it is expected that such a case occurs only when f has CM and f_γ is a critical slope eigenform. Indeed, if η and v_γ are linearly dependent, v_γ belongs to $\mathrm{Fil}^{k+1}\,\varepsilon_k H^*_{\mathrm{dR}}(W_k/\mathbb{Q})$. This implies that γ is divisible by p^{k+1} (hence f is ordinary at p and f_γ is a critical slope eigenform). Then it is easy to see that the local Galois representation attached to f at p is decomposable by the existence of the unit root subspace. In this case, it is conjectured that f has CM (see [CWE, (CG)]). For the case $k=0$, this is known by Zhao [Zh].

Conjecture 3.1 (**p-adic Beilinson conjecture for modular forms [PR]**) *Assume that the polynomial $X^2 - a_p(f)X + p^{k+1}$ has distinct roots. Suppose that one of the following conditions holds:*

1. $\mathrm{ord}_p(\gamma) < k+1$,
2. $\mathrm{ord}_p(\gamma) = k+1$ *and the p-refinement f_γ is not θ-critical.*

Moreover we assume that η and υ_γ are linearly independent over $\mathbb{Q}_p$. Then for $n \geq 0$, there exists a non-zero element

$$\xi \in \varepsilon_k H^{k+2}_{\mathscr{M}}(W_k, \mathbb{Q}(n+k+2))_{\mathbb{Z}}$$

such that the ratio $\dfrac{L_{p,\gamma,\eta,u}(f, \omega^{-n-1}, -n)}{R_{p,\gamma,\eta}(M(f)(-n), \xi)}$ is a non-zero rational number and we have

$$\frac{L_{p,\gamma,\eta,u}(f, \omega^{-n-1}, -n)}{R_{p,\gamma,\eta}(M(f)(-n), \xi)} = \pm \frac{L'(f, -n)}{R_{\infty,u}(M(f)(-n), \xi)}.$$

Remark 3.2

1. *In the original formulation of the p-adic Beilinson conjecture by Perrin-Riou [PR, 4.2.2 Conjecture (CP(M, τ))], it is expected the existence of p-adic L-function satisfying the interpolation property in the critical range and the p-adic Beilinson formula in the non-critical range. For the case of elliptic modular forms, we already have an appropriate p-adic L-function $L_{p,\gamma}(f, \chi, s)$ satisfying the interpolation property in the critical range. Therefore we can formulate the conjecture as more accessible form in this case. Moreover, Perrin-Riou used the étale regulator map to formulate the conjecture. Therefore the formula of the conjecture has a slightly different form. In general, the comparison between the étale regulator and the syntomic regulator is explained by Besser [Be1]. By [Be1, Remark 8.6.3], the difference is given by the factor $(1 - \frac{\Phi}{p^n})$. Using this result, it is easy to see that our formulation is equivalent to Perrin-Riou's formulation.*

2. *Our normalization of the p-adic L-functions is slightly different from Perrin-Riou's normalization. In fact, Perrin-Riou's p-adic L-functions should be the half of our p-adic L-function. For details, see [PR, 4.2.2 Conjecture (CP(M, τ)) and 4.3.4].*

3. *For $c \in \mathbb{Q}^\times$, it is easy to see*

$$L_{p,\gamma,c\eta,u}(f, \omega^{-n-1}, -n) = c^{-1} \cdot L_{p,\gamma,\eta,u}(f, \omega^{-n-1}, -n)$$

and

$$R_{p,\gamma,c\eta}(M(f)(-n), \xi) = c^{-1} \cdot R_{p,\gamma,\eta}(M(f)(-n), \xi).$$

Therefore the ratio $\dfrac{L_{p,\gamma,\eta,u}(f, \omega^{-n-1}, -n)}{R_{p,\gamma,\eta}(M(f)(-n), \xi)}$ does not depend on the choice of η. Moreover, we can show that the conjecture 3.1 is also independent of the choice of $u = (u^+, u^-)$.

4. *Bannai-Kings [BK] proved a version of the p-adic Beilinson conjecture for Hecke motives over imaginary quadratic fields with class number one when* $\mathrm{ord}_p(\gamma) = 0$.

5. *In this paper, we only consider the case that p does not divide N as in Perrin-Riou's book [PR]. If p divides N exactly once, we also have a p-adic L-function [AV, Vi]. More generally, Rodrigues Jacinto [Ro] has constructed a (de Rham cohomology valued) p-adic L-function for the general case (including the case p^2 divides N). Therefore it should be interesting to consider the p-adic Beilinson conjecture for the case of bad reduction using his p-adic L-function.*

3.4 p-adic Beilinson conjecture for elliptic curves

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and let ω_E denote the Néron differential. Let $u^\pm$ be the generator of $H^1(E(\mathbb{C}), \mathbb{Z})^\pm$ satisfying $\int_{u^+} \omega_E > 0$ and $i^{-1} \int_{u^-} \omega_E > 0$. Then we put $\Omega_E^+ = \int_{u^+} \omega_E$ and $\Omega_E^- = \int_{u^-} \omega_E$. It is known that

$$\frac{L(E, \overline{\chi}, 1)}{\Omega_E^{\chi(-1)}} \in \mathbb{Q}(\chi)$$

for any Dirichlet character χ. Let $f = f_E$ be the newform associated to E. For $a, m \in \mathbb{Q}$, we put

$$\lambda_E^\pm(a, m) = \frac{\pi i}{\Omega_E^\pm} \left\{ \int_\infty^{\frac{a}{m}} f(z) dz \pm \int_\infty^{-\frac{a}{m}} f(z) dz \right\}.$$

Assume that E has good reduction at p. Let γ be a root of $X^2 - a_p(E)X + p$. Using the modular symbol $\lambda_E^\pm$, we can construct the p-adic distribution $\mu_{E,\gamma}$ for any choice of the root γ and we can define the p-adic L-function $L_{p,\gamma}(E, \chi, s) = \displaystyle\int_{\mathbb{Z}_p^\times} \chi(x) \langle x \rangle^{s-1} d\mu_{E,\gamma}$ for $s \in \mathbb{Z}_p$. In particular, we have the following interpolation property: For a primitive Dirichlet character $\chi : (\mathbb{Z}/p^\nu\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$,

$$L_{p,\gamma}(E, \chi, 1) = \begin{cases} (1 - \gamma^{-1})^2 \dfrac{L(E, 1)}{\Omega_E^+} & \text{if } \chi = \mathbf{1}, \\ \left(\dfrac{p}{\gamma}\right)^\nu \cdot \dfrac{1}{\tau(\overline{\chi})} \cdot \dfrac{L(E, \overline{\chi}, 1)}{\Omega_E^{\chi(-1)}} & \text{if } \chi \neq \mathbf{1}. \end{cases}$$

For elliptic curves over $\mathbb{Q}$, the statement of the p-adic Beilinson conjecture can be slightly simplified as follows. Note that γ is also an eigenvalue of the Frobenius Φ on $H^1_{\mathrm{dR}}(E/\mathbb{Q}_p) \cong H^1_{\mathrm{crys}}(E_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Q}_p$. Choose a non-zero eigenvector v_γ in $H^1_{\mathrm{dR}}(E/\mathbb{Q}_p) \otimes \overline{\mathbb{Q}}_p$ with eigenvalue γ. Let

$$\mathrm{reg}_{\mathrm{syn}} : H^2_{\mathscr{M}}(E, \mathbb{Q}(n+2))_{\mathbb{Z}} \to H^2_{\mathrm{syn}}(E, \mathbb{Q}_p(n+2)) \cong H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)$$

be the syntomic regulator map. Suppose $\Phi(\omega_E) \neq \gamma\omega_E$. Note that this is equivalent to the condition that ω_E and v_γ are linearly independent. Then we set

$$R_{p,\gamma}(h^1(E)(-n), \xi) = \Gamma^*(-n)(1 - p^{n+1}\gamma^{-1}) \frac{\mathrm{Tr}(\mathrm{reg}_{\mathrm{syn}}(\xi) \cup v_\gamma)}{\mathrm{Tr}(\omega_E \cup v_\gamma)} \in \mathbb{Q}_p(\gamma).$$

Let

$$\begin{aligned}\mathrm{reg}_{\mathscr{D}} : H^2_{\mathscr{M}}(E, \mathbb{Q}(n+2))_{\mathbb{Z}} &\to H^2_{\mathscr{D}}(E, \mathbb{R}(n+2)) \\ &\cong H^1_B(E(\mathbb{C}), \mathbb{R}(n+1))^+ \\ &\cong \mathrm{Hom}(H_1(E(\mathbb{C}), \mathbb{Q})^{(-1)^{n+1}}, \mathbb{R}(n+1)).\end{aligned}$$

be the Beilinson regulator map and we denote

$$R_\infty(h^1(E)(-n), \xi) = \frac{1}{(2\pi i)^{n+1}} \mathrm{reg}_{\mathscr{D}}(\xi)(u^{(-1)^{n+1}}) \in \mathbb{R}.$$

By the Beilinson conjecture, it is expected that the ratio $\dfrac{L'(E, -n)}{R_\infty(h^1(E)(-n), \xi)}$ is a rational number.

Conjecture 3.3 (p-adic Beilinson conjecture for elliptic curves over $\mathbb{Q}$) *Assume that one of the following conditions holds:*

1. $\mathrm{ord}_p(\gamma) < 1$,
2. $\mathrm{ord}_p(\gamma) = 1$ *and f_γ is not θ-critical.*

Moreover we suppose that $\Phi(\omega_E) \neq \gamma\omega_E$. Then for $n \geq 0$, there exists a non-zero element $\xi \in H^2_{\mathscr{M}}(E, \mathbb{Q}(n+2))_{\mathbb{Z}}$ such that

$$\frac{L_{p,\gamma}(E, \omega^{-n-1}, -n)}{R_{p,\gamma}(h^1(E)(-n), \xi)} = \pm \frac{L'(E, -n)}{R_\infty(h^1(E)(-n), \xi)}.$$

Remark 3.4

1. *The assumption $\Phi(\omega_E) \neq \gamma\omega_E$ excludes only the case that E has good ordinary reduction at p, E has complex multiplication and γ has critical slope.*
2. *By the rank part of the Beilinson conjecture, it is expected that*

$$\dim_{\mathbb{Q}} H^2_{\mathscr{M}}(E, \mathbb{Q}(n+2))_{\mathbb{Z}} = 1.$$

3. *Brunault [Br1] gave an explicit formula for étale regulator of Beilinson-Kato elements.*
4. *Bertolini-Darmon [BD] gave an explicit formula for syntomic regulator and Beilinson regulator of Beilinson-Kato elements in the case of weight two cuspforms under some technical assumptions. However, their result does not cover Conjecture 3.3.*
5. *In [A2], we proposed conjectures on the relation between the special values of p-adic L-functions $L_{p,\alpha}(E, \omega^{-1}, 0)$ and the special values of p-adic hypergeometric functions ([A2, Conjecture 4.30 – 4.35]). Using the results in [A2], it is easy to see that these conjectures follow from Conjecture 3.3 and the rank part of the Beilinson conjecture.*

For $n = 0$, we will denote $R_{p,\gamma}(E, \xi) = R_{p,\gamma}(h^1(E), \xi)$ and $R_\infty(E, \xi) = R_\infty(h^1(E), \xi)$ for simplicity.

4 Syntomic regulators and overconvergent functions

In this section we work over the Witt ring $W = W(\mathbb{F})$ of a perfect field $\mathbb{F}$ of characteristic $p > 0$ endowed with the Frobenius F. Put $K := \mathrm{Frac}(W)$ the fractional field. Let A be a faithfully flat W-algebra. We mean by a p^n-th Frobenius on A an endomorphism σ such that $\sigma(x) \equiv x^{p^n} \bmod pA$ for all $x \in A$ and that σ is compatible with the p^n-th Frobenius on W. We also write x^σ instead of $\sigma(x)$. For a W-algebra A of finite type, we denote by $A^\dagger$ the weak completion. Namely if $A = W[T_1, \dots, T_n]$, then $A^\dagger = W[T_1, \dots, T_n]^\dagger$ is the ring of power series $\sum a_\alpha T^\alpha$ such that for some $r > 1$, $|a_\alpha| r^{|\alpha|} \to 0$ as $|\alpha| \to \infty$, and if $A = W[T_1, \dots, T_n]/I$, then $A^\dagger = W[T_1, \dots, T_n]^\dagger / IW[T_1, \dots, T_n]^\dagger$.

4.1 Setting and Notation

Let C be a smooth projective scheme over W of relative dimension 1. Let

$$f : Y \longrightarrow C$$

be an elliptic fibration over W, which means that Y is a smooth W-scheme and f is a projective flat morphism with a section $e : C \to Y$ whose general fiber is an elliptic curve. Let $T \subset C$ be a closed subscheme such that f is smooth over $S := C \setminus T$. Put $X := f^{-1}(S)$ and $A := \mathscr{O}(S)$. Write $A_K := A \otimes_W K$, $Y_K := Y \times_W K$, $A_\mathbb{F} := A/pA$, $Y_\mathbb{F} := Y \times_W W/pW$ etc.

We suppose that the following conditions **A1**, **A2** and **A3** are satisfied.

A1 T is a disjoint union of W-rational points $\{P_i \in S(W)\}$, and each $D_i := f^{-1}(P_i)$ is a relative simple NCD (possibly a smooth fiber) over W.

A2 $H^0_{\mathrm{zar}}(X, \Omega^1_{X/A})$ is a free A-module of rank one.

The condition **A2** is equivalent to that $H^1_{\mathrm{zar}}(\mathscr{O}_X)$ is a free A-module of rank one by the Serre duality. Then $H^1_{\mathrm{dR}}(X/A)$ is a free A-module of rank 2 by Lemma 7.6.

A3 There is a point P_0 such that f has a split multiplicative reduction at P_0. Let λ be the uniformizer of the local ring at the point P_0 and let $\mathscr{E}_{W[[\lambda]]} := Y \times_C \operatorname{Spec} W[[\lambda]]$ be the Tate curve over the formal neighborhood of P_0. Let

$$q = q(\lambda) = \kappa\lambda^n(1 + a_1\lambda + \cdots) \in \lambda W[[\lambda]] \tag{4.1}$$

be the multiplicative period. Then $\kappa \in W^\times$ and n is prime to p.

The multiplicative period q is characterized by the functional j-invariant $j(\lambda)$, namely it is the unique power series satisfying

$$j(\lambda) = \frac{1}{q} + 744 + 196884q + \cdots .$$

Let $\tau(\lambda) \in \lambda K[[\lambda]]$ be defined by $q = \kappa\lambda^n \exp(\tau(\lambda))$,

$$\tau(\lambda) = \log(1 + a_1\lambda + \cdots) \in \lambda K[[\lambda]]. \tag{4.2}$$

See Proposition 7.4 (7.6) for differential equations of q and $\tau(\lambda)$.

4.2 Category of Filtered F-isocrystal

Let $A^\dagger$ be the weak completion of A. We write $A_K^\dagger = A^\dagger \otimes_W K$. Fix a p-th Frobenius σ on $A^\dagger$. In [AM, 2.1] we introduced a category Fil-F-MIC(S) = Fil-F-MIC(S,σ), which consists of collections of datum $(H_{\mathrm{dR}}, H_{\mathrm{rig}}, c, \Phi, \nabla, \mathrm{Fil}^\bullet)$ such that

- H_{dR} is a finitely generated A_K-module,
- H_{rig} is a finitely generated $A_K^\dagger$-module,
- $c\colon H_{\mathrm{rig}} \cong H_{\mathrm{dR}} \otimes_{A_K} A_K^\dagger$ is the comparison isomorphism,
- $\Phi\colon \sigma^* H_{\mathrm{rig}} \xrightarrow{\cong} H_{\mathrm{rig}}$ is an isomorphism of $A_K^\dagger$-module,
- $\nabla\colon H_{\mathrm{dR}} \to \Omega^1_{S/K} \otimes H_{\mathrm{dR}}$ is an integrable connection that satisfies $\Phi\nabla = \nabla\Phi$,
- $\mathrm{Fil}^\bullet$ is a finite descending filtration on H_{dR} of locally free A_K-module (i.e. each graded piece is locally free), that satisfies $\nabla(\mathrm{Fil}^i) \subset \Omega^1 \otimes \mathrm{Fil}^{i-1}$.

Let $\mathrm{Fil}^\bullet$ denote the Hodge filtration on the de Rham cohomology, and ∇ the Gauss-Manin connection. Then one has an object

$$H^i(X/S) = H^i(X/A) := (H^i_{\mathrm{dR}}(X_K/S_K), H^i_{\mathrm{rig}}(X_{\mathbb{F}}/S_{\mathbb{F}}), c, \Phi, \nabla, \mathrm{Fil}^\bullet) \tag{4.3}$$

in Fil-F-MIC(S). For an integer r, the Tate object $\mathscr{O}_S(r)$ or $A(r) \in$ Fil-F-MIC(S) is defined as follows,

$$\mathscr{O}_S(r) = (A_K, A_K^\dagger, c, p^{-r}, d, F(r)^\bullet)$$

where $c : A_K^\dagger \cong A_K \otimes_{A_K} A_K^\dagger$ is the natural isomorphism, and $F(r)^\bullet$ is the filtration such that $F(r)^{-r}(A_K) = A_K$ and $F(r)^{-r+1}(A_K) = 0$. We define the Tate twist $M(r)$ by

$$M(r) = M \otimes \mathscr{O}_S(r) = (M_{\mathrm{dR}}, M_{\mathrm{rig}}, c, p^{-r}\Phi, \nabla, \mathrm{Fil}^{\bullet+r}) \in \text{Fil-}F\text{-MIC}(S).$$

4.3 Tate curve

Let $\mathscr{E} = \mathscr{E}_{W[[\lambda]]}$ be the Tate curve as in **A3**. As is well-known, there is a uniformization

$$\rho : \widehat{\mathbb{G}}_m := \operatorname{Spec} \varprojlim_n \left(W/p^n[u, u^{-1}]\right) \longrightarrow \mathscr{E}$$

over $W[[\lambda]]$. Let $\widehat{R} := W((\lambda))^\wedge$ be the p-adic completion of the ring of Laurent power series $W((\lambda)) = W[[\lambda]][\lambda^{-1}]$. Write $\widehat{R}_K := \widehat{R} \otimes_W K$. The de Rham cohomology $H^1_{\mathrm{dR}}(\mathscr{E}/\widehat{R})$ is a free $\widehat{R}$-module of rank 2. Following [AM, §5.1], we define a basis (which we call the *de Rham symplectic basis*)

$$\widehat{\omega},\ \widehat{\eta} \in H^1_{\mathrm{dR}}(\mathscr{E}/\widehat{R}) = H^1_{\mathrm{dR}}(\mathscr{E} \times_{W[[\lambda]]} \widehat{R}/\widehat{R})$$

in the following way. Firstly $\widehat{\omega} \in \Gamma(\mathscr{E}, \Omega^1_{\mathscr{E}/\widehat{R}})$ is a differential form of first kind such that $\rho^*\widehat{\omega} = du/u$. Note that this depends on the uniformization ρ. Let $\nabla : H^1_{\mathrm{dR}}(\mathscr{E}/\widehat{R}_K) \to$

$\Omega^1_{\widehat{R}_K/K} \otimes H^1_{\mathrm{dR}}(\mathscr{E}/\widehat{R}_K)$ be the Gauss-Manin connection, where $\Omega^1_{\widehat{R}_K/K}$ denotes the module of continuous 1-forms. Since $\mathrm{Ker}(\nabla) \cong K$, one can define the unique element $\widehat{\eta}$ such that $\nabla(\widehat{\eta}) = 0$ and $\mathrm{Tr}(\widehat{\omega} \cup \widehat{\eta}) = 1$, where $\mathrm{Tr} : H^2_{\mathrm{dR}}(\mathscr{E}/\widehat{R}_K) \to \widehat{R}_K$ is the trace map.

We shall give an explicit description of $\{\widehat{\omega}, \widehat{\eta}\}$ in case that $p \geq 5$ and $\mathscr{E}$ is defined by $y^2 = 4x^3 - g_2x - g_3$ (Proposition 7.3, (7.4)).

Proposition 4.1

$$\begin{pmatrix} \nabla(\widehat{\omega}) & \nabla(\widehat{\eta}) \end{pmatrix} = \begin{pmatrix} \widehat{\omega} & \widehat{\eta} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{dq}{q} & 0 \end{pmatrix}$$

Proof. [AM, Proposition 5.1]. $\square$

Let $H^1(X/A) \in \text{Fil-}F\text{-MIC}(A_K)$ be the object in (4.3). Suppose that σ extends to a p-th Frobenius on $\widehat{R} = W((\lambda))^\wedge$ (e.g. $\sigma(\lambda) = c\lambda^p$ with $c \in 1 + pW$). Then, according to [AM, 5.2], it gives rise to an object

$$H^1(\mathscr{E}/\widehat{R}) = (H^1_{\mathrm{dR}}(\mathscr{E}/\widehat{R}) \otimes_W K, \nabla, \Phi)$$

in the category $F\text{-MIC}(\widehat{R}_K) = F\text{-MIC}(\widehat{R}_K, \sigma)$ of F-integrable connections of $\widehat{R}$-modules (see [AM, Definition 5.2] for the definition of $F\text{-MIC}(\widehat{R}_K)$).

Proposition 4.2 *Define*

$$\log^{(\sigma)}(x) := p^{-1} \log\left(\frac{x^p}{x^\sigma}\right) = -p^{-1} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \frac{x^p}{x^\sigma}\right)^n$$

for $x \in \widehat{R}^\times$. If $x \in W^\times$, then we also write $\log^{(F)}(x) = \log^{(\sigma)}(x)$. Put

$$\tau^{(\sigma)}(\lambda) := \log^{(\sigma)}(q) = \log^{(F)}(\kappa) + n \log^{(\sigma)}(\lambda) + \tau(\lambda) - p^{-1}\tau(\lambda)^\sigma \in W[[\lambda]]. \tag{4.4}$$

Then we have

$$\begin{pmatrix} \Phi(\widehat{\omega}) & \Phi(\widehat{\eta}) \end{pmatrix} = \begin{pmatrix} \widehat{\omega} & \widehat{\eta} \end{pmatrix} \begin{pmatrix} p & 0 \\ -p\tau^{(\sigma)}(\lambda) & 1 \end{pmatrix}.$$

Proof. [AM, Theorem 5.6]. $\square$

Let $\omega, \eta \in H^1_{\mathrm{dR}}(X/A)$ be a free A-basis such that $\omega \in \varGamma(X, \Omega^1_{X/A}) \cong A$. Let $\widehat{\omega}, \widehat{\eta} \in H^1_{\mathrm{dR}}(\mathscr{E}/\widehat{R})$ be the de Rham symplectic basis as before. Define $F(\lambda) \in \widehat{R}$ and $f_i(\lambda) \in \widehat{R}$ by

$$\omega = F(\lambda)\widehat{\omega}, \quad \eta = f_1(\lambda)\widehat{\omega} + f_2(\lambda)\widehat{\eta}. \tag{4.5}$$

In case that $p \geq 5$ and $\mathscr{E}$ is defined by $y^2 = 4x^3 - g_2x - g_3$, we shall give an explicit description of $F(\lambda)$ and $f_i(\lambda)$ (Proposition 7.3 (7.4) together with (7.3)).

Proposition 4.3 *Let σ be a p-th Frobenius on $A^\dagger$ such that it extends on $\widehat{R}$. Let*

$$\begin{aligned}
F_{11} &= \frac{F(\lambda)^\sigma}{F(\lambda)}\left(1+\tau^{(\sigma)}(\lambda)\frac{f_1(\lambda)}{f_2(\lambda)}\right)\\
F_{12} &= \frac{1}{F(\lambda)}\left(pf_1(\lambda)^\sigma - f_1(\lambda)\frac{f_2(\lambda)^\sigma - p\tau^{(\sigma)}(\lambda)f_1(\lambda)^\sigma}{f_2(\lambda)}\right)\\
F_{21} &= -\tau^{(\sigma)}(\lambda)\frac{F(\lambda)^\sigma}{f_2(\lambda)}\\
F_{22} &= \frac{f_2(\lambda)^\sigma - p\tau^{(\sigma)}(\lambda)f_1(\lambda)^\sigma}{f_2(\lambda)}.
\end{aligned}$$

Then we have

$$\begin{pmatrix}\Phi(\omega) & \Phi(\eta)\end{pmatrix} = \begin{pmatrix}\omega & \eta\end{pmatrix}\begin{pmatrix}pF_{11} & F_{12}\\ pF_{21} & F_{22}\end{pmatrix}.$$

Moreover $F_{ij} \in A^\dagger$.

Proof. The representation matrix of Φ is straightforward from Proposition 4.2 together with (4.5). Then this implies that $F_{ij} \in A^\dagger_K = A^\dagger \otimes_W K$. We further show $F_{ij} \in A^\dagger$. By a commutative diagram

$$\begin{CD}
0 @>>> A^\dagger @>p>> A^\dagger @>>> A/pA @>>> 0\\
@. @VVV @VVV @V{\cap}VV @.\\
0 @>>> \widehat{R} @>p>> \widehat{R} @>>> \mathbb{F}((\lambda)) @>>> 0
\end{CD}$$

the W-module $\widehat{R}/A^\dagger$ is p-torsion free, and this implies that $A^\dagger = A^\dagger_K \cap \widehat{R}$. Therefore it is enough to show $F_{ij} \in \widehat{R}$. To do this, it is enough to show that $F(\lambda), f_2(\lambda) \in \widehat{R}^\times$. However, since $\{\omega, \eta\}$ is a free A-basis of $H^1_{\mathrm{dR}}(X/A)$, one has that $\mathrm{Tr}(\omega \cup \eta) = F(\lambda)f_2(\lambda) \in A^\times$ as required. $\square$

Corollary 4.4 *Let $a : \operatorname{Spec} W \to S$ be a W-rational point of S, and $\mathfrak{m} \subset A$ the corresponding prime ideal. Let X_a be the fiber at a, which is an elliptic curve over W. Put $X_{a,\overline{\mathbb{F}}_p} := X_a \times_W \overline{\mathbb{F}}_p$. Suppose $\sigma^{-1}(\mathfrak{m}A^\dagger) \supset \mathfrak{m}A^\dagger$. Then the representation matrix of the p-th Frobenius Φ on $H^1_{\mathrm{rig}}(X_{a,\overline{\mathbb{F}}_p}/\overline{\mathbb{F}}_p)$ is*

$$\begin{pmatrix}pF_{11}(a) & F_{12}(a)\\ pF_{21}(a) & F_{22}(a)\end{pmatrix},$$

where for $f \in A^\dagger_K$ we write by $f(a) \in A^\dagger_K/\mathfrak{m}A^\dagger_K \cong K$ the reduction modulo $\mathfrak{m}A^\dagger_K$.

4.4 Unit root fomula

Lemma 4.5 *Let σ, Φ and F_{ij} be as in Proposition 4.3. Put*

$$\widehat{B} := \varprojlim_n (A^\dagger/p^n A^\dagger[F_{22}^{-1}]) = \varprojlim_n (A/p^n A[F_{22}^{-1}])$$

and $\widehat{B}_K := \widehat{B} \otimes_W K$. Note that $\widehat{B} \neq \{0\}$ as $F_{22} \equiv f_2(\lambda)^{p-1} \not\equiv 0$ mod $p\widehat{R}$ [4]*. Then*

$$\frac{F(\lambda)}{F(\lambda)^\sigma}, \frac{E_1^{(\sigma)}(\lambda)}{F(\lambda)}, \frac{f_1(\lambda)}{F(\lambda)} \in \widehat{B}_K.$$

Proof. We prove our lemma by the same argument as that of [VdP, Proposition (7.12)]. For $f(\lambda) \in \widehat{B}$, one has

$$\Phi(f(\lambda)\omega + \eta) = (F_{12} + pf(\lambda)^\sigma F_{11})\omega + (F_{22} + pf(\lambda)^\sigma F_{21})\eta$$

by Proposition 4.3. Let $V := \{f(\lambda)\omega + \eta \mid f(\lambda) \in \widehat{B}\}$. Consider a map

$$S : V \longrightarrow V, \quad S(f(\lambda)\omega + \eta) = \frac{F_{12} + pf(\lambda)^\sigma F_{11}}{F_{22} + pf(\lambda)^\sigma F_{21}}\omega + \eta.$$

This is well-defined as $F_{22} + pf(\lambda)^\sigma F_{21} \in \widehat{B}^\times$ and we have

$$S(f_1(\lambda)\omega + \eta) - S(f_2(\lambda)\omega + \eta) = \frac{p(f_1(\lambda)^\sigma - f_2(\lambda)^\sigma)D}{(F_{22} + pf_1(\lambda)^\sigma F_{21})(F_{22} + pf_2(\lambda)^\sigma F_{21})}\omega,$$

where $D := F_{11}F_{22} - F_{21}F_{12} \in (A^\dagger)^\times$. This shows that S is a contraction map with respect to the norm $||\cdot||$ on $\widehat{R} = W((\lambda))^\wedge$ which is defined by $||\sum a_i\lambda^i|| := \sup\{|a_i| \mid i \in \mathbb{Z}\}$. Hence there is a unique fixed point $u_0 = f_0(\lambda)\omega + \eta \in V$, namely

$$S(u_0) = u_0 \quad \Longleftrightarrow \quad \Phi(u_0) = (F_{22} + pf_0(\lambda)^\sigma F_{21})u_0.$$

Replace $\widehat{B}$ with $\widehat{R}$ and consider the similar map $\widehat{S}$ on $\widehat{V} := \{f(\lambda)\omega + \eta \mid f(\lambda) \in \widehat{R}\}$. Then one also has a unique fixed point $\widehat{u}_0 = \widehat{f}_0(\lambda)\omega + \eta \in \widehat{V}$ of $\widehat{S}$, and then it turns out that $u_0 = \widehat{u}_0$ by the uniqueness. From (4.5), we have

$$f_2(\lambda)\widehat{\eta} = -\frac{f_1(\lambda)}{F(\lambda)}\omega + \eta.$$

By Proposition 4.2, one has that $\Phi(f_2(\lambda)\widehat{\eta}) = f_2(\lambda)^\sigma\widehat{\eta}$. This means that the above coincides with the unique fixed point $\widehat{u}_0$ of $\widehat{S}$, and hence

$$-\frac{f_1(\lambda)}{F(\lambda)} = \widehat{f}_0(\lambda) = f_0(\lambda).$$

[4]Indeed $F(\lambda)f_2(\lambda) \in A^\times$ as is in shown in the proof of Proposition 4.3. Note that $f_2(\lambda)^{p-1} \in \widehat{R}/p\widehat{R} = k((\lambda))$ belongs to A/pA as so does (F_{22} mod $pA^\dagger$). However this does not imply $f_2(\lambda)^{p-1} \in (A/pA)^\times$ because $F(\lambda)^{p-1} \notin A/pA$.

Moreover,

$$\Phi(f_2(\lambda)\widehat{\eta}) = f_2(\lambda)^\sigma\widehat{\eta} \quad\Longleftrightarrow\quad \Phi(f_0(\lambda)\omega+\eta) = \frac{f_2(\lambda)^\sigma}{f_2(\lambda)}(f_0(\lambda)\omega+\eta)$$

implies

$$\frac{f_2(\lambda)^\sigma}{f_2(\lambda)} = F_{22} + pf_0(\lambda)^\sigma F_{21}.$$

We thus have $f_1(\lambda)/F(\lambda), f_2(\lambda)^\sigma/f_2(\lambda) \in \widehat{B}$. Moreover since $\mathrm{Tr}(\omega\cup\eta) = F(\lambda)f_2(\lambda) \in A^\times$ (see the proof of Proposition 4.3), one also has $F(\lambda)/F(\lambda)^\sigma \in \widehat{B}$. Finally, we have

$$\mathrm{Tr}(E_1^{(\sigma)}(\lambda)\widehat{\omega} + E_2^{(\sigma)}(\lambda)\widehat{\eta}) \cup u_0) = \mathrm{Tr}(E_1^{(\sigma)}(\lambda)\widehat{\omega}\cup u_0) = E_1^{(\sigma)}(\lambda)/F(\lambda)\cdot\mathrm{Tr}(\omega\cup\eta) \in \widehat{B}_K$$

from (4.8). This implies $E_1^{(\sigma)}(\lambda)/F(\lambda) \in \widehat{B}_K$. □

Corollary 4.6 (Unit root formula) *Suppose $W = \mathbb{Z}_p$ for simplicity. Let $a : \mathrm{Spec}\,\mathbb{Z}_p \to \mathrm{Spec}\,A$ be a $\mathbb{Z}_p$-rational point of S, and $\mathfrak{m} \subset A$ the prime ideal. Let X_a be the fiber at a. Suppose $\sigma^{-1}(\mathfrak{m}A^\dagger) \supset \mathfrak{m}A^\dagger$. Suppose further that $X_{a,\mathbb{F}_p} := X_a \times_{\mathbb{Z}_p} \mathbb{F}_p$ is an ordinary elliptic curve. Then the unit root of $X_{a,\mathbb{F}_p}$ is*

$$\left.\frac{F(\lambda)}{F(\lambda)^\sigma}\right|_{\lambda=a}.$$

Proof. We first note that $X_{a,\mathbb{F}_p}$ is ordinary if and only if the absolute Frobenius on $H^1(\mathscr{O}_{X_{a,\mathbb{F}_p}})$ does not vanish, and this is equivalent to that $F_{22}(\lambda)|_{\lambda=a} \not\equiv 0 \mod p$ in Proposition 4.3. Hence a is a point of $\mathrm{Spec}\,\widehat{B}_K$ and the special value $F(\lambda)/F(\lambda)^\sigma|_{\lambda=a}$ makes sense. As is shown in the proof of Lemma 4.5, the unit root vector is $u_0|_{\lambda=a}$ and the unit root is

$$\left.\frac{f_2(\lambda)^\sigma}{f_2(\lambda)}\right|_{\lambda=a} = \left.\frac{F(\lambda)}{F(\lambda)^\sigma}\right|_{\lambda=a} \times \left.\frac{d(\lambda)^\sigma}{d(\lambda)}\right|_{\lambda=a} = \left.\frac{F(\lambda)}{F(\lambda)^\sigma}\right|_{\lambda=a},$$

where $d(\lambda) := F(\lambda)f_2(\lambda) \in A^\times$. □

4.5 Regulator formula for K_2 of elliptic curves

Recall from §4.1 the elliptic fibration $f : Y \to C$ and the singular fibers $D_i = f^{-1}(P_i)$. Let $U \subset X$ be an affine open set such that the following condition holds.

B1 There is an open immersion $S \hookrightarrow \overline{S}$ and a finite flat W-morphism $v : \overline{S}' \to \overline{S}$ that is étale over S where $\overline{S}$ is projective curve over W and $\overline{S}'$ is a smooth projective curve over W. Moreover $(U'/S', \overline{S}')$ satisfies the following conditions. Put $S' = v^{-1}(S)$, $T' = \overline{S}' \setminus S'$ and $U' = U \times_S S'$ and $f' : U' \to S'$.

- T' is finite étale over W.

- There is a commutative square

$$\begin{array}{ccc} U' & \xrightarrow{j} & \overline{X}' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \overline{f}'} \\ S' & \longrightarrow & \overline{S}' \end{array}$$

  where $\overline{X}'$ is a smooth projective W-scheme and j is an open immersion, that satisfies the following. Put $X' = (\overline{f}')^{-1}(S')$ and $F' = (\overline{f}')^{-1}(T')$ and $\overline{Z}' := \overline{X}' \setminus U'$. Then $X' \to S'$ is projective smooth, $\overline{Z}'$ is a relative simple NCD over W, the multiplicity of an arbitrary component of F' is prime to p, and $\overline{Z}' \cap X'$ a relative simple NCD over S'.

Then $R^i f'_{\mathrm{rig}} j^\dagger_{U'} \mathscr{O}_{(U'_K)^{\mathrm{an}}}$ is a coherent $j^\dagger_{S'} \mathscr{O}_{(S'_K)^{\mathrm{an}}}$-module by [AM, Lemma 4.1], and hence $R^i f_{\mathrm{rig}} j^\dagger_U \mathscr{O}_{U_K^{\mathrm{an}}}$ is a coherent $j^\dagger_S \mathscr{O}_{S_K^{\mathrm{an}}}$-module thanks to the finite flat base change theorem of rigid cohomology. Thus our setting $(U/S, f : Y \to C)$ is adapted to the setting in [AM, §4.1].

Let $K(X)$ (resp. $K'(X)$) denote the algebraic K-theory of locally free sheaves on X of finite ranks (resp. coherent sheaves on X). Let $\partial : K_2^M(\mathscr{O}(U)) \to K_1'(Z)$ be the composition of the natural map $K_2^M(\mathscr{O}(U)) \to K_2(U)$ and the boundary map (tame symbol) $K_2(U) \to K_1'(Z)$. Write the kernel by $K_2^M(\mathscr{O}(U))_{\partial=0}$. For an element $\xi \in K_2^M(\mathscr{O}(U))_{\partial=0}$, let $G_\xi(\lambda) \in \widehat{R}$ be defined by

$$\mathrm{dlog}(\xi) = G_\xi(\lambda)\frac{d\lambda}{\lambda} \wedge \widehat{\omega} \in \Gamma(\mathscr{E}, \Omega^2_{\mathscr{E}/W}).$$

Since this has at worst log pole and $\rho^*\widehat{\omega} = du/u$, we have

$$G_\xi(\lambda) = l + a_1\lambda + a_2\lambda^2 + \cdots \in W[[\lambda]]. \tag{4.6}$$

The constant term l is an integer (possibly zero). Indeed, let δ be the composition of the maps

$$K_2(\mathscr{E}) \xrightarrow{\text{tame}} K_1'(D) \xrightarrow{j^*} K_1(\mathbb{G}_{m,W}) \longrightarrow K_1(\mathbb{G}_{m,K}) \cong K[u, u^{-1}]^\times \xrightarrow{\mathrm{ord}_u} \mathbb{Z}\,,$$

where $D \subset \mathscr{E}$ is the central fiber at $\lambda = 0$ which is the Néron n-gon over W, and $j : \mathbb{G}_{m,W} \hookrightarrow D$ is the open immersion. Then $l = \pm\delta(\xi)$. According to [AM, Proposition 4.3], one can associate a 1-extension

$$0 \longrightarrow H^1(X/S)(2) \longrightarrow M_\xi(X/S) \longrightarrow \mathscr{O}_S \longrightarrow 0$$

in Fil-F-MIC(S, σ) to the element ξ. Let $e_\xi \in \mathrm{Fil}^0 M_\xi(X_K/A_K)_{\mathrm{dR}}$ be the unique lifting of $1 \in A_K$. Define $\varepsilon_i^{(\sigma)}(\lambda) \in A_K^\dagger$ by

$$e_\xi - \Phi(e_\xi) = \varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta. \tag{4.7}$$

Theorem 4.7 *Let $a : \operatorname{Spec} W \to \operatorname{Spec} A$ be a W-rational point of S, and $\mathfrak{m} \subset A$ the corresponding prime ideal. Let X_a be the fiber at a. Suppose $\sigma^{-1}(\mathfrak{m}A^\dagger) \supset \mathfrak{m}A^\dagger$. Let*

$$\mathrm{reg}_{\mathrm{syn}} : K_2(X_a) \longrightarrow H^2_{\mathrm{syn}}(X_a, \mathbb{Q}_p(2)) \cong H^1_{\mathrm{dR}}(X_a/K)$$

be the syntomic regulator map. Then

$$\mathrm{reg}_{\mathrm{syn}}(\xi|_{X_a}) = \varepsilon_1^{(\sigma)}(a)\omega + \varepsilon_2^{(\sigma)}(a)\eta.$$

Proof. This is immediate from [AM, Theorem 4.5]. □

Define $E_i^{(\sigma)}(\lambda) \in \widehat{R}_K$ by

$$\varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta = E_1^{(\sigma)}(\lambda)\widehat{\omega} + E_2^{(\sigma)}(\lambda)\widehat{\eta} \tag{4.8}$$

or explicitly

$$E_1^{(\sigma)}(\lambda) = F(\lambda)\varepsilon_1^{(\sigma)}(\lambda) + f_1(\lambda)\varepsilon_2^{(\sigma)}(\lambda) \tag{4.9}$$

$$E_2^{(\sigma)}(\lambda) = f_2(\lambda)\varepsilon_2^{(\sigma)}(\lambda), \tag{4.10}$$

where $F(\lambda)$ and $f_i(\lambda)$ are as in (4.5).

Theorem 4.8 *Let σ be a p-th Frobenius on $A^\dagger$ such that it extends on $\widehat{R}$. Define $f'(\lambda)$ by $f'(\lambda)d\lambda = df(\lambda)$. Then we have*

$$(E_1^{(\sigma)}(\lambda))' = \frac{G_\xi(\lambda)}{\lambda} - p^{-1}G_\xi(\lambda)^\sigma \frac{(\lambda^\sigma)'}{\lambda^\sigma} \tag{4.11}$$

$$(E_2^{(\sigma)}(\lambda))' = -E_1^{(\sigma)}(\lambda)\frac{q'}{q} + p^{-1}G_\xi(\lambda)^\sigma \frac{(\lambda^\sigma)'}{\lambda^\sigma}\log^{(\sigma)}(q). \tag{4.12}$$

If $\lambda^\sigma = c\lambda^p$ for some $c \in 1 + pW$, then $E_1^{(\sigma)}(\lambda), E_2^{(\sigma)}(\lambda) \in W[[\lambda]] \otimes_W K$ [5] *and*

$$E_1^{(\sigma)}(0) = \frac{l}{n}\log^{(F)}(\kappa) - lp^{-1}\log(c). \tag{4.13}$$

Proof. Apply ∇ on (4.7) and (4.8). Since $\Phi\nabla = \nabla\Phi$, one has

$$G_\xi(\lambda)\frac{d\lambda}{\lambda}\wedge\widehat{\omega} - \Phi\left(G_\xi(\lambda)\frac{d\lambda}{\lambda}\wedge\widehat{\omega}\right) = (E_1^{(\sigma)}(\lambda))'d\lambda\wedge\widehat{\omega} + \left(E_1^{(\sigma)}(\lambda)\frac{dq}{q} + (E_2^{(\sigma)}(\lambda))'d\lambda\right)\wedge\widehat{\eta},$$

where we apply Proposition 4.1 to the right hand side of (4.8). Apply Proposition 4.2 to the second term in the left hand side. Then we have

$$\frac{G_\xi(\lambda)}{\lambda}\widehat{\omega} - G_\xi(\lambda)^\sigma\frac{(\lambda^\sigma)'}{\lambda^\sigma}(p^{-1}\widehat{\omega} - p^{-1}\log^{(\sigma)}(q)\widehat{\eta}) = (E_1^{(\sigma)}(\lambda))'\widehat{\omega} + \left(E_1^{(\sigma)}(\lambda)\frac{q'}{q} + (E_2^{(\sigma)}(\lambda))'\right)\widehat{\eta}.$$

This finishes the proof of (4.11) and (4.12). Suppose that $\lambda^\sigma = c\lambda^p$ with $c \in 1 + pW$. It is straightforward to see $E_1^{(\sigma)}(\lambda) \in W[[\lambda]] \otimes_W K$ from (4.6) and (4.11). By (4.4), the coefficient of λ^{-1} in the right hand side of (4.12) is

$$-nE_1^{(\sigma)}(0) + p^{-1} \cdot lp(\log^{(F)}(\kappa) - np^{-1}\log(c)).$$

Since $E_2^{(\sigma)}(\lambda) \in \widehat{R}_K$, this must be zero. This shows (4.13). The right hand side of (4.12) belongs to $K[[\lambda]]$, so that we have $E_2^{(\sigma)}(\lambda) \in \widehat{R}_K \cap K[[\lambda]] = W[[\lambda]] \otimes_W K$. □

[5] More precisely, we shall see that $E_i^{(\sigma)}(\lambda) \in W[[\lambda]]$ by Lemma 4.10 below.

We have a complete description of the constant $E_1^{(\sigma)}(0)$, while the constant $E_2^{(\sigma)}(0)$ seems more delicate.

Conjecture 4.9 *Suppose $\lambda^\sigma = c\lambda^p$ for some $c \in 1 + pW$. Let $D \subset \mathscr{E}$ be the central fiber at $\lambda = 0$ which is the Néron n-gon over W, and $D_\bullet \to D$ the simplicial scheme. Then there exists a map (specialization map)*

$$\mathrm{Sp} : H^2_{\mathscr{M}}(\mathscr{E} \times_{W[[\lambda]]} W((\lambda)), \mathbb{Q}(2)) \longrightarrow H^2_{\mathscr{M}}(D_\bullet, \mathbb{Q}(2))$$

which is induced by the "nearby cycle functor with respect to a parameter $\kappa^{\frac{1}{n}}\lambda$", and we have

$$E_2^{(\sigma)}(0) = -\frac{l}{2n}(\log^{(F)}(\kappa) - np^{-1}\log(c))^2 + R_{\mathrm{syn}}(\mathrm{Sp}(\xi)),$$

where R_{syn} is the composition of the following arrows

$$H^2_{\mathscr{M}}(D_\bullet, \mathbb{Q}(2)) \xrightarrow{\mathrm{reg}_{\mathrm{syn}}} H^2_{\mathrm{syn}}(D_\bullet, \mathbb{Q}_p(2)) \xleftarrow{\cong} H^1_{\mathrm{dR}}(D_\bullet/W) \xrightarrow[\cong]{\widehat{\eta}\mapsto 1} W.$$

In particular, $R_{\mathrm{syn}}(\mathrm{Sp}(\xi))$ is a $\mathbb{Q}$-linear combination of $\ln_2^{(p)}(\epsilon)$'s with $\epsilon \in W \cap \overline{\mathbb{Q}}$, where $\ln_r^{(p)}(z) = \sum_{p\nmid n} z^n/n^r$ is the p-adic dilogarithm function.

4.6 Bounds on p-adic expansions of $\varepsilon_k^{(\sigma)}(\lambda)$

In this section we give effective bounds of rational functions $\varepsilon_k^{(\sigma)}(\lambda)$ modulo p^n according to [KT]. The main results are Theorems 4.13 and 4.14, which play key roles in later sections. We note that Kedlaya and Tuitman give a theorem in more general situation ([KT, Theorem 2.1]). However, as we need a little more precise statement, we give a self-contained proof for the sake of completeness.

We assume $p \geq 3$. Recall $f : Y \to C$ and an affine open set $U \subset Y$ that satisfies **A1**, **A2**, **A3** in §4.1 and **B1** in §4.5. Let

$$\xi \in K_2^M(\mathscr{O}(U))_{\partial=0}$$

be an arbitrary element. We fix a p-th Frobenius σ on $A^\dagger$ compatible with the Frobenius on W. For a W-rational point $P_i \in T$, let λ_i be a uniformizer of the local ring at P_i. Put $Y_i := Y \times_C \operatorname{Spec} W[[\lambda_i]]$ and $X_i = Y_i \setminus D_i$ where $D_i := f^{-1}(P_i)$. We further assume that the following condition holds.

B2 Put

$$H_i := \mathrm{Im}[H^1_{\mathrm{zar}}(Y_i, \Omega^\bullet_{Y_i/W[[\lambda_i]]}(\log D_i)) \to H^1_{\mathrm{dR}}(X_i/W((\lambda_i)))]$$

and

$$\mathrm{Fil}^1 H_i := \mathrm{Im}[H^0_{\mathrm{zar}}(Y_i, \Omega^1_{Y_i/W[[\lambda_i]]}(\log D_i)) \to H^1_{\mathrm{dR}}(X_i/W((\lambda_i)))].$$

Then, both of $\mathrm{Fil}^1 H_i$ and $H_i/\mathrm{Fil}^1 H_i$ are free $W[[\lambda_i]]$-modules of rank one.

We endow the log structures on Y_i and $\operatorname{Spec} W[[\lambda_i]]$ arising from the divisors D_i and $\lambda_i = 0$ respectively. Since the multiplicity of each component of D_i is prime to p (cf. **B1**), the morphism $(Y_i, D_i) \to (\operatorname{Spec} W[[\lambda_i]], (\lambda_i))$ is a smooth morphism of log schemes in the sense of [Ka2, §6]. By [Ka2, Theorem (6.4)], there is the natural isomorphism

$$H^k_{\mathrm{crys}}((Y_{i,\overline{\mathbb{F}}_p}, D_{i,\overline{\mathbb{F}}_p})/(W[[\lambda_i]], (\lambda_i))) \cong H^k_{\mathrm{zar}}(Y_i, \Omega^{\bullet}_{Y_i/W[[\lambda_i]]}(\log D_i)),$$

where the left hand side is the log crystalline cohomology. In particular, if $\sigma(\lambda_i) = c\lambda_i^p$ with $c \in 1 + pW$, then the σ-linear Frobenius on the log crystalline cohomology is defined.

Lemma 4.10 *Let $\{\omega_i, \eta_i\}$ be a $W[[\lambda_i]]$-basis of H_i. Suppose that σ satisfies $\lambda_i^\sigma = c\lambda_i^p$ for some $c \in 1 + pW$. Define $\varepsilon_j^{(\sigma)}(\lambda_i) \in A_K^\dagger$ ($j = 1, 2$) by*

$$e_\xi - \Phi(e_\xi) = \varepsilon_1^{(\sigma)}(\lambda_i)\omega_i + \varepsilon_2^{(\sigma)}(\lambda_i)\eta_i \in H^1_{\mathrm{dR}}(X_i/W((\lambda_i))) \otimes_{W((\lambda_i))} W((\lambda_i))^\wedge, \qquad (4.14)$$

where $W((\lambda_i))^\wedge$ denotes the p-adic completion. Then this element lies in the image of $H^1_{\mathrm{zar}}(Y_i, \Omega^{\bullet}_{Y_i/W[[\lambda_i]]}(\log D_i))$. Hence $\varepsilon_1^{(\sigma)}(\lambda_i)$ and $\varepsilon_2^{(\sigma)}(\lambda_i)$ belong to the subspace $W[[\lambda_i]] \cap A_K^\dagger = W[[\lambda_i]] \cap A^\dagger$.

Proof. Let $X' = X \cup D_i$ and $S' = S \cup \{P_i\}$ and let $Z' \subset \mathscr{X}$ be the closure of Z. Then $(X'/S', S, Z')$ is adapted to the setting in [AM, 4.1], so that the assertion follows from the diagram in [AM, Theorem 4.5]. □

Lemma 4.11 (Transformation of Frobenius) *Let σ be a p-th Frobenius on $A^\dagger$. Let F-MIC$(A_K^\dagger, \sigma)$ be the category of objects (H, ∇, Φ), where (H, ∇) is an integrable connection of free $A_K^\dagger$-module, and Φ is a p-th Frobenius on H compatible with σ. Let $\partial = \nabla_{d/d\lambda_i}$ be the differential operator which is defined to be the composition*

$$H \xrightarrow{\nabla} A \cdot d\lambda_i \otimes H \xrightarrow{d\lambda_i \otimes \omega \mapsto \omega} H.$$

Let σ' be another Frobenius on $A^\dagger$. Thanks to the natural equivalence F-MIC$(A_K^\dagger, \sigma) \cong F$-MIC$(A_K^\dagger, \sigma')$ one can associate $(H, \nabla, \Phi') \in F$-MIC$(A_K^\dagger, \sigma')$ to an object $(H, \nabla, \Phi) \in F$-MIC$(A_K^\dagger, \sigma)$. Then, for $x \in H$ we have

$$\Phi'(x) - \Phi(x) = \sum_{k=1}^{\infty} \frac{(\lambda_i^{\sigma'} - \lambda_i^\sigma)^k}{k!} \Phi \partial^k x.$$

Proof. See [EK, 6.1] or [Ke, 17.3.1]. □

Lemma 4.12 *Let $\{\omega_i, \eta_i\}$ be a $W[[\lambda_i]]$-basis of H_i such that $\omega_i \in \mathrm{Fil}^1 H_i$. Assume that $\lambda_i^\sigma \in pW[[\lambda_i]] + \lambda_i W[[\lambda_i]]$ so that it naturally induces a p-th Frobenius on $W[[\lambda_i]]$. Let $\varepsilon_1^{(\sigma)}(\lambda_i)$ and $\varepsilon_2^{(\sigma)}(\lambda_i)$ be defined by* (4.14)*. Put*

$$e_n := \max\{i \in \mathbb{Z}_{\geq 0} \mid \mathrm{ord}_p(p^i/i!) < n\}^6. \qquad (4.15)$$

[6] Notice that $e_n < \infty$ as we assumed $p > 2$.

Put $l_i := \min\{\mathrm{ord}_{\lambda_i}(\lambda_i^\sigma - \lambda_i^p), p\} \in \mathbb{Z}_{\geq 0}$*, where* $\mathrm{ord}_{\lambda_i} : K((\lambda_i)) \to \mathbb{Z}$ *is the order function with respect to* λ_i*. Then*

$$\varepsilon_1^{(\sigma)}(\lambda_i),\, \varepsilon_2^{(\sigma)}(\lambda_i) \in \lambda_i^{(l_i-p)e_{n+2}} W[[\lambda_i]] + p^n W[[\lambda_i, \lambda_i^{-1}]]$$

for any $n \geq 1$*.*

Proof. Let σ_0 be the p-th Frobenius on $W[[\lambda_i]]$ such that $\sigma_0(\lambda_i) = \lambda_i^p$, and Φ_0 the Frobenius with respect to σ_0. By Lemma 4.11, we have

$$\Phi(e_\xi) - \Phi_0(e_\xi) = \sum_{k=1}^{\infty} \frac{(\lambda_i^\sigma - \lambda_i^p)^k}{k!} \Phi_0 \partial^k e_\xi. \tag{4.16}$$

Write the right hand side by $G_1(\lambda_i)\omega_i + G_2(\lambda_i)\eta_i$, so that we have

$$\varepsilon_1^{(\sigma_0)}(\lambda_i) - \varepsilon_1^{(\sigma)}(\lambda_i) = G_1(\lambda_i), \quad \varepsilon_2^{(\sigma_0)}(\lambda_i) - \varepsilon_2^{(\sigma)}(\lambda_i) = G_2(\lambda_i).$$

It follows from Lemma 4.10 that $\varepsilon_1^{(\sigma_0)}(\lambda_i)$ and $\varepsilon_2^{(\sigma_0)}(\lambda_i)$ belong to $W[[\lambda_i]]$. Hence it is enough to show

$$G_1(\lambda_i),\, G_2(\lambda_i) \in \lambda_i^{(l_i-p)e_n} W[[\lambda_i]] + p^n W[[\lambda_i, \lambda_i^{-1}]]. \tag{4.17}$$

The differential operator ∂ is induced from the connection

$$\nabla : H^1(Y_i, \Omega^\bullet_{Y_i/W[[\lambda_i]]}(\log D_i)) \longrightarrow \frac{d\lambda_i}{\lambda_i} \otimes H^1(Y_i, \Omega^\bullet_{Y_i/W[[\lambda_i]]}(\log D_i)).$$

Moreover since $\nabla e_\xi = \mathrm{dlog}(\xi) \in \frac{d\lambda_i}{\lambda_i} \otimes \mathit{\Gamma}(Y_i, \Omega^1_{Y_i/W[[\lambda_i]]}(\log D_i))$, we have

$$\partial e_\xi \in \lambda_i^{-1} W[[\lambda_i]]\omega_i, \quad \partial^k e_\xi \in \lambda_i^{-k}(W[[\lambda_i]]\omega_i + W[[\lambda_i]]\eta_i)$$

for all $k \geq 2$. Since Φ_0 acts on the cohomology group $H^1(Y_i, \Omega^\bullet_{Y_i/W[[\lambda_i]]}(\log D_i))$, there are $u_k(\lambda_i), v_k(\lambda_i) \in W[[\lambda_i]]$ such that

$$\Phi_0 \partial^k e_\xi = \lambda_i^{-kp} p^{-2}(u_k(\lambda_i)\omega_i + v_k(\lambda_i)\eta_i) \tag{4.18}$$

for each $k \in \mathbb{Z}_{\geq 1}$, where "$p^{-2}$" comes from the Tate twist $H^1(X/A) \otimes A(2)$. Notice that since $\Phi_0\omega_i \equiv 0$ modulo p, one has

$$\Phi_0 \partial e_\xi \in \lambda_i^{-p} p^{-2} W[[\lambda_i]] \Phi_0 \omega_i \subset \lambda_i^{-p} p^{-1}(W[[\lambda_i]]\omega_i + W[[\lambda_i]]\eta_i)$$

and hence

$$u_1(\lambda_i),\, v_1(\lambda_i) \in pW[[\lambda_i]].$$

Applying (4.18) to (4.16), we have

$$\begin{aligned} G_1(\lambda_i)\omega_i + G_2(\lambda_i)\eta_i &= \sum_{k=1}^{\infty} \frac{(\lambda_i^\sigma - \lambda_i^p)^k}{k!} \lambda_i^{-kp} p^{-2}(u_k(\lambda_i)\omega_i + v_k(\lambda_i)\eta_i) \\ &= \sum_{k=1}^{\infty} \frac{p^{k-2}}{k!} \left(\frac{p^{-1}(\lambda_i^\sigma - \lambda_i^p)}{\lambda_i^p} \right)^k (u_k(\lambda_i)\omega_i + v_k(\lambda_i)\eta_i). \end{aligned}$$

Now (4.17) is immediate from this on noticing $p^{-1}(\lambda_i^\sigma - \lambda_i^p)/\lambda_i^p \in \lambda_i^{l_i-p} W[[\lambda_i]]$. □

Theorem 4.13 *Suppose that $C = \mathbb{P}^1_W$. Let λ be an inhomogeneous coordinate of $\mathbb{P}^1$ such that f has a split multiplicative reduction at $\lambda = 0$ as in* **A3**. *Let $\{a_0, a_1, \ldots, a_m\} \subset W$ be the subset such that $D_i := f^{-1}(a_i)$ is a singular fiber, and set $a_0 = 0$. Put $D_\infty := f^{-1}(\infty)$ (possibly a smooth fiber). Put $\lambda_i := \lambda - a_i$ for $i \in \{0, 1, \ldots, m\}$ and $\lambda_\infty := \lambda^{-1}$. Suppose further that the following condition is satisfied.*

B3 *There are $\omega \in \Gamma(\Omega^1_{X/A})$ and $\eta \in H^1_{\mathrm{dR}}(X/A)$ such that for each $i \in \{0, 1, \ldots, m, \infty\}$, $\{\lambda_i^{a_i}\omega, \lambda_i^{b_i}\eta\}$ is a $W[[\lambda_i]]$-basis of*

$$H_i = \mathrm{Im}\left[H^1(Y_i, \Omega^\bullet_{Y_i/W[[\lambda_i]]}(\log D_i)) \to H^1_{\mathrm{dR}}(X_i/W((\lambda_i)))\right]$$

for some $a_i, b_i \in \mathbb{Z}$.

Let σ be a p-th Frobenius on $A^\dagger$ such that $\lambda^\sigma = c\lambda^p$ with $c \in 1 + pW$. Define $\varepsilon_k^{(\sigma)}(\lambda) \in A_K^\dagger$ by

$$e_\xi - \Phi(e_\xi) = \varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta.$$

Put $l_i := \min\{\mathrm{ord}_{\lambda_i}(\lambda_i^\sigma - \lambda_i^p), p\}$. Let e_n be as in (4.15)*, and*

$$d_{1,n} := \sum_{i=0}^{m}(-a_i + (p - l_i)e_{n+2}) - a_\infty, \quad d_{2,n} := \sum_{i=0}^{m}(-b_i + (p - l_i)e_{n+2}) - b_\infty. \quad (4.19)$$

Then $\varepsilon_k^{(\sigma)}(\lambda) \in A^\dagger$ and there are polynomials $f_{k,n}(\lambda)$ of degree $\leq d_{k,n}$ such that

$$\varepsilon_1^{(\sigma)}(\lambda) \equiv \frac{f_{1,n}(\lambda)}{\prod_{i=0}^m \lambda_i^{-a_i+(p-l_i)e_{n+2}}} \mod p^n A^\dagger, \quad (4.20)$$

$$\varepsilon_2^{(\sigma)}(\lambda) \equiv \frac{f_{2,n}(\lambda)}{\prod_{i=0}^m \lambda_i^{-b_i+(p-l_i)e_{n+2}}} \mod p^n A^\dagger. \quad (4.21)$$

Proof. Put $\omega_i := \lambda_i^{a_i}\omega$ and $\eta_i := \lambda_i^{b_i}\eta$. Then

$$\varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta = \lambda_i^{-a_i}\varepsilon_1^{(\sigma)}(\lambda)\omega_i + \lambda_i^{-b_i}\varepsilon_2^{(\sigma)}(\lambda)\eta_i$$

for $i \in \{0, 1, \ldots, m, \infty\}$. Now the assertion is immediate from Lemma 4.12. □

In a similar way to the above, we can obtain an estimate of the p-adic expansion of the Frobenius matrix on $H^1_{\mathrm{rig}}(X_{\mathbb{F}}/S_{\mathbb{F}})$.

Theorem 4.14 *Let the notation and assumption be as in Theorem 4.13. Let*

$$\begin{pmatrix}\Phi(\omega) & \Phi(\eta)\end{pmatrix} = \begin{pmatrix}\omega & \eta\end{pmatrix}\begin{pmatrix}pF_{11} & F_{12} \\ pF_{21} & F_{22}\end{pmatrix}.$$

Let e_n *be as in* (4.15)*, and put*

$$d_{11,n} := \sum_{i=0}^{m}(pa_i - a_i + (p-l_i)e_n) + pa_\infty - a_\infty,$$

$$d_{21,n} := \sum_{i=0}^{m}(pa_i - b_i + (p-l_i)e_n) + pa_\infty - b_\infty,$$

$$d_{12,n} := \sum_{i=0}^{m}(pb_i - a_i + (p-l_i)e_n) + pb_\infty - a_\infty,$$

$$d_{22,n} := \sum_{i=0}^{m}(pb_i - b_i + (p-l_i)e_n) + pb_\infty - b_\infty.$$

Then there are polynomials $f_{ij,n}(\lambda) \in W[\lambda]$ *of degree* $\leq d_{ij,n}$ *such that*

$$pF_{11} \equiv \left(\prod_{i=0}^{m}(\lambda_i^\sigma)^{a_i}\lambda_i^{-a_i+(p-l_i)e_n}\right)^{-1} f_{11,n}(\lambda) \mod p^n A^\dagger, \tag{4.22}$$

$$pF_{21} \equiv \left(\prod_{i=0}^{m}(\lambda_i^\sigma)^{a_i}\lambda_i^{-b_i+(p-l_i)e_n}\right)^{-1} f_{21,n}(\lambda) \mod p^n A^\dagger, \tag{4.23}$$

$$F_{12} \equiv \left(\prod_{i=0}^{m}(\lambda_i^\sigma)^{b_i}\lambda_i^{-a_i+(p-l_i)e_n}\right)^{-1} f_{12,n}(\lambda) \mod p^n A^\dagger, \tag{4.24}$$

$$F_{22} \equiv \left(\prod_{i=0}^{m}(\lambda_i^\sigma)^{b_i}\lambda_i^{-b_i+(p-l_i)e_n}\right)^{-1} f_{22,n}(\lambda) \mod p^n A^\dagger. \tag{4.25}$$

Proof. Put $\omega_i := \lambda_i^{a_i}\omega$ and $\eta_i := \lambda_i^{b_i}\eta$ for $i \in \{0, 1, \ldots, m, \infty\}$. Let $F_{ab}^{(i)}$ be defined by

$$pF_{11}^{(i)}\omega_i + pF_{21}^{(i)}\eta_i := \Phi(\omega_i) = (\lambda_i^\sigma)^{a_i}\lambda_i^{-a_i}pF_{11}\cdot\omega_i + (\lambda_i^\sigma)^{a_i}\lambda_i^{-b_i}pF_{21}\cdot\eta_i,$$

$$F_{12}^{(i)}\omega_i + F_{22}^{(i)}\eta_i := \Phi(\eta_i) = (\lambda_i^\sigma)^{b_i}\lambda_i^{-a_i}F_{12}\cdot\omega_i + (\lambda_i^\sigma)^{b_i}\lambda_i^{-b_i}F_{22}\cdot\eta_i.$$

Let $\sigma_0(\lambda_i) = \lambda_i^p$, and Φ_0 the associated Frobenius on $H^1_{\mathrm{rig}}(X_0/S_0)$. Since Φ_0 acts on $H^1(Y_i, \Omega^\bullet_{Y_i/W[[\lambda_i]]}(\log D_i))$, we have the "integrality"

$$\Phi_0(\omega_i),\ \Phi_0(\eta_i) \in W[[\lambda_i]]\omega_i + W[[\lambda_i]]\eta_i. \tag{4.26}$$

It follows from Lemma 4.11 (transformation of Frobenius) that there are $u_k(\lambda_i), v_k(\lambda_i) \in W[[\lambda_i]]$ such that

$$\Phi(\omega_i) - \Phi_0(\omega_i) = \sum_{k=1}^{\infty}\frac{(\lambda_i^\sigma - \lambda_i^p)^k}{k!}\lambda_i^{-kp}(u_k(\lambda_i)\omega_i + v_k(\lambda_i)\eta_i) \tag{4.27}$$

$$= \sum_{k=1}^{\infty}\frac{p^k}{k!}\left(\frac{l(\lambda_i)}{\lambda_i^p}\right)^k(u_k(\lambda_i)\omega_i + v_k(\lambda_i)\eta_i) \tag{4.28}$$

as $\partial^k\omega_i \in \lambda_i^{-k}W[[\lambda_i]]\omega_i + \lambda_i^{-k}W[[\lambda_i]]\eta_i$, where $l(\lambda_i) := p^{-1}(\lambda_i^\sigma - \lambda_i^p) \in W[\lambda_i]$. By (4.26) and (4.28), we have "estimates"

$$pF_{11}^{(i)},\, pF_{21}^{(i)} \in \lambda_i^{(l_i-p)e_n}W[[\lambda_i]] + p^nW[[\lambda_i, \lambda_i^{-1}]]$$

or equivalently

$$pF_{11} \in \frac{1}{(\lambda_i^\sigma)^{a_i}\lambda_i^{-a_i+(p-l_i)e_n}}W[[\lambda_i]] + p^nW[[\lambda_i, \lambda_i^{-1}]]$$
$$pF_{21} \in \frac{1}{(\lambda_i^\sigma)^{a_i}\lambda_i^{-b_i+(p-l_i)e_n}}W[[\lambda_i]] + p^nW[[\lambda_i, \lambda_i^{-1}]]$$

for all $i \in \{0, 1, \ldots, m, \infty\}$. Now (4.22) and (4.23) follows from this. In the same way, we have

$$F_{12} \in \frac{1}{(\lambda_i^\sigma)^{b_i}\lambda_i^{-a_i+(p-l_i)e_n}}W[[\lambda_i]] + p^nW[[\lambda_i, \lambda_i^{-1}]]$$
$$F_{22} \in \frac{1}{(\lambda_i^\sigma)^{b_i}\lambda_i^{-b_i+(p-l_i)e_n}}W[[\lambda_i]] + p^nW[[\lambda_i, \lambda_i^{-1}]]$$

for all $i \in \{0, 1, \ldots, m, \infty\}$, and hence (4.24) and (4.25) follow. This completes the proof. □

5 Algorithm for computing p-adic regulators

In this section, we demonstrate how to compute the p-adic regulators of K_2 of elliptic curves. The key is Theorem 4.13. In what follows, $p \geq 3$ and $W = W(\mathbb{F}_{p^r})$ is the Witt ring of the finite field $\mathbb{F}_{p^r}$ with p^r elements. Put $K := \mathrm{Frac}(W)$.

5.1 Computing Syntomic Regulators

For $f(\lambda) \in K((\lambda))$, let $f'(\lambda) = \frac{df(\lambda)}{d\lambda}$ denote the derivative.

(Setting). Let λ be an inhomogeneous coordinate of $\mathbb{P}^1$. Let $g_2,\, g_3 \in W[\lambda]$ satisfy that $g_2g_3 \not\equiv 0 \bmod (p, \lambda)$. Let $f : Y \to \mathbb{P}^1$ be an elliptic fibration whose generic fiber is defined by an affine equation

$$y^2 = 4x^3 - g_2x - g_3.$$

We suppose that f satisfies **A3** in §4.1 at $\lambda = 0$. Put $n := \mathrm{ord}_\lambda(\Delta) > 0$, where $\Delta := g_2^3 - 27g_3^2$. We further assume that f satisfies the conditions **A1** and **A2** in §4.1, and the multiplicity of each component of an arbitrary singular fiber is prime to p. Then the conditions **B2** and **B3** in §4.6 are satisfied by Proposition 7.7. Put

$$\omega = \frac{dx}{y}, \quad \eta = \frac{xdx}{y}.$$

Let $U \subset Y$ be an affine open set that satisfies **B1** in §4.5. Let $\{a_0, a_1, \ldots, a_m\} \subset W$ with $a_0 = 0$ be the subset such that $D_i := f^{-1}(a_i)$ is a singular fiber ($f^{-1}(\infty)$ may be a singular fiber). Put $S := \operatorname{Spec} W[\lambda, \prod_i(\lambda - a_i)^{-1}]$ and $X := f^{-1}(S)$.

Let $\xi \in K_2^M(\mathscr{O}(U))$. Let σ be a p-th Frobenius on $W[\lambda, \prod(\lambda - a_i)^{-1}]^\dagger$ given by

$$\sigma(\lambda) = c\lambda^p, \quad c \in 1 + pW. \tag{5.1}$$

Let

$$e_\xi - \Phi(e_\xi) = \varepsilon_1^{(\sigma)}(\lambda)\omega + \varepsilon_2^{(\sigma)}(\lambda)\eta.$$

Let $s \geq 1$ be an integer, and $a \in W$ an element such that $a \not\equiv a_i \bmod p$ for all i. Our goal is to compute

$$\varepsilon_1^{(\sigma)}(a),\, \varepsilon_2^{(\sigma)}(a) \mod p^sW \tag{5.2}$$

explicitly. In case $\sigma(\lambda) = a^{F-p}\lambda^p$, this gives the syntomic regulator

$$\operatorname{reg}_{\mathrm{syn}}(\xi|_{X_a}) = \varepsilon_1^{(\sigma)}(a)\omega + \varepsilon_2^{(\sigma)}(a)\eta \in H^2_{\mathrm{syn}}(X_a, \mathbb{Q}_p(2)) \cong H^1_{\mathrm{dR}}(X_a/K)$$

for the fiber $X_a := f^{-1}(a)$ (see Theorem 4.7).

(i) Put $\lambda_i := \lambda - a_i$ for $i \in \{0, 1, \ldots, m\}$ and $\lambda_\infty := \lambda^{-1}$. For $i \in \{0, 1, \ldots, m\}$, put $l_i := \min\{\operatorname{ord}_{\lambda_i}(\lambda_i^\sigma - \lambda_i^p), p\}$. It follows from Proposition 7.7 that for each $i \in \{0, 1, \ldots, m, \infty\}$, there are integers a_i, b_i such that

$$\lambda_i^{a_i}\omega, \quad \lambda_i^{b_i}\eta$$

is a free $W[[\lambda_i]]$-basis of H_i, where H_i is as in Theorem 4.13, **B2**.

(ii) (cf. (4.1)). Put $J := g_2^3/\Delta$. The j-invariant is $1728J$. Put

$$\kappa := \left.\frac{\lambda^{-n}}{1728J}\right|_{\lambda=0}, \quad \text{where } n := -\operatorname{ord}_\lambda(J),$$

which belongs to $W^\times$ by **A3**.

(iii) (cf. Proposition 7.2). We fix a square root

$$c_0 := \sqrt{-\frac{g_2(0)}{18g_3(0)}} \in W^\times.$$

Here we note that $g_2(0)g_3(0) \not\equiv 0 \bmod p$ by the assumption.

(iv) (cf. (4.5) and Proposition 7.2, (7.3)). Let $(12g_2)^{-1/4} \in W[[\lambda]]$ be the power series with the constant term c_0. Put

$$F(\lambda) := (12g_2)^{-\frac{1}{4}}{}_2F_1\left({\frac{1}{12}, \frac{5}{12} \atop 1}; J^{-1}\right).$$

(v) (cf. (4.2) and Proposition 7.4). Let $\tau(\lambda) \in \lambda K[[\lambda]]$ be the power series determined by

$$\frac{d\tau(\lambda)}{d\lambda} = \frac{3(2g_2g_3' - 3g_2'g_3)}{2\Delta F(\lambda)^2} - \frac{n}{\lambda}.$$

By Proposition 7.4, $q := \kappa\lambda^n \exp(\tau(\lambda))$ is the multiplicative period of the Tate curve $\mathscr{E} = X \times_A \operatorname{Spec} W((\lambda))$.

(vi) (cf. (4.4)). Put

$$\tau^{(\sigma)}(\lambda) := \log^{(\sigma)}(q) = \log^{(F)}(\kappa) - np^{-1}\log(c) + \tau(\lambda) - p^{-1}\tau(\lambda)^{\sigma}.$$

(vii) (cf. §4.5). For $\xi \in K_2(X)$, let $g_\xi(\lambda) \in A$ be defined by $\operatorname{dlog}\xi = g_\xi(\lambda)\frac{d\lambda}{\lambda} \wedge \frac{dx}{y}$. Put

$$G_\xi(\lambda) := g_\xi(\lambda)F(\lambda) = l + a_1\lambda + a_2\lambda^2 + \cdots \in W[[\lambda]].$$

Note that l is an integer.

(viii) (cf. Proposition 7.3). Put

$$H(\lambda) := \frac{2\Delta}{3(2g_2g_3' - 3g_2'g_3)}\left(F'(\lambda) + \frac{\Delta'}{12\Delta}F(\lambda)\right).$$

Step 1 (Computing $E_i^{(\sigma)}(\lambda)$). According to Theorem 4.8 (4.11), (4.12), let $E_i^{(\sigma)}(\lambda) \in W[[\lambda]]$ be defined by

$$\frac{d}{d\lambda}E_1^{(\sigma)}(\lambda) = \frac{G_\xi(\lambda)}{\lambda} - \frac{G_\xi(\lambda)^\sigma}{\lambda}, \quad E_1^{(\sigma)}(0) := \frac{l}{n}\log^{(F)}(\kappa) - \frac{l}{p}\log(c), \tag{5.3}$$

and

$$\begin{aligned} \frac{d}{d\lambda}E_2^{(\sigma)}(\lambda) &= -E_1^{(\sigma)}(\lambda)\frac{q'}{q} + \frac{G_\xi(\lambda)^\sigma}{\lambda}\tau^{(\sigma)}(\lambda) \\ &= -E_1^{(\sigma)}(\lambda)\left(\frac{n}{\lambda} + \tau'(\lambda)\right) + \frac{G_\xi(\lambda)^\sigma}{\lambda}\tau^{(\sigma)}(\lambda). \end{aligned} \tag{5.4}$$

Compute the series expansions

$$\begin{aligned} F(\lambda) &= f_0 + f_1\lambda + f_2\lambda^2 + \cdots + f_N\lambda^N + O(\lambda^{N+1}), \\ H(\lambda) &= h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_N\lambda^N + O(\lambda^{N+1}), \\ E_1^{(\sigma)}(\lambda) &= A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_N\lambda^N + O(\lambda^{N+1}), \\ E_2^{(\sigma)}(\lambda) &= B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_N\lambda^N + O(\lambda^{N+1}) \end{aligned}$$

except B_0. The constant B_0 shall be computed in the next step. Here we take N to be a large integer such that the Step 2 and 3 work. See Remark 5.1 (5.8) for an explicit lower bound of N.

Step 2 (Computing $E_2^{(\sigma)}(0)$). Let

$$\varepsilon_1^{(\sigma)}(\lambda) := \frac{E_1^{(\sigma)}(\lambda)}{F(\lambda)} - H(\lambda)E_2^{(\sigma)}(\lambda), \quad \varepsilon_2^{(\sigma)}(\lambda) := F(\lambda)E_2^{(\sigma)}(\lambda) \tag{5.5}$$

(cf. (4.9), (4.10) and (7.4)). We compute the constant $E_2^{(\sigma)}(0) = B_0$ modulo p^s (without assuming Conjecture 4.9). We first note that B_0 is uniquely determined by the fact that $\varepsilon_2^{(\sigma)}(\lambda) = F(\lambda)E_2^{(\sigma)}(\lambda)$ is an overconvergent function (i.e. $F(\lambda)E_2^{(\sigma)}(\lambda) \in A_K^\dagger$). Indeed, if there is another constant $B_0^*(\neq B_0)$ such that $F(\lambda)(B_0^* + B_1\lambda + \cdots)$ is an overconvergent function, then $F(\lambda)E_2^{(\sigma)}(\lambda) - F(\lambda)(B_0^* + B_1\lambda + \cdots) = (B_0 - B_0^*)F(\lambda)$ is also overconvergent. This is impossible. Indeed, let $\zeta \in W^\times$ be a root of unity such that $X_{\zeta,\mathbb{F}_{p^r}} := X \times_S \operatorname{Spec}\mathbb{F}_{p^r}[t]/(t-\zeta)$ is an ordinary elliptic curve. If $F(\lambda)$ were overconvergent, then the special value of $F(\lambda)$ at $\lambda = \zeta$ makes sense. Then one has

$$\gamma_\zeta := \left.\frac{F(\lambda)}{F(\lambda^{p^r})}\right|_{\lambda=\zeta} = \frac{F(\zeta)}{F(\zeta^{p^r})} = \frac{F(\zeta)}{F(\zeta)} = 1.$$

This contradicts with the fact that γ_ζ is the unit root of the p^r-th Frobenius of $X_{\zeta,\mathbb{F}_{p^r}}$ (Corollary 4.6). In a practical way, one can use Theorem 4.13 to compute B_0. Namely letting $e_s := \max\{i \in \mathbb{Z}_{\geq 0} \mid \operatorname{ord}_p(p^i/i!) < s\}$ be as in (4.15) and $d_{2,s} := \sum_{i=0}^m(-b_i + (p-l_i)e_{s+2}) - b_\infty$ as in (4.19), one finds the unique

$$B_0 \bmod p^s W$$

satisfying that for a large $s' \geq s$,

$$\varepsilon_2^{(\sigma)}(\lambda) \equiv \frac{\text{polynomial of degree } \leq d_{2,s'}}{\prod_{i=0}^m(\lambda - a_i)^{-b_i+(p-l_i)e_{s'+2}}} \mod p^{s'}A^\dagger$$

or equivalently, the power series

$$F(\lambda)(B_0 + B_1\lambda + \cdots)\prod_{i=0}^m(\lambda - a_i)^{-b_i+(p-l_i)e_{s'+2}} \in (W/p^{s'}W)[[\lambda]]$$

modulo $p^{s'}$ terminates at least at the degree $d_{2,s'}$. See Remark 5.1 (5.7) for an explicit lower bound of s'.

Step 3 (Computing $\varepsilon_k^{(\sigma)}(\lambda)$). We have the series expansions of $E_k^{(\sigma)}(\lambda)$ by **Steps 1,2**, and hence

$$\varepsilon_1^{(\sigma)}(\lambda) = A_0' + A_1'\lambda + A_2'\lambda^2 + \cdots + A_N'\lambda^N + O(\lambda^{N+1}),$$
$$\varepsilon_2^{(\sigma)}(\lambda) = B_0' + B_1'\lambda + B_2'\lambda^2 + \cdots + B_N'\lambda^N + O(\lambda^{N+1})$$

modulo $p^sW[[\lambda]]$ by (5.5). We again apply Theorem 4.13. The power series

$$f_{1,s}(\lambda) := \varepsilon_1^{(\sigma)}(\lambda)\prod_{i=0}^m(\lambda - a_i)^{-a_i+(p-l_i)e_{s+2}}, \quad f_{2,s}(\lambda) := \varepsilon_2^{(\sigma)}(\lambda)\prod_{i=0}^m(\lambda - a_i)^{-b_i+(p-l_i)e_{s+2}}$$

modulo $p^sW[[\lambda]]$ terminate at least at the degree $d_{1,s}$ and $d_{2,s}$ respectively. They provide the descriptions of $\varepsilon_k^{(\sigma)}(\lambda)$ modulo p^s as rational functions,

$$\varepsilon_1^{(\sigma)}(\lambda) \equiv \frac{f_{1,s}(\lambda)}{\prod_{i=0}^m (\lambda - a_i)^{-a_i+(p-l_i)e_{s+2}}} \mod p^sW,$$

$$\varepsilon_2^{(\sigma)}(\lambda) \equiv \frac{f_{2,s}(\lambda)}{\prod_{i=0}^m (\lambda - a_i)^{-b_i+(p-l_i)e_{s+2}}} \mod p^sW.$$

The values $\varepsilon_k^{(\sigma)}(a)$ are obtained by evaluating the right hand side at $\lambda = a$.

Remark 5.1 *Suppose that there are two constants B_0, B_0^* such that $B_0 \not\equiv B_0^*$ mod p^s and*

$$F(\lambda)(B_0 + B_1\lambda + \cdots)\prod_{i=0}^m (\lambda - a_i)^{-b_i+(p-l_i)e_{s'+2}} \in (W/p^{s'}W)[[\lambda]]$$

and

$$F(\lambda)(B_0^* + B_1\lambda + \cdots)\prod_{i=0}^m (\lambda - a_i)^{-b_i+(p-l_i)e_{s'+2}} \in (W/p^{s'}W)[[\lambda]]$$

terminate at the degree $d_{2,s'}$. Since $\mathrm{ord}_p(B_0 - B_0^*) < s$,

$$F(\lambda)\prod_{i=0}^m (\lambda - a_i)^{-b_i+(p-l_i)e_{s'+2}} \in (W/p^{s'-s+1}W)[[\lambda]] \tag{5.6}$$

terminates at the degree $d_{2,s'}$. This implies that the unit root γ_ζ of the p^r-th Frobenius on $X_{\zeta,\mathbb{F}_{p^r}}$ satisfies

$$\gamma_\zeta = \left.\frac{F(\lambda)}{F(\lambda^{p^r})}\right|_{\lambda=\zeta} \equiv 1 \mod p^{s'-s+1}W.$$

Conversely letting

$$N_\zeta := \min\{i \in \mathbb{Z}_{\geq 1} \mid \gamma_\zeta \not\equiv 1 \mod p^iW\},$$

for an arbitrary $s' \geq N_\zeta + s - 1$, the power series (5.6) *cannot terminate at the degree $d_{2,s'}$. Therefore, the constant B_0 mod p^s is characterized by the condition that*

$$F(\lambda)(B_0 + B_1\lambda + \cdots)\prod_{i=0}^m (\lambda - a_i)^{-b_i+(p-l_i)e_{s'+2}} \in (W/p^{s'}W)[[\lambda]]$$

terminates at least at the degree $d_{2,s'}$ for all $s' \geq N_\zeta + s - 1$.

Summing up the above, we get a bound of N in **Step 1** *and a bound of s' in* **Step 2** *as follows. Choose $\zeta \in W$ such that $X_{\zeta,\mathbb{F}_{p^r}}$ is ordinary. Then take s' such that*

$$s' \geq N_\zeta + s - 1. \tag{5.7}$$

Let $F(\lambda)\prod_{i=0}^m(\lambda - a_i)^{-b_i+(p-l_i)e_{s'+2}} = \sum_{i=0}^\infty C_i\lambda^i$. Find an integer $M > d_{2,s'}$ such that

$$C_M \not\equiv 0 \mod p^{s'-s+1}W.$$

Then take N such that

$$N > \max\{d_{1,s}, d_{2,s}, M\}. \tag{5.8}$$

5.2 Computing the Frobenius matrix on $H^1_{\rm rig}(X_{\mathbb{F}}/S_{\mathbb{F}})$

We keep the situation and notation in §5.1. Let

$$\begin{pmatrix}\Phi(\omega) & \Phi(\eta)\end{pmatrix} = \begin{pmatrix}\omega & \eta\end{pmatrix}\begin{pmatrix}pF_{11}(\lambda) & F_{12}(\lambda)\\ pF_{21}(\lambda) & F_{22}(\lambda)\end{pmatrix}$$

be the matrix of the p-th Frobenius on $H^1_{\rm rig}(X_{\mathbb{F}}/S_{\mathbb{F}})$. It follows from Proposition 4.3 together with Proposition 7.3 (7.4) that one has

$$\begin{aligned}
F_{11}(\lambda) &= \frac{F(\lambda)^\sigma}{F(\lambda)}\left(1+\tau^{(\sigma)}(\lambda)H(\lambda)F(\lambda)\right),\\
F_{12}(\lambda) &= \frac{pH(\lambda)^\sigma}{F(\lambda)} - \frac{H(\lambda)}{F(\lambda)^\sigma} + p\tau^{(\sigma)}(\lambda)H(\lambda)H(\lambda)^\sigma,\\
F_{21}(\lambda) &= -\tau^{(\sigma)}(\lambda)F(\lambda)F(\lambda)^\sigma,\\
F_{22}(\lambda) &= \frac{F(\lambda)}{F(\lambda)^\sigma} - p\tau^{(\sigma)}(\lambda)H(\lambda)^\sigma F(\lambda).
\end{aligned}$$

One can apply Theorem 4.14 to compute

$$\begin{pmatrix}pF_{11}(a) & F_{12}(a)\\ pF_{21}(a) & F_{22}(a)\end{pmatrix} \mod p^s.$$

Namely compute the power series

$$f_{11,s}(\lambda) := pF_{11}(\lambda)\prod_{i=0}^m (\lambda_i^\sigma)^{a_i}\lambda_i^{-a_i+(p-l_i)e_s} \in (W/p^sW)[[\lambda]]$$

with coefficients in W/p^sW which terminates at least at the degree $d_{11,s}$, where we note $\lambda_i^\sigma = (\lambda - a_i)^\sigma = c\lambda^p - a_i^F$ (see (5.1)). Then one computes $F_{11}(a)$ by

$$pF_{11}(a) \equiv f_{11,s}(a)\cdot\left(\prod_{i=0}^m (ca^p - a_i^F)^{a_i}(a-a_i)^{-a_i+(p-l_i)e_s}\right)^{-1} \mod p^sW.$$

One can also compute pF_{12}, F_{21} and F_{22} in the same way.

6 Numerical Verifications of the p-adic Beilinson conjecture for K_2

We apply the algorithm in §5.1 to compute the syntomic regulators, and give numerical verifications of the p-adic Beilinson conjecture for K_2 (Conjecture 3.3, $n = 0$) of the following two examples

- $y^2 = x^3 - 2x^2 + (1-a)x$ with $a \in \mathbb{Q}\setminus\{0,1\}$,
- $y^2 = x(1-x)(1-(1-a)x)$ with $a \in \mathbb{Q}\setminus\{0,1\}$.

6.1 $y^2 = x^3 - 2x^2 + (1-\lambda)x$

We first discuss the elliptic fibration in [A2, Theorem 4.22]. Let $p \geq 5$ be a prime. By a simple replacement of variables, it is a fibration $f : Y \to \mathbb{P}^1$ defined by a Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2 := 4\lambda + \frac{4}{3},\ g_3 := \frac{8}{3}\lambda - \frac{8}{27}$$

over the base ring $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$. Put $S = \operatorname{Spec}\mathbb{Z}_{(p)}[\lambda, (\lambda - \lambda^2)^{-1}]$ and $X = f^{-1}(S)$. Put

$$\omega = \frac{dx}{y}, \quad \eta = \frac{xdx}{y}.$$

The functional j-invariant is $64(3\lambda+1)^3/(\lambda(1-\lambda)^2)$. The fibration f has a multiplicative reduction of Kodaira type I_1 at $\lambda = 0$ which is split over $\mathbb{Z}_{(p)}$, a multiplicative reduction of type I_2 at $\lambda = 1$, and an additive reduction of type III^* at $\lambda = \infty$. One can check that f satisfies all the conditions **A1**, **A2** and **A3** in §4.1 and the multiplicity of any component of an arbitrary singular fiber is at most 4. Hence **B2** and **B3** in §4.6 are also satisfied by Proposition 7.7. Write $X_{\mathbb{Z}_p} = X \times_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$, $X_{\mathbb{F}_p} = X \times_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ etc. Let σ be the p-th Frobenius on $\mathbb{Z}_p[\lambda, (\lambda - \lambda^2)^{-1}]^\dagger$ given by $\sigma(\lambda) = c\lambda^p$ with $c \in 1 + p\mathbb{Z}_p$. Let $Z \subset X$ be the union of sections of $\{3y = \pm(6x-2)\}$, $\{3x+2=0\}$, $\{3x-1=0\}$ and $\{x=\infty\}$. Put $U = X \setminus Z$. Moreover the condition **B1** in §4.5 holds by taking $\overline{S} = \overline{S}' = \mathbb{P}^1$ and $v : \overline{S}' \to \overline{S}$ given by $v(\lambda) = \lambda^4$ (note that the elliptic fibration $X \times_S S' \to S'$ has multiplicative reductions at $\lambda = 0$ and $\lambda^4 = 1$ and a good reduction at $\lambda = \infty$). Let

$$\xi := \left\{ \frac{3y+6x-2}{3y-6x+2}, \frac{-9\lambda(3x+2)}{(3x-1)^3} \right\} \in K_2^M(\mathscr{O}(U)) \tag{6.1}$$

be the symbol in [A2, Theorem 4.22] [7]. A direct calculation yields

$$\operatorname{dlog}(\xi) = 2\frac{d\lambda}{\lambda}\frac{dx}{y}. \tag{6.2}$$

Put

$$F_{\frac{1}{4},\frac{3}{4}}(\lambda) = {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{matrix} ; \lambda \right).$$

The notation in §5.1 in our case is explicitly given as follows.

(i) $(a_0, b_0) = (a_1, b_1) = (0,0)$ and $(a_\infty, b_\infty) = (0,1)$.

(ii) Since $1728J = 64(3\lambda+1)^3/(\lambda(1-\lambda)^2))$, one has $n = -\operatorname{ord}_\lambda(J) = 1$ and $\kappa = 1/64$.

(iii) One has $c_0^2 = -g_2(0)/(18g_3(0)) = 1/4$. We take $c_0 = 1/2$.

[7] The symbol in [A2, Theorem 4.22] is $\left\{ \frac{y-(x-1)}{y+(x-1)}, \frac{-tx}{(x-1)^3} \right\}$ where the Weierstrass equation is $y^2 = x^3 - 2x^2 + (1-t)x$.

(iv) One can show

$$F(\lambda) := (12g_2)^{-\frac{1}{4}}{}_2F_1\left(\begin{matrix}\frac{1}{12},\frac{5}{12}\\1\end{matrix};J^{-1}\right) = \frac{1}{2}F_{\frac{1}{4},\frac{3}{4}}(\lambda)$$

in the following way. Recall from the proof of Proposition 7.2 that $F(\lambda)$ is characterized by

$$F(\lambda) = \frac{1}{2\pi i}\int_{\delta_\lambda}\omega.$$

By Theorem 7.1 we have

$$\begin{pmatrix}\partial_\lambda(\omega) & \partial_\lambda(\eta)\end{pmatrix} = \begin{pmatrix}\omega & \eta\end{pmatrix}\begin{pmatrix}\frac{3\lambda-1}{12(\lambda-\lambda^2)} & -\frac{3\lambda+1}{36(\lambda-\lambda^2)}\\ \frac{1}{4(\lambda-\lambda^2)} & -\frac{3\lambda-1}{12(\lambda-\lambda^2)}\end{pmatrix},$$

where $\partial_\lambda := \nabla_{d/d\lambda}$. Hence one has

$$((\lambda-\lambda^2)\partial_\lambda^2 + (1-2\lambda)\partial_\lambda - \frac{3}{4})(\omega) = 0. \tag{6.3}$$

This implies that $F(\lambda)$ satisfies

$$(\lambda-\lambda^2)\frac{d^2}{d\lambda^2}F(\lambda) + (1-2\lambda)\frac{d}{d\lambda}F(\lambda) - \frac{3}{4}F(\lambda) = 0. \tag{6.4}$$

This is the hypergeometric differential equation (e.g. [NIST, 15.10]). Hence $F(\lambda) = cF_{\frac{1}{4},\frac{3}{4}}(\lambda)$ for some constant c. Since the constant term of $F(\lambda)$ is $c_0 = 1/2$ by definition, one has $c = 1/2$.

(v) The multiplicative period is $q = 64^{-1}\lambda\exp(\tau(\lambda))$ with

$$\lambda\frac{d}{d\lambda}\tau(\lambda) = -1 + (1-\lambda)^{-1}F_{\frac{1}{4},\frac{3}{4}}(\lambda)^{-2}, \quad \tau(0) = 0.$$

Explicitly,

$$\tau(\lambda) = -\frac{5}{8}\lambda - \frac{269}{1024}\lambda^2 - \frac{1939}{12288}\lambda^3 - \frac{922253}{8388608}\lambda^4 + \cdots.$$

(vi) $\tau^{(\sigma)}(\lambda) = -p^{-1}\log(64^{p-1}c) + \tau(\lambda) - p^{-1}\tau(\lambda)^\sigma$.

(vii) By (6.2), one has $g_\xi(\lambda) = 2$ and $G_\xi(\lambda) = 2F(\lambda) = F_{\frac{1}{4},\frac{3}{4}}(\lambda)$ (hence $l = 1$).

(viii)

$$\begin{aligned}H(\lambda) &= 2(\lambda-\lambda^2)F'_{\frac{1}{4},\frac{3}{4}}(\lambda) + \frac{1}{6}(1-3\lambda)F_{\frac{1}{4},\frac{3}{4}}(\lambda)\\ &= \frac{1}{6} - \frac{3}{32}\lambda - \frac{85}{2048}\lambda^2 - \frac{875}{32768}\lambda^3 - \frac{165165}{8388608}\lambda^4 + \cdots.\end{aligned}$$

(ix) Let $E_1^{(\sigma)}(\lambda) \in \mathbb{Z}_p[[\lambda]]$ be defined by

$$(E_1^{(\sigma)}(\lambda))' = \frac{G_\xi(\lambda)}{\lambda} - \frac{G_\xi(\lambda)^\sigma}{\lambda}, \quad E_1^{(\sigma)}(0) := -p^{-1}\log(64^{p-1}c).$$

Let $E_2^{(\sigma)}(\lambda) = C + a_1\lambda + \cdots$ be the power series satisfying

$$\begin{aligned}\frac{d}{d\lambda}E_2^{(\sigma)}(\lambda) &= -E_1^{(\sigma)}(\lambda)\left(\frac{1}{\lambda}+\tau'(\lambda)\right) + \frac{G_\xi(\lambda)^\sigma}{\lambda}\tau^{(\sigma)}(\lambda)\\ &= -\frac{E_1^{(\sigma)}(\lambda)}{\lambda}\left((1-\lambda)^{-1}F_{\frac{1}{4},\frac{3}{4}}(\lambda)^{-2}\right) + \frac{G_\xi(\lambda)^\sigma}{\lambda}\tau^{(\sigma)}(\lambda).\end{aligned}$$

This determines $E_2^{(\sigma)}(\lambda)$ except the constant term C. We expect

$$C = -\frac{1}{2}\left(p^{-1}\log(64^{p-1}c)\right)^2$$

according to Conjecture 4.9. The authors do not know a proof of this, while it does not matter toward numerical computation thanks to the method in **Step 2** in §5.1.

Let $a \in \mathbb{Z}_{(p)}$ such that $a \not\equiv 0, 1 \bmod p$, and X_a the fiber at $\lambda = a$. Let $\sigma_a(\lambda) = a^{1-p}\lambda^p$, and

$$\mathrm{reg}_{\mathrm{syn}}(\xi|_{\lambda=a}) = \varepsilon_1^{(\sigma_a)}(a)\frac{dx}{y} + \varepsilon_2^{(\sigma_a)}(a)\frac{xdx}{y}.$$

We use MACAULAY2 for computing the series expansions of $\varepsilon_i^{(\sigma_a)}(\lambda)$. The following code computes $\varepsilon_i^{(\sigma_a)}(\lambda) \bmod \lambda^{151}$ in case that $p = 5$, $a = 2$ ($c = 2^{1-p}$) under that the precision of $p^{-1}\log(64^{p-1}c)$ is mod p^{20}.

```
p=5
r=20
aa=2
c=1/aa^(p-1)
n=150

AA=QQ[t]/ideal(t^(n+1))
Pochhammer=(a,i)->product(1..i,j->(a+j-1))
tIntegral=f->sum(1..n,i->(f_(t^i))*t^i/i)
modps=(i,m)->(numerator(i/1)*(gcdCoefficients(m,denominator(i/1)))_2)%m
Log=c->-sum(1..(2*r),i->(1-c)^i/i)
HG=sum(0..n,i->Pochhammer(1/4,i)*Pochhammer(3/4,i)/i!^2*t^i)
F=1/2*HG
Gxi=HG
H=2*t*(1-t)*diff(t,HG)+1/6*(1-3*t)*HG
g5=(1-HG)^5
g25=g5^5
g125=g25^5
```

```
inv0=sum(0..4,i->(1-HG)^i)
inv1=sum(0..4,i->g5^i)
inv2=sum(0..4,i->g25^i)
inv3=sum(0..4,i->g125^i)
invHG=inv0*inv1*inv2*inv3
tau=tIntegral(-1+invHG^2*sum(0..n,i->t^i))
const=modps(-1/p*Log(64^(p-1)*c),p^r)
taus=const+tau-1/p*sub(tau,t=>c*t^p)
E1=const+tIntegral(Gxi-sub(Gxi,t=>c*t^p))
E2=-1/2*const^2+tIntegral(-E1*(1+t*diff(t,tau))+sub(Gxi,t=>c*t^p)*taus)
ve1=2*invHG*E1-H*E2
ve2=F*E2
```

If $a = 2, 4, 8$, then the symbol ξ is integral. By Theorem 4.13 (see also §5.1 **Step 3**), we can compute the values $\varepsilon_i^{(\sigma_a)}(a)$. Here we attach the code of MACAULAY2 for computing the values modulo p^{15} which is a sequel of the above code[8].

```
max select(2..50,i->(i-sum(1..50,k->floor(i/p^k)))<15+2)
d=(p-1)*oo
e1=modps(sub(ve1*(1-t)^d/(1-aa)^d,t=>aa),p^15)
e2=modps(sub(ve2*(1-t)^d/(1-aa)^d,t=>aa),p^15)
```

$a = 2$

p	$\varepsilon_1^{(\sigma_a)}(a)$	$\varepsilon_2^{(\sigma_a)}(a)$
5	$19674787784(p^{15})$	$14084682764(p^{15})$
7	$2590802242826(p^{15})$	$1117519637850(p^{15})$
11	$7965734565(p^{10})$	$13678445088(p^{10})$
13	$11926785562(p^{10})$	$72813360585(p^{10})$
17	$278698244(p^{8})$	$3681687094(p^{8})$

"(p^s)" means "mod p^s".

$a = 4$

p	$\varepsilon_1^{(\sigma_a)}(a)$	$\varepsilon_2^{(\sigma_a)}(a)$
5	$10899377190(p^{15})$	$18907289609(p^{15})$
7	$1079749670299(p^{15})$	$3800305332560(p^{15})$
11	$1981716464(p^{10})$	$23499057595(p^{10})$
13	$99992661864(p^{10})$	$54272943535(p^{10})$
17	$5624041300(p^{8})$	$6557787442(p^{8})$

$a = 8$

p	$\varepsilon_1^{(\sigma_a)}(a)$	$\varepsilon_2^{(\sigma_a)}(a)$
5	$25874535623(p^{15})$	$28249557885(p^{15})$
7	bad reduction	
11	$11113392970(p^{10})$	$10168663913(p^{10})$
13	$50210889322(p^{10})$	$1878713357(p^{10})$
17	$4803397460(p^{8})$	$6578109195(p^{8})$

[8] For computing the values with precision p^s, we need to take n to be greater than $d_{1,s}$ and $d_{2,s}$ in (4.19), and r to be greater than $s + \max\{\mathrm{ord}_p(i) \mid 0 < i \leq n\}$ in the code. Approximately, $n \sim (s+2)(p-1)^2/(p-2)$ and $r \sim s + \log_p(n)$, so the bigger sp is, the computational complexity will increase.

Next we compute the p-adic regulators

$$R_{p,\gamma}(X_a,\xi) = R_{p,\gamma}(h^1(X_a),\xi) = (1-p\gamma^{-1})\frac{\mathrm{Tr}(\mathrm{reg}_{\mathrm{syn}}(\xi|_{\lambda=a}) \cup v_\gamma)}{\mathrm{Tr}(\omega_{X_a} \cup v_\gamma)}$$

introduced in §3.4, where γ is a unit or non-unit root of X_a, $v_\gamma \in H^1_{\mathrm{dR}}(X_a/\mathbb{Q}_p) \otimes \overline{\mathbb{Q}}_p$ is the eigenvector of $\Phi \otimes 1_{\overline{\mathbb{Q}}_p}$ with eigenvalue γ and ω_{X_a} is the Néron differential. To do this, we first compute the representation matrix

$$\begin{pmatrix}\Phi(\omega) & \Phi(\eta)\end{pmatrix} = \begin{pmatrix}\omega & \eta\end{pmatrix}\begin{pmatrix}pF_{11}(\lambda) & F_{12}(\lambda)\\ pF_{21}(\lambda) & F_{22}(\lambda)\end{pmatrix}\Bigg|_{\lambda=a}$$

of the p-th Frobenius on $H_{\mathrm{crys}}(X_{a,\mathbb{F}_p}/\mathbb{Z}_p)$. The code of MACAULAY2 for computing the power series F_{ij} is as follows,

```
Hp=sub(H,t=>c*t^p)
F11=sub(HG,t=>c*t^p)*invHG*(1+1/2*taus*H*HG)
F12=2*p*Hp*invHG-2*H*sub(invHG,t=>c*t^p)+p*taus*H*Hp
F21=-1/4*taus*HG*sub(HG,t=>c*t^p)
F22=HG*sub(invHG,t=>c*t^p)-1/2*p*taus*Hp*HG
```

Let us see the case $p=5$ and $a=4$ (the other cases are done in the same way). Let α be the unit root, and $\beta = p\alpha^{-1}$ the non-unit root. By counting the $\mathbb{F}_5$-rational points of $X_{a,\mathbb{F}_5}$, one has that the trace of the Frobenius is -2, and hence $\alpha = -1+2\sqrt{-1} \equiv 22219310363 \mod 5^{15}$. Similarly to the computations of $\varepsilon_i^{(\sigma_a)}(a)$, it follows from Theorem 4.14 (see also §5.2) that one can compute the matrix

$$\begin{pmatrix}pF_{11}(a) & F_{12}(a)\\ pF_{21}(a) & F_{22}(a)\end{pmatrix} \equiv \begin{pmatrix}30465757535 & 3584355845\\ 11560944585 & 51820588\end{pmatrix} \mod 5^{15}. \tag{6.5}$$

Hence the eigenvector v_α satisfies

$$v_\alpha \equiv 710786365\omega + \eta \mod 5^{14},$$

and the eigenvector v_β for the non-unit root β satisfies

$$v_\beta \equiv \omega + 28450379180\eta \mod 5^{15}.$$

In cases $a = 2,4,8$, it follows by computing the minimal Weierstrass equation (e.g. MAGMA [Magma]) that one has $\omega_{X_a} = \omega$. We thus have

$$\begin{aligned}\frac{\mathrm{Tr}(\mathrm{reg}_{\mathrm{syn}}(\xi|_{\lambda=a}) \cup v_\alpha)}{\mathrm{Tr}(\omega_{X_a} \cup v_\alpha)} &\equiv (\varepsilon_1^{(\sigma_a)}(a) - 710786365\varepsilon_2^{(\sigma_a)}(a))\frac{\mathrm{Tr}(\omega \cup \eta)}{\mathrm{Tr}(\omega_{X_a} \cup \eta)}\\ &\equiv 3956230280 \mod 5^{14},\end{aligned}$$

$$\frac{\mathrm{Tr}(\mathrm{reg}_{\mathrm{syn}}(\xi|_{\lambda=a})\cup v_\beta)}{\mathrm{Tr}(\omega_{X_a}\cup v_\beta)} \equiv \frac{28450379180\varepsilon_1^{(\sigma_a)}(a)-\varepsilon_2^{(\sigma_a)}(a)}{28450379180}\frac{\mathrm{Tr}(\omega\cup\eta)}{\mathrm{Tr}(\omega_{X_a}\cup\eta)}$$
$$\equiv 5^{-1}\cdot 5576309542 \mod 5^{13},$$

where $a \equiv b$ mod p^s for $a, b \in \mathbb{Q}_p$ means that $a - b \in p^s\mathbb{Z}_p$. Hence

$$R_{5,\alpha}(X_a,\xi) \equiv 5110910605 \mod 5^{14},$$
$$R_{5,5\alpha^{-1}}(X_a,\xi) \equiv 5^{-1}\cdot 2495229428 \mod 5^{13}.$$

If $a = 4$ and $p = 7$, then X_a has a supersingular reduction. In this case the Frobenius is

$$\begin{pmatrix} pF_{11}(a) & F_{12}(a) \\ pF_{21}(a) & F_{22}(a)\end{pmatrix} \equiv \begin{pmatrix} 3603314253994 & 2631033376372 \\ 899507565369 & 1144247255949\end{pmatrix} \mod 7^{15}, \tag{6.6}$$

and the eigenvector $v_{\sqrt{-7}}$ for $\sqrt{-7}$ satisfies

$$v_{\sqrt{-7}} \equiv \omega + \overbrace{(2066673311059 + 1689081159560\sqrt{-7})}^{e}\,\eta \mod 7^{15}\mathbb{Z}_p[\sqrt{-7}].$$

Hence

$$R_{7,\sqrt{-7}}(X_a,\xi) = (1+\sqrt{-7})\frac{\mathrm{Tr}(\mathrm{reg}_{\mathrm{syn}}(\xi|_{t=a})\cup v_{\sqrt{-7}})}{\mathrm{Tr}(\omega_{X_a}\cup v_{\sqrt{-7}})}$$
$$\equiv (1+\sqrt{-7})\frac{e\cdot\varepsilon_1^{(\sigma_a)}(a)-\varepsilon_2^{(\sigma_a)}(a)}{e}$$
$$\equiv 224156767500 + 7^{-1}\cdot 577165355498\sqrt{-7}$$

modulo $7^{13}\mathbb{Z}_p[\sqrt{-7}]$. Other results are given in the following table[9], where α denotes the unit root or $\sqrt{-p}$.

Table of p-adic regulators

$(p^s) = \mod p^s$

$a=2$	p	$R_{p,\alpha}(X_a,\xi)$	$R_{p,\beta}(X_a,\xi)$
	5	$5162207171(p^{15})$	$5^{-1}\cdot 4698651004(p^{13})$
	7	$1898050197813(p^{15})$	$47494881888(p^{13})$
	11	$4688873516(p^{10})$	$4077822(p^8)$
	13	$109756622389(p^{10})$	$13^{-1}\cdot 1014980225(p^8)$
	17	$1874620233(p^8)$	$17^{-1}\cdot 239203868(p^6)$

$a=4$	p	$R_{p,\alpha}(X_a,\xi)$	$R_{p,\beta}(X_a,\xi)$
	5	$5110910605(p^{14})$	$5^{-1}\cdot 2495229428(p^{13})$
	7	$224156767500 + 7^{-1}\cdot 577165355498\sqrt{-7}(p^{13})$	(supersingular)
	11	$23123035879(p^{10})$	$11^{-1}\cdot 1673260489(p^8)$
	13	$131267814396(p^{10})$	$13^{-1}\cdot 8897873775(p^8)$
	17	$4659831644(p^8)$	$17^{-1}\cdot 401683799(p^6)$

[9] It takes about only a few minutes for computing each value by our implementation. One can raise the precision as long as "sp" is not too large.

	p	$R_{p,\alpha}(X_a,\xi)$	$R_{p,\beta}(X_a,\xi)$
$a=8$	5	$299743657+14397739353\sqrt{-5}(p^{14})$	(supersingular)
	7	bad reduction	
	11	$7356580887(p^{10})$	$11^{-1}\cdot 1944330879(p^{8})$
	13	$89811548583(p^{10})$	$13^{-1}\cdot 7256384468(p^{8})$
	17	$4687529300(p^{8})$	$17^{-1}\cdot 270140891(p^{6})$

We compare the above regulators with the p-adic L-values. One can compute the special values $L_{p,\alpha}(X_a,\omega^{-1},0)$ by SAGE or Pari/GP. The following is the code of Pari/GP for the case $a=2$ and $p=5$ (the other cases are similar):

```
E = ellinit([0,-2,0,-1,0]);
[M,phi] = msfromell(E,-1);
Mp = mspadicinit(M, 5, 10, 0);
mu = mspadicmoments(Mp, phi);
mspadicL(mu, [-1,-1])
```

		$L_{p,\alpha}(X_a,\omega^{-1},0)$
$a=2$	$p=5$	$3808454(p^{10})$
	$p=7$	$183475467(p^{10})$
	$p=11$	$21248551085(p^{10})$
	$p=13$	$28101869460(p^{10})$
	$p=17$	$5101137208(p^{8})$

	$a=4$	$L_{p,\alpha}(X_a,\omega^{-1},0)$
$a=4$	$p=5$	$8021260(p^{10})$
	$p=7$	$64290103+7^{-1}\cdot 107026729\sqrt{-7}(p^{10})$
	$p=11$	$1407194361(p^{10})$
	$p=13$	$72224584651(p^{10})$
	$p=17$	$4645841619(p^{8})$

	$a=8$	$L_{p,\alpha}(X_a,\omega^{-1},0)$
$a=8$	$p=5$	$3784189+6416206\sqrt{-5}(p^{10})$
	$p=11$	$14713161774(p^{10})$
	$p=13$	$41764605317(p^{10})$
	$p=17$	$2399301159(p^{8})$

The values $L_{p,\beta}(X_a,\omega^{-1},0)$ of the critical slope p-adic L-function are kindly provided by Professor Robert Pollack using the theory of overconvergent modular symbols (see [PS1, Section 8] for the algorithm to compute the critical slope p-adic L-functions). The code is available in [P]:

		$L_{p,\beta}(X_a,\omega^{-1},0)$
$a=2$	$p=5$	$2208371/5(p^{10})$
	$p=7$	$1125190415157500314434008856467880(p^{30})$
	$p=11$	$883505012781211369302158491710196003210 77(p^{30})$

$\boxed{a=4}$

$a=4$	$L_{p,\beta}(X_a,\omega^{-1},0)$
$p=5$	$19846135286/5(p^{15})$
$p=11$	$3126248589067757551902632317230458796086\,90/11(p^{30})$
$p=13$	$4756707413775524290155098357313521298569\,03404/13(p^{30})$

The above indicate the following:

a	$R_{p,\alpha}/L_{p,\alpha}(X_a,\omega^{-1},0)$	$R_{p,\beta}/L_{p,\beta}(X_a,\omega^{-1},0)$
2	-1	-1
4	-2	-2
8	$1/2$	——

Next we compute the Beilinson regulators. Let $X_\lambda := f^{-1}(\lambda)$ be the fiber at $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$, an elliptic curve over $\mathbb{R}$. Let F_∞ denote the infinite Frobenius on $X_\lambda(\mathbb{C})$. Fix generators $u^\pm \in H_1(X_\lambda(\mathbb{C}),\mathbb{Z})^{F_\infty=\pm 1} \cong \mathbb{Z}$. Since the elliptic curve X_λ is the rectangle case (i.e. $4x^3-g_2x-g_3$ has 3 distinct real roots), one has $H_1(X_\lambda(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}u^+ + \mathbb{Z}u^-$. Let

$$\mathrm{reg}_{\mathbb{R}} : K_2(X_\lambda) \longrightarrow H^2_{\mathscr{D}}(X_\lambda,\mathbb{R}(2)) \cong \mathrm{Hom}(H_1(X_\lambda(\mathbb{C}),\mathbb{Z}),\mathbb{R}(1))$$

be the Beilinson regulator map. Then

$$R_\infty(X_\lambda,\xi) = \frac{1}{2\pi i}\langle \mathrm{reg}_{\mathbb{R}}(\xi), u^-\rangle.$$

Note that in general

$$\langle \mathrm{reg}_{\mathbb{R}}(\xi), u^+\rangle \in \mathbb{R}, \quad \langle \mathrm{reg}_{\mathbb{R}}(\xi), u^-\rangle \in \mathbb{R}(1).$$

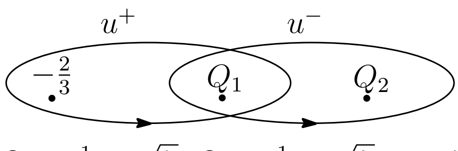
We claim

$$\frac{1}{2\pi i}\langle \mathrm{reg}_{\mathbb{R}}(\xi), u^-\rangle = \pm \mathrm{Re}\left[-\log 64 + \log\lambda + \frac{3\lambda}{16}\,{}_4F_3\left(\begin{matrix}\frac{5}{4},\frac{7}{4},1,1\\ 2,2,2\end{matrix};\lambda\right)\right]. \tag{6.7}$$

If $0<\lambda<1$, the equation $4x^3-g_2x-g_3$ has 3 distinct real roots

$$-\frac{2}{3} < \frac{1}{3}-\sqrt{\lambda} < \frac{1}{3}+\sqrt{\lambda}.$$

Hence $u^\pm$ are the cycles in the x-plane displayed as follows:

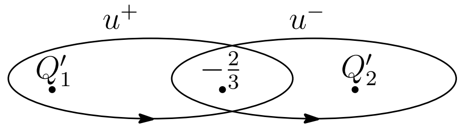


$Q_1 = \frac{1}{3}-\sqrt{\lambda}$, $Q_2 = \frac{1}{3}+\sqrt{\lambda}$, $0<\lambda<1$

If $\lambda>1$, the equation $4x^3-g_2x-g_3$ has 3 distinct real roots

$$\frac{1}{3}-\sqrt{\lambda} < -\frac{2}{3} < \frac{1}{3}+\sqrt{\lambda},$$

and then the generator $u^\pm$ are as follows,



$$Q_1' = \frac{1}{3} - \sqrt{\lambda},\ Q_2' = \frac{1}{3} + \sqrt{\lambda},\ \lambda > 1$$

Let δ_λ be the vanishing cycle at $\lambda = 0$. If $0 < \lambda < 1$, then u^- is the vanishing cycle δ_λ, and u^+ is the vanishing cycle at $\lambda = 1$. Let T_0 (resp. T_1) denote the local monodromy at $\lambda = 0$ (resp. $\lambda = 1$). Then $T_0\delta_\lambda = \delta_\lambda$ and $T_1\delta_\lambda = \delta_\lambda \pm u^+$. This shows that the minus-part of δ_λ is invariant under the monodromy, so that one has a single valued real analytic function

$$\mathrm{Re}\left(\frac{1}{2\pi i}\langle \mathrm{reg}_{\mathbb{R}}(\xi), \delta_\lambda\rangle\right) = \frac{1}{2\pi i}\langle \mathrm{reg}_{\mathbb{R}}(\xi), u^-\rangle \tag{6.8}$$

on a region $\{\lambda > 1,\ \lambda \neq 1\}$. By (6.2), one has

$$f(\lambda) := \lambda\frac{d}{d\lambda}\langle \mathrm{reg}_{\mathbb{R}}(\xi), \delta_\lambda\rangle = 2\left\langle \frac{dx}{y}, \delta_\lambda\right\rangle = 2\int_{\delta_\lambda}\frac{dx}{y} = 4\int_{\frac{1}{3}-\sqrt{\lambda}}^{\frac{1}{3}+\sqrt{\lambda}}\frac{dx}{y}$$

which is a single valued analytic function on $0 \le \arg(z-1) < 2\pi$. Since δ_λ is the vanishing cycle, $f(\lambda)$ is meromorphic at $\lambda = 0$. By (6.3), it satisfies the hypergeometric differential equation (6.4). Therefore $f(\lambda) = cF_{\frac{1}{4},\frac{3}{4}}(\lambda)$ with some constant c. Since

$$f(0) = \lim_{\lambda\to+0} 4\int_{\frac{1}{3}-\sqrt{\lambda}}^{\frac{1}{3}+\sqrt{\lambda}}\frac{dx}{y} = \pm 2\pi i,$$

one concludes $f(\lambda) = \pm 2\pi i\, F_{\frac{1}{4},\frac{3}{4}}(\lambda)$. Hence we have

$$\frac{1}{2\pi i}\langle \mathrm{reg}_{\mathbb{R}}(\xi), u^-\rangle \overset{(6.8)}{=} \mathrm{Re}\left(\frac{1}{2\pi i}\langle \mathrm{reg}_{\mathbb{R}}(\xi), \delta_\lambda\rangle\right) = \pm\mathrm{Re}\left[C + \log\lambda + \frac{3\lambda}{16}\,{}_4F_3\left(\begin{matrix}\frac{5}{4},\frac{7}{4},1,1\\2,2,2\end{matrix};\lambda\right)\right]$$

with some constant C. Finally, one can determine C in the following way. Let T_∞ denote the local monodromy at $\lambda = \infty$, One can show (cf. the proof of [A1, Theorem 3.6])

$$(T_\infty - e^{\frac{\pi i}{2}})(T_\infty - e^{\frac{3\pi i}{2}})\langle \mathrm{reg}_{\mathbb{R}}(\xi), \delta_\lambda\rangle \in \mathbb{R}(2) = \mathbb{R}$$

and hence

$$(T_\infty - e^{\frac{\pi i}{2}})(T_\infty - e^{\frac{3\pi i}{2}})\mathrm{Re}\left[C + \log\lambda + \frac{3\lambda}{16}\,{}_4F_3\left(\begin{matrix}\frac{5}{4},\frac{7}{4},1,1\\2,2,2\end{matrix};\lambda\right)\right] = 0.$$

On the other hand, it follows from [A1, Lemma 3.5] that one has

$$(T_\infty - e^{\frac{\pi i}{2}})(T_\infty - e^{\frac{3\pi i}{2}})\left[-\log 64 + \log\lambda + \frac{3\lambda}{16}\,{}_4F_3\left(\begin{matrix}\frac{5}{4},\frac{7}{4},1,1\\2,2,2\end{matrix};\lambda\right)\right] \in \mathbb{Q}(1) = 2\pi i\mathbb{Q}.$$

Therefore

$$(T_\infty - e^{\frac{\pi i}{2}})(T_\infty - e^{\frac{3\pi i}{2}})(C + \log 64) = 0 \iff C = -\log 64.$$

This completes the proof of (6.7).

We may take

$$R_\infty(X_\lambda, \xi) = \mathrm{Re}\left[-\log 64 + \log\lambda + \frac{3\lambda}{16}\,{}_4F_3\left(\begin{matrix}\frac{5}{4}, \frac{7}{4}, 1, 1\\ 2, 2, 2\end{matrix}; \lambda\right)\right]$$

by replacing u^- with $-u^-$ if necessary. Here is the table of numerical computations (of precision 20) by MATHEMATICA for ${}_4F_3$ and MAGMA for L-values:

Table of Beilinson regulators and L-values

a	$R_\infty(X_a, \xi)$	$L'(X_a, 0)$	$R_\infty/L'(X_a, 0)$
2	-3.01582754200021	3.01582754200021	-1
4	-2.47055987085139	1.23527993542569	-2
8	-2.04611026378501	-4.09222052757003	0.5

We thus have numerical verifications on the p-adic Beilinson conjecture for K_2 of the above elliptic curves.

Remark 6.1 *Let $z_{\mathrm{BK}} \in \varprojlim_n K_2(Y(Np^n))_{\mathbb{Q}}$ be the Beilinson-Kato element. We denote by $z_{\mathrm{BK}}(2) \in K_2(E)_{\mathbb{Q}}$ its image for E an elliptic curve of conductor N. Brunault[10] showed that if E does not have CM, then*

$$R_{p,\alpha}(E, z_{\mathrm{BK}}(2)) = \prod_{l|N}(1 - a_l(E))\frac{L(E,1)\Omega_E^-}{i\langle f, f\rangle}L_{p,\alpha}(E, \omega^{-1}, 0). \tag{6.9}$$

In particular the p-adic regulator vanishes when $a_l(E) = 1$ for some $l|N$ or $L(E,1) = 0$ (he further expects $z_{\mathrm{BK}}(2) = 0$, loc.cit. Question 12). Concerning $y^2 = x^3 - 2x^2 + (1-a)x$, if $a = 4$ then $a_3(E) = 1$ and if $a = 8$ then $L(E,1) = 0$. In these cases one cannot use z_{BK} to discuss the p-adic Beilinson conjecture for K_2. Instead we use the symbol ξ in (6.1)*, and obtain the numerical verifications.*

6.2 Legendre family $y^2 = x(1-x)(1-(1-\lambda)x)$

Next example is the Legendre family given by a Weierstrass equation $y_0^2 = x_0(1-x_0)(1-(1-\lambda)x_0)$. Let $p \geq 5$ be a prime. Replace the variables $(x, y) = (x_0^{-1} + (\lambda-2)/3, 2y_0x_0^{-2})$. We have a family

$$f : Y \longrightarrow \mathbb{P}^1$$

[10] Brunault [Br1, Théorème 38 (115)] obtains a formula in terms of the composition of the étale regulator map and the Bloch-Kato exponential map

$$\log \mathrm{reg}_f : K_2(E) \longrightarrow H^1_{\mathrm{dR}}(E/\mathbb{Q}_p).$$

By [Be1, Proposition 9.11], this agrees with the syntomic regulator map, more precisely $(1-p^{-2}\Phi)\log\mathrm{reg}_f = \mathrm{reg}_{\mathrm{syn}}$, where Φ is the p-th Frobenius on $H^1_{\mathrm{dR}}(E/\mathbb{Q}_p) \cong H^1_{\mathrm{crys}}(E_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes \mathbb{Q}$ (see also §3.2). Hence one has $\mathrm{Tr}(\mathrm{reg}_{\mathrm{syn}}(z) \cup v_\alpha) = (1 - p^{-1}\alpha^{-1})\mathrm{Tr}(\log\mathrm{reg}_f(z) \cup v_\alpha)$ and then [Br1, Théorèm 38 (115)] reads (6.9).

of elliptic curves over $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$ whose generic fiber is given by a Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3, \quad g_2 := \frac{4}{3}(\lambda^2 - \lambda + 1),\ g_3 := -\frac{4}{27}(\lambda+1)(\lambda-2)(2\lambda-1).$$

Put $S := \operatorname{Spec} \mathbb{Z}_{(p)}[\lambda, (\lambda-\lambda^2)^{-1}]$ and $X := f^{-1}(S)$. The functional j-invariant is $256(\lambda^2 - \lambda+1)^3/(\lambda^2(1-\lambda)^2)$. The singular fibers appear at $\lambda = 0, 1, \infty$, and the Kodaira symbols are I_2, I_2 and I_2^* respectively. Moreover the fiber at $\lambda = 0$ is split multiplicative over $\mathbb{Z}_{(p)}$, while so is not the fiber at $\lambda = 1$. One can see that f satisfies all the conditions **A1**, **A2** and **A3** in §4.1 and **B2**, **B3** in §4.6 hold as the multiplicity of any component of an arbitrary singular fiber is at most 2 (Proposition 7.7). Let

$$\omega := \frac{dx}{y} = -\frac{dx_0}{2y_0}, \quad \eta := \frac{xdx}{y} = -\left(x_0^{-1} + \frac{\lambda-2}{3}\right)\frac{dx_0}{2y_0}$$

be a $\mathscr{O}(S)$-basis of $H^1_{\mathrm{dR}}(X/S)$. Let $Z \subset X$ be the union of sections $\{x_0 = 0\}$, $\{x_0 = 1\}$ and $\{x_0 = \infty\}$ and a divisor $\{y_0 = \pm(1-x_0)\}$. Put $U = X \setminus Z$. By taking $\overline{S} = \overline{S}' = \mathbb{P}^1$ and $v : \overline{S}' \to \overline{S}$ given by $v(\lambda) = \lambda^2$, one can see that the condition **B1** in §4.5 holds (note that the elliptic fibration $X \times_S S' \to S'$ has multiplicative reductions at $\lambda = 0, \pm 1$ and a good reduction at $\lambda = \infty$). Let

$$\xi = \left\{ -1 + \frac{18y}{9y - 2(3x-\lambda-1)(3x-\lambda+2)}, \lambda\left(x - \frac{\lambda+1}{3}\right)^{-2} \right\} \in K_2^M(\mathscr{O}(U)) \quad (6.10)$$

be the symbol in [A2, (5.1)][11]. One immediately has

$$\operatorname{dlog}(\xi) = -\frac{d\lambda}{\lambda}\frac{dx_0}{y_0} = 2\frac{d\lambda}{\lambda}\frac{dx}{y}.$$

Put $\Delta := g_2^3 - 27g_3^2 = 16\lambda^2(1-\lambda)^2$, and

$$F_{\frac12,\frac12}(\lambda) = {}_2F_1\left({\frac12, \frac12 \atop 1}; \lambda\right).$$

The notation in §5.1 in our case is explicitly given as follows.

(i) $(a_0, b_0) = (a_1, b_1) = (0, 0)$ and $(a_\infty, b_\infty) = (0, 1)$.

(ii) $n = -\operatorname{ord}_\lambda(J) = 2$, $\kappa = 1/256$.

(iii) One has $c_0^2 = -g_2(0)/(18g_3(0)) = 1/4$. We take $c_0 = 1/2$.

(iv) One can show
$$F(\lambda) = \frac{1}{2}F_{\frac12,\frac12}(\lambda)$$
in the same way as in §6.1 (iv).

[11] The symbol in [A2, (5.1)] is $\left\{\frac{y_0 - 1 + x_0}{y_0 + 1 - x_0}, \frac{\lambda x_0^2}{(1-x_0)^2}\right\}$.

(v) The multiplicative period is $q = 256^{-1}t^2 \exp(\tau(\lambda))$ with

$$\lambda \frac{d}{d\lambda}\tau(\lambda) = -2 + 2(1-\lambda)^{-1} F_{\frac{1}{2},\frac{1}{2}}(\lambda)^{-2}, \quad \tau(0) = 0.$$

Explicitly,

$$\tau(\lambda) = -\lambda - \frac{13}{32}\lambda^2 - \frac{23}{96}\lambda^3 - \frac{2701}{16384}\lambda^4 + \cdots .$$

(vi) $\tau^{(\sigma)}(\lambda) = -p^{-1}\log(256^{p-1}c^2) + \tau(\lambda) - p^{-1}\tau(\lambda)^\sigma$.

(vii) By (6.2), one has $g_\xi(\lambda) = 2$ and $G_\xi(\lambda) = 2F(\lambda) = F_{\frac{1}{2},\frac{1}{2}}(\lambda)$ (hence $l = 1$).

(viii)

$$\begin{aligned} H(\lambda) &= (\lambda - \lambda^2)F'_{\frac{1}{2},\frac{1}{2}}(\lambda) + \frac{1}{6}(1-2\lambda)F_{\frac{1}{2},\frac{1}{2}}(\lambda) \\ &= \frac{1}{6} - \frac{1}{24}\lambda - \frac{11}{384}\lambda^2 - \frac{29}{1536}\lambda^3 - \frac{1375}{98304}\lambda^4 + \cdots . \end{aligned}$$

(ix) Let $E_1^{(\sigma)}(\lambda) \in \mathbb{Z}_p[[\lambda]]$ be defined by

$$(E_1^{(\sigma)}(\lambda))' = \frac{G_\xi(\lambda)}{\lambda} - \frac{G_\xi(\lambda)^\sigma}{\lambda}, \quad E_1^{(\sigma)}(0) := -p^{-1}\log(16^{p-1}c).$$

Let $E_2^{(\sigma)}(\lambda) = C + a_1\lambda + \cdots$ be the power series satisfying

$$\begin{aligned} \frac{d}{d\lambda}E_2^{(\sigma)}(\lambda) &= -E_1^{(\sigma)}(\lambda)\left(\frac{2}{\lambda} + \tau'(\lambda)\right) + \frac{G_\xi(\lambda)^\sigma}{\lambda}\tau^{(\sigma)}(\lambda) \\ &= -2\frac{E_1^{(\sigma)}(\lambda)}{\lambda}\left((1-\lambda)^{-1}F_{\frac{1}{2},\frac{1}{2}}(\lambda)^{-2}\right) + \frac{G_\xi(\lambda)^\sigma}{\lambda}\tau^{(\sigma)}(\lambda). \end{aligned}$$

This determines $E_2^{(\sigma)}(\lambda)$ except the constant term C. We expect

$$C = -\frac{1}{4}\left(p^{-1}\log(256^{p-1}c^2)\right)^2 = -\left(p^{-1}\log(16^{p-1}c)\right)^2$$

according to Conjecture 4.9, while it does not matter toward numerical computation thanks to the method in **Step 2** in §5.1. Let X_a be the fiber at $\lambda = a$. Let $\sigma_a(\lambda) = a^{1-p}\lambda^p$, and

$$\mathrm{reg}_{\mathrm{syn}}(\xi|_{\lambda=a}) = \varepsilon_1^{(\sigma_a)}(a)\frac{dx}{y} + \varepsilon_2^{(\sigma_a)}(a)\frac{xdx}{y}.$$

We now discuss a particular case $a = 4$. In this case, X_4 is isogenous to $X_0(24)$ and the Néron differential ω_{X_a} is dx/y.

Theorem 6.2 *Let p be a prime at which X_4 has a good reduction (including a supersingular reduction). Then Conjecture 3.3 for X_4 is true in case $n = 0$ and $\mathrm{ord}_p(\gamma) < 1$.*

Proof. Noticing

$$\lim_{\lambda\to+0}\lambda\frac{d}{d\lambda}\langle\mathrm{reg}_{\mathbb{R}}(\xi),u^-\rangle=2\lim_{\lambda\to+0}\int_1^{(1-\lambda)^{-1}}\frac{dx_0}{y_0}=2\pi i$$

one can show that the Beilinson regulator is

$$R_\infty(X_a,\xi)=\frac{1}{2\pi i}\langle\mathrm{reg}_{\mathbb{R}}(\xi),u^-\rangle=\mathrm{Re}\left[-\log 16+\log a+\frac{a}{4}\,{}_4F_3\left({\frac32,\frac32,1,1\atop 2,2,2};a\right)\right]$$

in the same way as in §6.1 (the orientation of u^- is chosen suitably). By [A1, Lemma 3.5] (or [A3, Example 5.3]), this is replaced with

$$-2\mathrm{Re}\left[a^{-\frac12}\,{}_3F_2\left({\frac12,\frac12,\frac12\atop 1,\frac32};\frac1a\right)\right]$$

when $a\in\mathbb{R}_{>0}$. A formula of Rogers and Zudilin ([RZ, Theorem 2, p.399 and (6), p.386]) yields

$$\frac{\pi^2}{12}\,{}_3F_2\left({\frac12,\frac12,\frac12\atop \frac32,1};\frac14\right)=L(E_{24},2)=\frac{\pi^2}{6}L'(E_{24},0).$$

We thus have

$$R_\infty(X_4,\xi)/L'(X_4,0)=-2. \tag{6.11}$$

On the other hand, Theorem 8.6 in Appendix B yields that ξ agrees with the Beilinson-Kato element $4z_{\mathrm{BK},X_4}$. Since X_4 does not have CM, one can apply Brunault's formula (6.9) ([Br1, Théorème 38 (115)]). We thus have

$$R_{p,\gamma}(X_4,\xi)/L_{p,\gamma}(X_4,\omega^{-1},0)=-2. \tag{6.12}$$

Hence Conjecture 3.3 for the case $n=0$ and $\mathrm{ord}_p(\gamma)<1$ follows. □

Remark 6.3 (6.12) *also gives an example of [A2, Conjecture 5.2],*

$$(1-p\alpha^{-1})\mathscr{F}^{(\sigma_t)}_{\frac12,\frac12}(t)|_{t=4}=-L_{p,\alpha}(X_4,\omega^{-1},0)$$

for p such that X_4 has a good ordinary reduction.

We also discuss the critical slope case (i.e. $\gamma=\beta$, where β is a non-unit root with $\mathrm{ord}_p(\beta)=1$) for the elliptic curve X_4. In the same way as in §6.1, one has the table:

p	$\varepsilon_1^{(\sigma_a)}(a)\|_{a=4}$	$\varepsilon_2^{(\sigma_a)}(a)\|_{a=4}$
5	$14203707162(p^{15})$	$2689502325(p^{15})$
7	$3834481697564(p^{15})$	$1564968189963(p^{15})$
11	$19820018884(p^{10})$	$8395655104(p^{10})$
13	$114689392802(p^{10})$	$107876860898(p^{10})$
17	$159074202(p^{8})$	$5132326047(p^{8})$

The p-adic regulators

$$R_{p,\beta}(X_4,\xi) = (1-p\beta^{-1})\frac{\mathrm{Tr}(\mathrm{reg}_{\mathrm{syn}}(\xi|_{\lambda=4})\cup v_\beta)}{\mathrm{Tr}(\omega_{X_4}\cup v_\beta)}$$

are as follows:

p	$R_{p,\beta}(X_4,\xi)$
5	$276342201(p^{13})$
7	(supersingular)
11	$11^{-1}\cdot 114484571(p^{8})$
13	$13^{-1}\cdot 12923433(p^{8})$
17	$17^{-1}\cdot 23698690(p^{6})$

On the other hand, $L_{5,\beta}(X_a,\omega^{-1},0) = 305462 \bmod 5^8$ and $L_{11,\beta}(E,\omega,0) = 1453013467/11 \bmod 11^8$ (computed by Robert Pollack). This indicates

$$R_{p,\beta}(X_4,\xi)/L_{p,\beta}(X_4,\omega^{-1},0) = -2 \tag{6.13}$$

as desired.

7 Appendix A: Gauss-Manin connection for elliptic fibrations

7.1 Gauss-Manin connection for elliptic fibrations

Let S be a smooth affine curve and $f : U \to S$ a projective smooth family of elliptic curves whose affine form is given by a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ with $\Delta := g_2^3 - 27g_3^2 \in \mathscr{O}_S(S)^\times$. We fix a description of elements of the de Rham cohomology $H^\bullet_{\mathrm{dR}}(U/S)$ in the following way. Let $U_0 = \mathrm{Spec}\,\mathscr{O}(S)[x,y]/(y^2-4x^3+g_2x+g_3)$ and $U_\infty = \mathrm{Spec}\,\mathscr{O}(S)[z,u]/(u^2-4z+g_2z^3+g_3z^4)$ be affine open sets of U with $z=1/x$ and $u = y/x^2$. Put

$$\check{C}^0(\mathscr{F}) := \varGamma(U_0,\mathscr{F})\oplus\varGamma(U_\infty,\mathscr{F}),\quad \check{C}^1(\mathscr{F}) := \varGamma(U_0\cap U_\infty,\mathscr{F})$$

a Cech complex for a (Zariski) sheaf $\mathscr{F}$. The double complex

$$\begin{array}{ccccccc}
\check{C}^0(\mathscr{O}_U) & \xrightarrow{d} & \check{C}^0(\Omega^1_{U/S}) & \qquad & (x_0,x_\infty) & \xrightarrow{d} & (dx_0,dx_\infty)\\
\downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
\check{C}^1(\mathscr{O}_U) & \xrightarrow{d} & \check{C}^1(\Omega^1_{U/S}) & & x_0-x_\infty & &
\end{array}$$

gives rise to the total complex

$$\check{C}^\bullet(U/S) : \check{C}^0(\mathscr{O}_U) \xrightarrow{\delta\times d} \check{C}^1(\mathscr{O}_U)\times\check{C}^0(\Omega^1_{U/S}) \xrightarrow{(-d)\times\delta} \check{C}^1(\Omega^1_{U/S})$$

of $\mathscr{O}(S)$-modules starting from degree 0, and the cohomology provides the de Rham cohomology $H^\bullet_{\mathrm{dR}}(U/S)$:

$$H^q_{\mathrm{dR}}(U/S) = H^q(\check{C}^\bullet(U/S)), \quad q \geq 0.$$

Elements of $H^1_{\mathrm{dR}}(U/S)$ are represented by cocycles

$$(f) \times (x_0, x_\infty) \quad \text{with } df = x_0 - x_\infty.$$

The following formula is well-known.

Theorem 7.1 *Suppose that Ω^1_S is a free $\mathscr{O}_S$-module with a base $d\lambda \in \Gamma(S, \Omega^1_S)$. For $f \in \mathscr{O}_S(S)$, we define $f' \in \mathscr{O}_S(S)$ by $df = f' d\lambda$. Let*

$$\omega := (0) \times (\frac{dx}{y}, -\frac{dz}{u}) \tag{7.1}$$

$$\eta := (\frac{y}{2x}) \times (\frac{xdx}{y}, \frac{(g_2 z + 2g_3 z^2)dz}{4u}) \tag{7.2}$$

be elements in $H^1_{\mathrm{dR}}(U/S)$. Then we have

$$\nabla\omega = -\frac{\Delta'}{12\Delta} d\lambda \otimes \omega + \frac{3(2g_2 g_3' - 3g_2' g_3)}{2\Delta} d\lambda \otimes \eta,$$

$$\nabla\eta = -\frac{g_2(2g_2 g_3' - 3g_2' g_3)}{8\Delta} d\lambda \otimes \omega + \frac{\Delta'}{12\Delta} d\lambda \otimes \eta.$$

Proposition 7.2 *Let $g_2, g_3 \in K[[\lambda]]$ be power series such that $g_2(0) \neq 0$, $g_3(0) \neq 0$ and $\Delta = g_2^3 - 27g_3^2 \in \lambda K[[\lambda]]$. Let $f : U \to S = \operatorname{Spec} K[[\lambda]]$ be the family of elliptic curves given by an affine equation*

$$y^2 = 4x^3 - g_2 x - g_3$$

with semistable reduction at $\lambda = 0$. Let $(12g_2)^{-1/4} = c_0 + c_1\lambda + \cdots \in \overline{K}[[\lambda]]$ denote the power series such that the initial term satisfies $c_0^2 = -18^{-1} g_2(0) g_3(0)^{-1}$. Let $F(\lambda)$ be defined by

$$\rho^*\omega = F(\lambda)\frac{du}{u}.$$

Then

$$F(\lambda) = (12g_2)^{-\frac{1}{4}} {}_2F_1\left({\frac{1}{12}, \frac{5}{12} \atop 1}; J^{-1}\right), \quad \text{where } J := g_2^3/(g_2^3 - 27g_3^2). \tag{7.3}$$

Proof. We may assume $K = \mathbb{C}$. Denote by $U_\lambda^{an} = f^{-1}(\lambda)$ the analytic torus for $|\lambda| \ll 1$. Let $\delta_\lambda \in H_1(U_\lambda^{an}, \mathbb{Z})$ be the vanishing cycle. Then

$$F(\lambda) = \frac{1}{2\pi i}\int_{\delta_\lambda} \omega.$$

This is the unique solution of the Picard-Fuchs equation [SB, (1.3)], and then (7.3) is proven in [SB, (1.5)]. □

Proposition 7.3 *Let the notation be as in Proposition 7.2. Put*

$$H(\lambda) := \frac{2\Delta}{3(2g_2g_3' - 3g_2'g_3)}\left(F'(\lambda) + \frac{\Delta'}{12\Delta}F(\lambda)\right)$$

and

$$\widehat{\omega} := \frac{1}{F(\lambda)}\omega, \quad \widehat{\eta} := -H(\lambda)\omega + F(\lambda)\eta \tag{7.4}$$

$$\left(\Longleftrightarrow \omega = F(\lambda)\widehat{\omega}, \quad \eta = H(\lambda)\widehat{\omega} + \frac{1}{F(\lambda)}\widehat{\eta}\right).$$

Then $\{\widehat{\omega}, \widehat{\eta}\}$ *forms a de Rham symplectic basis which satisfies*

$$\nabla(\widehat{\omega}) = \frac{dq}{q} \otimes \widehat{\eta},$$

where $q = \kappa\lambda^n + \cdots \in \lambda K[[\lambda]]$ *is the multiplicative period.*

Proof. By (7.1) and (7.2), it is straightforward to see that

$$\nabla(\widehat{\eta}) = 0, \quad \nabla(\widehat{\omega}) = \frac{3(2g_2g_3' - 3g_2'g_3)}{2\Delta F(\lambda)^2}d\lambda \otimes \widehat{\eta}.$$

Since

$$\nabla(\omega \cup \eta) = \nabla(\omega) \cup \eta + \omega \cup \nabla(\eta) = 0,$$

$\widehat{\omega} \cup \widehat{\eta} = C \neq 0$ is a nonzero constant. Hence $\{\widehat{\omega}, C^{-1}\widehat{\eta}\}$ forms a de Rham symplectic basis. It follows from Proposition 4.1 that one has

$$\frac{dq}{q} = C\frac{1}{F(\lambda)^2}\frac{3(2g_2g_3' - 3g_2'g_3)}{2\Delta}d\lambda = -C\frac{3g_2g_3}{2F(\lambda)^2\Delta}\left(3\frac{g_2'}{g_2} - 2\frac{g_3'}{g_3}\right)d\lambda. \tag{7.5}$$

Recall the formula (7.3)

$$F(\lambda) = (12g_2)^{-\frac{1}{4}}{}_2F_1\left(\begin{matrix}\frac{1}{12}, \frac{5}{12} \\ 1\end{matrix}; J^{-1}\right) = c_0 + c_1\lambda + \cdots, \quad J := g_2^3/(g_2^3 - 27g_3^2)$$

with $c_0^2 = -18^{-1}g_2(0)g_3(0)^{-1}$. Take the residue of (7.5) at $\lambda = 0$. Since

$$\mathrm{dlog}(g_3^{-2}\Delta + 27) = \mathrm{dlog}\frac{g_2^3}{g_3^2} = \left(3\frac{g_2'}{g_2} - 2\frac{g_3'}{g_3}\right)d\lambda = \left(\frac{cng_3(0)^{-2}}{27}\lambda^{n-1} + (\text{higher terms})\right)d\lambda$$

where $\Delta = c\lambda^n + \cdots$ with $c \neq 0$, one has

$$-C\frac{3g_2(0)g_3(0)}{2c_0^2}\frac{ng_3(0)^{-2}}{27} = n \quad \Longleftrightarrow \quad C = 1.$$

□

In the above proof, we showed the following.

Proposition 7.4 *Let* $\tau(\lambda) \in \lambda K[[\lambda]]$ *be defined by* $q = \kappa\lambda^n\exp(\tau(\lambda))$. *Then*

$$\frac{dq}{q} = \frac{3(2g_2g_3' - 3g_2'g_3)}{2\Delta F(\lambda)^2}d\lambda, \quad \frac{d\tau(\lambda)}{d\lambda} = \frac{3(2g_2g_3' - 3g_2'g_3)}{2\Delta F(\lambda)^2} - \frac{n}{\lambda}. \tag{7.6}$$

7.2 Deligne's canonical extension

Let $\overline{S}$ be a smooth curve over $\mathbb{C}$, and $j : S := \overline{S} - \{P\} \hookrightarrow \overline{S}$ an inclusion. Let $(\mathscr{H}, \nabla)$ be a vector bundle with integrable connection over S. Then there is unique subbundle $\mathscr{H}_e \subset j_*^{an}\mathscr{H}$ satisfying the following conditions (cf. [Zu] (17)):

- The connection extends to a map $\nabla : \mathscr{H}_e \to \Omega^1_{\overline{S}}(\log P) \otimes \mathscr{H}_e$,
- each eigenvalue α of $\mathrm{Res}_P(\nabla)$ satisfies $0 \leq \mathrm{Re}(\alpha) < 1$.

The extended bundle $(\mathscr{H}_e, \nabla)$ is called *Deligne's canonical extension*. The inclusion map

$$[\mathscr{H}_e \xrightarrow{\nabla} \Omega^1_{\overline{S}}(\log P) \otimes \mathscr{H}_e] \longrightarrow [j_*\mathscr{H} \xrightarrow{\nabla} \Omega^1_S \otimes j_*\mathscr{H}]$$

is a quasi-isomorphism of complexes of sheaves. Besides $\exp(-2\pi i \mathrm{Res}_P(\nabla))$ coincides with the monodromy operator on $H_{\mathbb{C}} = \mathrm{Ker}(\nabla^{an})$ around P (cf. [S], (2.21)).

Let $f : \overline{U} \to \overline{S}$ be a projective flat morphism of nonsingular algebraic varieties such that f is smooth over S and $D = f^{-1}(P)$ is a NCD in $\overline{U}$. Let $U = f^{-1}(S)$. Let $(\mathscr{H}, \nabla)$ be the higher direct image $(R^i f_* \Omega^\bullet_{U/S}, \nabla)$. Then all eigenvalues of $\mathrm{Res}_P(\nabla)$ are rational numbers. There is the natural isomorphism

$$R^i f_* \Omega^\bullet_{\overline{U}/\overline{S}}(\log D) \cong \mathscr{H}_e. \tag{7.7}$$

In particular, the left hand side is a locally free $\mathscr{O}_{\overline{S}}$-module. If f is a morphism of smooth K-schemes, then Deligne's extension $(R^i f_* \Omega^\bullet_{\overline{U}/\overline{S}}(\log D), \nabla)$ can be obviously defined over K.

We have seen how to compute a connection matrix of the Gauss-Manin connection for a family of hyperelliptic curves. In case of an elliptic fibration, they are simply given as follows.

Proposition 7.5 *Let $f : \mathscr{E} \to \mathrm{Spec}\, K[[\lambda]]$ be an elliptic fibration defined by a minimal Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ with $g_2, g_3 \in K[[\lambda]]$, $\Delta := g_2^3 - 27g_3^2 \neq 0$. Then $\mathscr{H}_e$ has a basis $\{\omega, \eta\}$ (resp. $\{\lambda\omega, \eta\}$) if f has a semistable or smooth reduction (resp. additive reduction).*

Proof. Since we have Theorem 7.1, we can show the above by case-by-case analysis based on $(\mathrm{ord}(g_2), \mathrm{ord}(g_3))$. The detail is left to the reader. □

7.3 The de Rham cohomology of elliptic curves over a ring

Let U be a smooth scheme over a commutative ring A. For a point s of $\mathrm{Spec}\, A$, we denote the residue field by $k(s)$, and write $U_s := U \times_A k(s)$. We define the de Rham cohomology $H^\bullet_{\mathrm{dR}}(U/A)$ in the same way as before:

$$H^i_{\mathrm{dR}}(U/A) := \mathbb{H}^i_{\mathrm{zar}}(U, \Omega^\bullet_{U/A}), \quad i \in \mathbb{Z}_{\geq 0}.$$

Lemma 7.6 *Let A be a noetherian integral domain. Let $U \to \operatorname{Spec} A$ be a smooth projective morphism of relative dimension one with geometrically connected fibers. Then, for any $i, j, l \geq 0$, $H^i(U, \Omega^j_{U/A})$ and $H^l_{\mathrm{dR}}(U/A)$ are locally free A-modules and there are the canonical isomorphisms*

$$H^i(U, \Omega^j_{U/A}) \otimes_A k(s) \xrightarrow{\cong} H^i(U_s, \Omega^j_{U_s/k(s)}), \tag{7.8}$$

$$H^l_{\mathrm{dR}}(U/A) \otimes_A k(s) \xrightarrow{\cong} H^l_{\mathrm{dR}}(U_s/k(s)) \tag{7.9}$$

for all points $s \in \operatorname{Spec} A$. Moreover there is a (non-canonical) isomorphism

$$H^1_{\mathrm{dR}}(U/A) \cong H^0(\Omega^1_{U/A}) \oplus H^1(\mathscr{O}_U). \tag{7.10}$$

Proof. Since $\dim_{k(s)} H^i(\Omega^j_{U_s/k(s)})$ is constant with respect to s, one can apply [Ha1, III,12.9], so that $H^i(U, \Omega^j_{U/A})$ is a locally free A-module and the isomorphism (7.8) follows. The natural map $H^1(\Omega^1_{U/A}) \to H^2_{\mathrm{dR}}(U/A)$ is surjective as $H^2(\mathscr{O}_U) = 0$. There are the trace maps $\operatorname{Tr} : H^1(\Omega^1_{U_s/k(s)}) \to k(s)$ (cf. [Ha2, VII,4.1]) and $\operatorname{Tr} : H^2_{\mathrm{dR}}(U_s/k(s)) \to k(s)$ (cf. [Ha3, Proposition (2.2)]) which are compatible. This implies that the map $H^1(\Omega^1_{U_s/k(s)}) \to H^2_{\mathrm{dR}}(U_s/k(s))$ is a non-zero surjective map, and hence bijective as $\dim_{k(s)} H^1(\Omega^1_{U_s/k(s)}) = 1$. Let

$$\begin{array}{ccc} H^1(\Omega^1_{U/A}) \otimes_A k(s) & \xrightarrow{\text{surj.}} & H^2_{\mathrm{dR}}(U/A) \otimes_A k(s) \\ \cong\downarrow & & \downarrow \\ H^1(\Omega^1_{U_s/k(s)}) & \xrightarrow{\cong} & H^2_{\mathrm{dR}}(U_s/k(s)) \end{array}$$

be a commutative diagram. Hence all arrows are bijective and all terms are one-dimensional for all points $s \in \operatorname{Spec} A$. We thus have (7.9) in case $l = 2$ and that $H^2_{\mathrm{dR}}(U/A)$ is locally free of rank 1, since A is a noetherian integral domain ([Ha1, II, 8.9]). Moreover we have an isomorphism $H^1(\Omega^1_{U/A}) \xrightarrow{\cong} H^2_{\mathrm{dR}}(U/A)$ by Nakayama's lemma, so that we have an exact sequence

$$0 \longrightarrow H^0(\Omega^1_{U/A}) \longrightarrow H^1_{\mathrm{dR}}(U/A) \longrightarrow H^1(\mathscr{O}_U) \longrightarrow 0. \tag{7.11}$$

Since the right term is a projective A-module, the sequence splits. Hence one has a (non-canonical) isomorphism (7.10), and also that $H^1_{\mathrm{dR}}(U/A)$ is a locally free A-module. Finally, the isomorphism (7.9) in case $l = 1$ follows from a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\Omega^1_{U/A}) \otimes k(s) & \longrightarrow & H^1_{\mathrm{dR}}(U/A) \otimes k(s) & \longrightarrow & H^1(\mathscr{O}_U) \otimes k(s) & \longrightarrow & 0 \\ & & \cong\downarrow & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & H^0(\Omega^1_{U_s/k(s)}) & \longrightarrow & H^1_{\mathrm{dR}}(U_s/k(s)) & \longrightarrow & H^1(\mathscr{O}_{U_s}) & \longrightarrow & 0, \end{array}$$

where the right and left isomorphisms follow from (7.8). □

Let W be a commutative ring of characteristic zero, and let $f : U \to S$ be a projective smooth morphism of smooth W-schemes. Then one can define the Gauss-Manin connection

$$\nabla : H^i_{\mathrm{dR}}(U/S) \longrightarrow \Omega^1_S \otimes H^i_{\mathrm{dR}}(U/S)$$

in the same way as before, where $\Omega^\bullet_S = \Omega^\bullet_{S/W}$. Moreover if f extends to a projective flat family $\overline{f} : \overline{U} \to \overline{S} = S \cup \{P\}$ such that P is a W-rational point and $D = f^{-1}(P)$ is a relative NCD, then one can also define Deligne's canonical extension

$$(R^i f_* \Omega^\bullet_{\overline{U}/\overline{S}}(\log D), \nabla).$$

The sheaf $R^i f_* \Omega^\bullet_{\overline{U}/\overline{S}}(\log D)$ is a coherent $\mathscr{O}_{\overline{S}}$-module, but not necessarily a locally free $\mathscr{O}_{\overline{S}}$-module. It seems difficult to ask whether it is locally free or not in a general situation. The following gives a partial answer in case of elliptic fibrations.

Proposition 7.7 *Suppose that W is a noetherian integral domain of characteristic zero such that 2 is invertible in W. Put $K := \mathrm{Frac}\, W$. Let $f : \mathscr{E} \to \mathrm{Spec}\, W[[\lambda]]$ be an elliptic fibration given by a Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ with $g_2, g_3 \in W[[\lambda]]$, $\Delta := g_2^3 - 27 g_3^2 \in W((\lambda))^\times$, such that the central fiber $D = f^{-1}(0)$ is a relative NCD. Suppose that $y^2 = 4x^3 - g_2 x - g_3$ is minimal as a Weierstrass equation over $K[[\lambda]]$. Let $E := \mathscr{E} \times_{W[[\lambda]]} W((\lambda))$ and put $\omega := dx/y$ and $\eta := x dx/y$. Then*

$$H^1_{\mathrm{dR}}(E/W((\lambda))) = W((\lambda))\omega + W((\lambda))\eta \tag{7.12}$$

is a free $W((\lambda))$-module of rank 2. Put

$$H := \mathrm{Im}[H^1(\mathscr{E}, \Omega^\bullet_{\mathscr{E}/W[[\lambda]]}(\log D)) \to H^1_{\mathrm{dR}}(E/W((\lambda)))].$$

Let $\omega_0 = \omega = dx/y$ (resp. $\omega_0 = \lambda\omega$) if f has a smooth or multiplicative reduction (resp. an additive reduction). Then

$$H \subset W[[\lambda]]\omega_0 + W[[\lambda]]\eta. \tag{7.13}$$

The equality holds if the multiplicity of any component of D is invertible in W.

Proof. Write $E_s := E \times_{W((\lambda))} k(s)$ for $s \in \mathrm{Spec}\, W((\lambda))$. Then $\{dx/y|_{E_s}, x dx/y|_{E_s}\}$ is a $k(s)$-basis of $H^1_{\mathrm{dR}}(E_s/k(s))$. Therefore it follows from Lemma 7.6 (7.9) and Nakayama's lemma that $H^1_{\mathrm{dR}}(E/W((\lambda)))$ is a free $W((\lambda))$-module with basis $\{\omega, \eta\}$. Next we show (7.13). Write $\mathscr{E}_K := \mathscr{E} \times_{W[[\lambda]]} K[[\lambda]]$, $E_K := E \times_{W[[\lambda]]} K[[\lambda]]$ etc. Put

$$H_K := \mathrm{Im}[H^1(\mathscr{E}_K, \Omega^\bullet_{\mathscr{E}_K/K[[\lambda]]}(\log D)) \to H^1_{\mathrm{dR}}(E_K/K((\lambda)))].$$

By Proposition 7.5, H_K is a free $K[[\lambda]]$-module with basis $\{\omega_0, \eta\}$. Hence

$$H \subset \Big(W((\lambda))\omega_0 + W((\lambda))\eta\Big) \cap H_K = W[[\lambda]]\omega_0 + W[[\lambda]]\eta$$

as required. The last statement is more delicate. Recall the descriptions (7.1) and (7.2)

$$\omega_0 = (0) \times (\lambda^e \frac{dx}{y}, -\lambda^e \frac{dz}{u}), \quad \eta = (\frac{y}{2x}) \times (\frac{xdx}{y}, \frac{(g_2 z + 2g_3 z^2)dz}{4u}) \in H^1_{\mathrm{dR}}(\mathscr{E}/W((\lambda)))$$

in terms of Cech cocycles, where we put $e = 0$ if D is multiplicative and $e = 1$ if additive. Let $U_0^* := \operatorname{Spec} W[[\lambda]][x, y]/(y^2 - 4x^3 + g_2 x + g_3)$ and $U_\infty^* := \operatorname{Spec} W[[\lambda]][z, u]/(u^2 - 4z + g_2 z^3 + g_3 z^4)$, and $\rho : \mathscr{E} \to \mathscr{E}^* := U_0^* \cup U_\infty^*$ the blow-ups. Let $U_0 := \rho^{-1}(U_0^*)$ and $U_\infty := \rho^{-1}(U_\infty^*)$. To prove $\omega_0, \eta \in H$, we show that the above cocycles belong to

$$\varGamma(U_0 \cap U_\infty, \mathscr{O}_{\mathscr{E}}) \times \varGamma(U_0, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)) \oplus \varGamma(U_\infty, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)).$$

Since $y/2x \in \varGamma(U_0 \cap U_\infty, \mathscr{O}_{\mathscr{E}})$, it is enough to show that

$$\lambda^e \frac{dx}{y} \in \varGamma(U_0, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)), \quad \lambda^e \frac{dz}{u} \in \varGamma(U_\infty, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)), \tag{7.14}$$

and

$$\frac{xdx}{y} \in \varGamma(U_0, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)), \quad \frac{(g_2 z + 2g_3 z^2)dz}{u} \in \varGamma(U_\infty, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)). \tag{7.15}$$

We first show (7.14). We know that

$$\lambda^e \frac{dx}{y} \in \varGamma(U_0, \Omega^1_{\mathscr{E}_K/K[[\lambda]]}(\log D)), \quad \lambda^e \frac{dz}{u} \in \varGamma(U_\infty, \Omega^1_{\mathscr{E}_K/K[[\lambda]]}(\log D)) \tag{7.16}$$

by Proposition 7.5. On the other hand, since $\varGamma(\mathscr{E}, \Omega^1_{\mathscr{E}/W((\lambda))})$ is a free $W((\lambda))$-module with basis $dx/y = -dz/u$, one has

$$\lambda^{e+m} \frac{dx}{y} \in \varGamma(U_0, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)), \quad \lambda^{e+m} \frac{dz}{u} \in \varGamma(U_\infty, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)) \tag{7.17}$$

for some large integer m. By the assumption, the multiplicity of any component of D is invertible in W. One sees that $\Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)$ is locally free as f is locally given by $\lambda \mapsto x_1^{r_1} x_2^{r_2}$, where r_i denotes the multiplicity. One can check that the map a in the following diagram is injective:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D) & \xrightarrow{\lambda^m} & \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D) & \longrightarrow & \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)/\lambda^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} & & \\
0 & \longrightarrow & \Omega^1_{\mathscr{E}_K/K[[\lambda]]}(\log D) & \xrightarrow{\lambda^m} & \Omega^1_{\mathscr{E}_K/K[[\lambda]]}(\log D) & \longrightarrow & \Omega^1_{\mathscr{E}_K/K[[\lambda]]}(\log D)/\lambda^m & \longrightarrow & 0.
\end{array}$$

Now the diagram chase with use of (7.16) and (7.17) implies (7.14).

Next we show (7.15). It follows from (7.14) that

$$\lambda^e \frac{xdx}{y} \in \varGamma(U_0, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)), \quad \lambda^e \frac{(g_2 z + 2g_3 z^2)dz}{u} \in \varGamma(U_\infty, \Omega^1_{\mathscr{E}/W[[\lambda]]}(\log D)).$$

If we show

$$\frac{xdx}{y} \in \Gamma(U_{0,K}, \Omega^1_{\mathscr{E}_K/K[[\lambda]]}(\log D_K)), \qquad \frac{(g_2 z + 2g_3 z^2)dz}{u} \in \Gamma(U_{\infty,K}, \Omega^1_{\mathscr{E}_K/K[[\lambda]]}(\log D_K)),$$

then, in the same way as above, one can show (7.15) by the diagram chase, which completes the proof of Proposition 7.7. Since $U_\infty \to \operatorname{Spec} W[[\lambda]]$ is smooth, one has $dz/u \in \Gamma(U_\infty, \Omega^1_{\mathscr{E}/W[[\lambda]]})$, and hence the latter follows. The rest is to show

$$\frac{xdx}{y} \in \Gamma(U_{0,K}, \Omega^1_{\mathscr{E}_K/K[[\lambda]]}(\log D_K)). \tag{7.18}$$

If $e = 0$, there is nothing to prove. Suppose $e = 1$, namely D_K is additive. Let T_K be the singular points of D_K and put $\mathscr{E}_K^\circ := \mathscr{E}_K \setminus T_K$. It is enough to show

$$\frac{xdx}{y} \in \Gamma(U_{0,K} \setminus T_K, \Omega^1_{\mathscr{E}_K^\circ/K[[\lambda]]}(\log D_K)) \tag{7.19}$$

as $\operatorname{codim}(T_K) = 2$. Let $\operatorname{Spec} K[[\lambda_0]] \to \operatorname{Spec} K[[\lambda]]$ be given by $\lambda_0^n = \lambda$ and consider a cartesian diagram

$$\begin{array}{ccccc}
\widetilde{\mathscr{E}}^\circ_{0,K} & \xrightarrow{j} & \mathscr{E}^\circ_{0,K} & \longrightarrow & \mathscr{E}^\circ_K \\
 & \searrow^{\widetilde{f}} & \downarrow & \square & \downarrow f \\
 & & K[[\lambda_0]] & \longrightarrow & K[[\lambda]]
\end{array}$$

with j the normalization. The morphism f is locally given by $(u,v) \mapsto \lambda = u^{n_i}$ around a component of the central fiber, where n_i is the multiplicity. Take $n = \operatorname{lcm}(n_i)_i$. Then $\mathscr{E}^\circ_{0,K}$ is locally defined by $u^{n_i} = \lambda^n$ in (u,v,λ)-space, and hence $\widetilde{\mathscr{E}}^\circ_{0,K}$ is regular and $\widetilde{f}$ is smooth. Let $\widetilde{U}_{0,K} \subset \widetilde{\mathscr{E}}^\circ_{0,K}$ be the inverse image of $U_{0,K}$, and $\widetilde{D}_K$ the central fiber of $\widetilde{f}$. Then, to show (7.19), it is enough to show

$$\frac{xdx}{y} \in \Gamma(\widetilde{U}_{0,K}, \Omega^1_{\widetilde{\mathscr{E}}^\circ_{0,K}/K[[\lambda_0]]}(\log \widetilde{D}_K)).$$

Let $x_0 = x/\lambda_0^m$, $y_0 = y/\lambda_0^l$, $g_{2,0} = g_2/\lambda_0^{2m}$ and $g_{3,0} = g_3/\lambda_0^{3m}$ with $3m = 2l$ such that $y_0^2 = 4x_0^3 - g_{2,0}x_0 - g_{3,0}$ is minimal over $K[[\lambda_0]]$. Since the Kodaira type of $\widetilde{f}$ is I_s with $s \geq 0$, one has

$$\frac{xdx}{y} = \lambda_0^{2m-l}\frac{x_0 dx_0}{y_0} \in \Gamma(\widetilde{U}_{0,K}, \Omega^1_{\widetilde{\mathscr{E}}_{0,K}/K[[\lambda_0]]}) \subset \Gamma(\widetilde{U}_{0,K}, \Omega^1_{\widetilde{\mathscr{E}}_{0,K}/K[[\lambda_0]]}(\log \widetilde{D}_K))$$

as $2m - l > 0$. This completes the proof. □

8 Appendix B : Comparing elements in K_2 of elliptic curves

by François Brunault

In this appendix, we use results of Goncharov and Levin [GL] to compare two elements in K_2 of the elliptic curve $X_0(24)$: the element ξ in (6.10) and the Beilinson–Kato element. This comparison is used in the proof of Theorem 6.2.

The proof of this relation builds on PARI/GP scripts freely available at [Br2]. More precisely, launching the PARI/GP file `compareK2E.gp` will run all the computations involved in the proof.

8.1 Describing K_2 of elliptic curves

Goncharov and Levin gave an explicit description of K_2 of an elliptic curve using the so-called elliptic Bloch group. In this section we recall this construction and state the result needed later to compare the two elements in K_2 (see Theorem 8.5).

Let E be an elliptic curve defined over a number field k. To describe Quillen's K-group $K_2(E)$, our starting point is the localisation map $K_2(E) \to K_2(k(E))$, where $k(E)$ is the function field of E. The group $K_2(k(E))$ can be described using Matsumoto's theorem: for any field F, we have an isomorphism

$$K_2(F) \cong \frac{F^\times \otimes_{\mathbb{Z}} F^\times}{\langle x \otimes (1-x) : x \in F \setminus \{0,1\} \rangle}.$$

The class of $x \otimes y$ in $K_2(F)$ is denoted by $\{x,y\}$ and is called a Milnor symbol. The relations $\{x, 1-x\} = 0$ are called the Steinberg relations.

Let $\mathbb{Z}[E(\overline{k})]$ be the group algebra of $E(\overline{k})$. Consider the Bloch map

$$\begin{aligned} \beta : \overline{k}(E)^\times \times \overline{k}(E)^\times &\longrightarrow \mathbb{Z}[E(\overline{k})] \\ (f,g) &\longmapsto \sum_{i,j} m_i n_j (p_i - q_j), \end{aligned}$$

where $\operatorname{div}(f) = \sum_i m_i(p_i)$ and $\operatorname{div}(g) = \sum_j n_j(q_j)$ are the divisors of f and g. The map β is bilinear, so it induces a map

$$\overline{k}(E)^\times \otimes_{\mathbb{Z}} \overline{k}(E)^\times \longrightarrow \mathbb{Z}[E(\overline{k})],$$

which we still denote by β.

Let I be the augmentation ideal of $\mathbb{Z}[E(\overline{k})]$. The group P of principal divisors on $E/\overline{k}$ is generated by the divisors

$$(p+q) - (p) - (q) + (0) = \big((p) - (0)\big) \cdot \big((q) - (0)\big) \qquad \text{with } p, q \in E(\overline{k}),$$

so we have $P = I^2$ and $I/I^2 \cong E(\overline{k})$. It follows that β takes values in I^4, and the image of β generates I^4. Following Goncharov and Levin, we now define the elliptic Bloch group of E.

Definition 8.1 *Let $R_3^*(E/\overline{k})$ be the subgroup of $\mathbb{Z}[E(\overline{k})]$ generated by the divisors $\beta(f, 1-f)$ with $f \in \overline{k}(E)$, $f \neq 0, 1$. The elliptic Bloch group of $E/\overline{k}$ is*

$$B_3^*(E/\overline{k}) = \frac{I^4}{R_3^*(E/\overline{k})}.$$

The elliptic Bloch group of E/k is

$$B_3^*(E) = B_3^*(E/\overline{k})^{\mathrm{Gal}(\overline{k}/k)}.$$

By Matsumoto's theorem and the definition of the elliptic Bloch group, the map β gives rise to a commutative diagram

$$\begin{array}{ccc} K_2(\overline{k}(E)) & \longrightarrow & B_3^*(E/\overline{k}) \\ \uparrow & & \uparrow \\ K_2(k(E)) & \longrightarrow & B_3^*(E). \end{array}$$

Goncharov and Levin [GL] proved the following fundamental result.

Theorem 8.2 *The composite map*

$$K_2(E) \otimes \mathbb{Q} \longrightarrow K_2(k(E)) \otimes \mathbb{Q} \longrightarrow B_3^*(E) \otimes \mathbb{Q}$$

is injective.

In fact, Goncharov and Levin showed that modulo torsion, $K_2(E)$ is isomorphic to the kernel of an explicit map

$$\delta_3 : B_3^*(E) \longrightarrow \left(\overline{k}^\times \otimes E(\overline{k})\right)^{\mathrm{Gal}(\overline{k}/k)}.$$

Proof of Theorem 8.2. By Quillen's localisation theorem, there is a long exact sequence

$$\cdots \longrightarrow \bigoplus_{p\in E} K_2(k(p)) \longrightarrow K_2(E) \longrightarrow K_2(k(E)) \xrightarrow{\partial} \bigoplus_{p\in E} k(p)^\times \longrightarrow \cdots,$$

where p runs over the closed points of E; see the exact sequence in the proof of [Q, Theorem 5.4], with $i = 2$ and $p = 0$. Tensoring with $\mathbb{Q}$ and using the fact that K_2 of a number field is a torsion group [Ga], we get an isomorphism $K_2(E) \otimes \mathbb{Q} \cong \mathrm{Ker}(\partial) \otimes \mathbb{Q}$.

By [GL, Theorem 3.8], the natural map $K_2(k(E)) \to B_3^*(E)$ induces an isomorphism

$$\left(\frac{H^0(E, \mathcal{K}_2)}{\mathrm{Tor}(k^\times, E(k)) + K_2(k)}\right) \otimes \mathbb{Q} \cong \mathrm{Ker}(\delta_3) \otimes \mathbb{Q},$$

where $H^0(E, \mathcal{K}_2) = \mathrm{Ker}(\partial)$. Since $\mathrm{Tor}(k^\times, E(k))$ is a torsion group [W, Proposition 3.1.2(a)] and $K_2(k)$ is also torsion, we get the result. □

We will need to work with the full group of divisors $\mathbb{Z}[E(\overline{k})]$, using (a modification of) the group $B_3(E)$ introduced in [GL, Definition 3.1]. The difference is that we use the m-distribution relations only for $m = -1$.

Definition 8.3 *Let $R_3(E/\overline{k})$ be the subgroup of $\mathbb{Z}[E(\overline{k})]$ generated by the divisors $\beta(f, 1-f)$ with $f \in \overline{k}(E)$, $f \neq 0, 1$ as well as the divisors $(p) + (-p)$ with $p \in E(\overline{k})$. We define*

$$B_3(E/\overline{k}) = \frac{\mathbb{Z}[E(\overline{k})]}{R_3(E/\overline{k})}, \qquad \text{and} \qquad B_3(E) = B_3(E/\overline{k})^{\mathrm{Gal}(\overline{k}/k)}.$$

Goncharov and Levin proved the following result (compare [GL, Proposition 3.19(a)]).

Proposition 8.4 *The canonical map $B_3^*(E) \otimes \mathbb{Q} \to B_3(E) \otimes \mathbb{Q}$ is injective.*

Proof. It suffices to establish the result for $E/\overline{k}$. Let $D = \sum n_i[p_i] \in I^4$ be a divisor belonging to $R_3(E/\overline{k})$. Write $D = D' + D''$ with $D' \in R_3^*(E/\overline{k})$ and $D'' = \sum_j m_j((q_j) + (-q_j))$. The divisor D'' belongs to I^4 and is invariant under the elliptic involution $\sigma : p \mapsto -p$ on E. Thus we can write

$$2D'' = D'' + \sigma(D'') = \beta\Big(\sum_{\ell} (f_\ell \otimes g_\ell) + \sigma^*(f_\ell \otimes g_\ell)\Big)$$

for some rational functions f_ℓ, g_ℓ. By [GL, Lemma 3.21], for any rational functions f, g on $E/\overline{k}$, we have $\sigma^*\{f, g\} = -\{f, g\}$ in $K_2(\overline{k}(E))/\{\overline{k}^\times, \overline{k}(E)^\times\}$. It follows that $(f \otimes g) + \sigma^*(f \otimes g)$ is a linear combination of Steinberg relations and of elements $\lambda \otimes h$ with $\lambda \in \overline{k}^\times$ and $h \in \overline{k}(E)^\times$. Applying the map β and noting that $\beta(\lambda \otimes h) = 0$, we get $2D'' \in R_3^*(E/\overline{k})$ as desired. ☐

Putting together Theorem 8.2 and Proposition 8.4, we get the following result.

Theorem 8.5 *The composite map*

$$\overline{\beta} : \quad K_2(E) \otimes \mathbb{Q} \longrightarrow K_2(k(E)) \otimes \mathbb{Q} \longrightarrow B_3(E) \otimes \mathbb{Q},$$

sending an element $\sum_i \{f_i, g_i\}$ to the class of the divisor $\sum_i \beta(f_i, g_i)$, is injective.

Thanks to Theorem 8.5, any equality in $K_2(E) \otimes \mathbb{Q}$ can be proved by applying the map $\overline{\beta}$ and comparing the divisors. Of course, the difficult part is to find the necessary Steinberg relations. In the following sections, we use this strategy to compare an "elliptic" and a "modular" element in K_2 of the elliptic curve $X_0(24)$.

8.2 Special elements in K_2 of $X_0(24)$

8.2.1 The minimal model

The curve E_4 in the Legendre family is given by the equation $y^2 = x(1-x)(1+3x)$. A minimal Weierstrass equation is

$$E : Y^2 = X^3 - X^2 - 4X + 4,$$

obtained with the change of variables $(X, Y) = (1-3x, -3y)$. This is the curve $24a1$ in the Cremona database [Cr]. The Néron differential is (up to sign)

$$\omega_E = \frac{dX}{2Y} = \frac{dx}{2y}.$$

The torsion subgroup of E is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by the points $p_1 = (0, 2)$ of order 4, and $p_2 = (1, 0)$ of order 2.

8.2.2 The modular parametrisation

The curve E is in fact isomorphic to the modular curve $X_0(24)$. We denote by

$$\varphi : X_0(24) \to E$$

the modular parametrisation, normalised so that $\varphi(\infty) = 0$ and $\varphi^*(\omega_E) = \omega_f = 2\pi i f(z)dz$, where f is the unique newform of weight 2 and level $\Gamma_0(24)$.

The modular curve $X_0(24)$ has 8 cusps: $\infty, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}$. They are all rational, and therefore correspond via φ to the 8 rationals points on E. We now make explicit this bijection. Let $\Lambda \subset \mathbb{C}$ be the lattice of periods of $\omega_E = dX/(2Y)$. We have a canonical isomorphism

$$\gamma : \mathbb{C}/\Lambda \xrightarrow{\cong} E(\mathbb{C})$$

such that $\gamma^*(\omega_E) = dz$ (so that $\gamma^{-1}(p) = \int_0^p \omega_E$). The idea is the following: given a point $\tau \in \mathcal{H} \cup \mathbf{P}^1(\mathbb{Q})$, we have

$$\varphi(\tau) = \gamma\left(\int_0^{\varphi(\tau)} \omega_E\right) = \gamma\left(\int_\infty^\tau \omega_f\right).$$

The last integral, as well as the map γ, can be computed using PARI/GP [PARI]. Note that although the computation is numerical, we know that if τ is a cusp, then $\int_\infty^\tau \omega_f$ belongs to the lattice $\frac{1}{4}\Lambda$, hence its value can be ascertained. Using the PARI/GP code in [Br2], we obtain the following table for the images by φ of the cusps of $X_0(24)$ by φ.

c	∞	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$
$\varphi(c)$	0	$3p_1 + p_2$	$p_1 + p_2$	$3p_1$	$2p_1 + p_2$	p_1	p_2	$2p_1$
(X, Y)	$(0:1:0)$	$(4, -6)$	$(4, 6)$	$(0, -2)$	$(-2, 0)$	$(0, 2)$	$(1, 0)$	$(2, 0)$

The sign of the functional equation of the L-function $L(E,s)$ is $+1$, hence the Atkin-Lehner involution $W_{24} : \tau \mapsto -1/(24\tau)$ satisfies $W_{24}(f) = -f$. This implies that the map $W_{24} : E \to E$ has the form $p \mapsto p_0 - p$ for some rational point p_0 on E, and the table above gives $p_0 = 3p_1 + p_2$.

8.2.3 The Beilinson–Kato element

Recall the definition of the Beilinson–Kato element z_E in $K_2(E) \otimes \mathbb{Q}$ (see [Br1, Définition 9.3]):

$$z_E = \varphi_*\Big(\frac{1}{2}\{u_{24}, W_{24}(u_{24})\}'\Big),$$

where u_N is the following product of Siegel units

$$u_N = \prod_{\substack{a \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b \in \mathbb{Z}/N\mathbb{Z}}} g_{a,b},$$

and the superscript $'$ means addition of Milnor symbols $\{\lambda, v\}$ with $\lambda \in \mathbb{Q}^\times$ and $v \in \mathcal{O}(Y_0(24))^\times$ in order to obtain an element of $K_2(X_0(24)) \otimes \mathbb{Q}$. Since the symbols $\{\lambda, v\}$ are killed by β, we can safely ignore them in the computation.

We wish to compute the divisor of the modular unit u_{24}. Let us work more generally with u_N. From the definition of Siegel units as infinite products, we know that the order of vanishing of $g_{a,b}$ at the cusp ∞ of $X(N)$ is equal to $NB_2(\{\frac{a}{N}\})/2$, where $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial and $\{t\} = t - \lfloor t \rfloor$ is the fractional part of t. Moreover, we have the transformation formula $g_{a,b} \circ \alpha = g_{(a,b)\alpha}$ in $\mathcal{O}(Y(N))^\times \otimes \mathbb{Q}$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Using this, we can compute the order of vanishing of $g_{a,b}$ at any cusp.

Since we are working with $X_0(N)$ instead of $X(N)$, we need to take into account the widths of the cusps of $X_0(N)$. The width of the cusp $1/d \in X_0(N)$ is

$$w(1/d) = \frac{N}{d \cdot \gcd(d, N/d)}.$$

Since $1/d = \left(\begin{smallmatrix} 1 & 0 \\ d & 1 \end{smallmatrix}\right)\infty$, we have

$$\begin{aligned}
\mathrm{ord}_{1/d}(u_N) &= \sum_{\substack{a \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b \in \mathbb{Z}/N\mathbb{Z}}} \mathrm{ord}_{1/d}(g_{a,b}) \\
&= \sum_{\substack{a \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b \in \mathbb{Z}/N\mathbb{Z}}} w(1/d)\mathrm{ord}_\infty(g_{a+db,b}) \\
&= \frac{w(1/d)}{2} \sum_{\substack{a \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b \in \mathbb{Z}/N\mathbb{Z}}} B_2\Big(\Big\{\frac{a+db}{N}\Big\}\Big) \\
&= \frac{d\varphi(N)}{2\gcd(d, N/d)\varphi(d)} \sum_{a \in (\mathbb{Z}/d\mathbb{Z})^\times} B_2\Big(\Big\{\frac{a}{d}\Big\}\Big).
\end{aligned}$$

Here we used the distribution relation for the periodic Bernoulli polynomials,

$$B_n(\{mt\}) = m^{n-1} \sum_{k=0}^{m-1} B_n\big(\{t + \frac{k}{m}\}\big) \qquad (m \geq 1).$$

We deduce the order of vanishing of u_{24} at each cusp of $X_0(24)$, and therefore its divisor:

$$\mathrm{div}(u_{24}) = \frac{1}{6}(\infty) + \frac{2}{3}(0) - \frac{1}{3}(1/2) - \frac{2}{3}(1/3) - \frac{1}{6}(1/4) + \frac{1}{3}(1/6) - \frac{1}{6}(1/8) + \frac{1}{6}(1/12).$$

The fractions appearing here mean that u_{24} is only an element of $\mathcal{O}(Y_0(24))^\times \otimes \mathbb{Q}$, in other words some power of u_{24} is a modular unit.

Applying the modular parametrisation φ, we get

$$\mathrm{div}(u_{24}) = \frac{1}{6}\big((0)+4(3p_1+p_2)-2(p_1+p_2)-4(3p_1)+2(p_1)-(2p_1+p_2)-(p_2)+(2p_1)\big). \quad (8.1)$$

Applying the Atkin-Lehner involution, we also have

$$\mathrm{div}(W_{24}(u_{24})) = \frac{1}{6}\big((3p_1+p_2)+4(0)-2(2p_1)-4(p_2)+2(2p_1+p_2)-(p_1)-(3p_1)+(p_1+p_2)\big). \quad (8.2)$$

8.2.4 The symbol ξ in (6.10)

Recall

$$\xi = \{f, g\} \in K_2(E_4) \otimes \mathbb{Q}, \qquad f = \frac{y-x+1}{y+x-1}, \qquad g = -\frac{(x-1)^2}{4x^2}. \quad (8.3)$$

Using MAGMA,we can find the divisors of f and g. Here is the code:

```
A2<x,y> := AffinePlane(Rationals());
C := Curve(A2, y^2-x*(1-x)*(1+3*x));
Cbar := ProjectiveClosure(C);
E, phi := EllipticCurve(Cbar);
Emin, psi := MinimalModel(E);
F<x,y> := FunctionField(Cbar);
f := (y-x+1)/(y+x-1);
g := -(x-1)^2/(4*x^2);
div_f := Decomposition(Divisor(f));
div_g := Decomposition(Divisor(g));
print "div(f) =", [<p[2], psi(phi(RepresentativePoint(p[1])))> : p in div_f];
print "div(g) =", [<p[2], psi(phi(RepresentativePoint(p[1])))> : p in div_g];
```

We obtain

$$\begin{aligned} \mathrm{div}(f) &= -(3p_1) + (p_1) - (3p_1 + p_2) + (p_1 + p_2) \quad (8.4) \\ \mathrm{div}(g) &= 4(2p_1 + p_2) - 4(p_2). \end{aligned}$$

8.3 Comparing the divisors

We are now going to apply $\overline{\beta}$ to the two elements in $K_2(E) \otimes \mathbb{Q}$, and compare the results.

For the Beilinson–Kato element, we find using (8.1) and (8.2) that

$$\beta(u_{24}, W_{24}(u_{24})) = \frac{1}{36}\big(8(0) - 8(p_2) + 28(p_1) - 28(p_1 + p_2) \\ + 8(2p_1) - 8(2p_1 + p_2) - 44(3p_1) + 44(3p_1 + p_2)\big).$$

In the group $B_3(E) \otimes \mathbb{Q}$, we have the relation $(p) + (-p) = 0$ for any point p, hence $(p) = 0$ if p is 2-torsion. So we can remove the 2-torsion points from the above divisor. In fact, we can express everything in terms of p_1 and $p_1 + p_2$ alone. We find

$$\overline{\beta}\big(\{u_{24}, W_{24}(u_{24})\}\big) = 2(p_1) - 2(p_1 + p_2),$$

and thus

$$\overline{\beta}(z_E) = (p_1) - (p_1 + p_2). \tag{8.5}$$

We proceed similarly for the element ξ in (6.10). Using (8.4), we compute

$$\beta(f, g) = -8(p_1) - 8(p_1 + p_2) + 8(3p_1) + 8(3p_1 + p_2),$$

which gives

$$\overline{\beta}(\xi) = -16(p_1) - 16(p_1 + p_2). \tag{8.6}$$

The divisors $\overline{\beta}(z_E)$ and $\overline{\beta}(\xi)$ are not proportional, which suggests a non-trivial Steinberg relation involving p_1 and $p_1 + p_2$. We can determine it experimentally by computing the elliptic dilogarithm of these points. Let us denote by $D_E : E(\mathbb{C}) \to \mathbb{R}$ the Bloch elliptic dilogarithm. Using the PARI/GP code in [Br2], we find numerically

$$5D_E(p_1) + 3D_E(p_1 + p_2) \approx 0. \tag{8.7}$$

This means that we should have $5(p_1) + 3(p_1 + p_2) = 0$ in the group $B_3(E) \otimes \mathbb{Q}$. We will prove that this is indeed the case, by exhibiting a Steinberg relation.

We search for a rational function h on E such that the zeros and poles of both h and $1-h$ are among the 8 torsion points of E. To do this, we use Mellit's technique of *incident lines* [Me]; see also [GL, Proof of Lemma 3.29].

We view E as a non-singular plane cubic. We generate all the lines passing only through the 8 torsion points of E. Say we have found three such lines ℓ_1, ℓ_2, ℓ_3 which, moreover, meet at a point p_0 of $\mathbb{P}^2$. If the lines are pairwise distinct, we may choose equations f_1, f_2, f_3 for these lines satisfying $f_1 + f_2 = f_3$. Then $h = f_1/f_3$ has the property that the divisors of h and $1 - h$ are supported at the torsion points of E. In particular $\beta(h, 1 - h)$ is also supported at the torsion points, which gives a relation in $B_3(E) \otimes \mathbb{Q}$.

If the intersection point p_0 lies on the curve, then the above relation is trivial: it is contained in the subgroup generated by the divisors $(p) + (-p)$. If, however, p_0 does not lie on the curve, then we usually get something interesting.

It turns out that this method of incident lines works very well in practice. We implemented it in PARI/GP [Br2]. The idea is to search for triples of incident lines, and determine the associated Steinberg relations. More precisely, a line ℓ is encoded by the effective divisor $\ell \cap E$, which has degree 3 and is supported in the finite group $E(\mathbb{Q})$ generated by p_1 and p_2. There are finitely many such divisors. Moreover, we check whether lines ℓ_1, ℓ_2, ℓ_3 with equations $\ell_i : a_i X + b_i Y + c_i = 0$ are incident by computing the determinant of the vectors (a_i, b_i, c_i) with $i = 1, 2, 3$. Also, we keep only those triples for which the intersection point p_0 does not lie on E. In the present situation, running through all possibilities, the computer finds for example the following triple of incident lines:

$$\ell_1 \cap E = 2(p_1) + (2p_1) \qquad \ell_2 \cap E = (3p_1) + (2p_1 + p_2) + (3p_1 + p_2) \qquad \ell_3 \cap E = 3(0).$$

The line ℓ_3 is just the line at infinity, so that the lines ℓ_1 and ℓ_2 are parallel. They are defined respectively by the equations

$$f_1 = -\frac{1}{4}(X + Y - 2) \qquad f_2 = \frac{1}{4}(X + Y + 2).$$

Note that $\ell_1 \cap \ell_2 = (1 : -1 : 0) \notin E$. We have $f_1 + f_2 = 1$, and the divisors of f_1 and f_2 are given by

$$\begin{aligned} \operatorname{div}(f_1) &= 2(p_1) + (2p_1) - 3(0), \\ \operatorname{div}(f_2) &= (3p_1) + (2p_1 + p_2) + (3p_1 + p_2) - 3(0). \end{aligned}$$

The associated Steinberg relation is

$$\begin{aligned} \beta(f_1, f_2) &= 9(0) + (p_2) - 9(p_1) - 3(p_1 + p_2) - (2p_1) - (2p_1 + p_2) + (3p_1) + 3(3p_1 + p_2) \\ &\equiv -10(p_1) - 6(p_1 + p_2) \qquad \text{in } B_3(E) \otimes \mathbb{Q}. \end{aligned}$$

So we get the non-trivial relation $5(p_1) + 3(p_1 + p_2) = 0$ in $B_3(E) \otimes \mathbb{Q}$. As a consequence, (8.5) and (8.6) simplify:

$$\overline{\beta}(z_E) = (p_1) + \frac{5}{3}(p_1) = \frac{8}{3}(p_1)$$

and

$$\overline{\beta}(\xi) = -16(p_1) - 16 \times -\frac{5}{3}(p_1) = \frac{32}{3}(p_1).$$

Using Theorem 8.5, we deduce:

Theorem 8.6 $\xi = 4z_E$ *in* $K_2(E) \otimes \mathbb{Q}$.

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