On fermion grading symmetry for quasi-local systems

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Abstract

We discuss fermion grading symmetry for quasi-local systems with graded commutation relations. We introduce a criterion of spontaneously symmetry breaking (SSB) for general quasi-local systems. It is formulated based on the idea that each pair of distinct phases (appeared in spontaneous symmetry breaking) should be disjoint not only for the total system but also for every complementary outside system of a local region specified by the given quasi-local structure. Under a completely model independent setting, we show the absence of SSB for fermion grading symmetry in the above sense.

We obtain some structural results for equilibrium states of lattice systems. If there would exist an even KMS state for some even dynamics that is decomposed into noneven KMS states, then those noneven states inevitably violate our local thermal stability condition.

1 Introduction

The univalence super-selection rule that forbids the superposition of two states whose total angular momenta are integers and half-integers is regarded as a natural law [WicWigWign], see also e.g. § 6.1 of [A8], III.1 of [H], § 2.2 of [We]. However, if we take a more fundamental standpoint, there are subtle points in deciding whether a (conserved) quantity satisfies the superselection rule, see e.g. [A6]. Particularly, if the number of degrees of freedom is infinite as usually considered in statistical mechanics and quantum field theory, it is not obvious that symmetry assumed for kinematics leads its preservation in the state level, that is, the absence of spontaneously symmetry breaking. In this note we try to justify the univalence superselection, i.e. unbroken symmetry of fermion grading transformations that multiply fermion fields by -1. We clarify how fermion grading symmetry is different from other symmetries and is hardly broken.

We shall review some relevant results. First, if a state is invariant under some asymptotically abelian group of automorphisms like space-translations, then fermion grading symmetry is *perfectly* preserved. That is, any such state has zero expectation value for every odd element [LR] [P]. (See also e.g. 7.1.6 of [Ru], Exam. 5.2.21 of [BR]. The same statement for quantum field theory is given in [DSu].) We shall refer to [NTh] that discusses (possible) forms of symmetry breaking of fermion grading transformations for dynamics that commutes with some asymptotically abelian group of automorphisms. But the status of broken and unbroken symmetry of fermion grading is not given there.

It seems not unreasonable to expect unbroken symmetry of fermion grading irrespective of such translation invariant assumptions. It has been shown however that non-factor quasi-free states of the CAR algebra have odd elements in their centers and give an example of the breakdown of fermion grading symmetry, though being rather technical and not coming from a physical model [MaV]. We note that two mutually disjoint noneven states in the factor decomposition of each non-factor quasi-free state have a common state restriction outside of some local region. It can be said that those noneven states are not macroscopically distinguishable.

We are led to consider that the conventional criterion of spontaneously symmetry breaking based on the center merely for the total system is too weak to be an appropriate formula for general quasi-local systems. We introduce a more demanding criterion of SSB for general quasi-local systems, which turns to be equivalent to the usual one for tensor-product systems. A pair of states are said to be *disjoint with respect to the given quasi-local structure* if for every local region, their state restrictions to its complementary outside system induce disjoint GNS representations (Definition 1). Using this notion, we propose a criterion of spontaneously symmetry breaking (Definition 2).

We show the absence of spontaneously symmetry breaking in the above sense for fermion grading symmetry for general graded quasi-local systems that encompass lattice and continuous systems (Proposition 1). This proposition may be similar to the following statement in [R]: No odd element exists in observable at infinity [LRu].

We study temperature states (Gibbs states and KMS states) of lattice systems with graded commutation relations. For every even Gibbs state, we have a grading preserving isomorphism from its center onto that of its state restriction to the complementary outside system of each local region (Proposition 3).

For now, we cannot provide a definite answer whether fermion grading symmetry is perfectly preserved or not for temperature states of those lattice systems. We only claim that if a KMS state breaks the fermion grading symmetry, then it is not thermodynamically stable. More precisely, suppose that the odd part of the center of an even KMS state for even dynamics is not empty. Then in the factor decomposition of its perturbed state by a local Hamiltonian multiplied by the inverse temperature, there are noneven KMS states that violate the local thermal stability condition (a minimum free energy condition for open systems) with respect to the perturbed dynamics acting trivially on the specified local region (Proposition 5). We give a remark upon our choice of the local thermal stability condition. In [AM1] we introduced two versions of local thermal stability — LTS-M and LTS-P. We make use of the latter that will be simply called LTS here. (See Appendix for the details.) Though we have no example of such breaking nor disprove its existence, we may say that the violation of the univalence superselection rule, if it would occur, is pathological from a thermodynamical viewpoint.

2 Notation and some known results

We recall the definition of quasi-local $\mathbf{C}^*$-systems. (For references, we refer e.g. to § 2 of [R], § 2.6 of [BR], and § 7.1 of [Ru].) Let $\mathfrak{F}$ be a directed set with a partial order relation $\geq$ and an orthogonal relation $\perp$ satisfying the following conditions:
a) If $\alpha \leq \beta$ and $\beta \perp \gamma$, then $\alpha \perp \gamma$.
b) For each $\alpha, \beta \in \mathfrak{F}$, there exists a unique upper bound $\alpha \vee \beta \in \mathfrak{F}$ which satisfies $\gamma \geq \alpha \vee \beta$ for any $\gamma \in \mathfrak{F}$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$.
c) For each $\alpha \in \mathfrak{F}$, there exists a unique α_c in $\mathfrak{F}$ satisfying $\alpha_c \perp \alpha$ and $\alpha_c \geq \beta$ for any $\beta \in \mathfrak{F}$ such that $\beta \perp \alpha$.

We consider a $\mathbf{C}^*$-algebra $\mathcal{A}$ furnished with the following structure. Let $\{\mathcal{A}_\alpha;\ \alpha \in \mathfrak{F}\}$ be a family of $\mathbf{C}^*$-subalgebras of $\mathcal{A}$ with the index set $\mathfrak{F}$. Let Θ be an involutive $*$-automorphism that determines the grading on $\mathcal{A}$ as

$$\mathcal{A}^e := \{A \in \mathcal{A} \mid \Theta(A) = A\}, \quad \mathcal{A}^o := \{A \in \mathcal{A} \mid \Theta(A) = -A\}. \tag{1}$$

These $\mathcal{A}^e$ and $\mathcal{A}^o$ are called the even and the odd parts of $\mathcal{A}$. For $\alpha \in \mathfrak{F}$

$$\mathcal{A}_\alpha^e := \mathcal{A}^e \cap \mathcal{A}_\alpha, \quad \mathcal{A}_\alpha^o := \mathcal{A}^o \cap \mathcal{A}_\alpha. \tag{2}$$

The above grading structure is referred to as fermion grading (by the condition L4 defined below). For a given state ω on $\mathcal{A}$, its restriction to $\mathcal{A}_\alpha$ is denoted ω_α. If a state takes zero for all odd elements, it is called even.

Let $\mathfrak{F}_{\rm loc}$ be a subset of $\mathfrak{F}$ corresponding to the set of indices of all local subsystems and set $\mathcal{A}_{\rm loc} := \bigcup_{\alpha \in \mathfrak{F}_{\rm loc}} \mathcal{A}_{\alpha}$. We assume L1, L2, L3, L4 as follows:

L1. $\mathcal{A}_{\rm loc} \cap \mathcal{A}_{\delta}$ is norm-dense in $\mathcal{A}_{\delta}$ for any $\delta \in \mathfrak{F}$.
L2. If $\alpha \geq \beta$, then $\mathcal{A}_{\alpha} \supset \mathcal{A}_{\beta}$.
L3. $\Theta(\mathcal{A}_{\alpha}) = \mathcal{A}_{\alpha}$ for all $\alpha \in \mathfrak{F}$.
L4. For $\alpha \perp \beta$ the following graded commutation relations hold

$$[\mathcal{A}^{e}_{\alpha},\ \mathcal{A}^{e}_{\beta}] = 0, \qquad [\mathcal{A}^{e}_{\alpha},\ \mathcal{A}^{o}_{\beta}] = [\mathcal{A}^{o}_{\alpha},\ \mathcal{A}^{e}_{\beta}] = 0,$$
$$\{\mathcal{A}^{o}_{\alpha},\ \mathcal{A}^{o}_{\beta}\} = 0,$$

where $[A,\ B] = AB - BA$ is the commutator and $\{A,\ B\} = AB + BA$ is the anti-commutator.

Our $\mathfrak{F}_{\rm loc}$ may correspond to the set of all bounded open subsets of a space(-time) region or the set of all finite subsets of a lattice. About the condition c), α_c will indicate the complement of α in the total region. We set L1 as it is for the necessity in the proof of Proposition 1.

For $A \in \mathcal{A}$ (and also for $A \in \mathcal{A}_{\alpha}$ due to the condition L3), we have the following unique decomposition:

$$A = A_{+} + A_{-}, \quad A_{+} := \frac{1}{2}\big(A + \Theta(A)\big) \in \mathcal{A}^{e}(\mathcal{A}^{e}_{\alpha}), \quad A_{-} := \frac{1}{2}\big(A - \Theta(A)\big) \in \mathcal{A}^{o}(\mathcal{A}^{o}_{\alpha}). \quad (3)$$

In order to ensure that the fermion grading involution Θ acts non-trivially on $\mathcal{A}$, we may assume, for example, that $\mathcal{A}^{o}_{\alpha}$ is not empty for all $\alpha \in \mathfrak{F}$. However, all our results below obviously hold for any trivial cases where fermions do not or rarely exist.

3 A criterion of spontaneous symmetry breaking appropriate for general quasi-local systems and Fermion grading symmetry

A pair of states will be called disjoint with each other if their GNS representations are disjoint, see e.g. § 2.4.4 and § 4.2.2 of [BR]. We shall employ the following more demanding condition for disjointness of two states.

Definition 1. *Let ω_1 and ω_2 be states of a quasi-local system $(\mathcal{A},\ \{\mathcal{A}_{\alpha}\}_{\alpha \in \mathfrak{F}_{\rm loc}})$. If for every $\gamma \in \mathfrak{F}_{\rm loc}$, their restrictions to the complementary outside system of γ, i.e., ${\omega_1}_{\gamma_c}$ and ${\omega_2}_{\gamma_c}$ are disjoint with each other, then ω_1 and ω_2 are said to be disjoint with respect to the quasi-local structure $\{\mathcal{A}_{\alpha}\}_{\alpha \in \mathfrak{F}_{\rm loc}}$.*

We shall give a criterion of spontaneously symmetry breaking based on Definition 1 as follows. Let G be a group and $\tau_g (g \in G)$ be its action of $*$-automorphisms on a quasi-local system $(\mathcal{A},\ \{\mathcal{A}_\alpha\}_{\alpha \in \mathfrak{F}_{\rm loc}})$. Suppose that τ_g commutes with a given (Hamiltonian) dynamics for every $g \in G$. Let Λ denote some set of physical states (e.g. the set of all ground states or all equilibrium states at some temperature for the given dynamics), and Λ^G denote the set of all G-invariant states in Λ. Let ω be an extremal point in Λ^G. Suppose that ω has a factor state decomposition in Λ in the form of $\omega = \int d\mu(g)\omega_g$ with $\omega_g := \tau_g^* \omega_0 (= \omega_0 \circ \tau_g)$, where ω_0 is a factor state in Λ (but not in Λ^G) and so is each ω_g, and μ denotes some probability measure on G. With the above setting, we define the following.

Definition 2. *If for each $g \neq g'$ of G a pair of factor states ω_g and $\omega_{g'}$ are disjoint with respect to the given quasi-local structure, then it is said that the G-symmetry is macroscopically broken.*

Let ω be a state of a quasi-local system $(\mathcal{A},\ \{\mathcal{A}_\alpha\}_{\alpha \in \mathfrak{F}_{\rm loc}})$. It is said that ω satisfies the cluster property (with respect to the quasi-local structure) if for any given $\varepsilon > 0$ and any $A \in \mathcal{A}$ there exists an $\alpha \in \mathfrak{F}_{\rm loc}$ such that

$$\big|\omega(AB) - \omega(A)\omega(B)\big| < \varepsilon \|B\| \tag{4}$$

for all $B \in \bigcup_{\beta \perp \alpha} \mathcal{A}_\beta$. It is shown in [R] and Theorem 2.6.5 [BR] that every factor state satisfies this cluster property. However, the converse does not always hold; non-factor quasi-free states of the CAR algebra satisfy the cluster property with respect to the quasi-local (lattice) structure used for their construction, see [MaV] for details.

The following proposition asserts that fermion grading symmetry cannot be broken in the sense of Definition 2. A remarkable thing is that it makes no reference to the dynamics. We are using essentially no more than the canonical anticommutation relations (CAR) for its proof. (The idea of the proof comes from our study on state correlation for composite fermion systems done in [AM3] [M].)

Proposition 1. *Let ω be a state of a quasi-local system $(\mathcal{A},\ \{\mathcal{A}_\alpha\}_{\alpha \in \mathfrak{F}_{\rm loc}})$ and Θ denote the fermion grading involution of $\mathcal{A}$. Suppose that ω satisfies the cluster property with respect to the quasi-local structure. Then ω and $\omega\Theta$ cannot be disjoint with respect to the quasi-local structure $\{\mathcal{A}_\alpha\}_{\alpha \in \mathfrak{F}_{\rm loc}}$. Accordingly spontaneously symmetry breaking in the sense of Definition 2 does not exist for fermion grading symmetry.*

Proof. Suppose that ω and $\omega\Theta$ are disjoint with respect to the quasi-local structure $\{\mathcal{A}_\alpha\}_{\alpha\in\mathfrak{F}_{\rm loc}}$. Then ω and $\omega\Theta$ restricted to $\mathcal{A}_{\alpha_c}$ are disjoint for each $\alpha \in \mathfrak{F}_{\rm loc}$. Hence it follows that

$$\left\|\omega_{\alpha_c} - \omega\Theta_{\alpha_c}\right\| = 2. \tag{5}$$

This is equivalent to the existence of an odd element $A_- \in \mathcal{A}^o_{\alpha_c}$ such that $\|A_-\| \le 1$ and $|\omega(A_-) - \omega(\Theta(A_-))| = |\omega(A_-) - \omega(-A_-)| = 2|\omega(A_-)| = 2$, namely,

$$\left|\omega(A_-)\right| = 1. \tag{6}$$

By (3) and L1, we have that $\mathcal{A}_{\rm loc}\cap\mathcal{A}^e_\delta$ is norm dense in $\mathcal{A}^e_\delta$ and so is $\mathcal{A}_{\rm loc}\cap\mathcal{A}^o_\delta$ in $\mathcal{A}^o_\delta$ for any $\delta\in\mathfrak{F}$. Hence from (6), we have some A_- in $\mathcal{A}^o_\gamma$ for some $\gamma\in\mathfrak{F}_{\rm loc}$ such that $\gamma\le\alpha_c$, $\|A_-\|\le 1$ and

$$\left|\omega(A_-)\right| > 0.999. \tag{7}$$

(We use a sloppy notation for A_- in the above; A_- in (6) belonging to $\mathcal{A}_{\alpha_c}$ is approximated by A_- in (7) belonging to $\mathcal{A}^o_\gamma$.) By the decomposition of A_- into hermitian elements

$$A_- = 1/2(A_- + A_-^*) - i\big(i/2(A_- - A_-^*)\big),$$

we have

$$\left|\omega\big(1/2(A_- + A_-^*)\big) - i\omega\big(i/2(A_- - A_-^*)\big)\right| > 0.999.$$

Since $(A_- + A_-^*)$ and $i(A_- - A_-^*)$ are both self-adjoint, we have

$$\left|\omega\left(1/2(A_- + A_-^*)\right)\right|^2 + \left|\omega\big(i/2(A_- - A_-^*)\big)\right|^2 > 0.999^2.$$

Hence we have

$$\left|\omega\left(1/2(A_- + A_-^*)\right)\right| > \frac{0.999}{\sqrt{2}} \quad\text{or}\quad \left|\omega\big(i/2(A_- - A_-^*)\big)\right| > \frac{0.999}{\sqrt{2}}. \tag{8}$$

From (8), $\left\|1/2(A_- + A_-^*)\right\| \le 1$ and $\left\|i/2(A_- - A_-^*)\right\| \le 1$, we can choose $A_- = A_-^* \in \mathcal{A}^o_\gamma$ (by adjusting ± 1) such that $\|A_-\|\le 1$ and

$$\omega(A_-) > \frac{0.999}{\sqrt{2}}. \tag{9}$$

By the cluster property assumption (4) on ω, for a sufficiently small $\varepsilon > 0$ and the above specified $A_- \in \mathcal{A}^o_\gamma$ there exists an $\alpha' \in \mathfrak{F}_{\rm loc}$ such that

$$\left|\omega(A_- B) - \omega(A_-)\omega(B)\right| < \varepsilon \|B\| \tag{10}$$

for all $B \in \bigcup_{\beta \perp \alpha'} \mathcal{A}_\beta$.

By (5) with $\alpha = \gamma \vee \alpha'$, the same argument leading to (9) implies that there exists $B_- = B_-^* \in \mathcal{A}^o_\zeta$ such that $\zeta \perp (\gamma \vee \alpha')$, $\|B_-\| \leq 1$ and

$$\omega(B_-) > \frac{0.999}{\sqrt{2}}. \tag{11}$$

Substituting the above B_- to B in (10), and using (9) and (11), we have

$$\begin{aligned} \left|\mathbf{Im}\left(\omega(A_- B_-)\right)\right| &< \varepsilon, \\ \mathbf{Re}\big(\omega(A_- B_-)\big) &> \frac{0.999^2}{2} - \varepsilon. \end{aligned} \tag{12}$$

Due to $A_- = A_-^* \in \mathcal{A}^o_\gamma$, $B_- = B_-^* \in \mathcal{A}^o_\zeta$, and $\gamma \perp \zeta$, $A_- B_-$ is skew-self-adjoint, i.e. $(A_- B_-)^* = -A_- B_-$. Therefore $\omega(A_- B_-)$ is a purely imaginary number, which however contradicts with (12). Thus we have shown that ω and $\omega\Theta$ cannot be disjoint with respect to $\{\mathcal{A}_\alpha\}_{\alpha \in \mathfrak{F}_{\rm loc}}$.

Since any factor state satisfies the cluster property, the possibility of SSB of Definition 2 for the symmetry Θ is negated. $\square$

4 On the centers of temperature states of lattice systems

From now on, we consider lattice fermion systems [AM2] and also the lattice systems with graded commutation relations [A7] satisfying the translation uniformity to be specified. Take $\mathbb{Z}^\nu$, $\nu (\in \mathbb{N})$-dimensional cubic integer lattice. Let $\mathfrak{F}_{\rm loc}$ be a set of all finite subsets of the lattice. We assume that there is a finite number of degrees of freedom (spins) on each site of the lattice. For general graded lattice systems, we further assume that the subalgebra $\mathcal{A}_{\{i\}}$ on each site i on the lattice is isomorphic to a $d \times d$ full matrix algebra, $d \in \mathbb{N}$ being independent of i. Hence for each $\mathrm{I} \in \mathfrak{F}_{\rm loc}$, $\mathcal{A}_{\mathrm{I}}$ is isomorphic to a $d^{|\mathrm{I}|} \times d^{|\mathrm{I}|}$ full matrix algebra, and $\mathcal{A}$ is a UHF algebra of type d^∞ by Lemma 2.1 of [A7]. (As an example of such systems, $\mathcal{A}_{\{i\}}$ is generated by fermion operators a_i, a_i^*, and spin operators represented by the Pauli matrices σ_i^x, σ_i^y, σ_i^z which are even elements commuting with all fermion operators.)

We denote the conditional expectation of the tracial state from $\mathcal{A}$ onto $\mathcal{A}_{\mathrm{J}}$ by E_{J}. The interaction among sites is determined by the potential Φ, a map from $\mathfrak{F}_{\mathrm{loc}}$ to $\mathcal{A}$ satisfying the following conditions:

(Φ-a) $\Phi(\mathrm{I}) \in \mathcal{A}_{\mathrm{I}}$, $\Phi(\emptyset) = 0$.
(Φ-b) $\Phi(\mathrm{I})^* = \Phi(\mathrm{I})$.
(Φ-c) $\Theta\big(\Phi(\mathrm{I})\big) = \Phi(\mathrm{I})$.
(Φ-d) $E_{\mathrm{J}}\big(\Phi(\mathrm{I})\big) = 0$ if $\mathrm{J} \subset \mathrm{I}$ and $\mathrm{J} \neq \mathrm{I}$.
(Φ-e) For each fixed $\mathrm{I} \in \mathfrak{F}_{\mathrm{loc}}$, the net $\{H_{\mathrm{J}}(\mathrm{I})\}_{\mathrm{J}}$ with $H_{\mathrm{J}}(\mathrm{I}) := \sum_{\mathrm{K}}\big\{\Phi(\mathrm{K});\ \mathrm{K}\cap \mathrm{I} \neq \emptyset,\ \mathrm{K} \subset \mathrm{J}\big\}$ is a Cauchy net for $\mathrm{J} \in \mathfrak{F}_{\mathrm{loc}}$ in the norm topology converging to a local Hamiltonian $H(\mathrm{I}) \in \mathcal{A}$.

Let $\mathcal{P}$ denote the real vector space of all Φ satisfying the above all conditions. The set of all $*$-derivations on the domain $\mathcal{A}_{\mathrm{loc}}$ commuting with Θ is denoted $D(\mathcal{A}_{\mathrm{loc}})$. There exists a bijective real linear map from $\Phi \in \mathcal{P}$ to $\delta \in D(\mathcal{A}_{\mathrm{loc}})$ for the lattice fermion systems (Theorem 5.13 of [AM2]), and similarly for the graded lattice systems (Theorem 4.2 of [A7]). The connection between $\delta \in D(\mathcal{A}_{\mathrm{loc}})$ and its corresponding $\Phi \in \mathcal{P}$ is given by

$$\delta(A) = i[H(\mathrm{I}),\, A], \quad A \in \mathcal{A}_{\mathrm{I}} \tag{13}$$

for every $\mathrm{I} \in \mathfrak{F}_{\mathrm{loc}}$, where the local Hamiltonian $H(\mathrm{I})$ is determined by (Φ-e) for this Φ.

The condition (Φ-d) is called the standardness which is for fixing ambiguous terms (such as scalars) irrelevant to the dynamics given by (13). We remark that any product state, for example the Fock state, can be used in place of the tracial state for E_{J} to obtain a similar one-to-one correspondence between δ and Φ. Furthermore, characterizations of equilibriums states, such as LTS, Gibbs (and also the variational principle for translation invariant states), have been all shown to be independent of the choice of those product states [A7].

The above-mentioned Gibbs condition was defined for the quantum spin lattice systems [AI], and then extended to the lattice fermion systems in § 7.3 of [AM2], and to the graded lattice systems under consideration [A7]. Let Ω be a cyclic and separating vector of a von Neumann algebra $\mathfrak{M}$ on $\mathcal{H}$ and Δ denote the modular operator for $(\mathfrak{M}, \Omega)$, see [T]. The state ω on $\mathfrak{M}$ given by $\omega(A) = (\Omega, A\Omega)$ for $A \in \mathfrak{M}$ satisfies the KMS condition for the modular automorphism group $\sigma_t := \mathrm{Ad}(\Delta^{it})$, $t \in \mathbb{R}$, at the inverse temperature $\beta = -1$ and is called the *modular state* with respect to σ_t. The following definition works for any lattice system under consideration.

Definition 3. *Let φ be a state of $\mathcal{A}$ and $\left(\mathcal{H}_\varphi,\ \pi_\varphi,\ \Omega_\varphi\right)$ be its GNS triplet. It is said that φ satisfies the Gibbs condition for $\delta \in D(\mathcal{A}_{\mathrm{loc}})$ at inverse temperature $\beta \in \mathbb{R}$, for short (δ, β)-Gibbs condition, if and only if the following conditions are satisfied :*
(Gibbs-1) The GNS vector Ω_φ is separating for $\mathfrak{M}_\varphi := \pi_\varphi(\mathcal{A})''$.
For $(\mathfrak{M}_\varphi, \Omega_\varphi, \mathcal{H}_\varphi)$, the modular operator Δ_φ and the modular automorphism group $\sigma_{\varphi,t}$ are defined. Let $\sigma_{\varphi,t}^{\beta H(\mathrm{I})}$ denote the one-parameter group of $*$-automorphisms determined by the generator $\delta_\varphi + \delta_{\pi_\varphi(\beta H(\mathrm{I}))}$, where δ_φ denotes the generator for $\sigma_{\varphi,t}$ and $\delta_{\pi_\varphi(\beta H(\mathrm{I}))}(A) := i[\beta\pi_\varphi(H(\mathrm{I})),\ A]$ for $A \in \mathfrak{M}_\varphi$.
(Gibbs-2) For every $\mathrm{I} \in \mathfrak{F}_{\mathrm{loc}}$, $\sigma_{\varphi,t}^{\beta H(\mathrm{I})}$ fixes the subalgebra $\pi_\varphi(\mathcal{A}_\mathrm{I})$ elementwise.

The modular state for $\sigma_{\varphi,t}^{\beta H(\mathrm{I})}$ is given as the vector state of a (uniquely determined) unit vector $\Omega_\varphi^{\beta H(\mathrm{I})}$ lying in the natural cone for $(\mathfrak{M}_\varphi, \Omega_\varphi)$ and is denoted $\varphi^{\beta H(\mathrm{I})}$. We use the same symbol for its restriction to $\mathcal{A}$, namely, $\varphi^{\beta H(\mathrm{I})}(A) := \left(\pi(A)\Omega_\varphi^{\beta H(\mathrm{I})}, \Omega_\varphi^{\beta H(\mathrm{I})}\right)$ for $A \in \mathcal{A}$. We remark that $\Omega_\varphi^{\beta H(\mathrm{I})}$ is normalized and $\varphi^{\beta H(\mathrm{I})}(\mathbf{1}) = 1$ in our notation. (For the general references of the perturbed states, the relative modular automorphisms, and their application to quantum statistical mechanics, see [A1] [A2] and § 5.4 of [BR].)

We next show the product property of $\varphi^{\beta H(\mathrm{I})}$ in the following sense.

Lemma 2. *Let φ be a (δ, β)-Gibbs state for $\delta \in D(\mathcal{A}_{\mathrm{loc}})$ and $\beta \in \mathbb{R}$. If it is even, then for each $\mathrm{I} \in \mathfrak{F}_{\mathrm{loc}}$, $\varphi^{\beta H(\mathrm{I})}$ is a product state extension of the tracial state tr_I on $\mathcal{A}_\mathrm{I}$ and its restriction to $\mathcal{A}_{\mathrm{I_c}}$, as denoted*

$$\varphi^{\beta H(\mathrm{I})} = \mathrm{tr}_\mathrm{I} \circ \varphi^{\beta H(\mathrm{I})}\big|_{\mathcal{A}_{\mathrm{I_c}}}. \tag{14}$$

Proof. It has been already shown in Proposition 7.7 of [AM2] for the lattice fermion systems, and we can easily verify this statement for the graded lattice systems as well. But we shall provide a slightly simpler proof.

In Theorem 9.1 of [A2] it is shown that

$$\varphi^{\beta H(\mathrm{I})}([Q_1, Q_2]Q) = 0 \tag{15}$$

for every $Q_1, Q_2 \in \mathcal{A}_\mathrm{I}$ and $Q \in \mathcal{A}_\mathrm{I}'$, the commutant of $\mathcal{A}_\mathrm{I}$ in $\mathcal{A}$. From this we see that $\varphi^{\beta H(\mathrm{I})}$ is a product state extension of the tracial state tr_I on $\mathcal{A}_\mathrm{I}$ and its restriction to $\mathcal{A}_\mathrm{I}'$.

Since φ is an even state and $H(\mathrm{I})$ is an even self-adjoint element, $\varphi^{\beta H(\mathrm{I})}$ is also even. It is easy to see

$$\mathcal{A}_\mathrm{I}' = \mathcal{A}_{\mathrm{I_c}}^e + v_\mathrm{I}\,\mathcal{A}_{\mathrm{I_c}}^o,$$

where

$$v_i := a_i^* a_i - a_i a_i^*, \quad v_{\mathrm{I}} := \prod_{i \in \mathrm{I}} v_i. \tag{16}$$

This v_{I} is a self-adjoint unitary implementing Θ on $\mathcal{A}_{\mathrm{I}}$. For $A_+ \in \mathcal{A}_{\mathrm{I}}^e$, $A_- \in \mathcal{A}_{\mathrm{I}}^o$, $B_+ \in \mathcal{A}_{\mathrm{I_c}}^e$ and $B_- \in \mathcal{A}_{\mathrm{I_c}}^o$, computing the expectation values of all $A_\sigma B_{\sigma'}$ with $\sigma = \pm$ and $\sigma' = \pm$ for $\varphi^{\beta H(\mathrm{I})}$, we obtain

$$\varphi^{\beta H(\mathrm{I})}(A_+ B_+) = \mathrm{tr}_{\mathrm{I}}(A_+) \varphi^{\beta H(\mathrm{I})}(B_+),$$

and zeros for the others, i.e $A_+ B_-$, $A_- B_+$ and $A_- B_-$. Therefore $\varphi^{\beta H(\mathrm{I})}$ is equal to the product state extension of the tracial state tr_{I} on $\mathcal{A}_{\mathrm{I}}$ and $\varphi^{\beta H(\mathrm{I})}|_{\mathcal{A}_{\mathrm{I_c}}}$. $\square$

We provide a grading structure with von Neumann algebras generated by even states and with their Θ-invariant subalgebras. For an even state ω of a quasi-local system, let $(\mathcal{H}_\omega,\ \pi_\omega,\ \Omega_\omega)$ be a GNS triplet of ω and let $\mathfrak{M}_\omega$ denote the von Neumann algebra generated by this representation. Let $U_{\Theta,\omega}$ be a unitary operator of $\mathcal{H}_\omega$ implementing the grading involution Θ, and $\Theta_\omega := \mathrm{Ad}(U_{\Theta,\omega})$. Then even and odd parts of $\mathfrak{M}_\omega$ are given by

$$\mathfrak{M}_\omega^e := \{A \in \mathfrak{M}_\omega \mid \Theta_\omega(A) = A\}, \quad \mathfrak{M}_\omega^o := \{A \in \mathfrak{M}_\omega \mid \Theta_\omega(A) = -A\}. \tag{17}$$

Let $\mathfrak{N}$ be a Θ-invariant subalgebra of $\mathfrak{M}_\omega$. We give its grading as

$$\mathfrak{N}^{e(\Theta_\omega)} := \mathfrak{N} \cap \mathfrak{M}_\omega^e, \quad \mathfrak{N}^{o(\Theta_\omega)} := \mathfrak{N} \cap \mathfrak{M}_\omega^o, \tag{18}$$

where the superscripts $e(\Theta_\omega)$ and $o(\Theta_\omega)$ indicate that the grading is determined by Θ_ω. For any $A \in \mathfrak{M}_\omega$ (also $A \in \mathfrak{N}$), we have its unique decomposition $A = A_+ + A_-$ such that $A_+ \in \mathfrak{M}_\omega^e (\mathfrak{N}^{e(\Theta_\omega)})$ and $A_- \in \mathfrak{M}_\omega^o (\mathfrak{N}^{o(\Theta_\omega)})$ in the same manner as (3).

Let ω_1 and ω_2 be even states on $\mathcal{A}$. Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be some Θ-invariant subalgebras of $\mathfrak{M}_{\omega_1}$ and $\mathfrak{M}_{\omega_2}$, respectively. If there is an isomorphism η from $\mathfrak{N}_1$ onto $\mathfrak{N}_2$, that is, $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are isomorphic, then we denote this relationship by $\mathfrak{N}_1 \sim \mathfrak{N}_2$. If there is a grading preserving isomorphism η from $\mathfrak{N}_1$ onto $\mathfrak{N}_2$, that is, η maps the even part to the even, the odd to the odd, then we write $\mathfrak{N}_1 \sim^\Theta \mathfrak{N}_2$. Obviously each of '$\sim$' and '$\sim^\Theta$' is an equivalence relation.

We recall relative entropy, which will be used in the proof of the next Proposition and also for the formulation of our local thermal stability condition in the next section and Appendix. For two states ω_1 and ω_2 of a finite-dimensional system, it is defined by

$$\begin{aligned} S(\omega_1,\ \omega_2) \ &:= \ \omega_2\left(\log D_2 - \log D_1\right),\ \text{if}\ \ker D_1 \subset \ker D_2, \\ &:= \ +\infty,\ \text{otherwise}, \end{aligned} \tag{19}$$

where D_i is the density matrix for ω_i $(i = 1, 2)$. It is positive, and zero if and only if $\omega_1 = \omega_2$. Its generalization to von Neumann algebras is given in [A4] [A5]. (Note that the order of two states and the sign convention of relative entropy are both reversed in [BR].)

In the following discussion we are interested in centers. Let us denote the center of $\mathfrak{M}_\omega$ by $\mathfrak{Z}_\omega$. It is immediate to see that $\mathfrak{Z}_\omega$ is Θ-invariant for an even state ω. We shall use shorthanded $\mathfrak{Z}^e_\omega$ and $\mathfrak{Z}^o_\omega$ for $\mathfrak{Z}^{e(\Theta_\omega)}_\omega$ and $\mathfrak{Z}^{o(\Theta_\omega)}_\omega$, respectively.

Proposition 3. *Let φ be an even (δ, β)-Gibbs state. For $\mathrm{I} \in \mathfrak{F}_{\mathrm{loc}}$, let $\varphi_{\mathrm{I_c}}$ denote the state restriction of φ onto $\mathcal{A}_{\mathrm{I_c}}$. Then for any $\mathrm{I} \in \mathfrak{F}_{\mathrm{loc}}$ there is a grading preserving isomorphism between the centers of the von Neumann algebras generated by the GNS representation of φ and by that of $\varphi_{\mathrm{I_c}}$. Especially, φ is a factor state if and only if so is $\varphi_{\mathrm{I_c}}$.*

Proof. Let $\left(\mathcal{H}_\varphi,\ \pi_\varphi,\ \Omega_\varphi\right)$ be a GNS triplet of φ, and $\Omega^{\beta H(\mathrm{I})}_\varphi$ denote the normalized vector representing its perturbed state $\varphi^{\beta H(\mathrm{I})}$ as in Definition 3. By Theorem 3.10 of [A5] (also by the discussion below Definition 6.2.29 of [BR]),

$$\begin{aligned} S(\varphi,\ \varphi^{\beta H(\mathrm{I})}) \le 2\|\beta H(\mathrm{I})\|, \\ S(\varphi^{\beta H(\mathrm{I})},\ \varphi) \le 2\|\beta H(\mathrm{I})\|. \end{aligned} \tag{20}$$

Since the relative entropy is not increasing by restriction onto any subsystem, taking the restrictions of φ and $\varphi^{\beta H(\mathrm{I})}$ onto $\mathcal{A}_{\mathrm{I_c}}$ denoted $\varphi_{\mathrm{I_c}}$ and ${\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}$ respectively, we have

$$S(\varphi_{\mathrm{I_c}},\ {\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}) \le 2\|\beta H(\mathrm{I})\|, \tag{21}$$

$$S({\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}},\ \varphi_{\mathrm{I_c}}) \le 2\|\beta H(\mathrm{I})\|. \tag{22}$$

By applying the argument in § 2 and 3 of [A3] to the present case, (21) implies that $\varphi_{\mathrm{I_c}}$ quasi-contains ${\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}$, and also (22) the vice-versa. (The notion of quasi-containment given in this reference is as follows. For a pair of representations π_1 and π_2 of a $\mathbf{C}^*$-algebra, if there is a subrepresentation of π_1 which is quasi-equivalent to π_2, then π_1 is said to quasi-contain π_2.) Therefore $\varphi_{\mathrm{I_c}}$ and ${\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}$ are quasi-equivalent. Let $\left(\mathcal{H}_{\varphi_{\mathrm{I_c}}},\ \pi_{\varphi_{\mathrm{I_c}}},\ \Omega_{\varphi_{\mathrm{I_c}}}\right)$

and $\left(\mathcal{H}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}},\ \pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}},\ \Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}\right)$ be GNS representations for $\varphi_{\mathrm{I_c}}$ and $\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}$, $\mathfrak{M}_{\varphi_{\mathrm{I_c}}}$ and $\mathfrak{M}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}$ be von Neumann algebras generated by those representations of $\mathcal{A}_{\mathrm{I_c}}$. By taking the restriction of the canonical isomorphism between the von Neumann algebras $\mathfrak{M}_{\varphi_{\mathrm{I_c}}}$ and $\mathfrak{M}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}$ which maps $\pi_{\varphi_{\mathrm{I_c}}}(A)$ to $\pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A)$ for $A \in \mathcal{A}$ onto their centers $\mathfrak{Z}_{\varphi_{\mathrm{I_c}}} := \mathfrak{M}_{\varphi_{\mathrm{I_c}}} \cap \mathfrak{M}'_{\varphi_{\mathrm{I_c}}}$ and $\mathfrak{Z}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}} := \mathfrak{M}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}} \cap \mathfrak{M}'_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}$, we have

$$\mathfrak{Z}_{\varphi_{\mathrm{I_c}}} \sim^{\Theta} \mathfrak{Z}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}. \tag{23}$$

In the above derivation, we have noted that even and odd parts of von Neumann algebras generated by a GNS representation are weak limits of even and odd parts of a underlying $\mathbf{C}^*$-system (mapped onto the GNS space), and hence the canonical isomorphism conjugating a pair of quasi-equivalent representations and its restriction to Θ-invariant subalgebras are grading preserving.

We shall construct a GNS representation of $\varphi^{\beta H(\mathrm{I})}$ (on $\mathcal{A}$) from the above $\left(\mathcal{H}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}},\ \pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}},\ \Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}\right)$ on $\mathcal{A}_{\mathrm{I_c}}$ and a GNS representation of the tracial state tr_{I} on $\mathcal{A}_{\mathrm{I}}$ denoted $\left(\mathcal{K}_{\mathrm{I}}, \kappa_{\mathrm{I}}, \Omega_{\mathrm{I}}\right)$. Define

$$\begin{aligned}
\mathcal{K} &:= \mathcal{K}_{\mathrm{I}} \otimes \mathcal{H}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}, \\
\Psi &:= \Omega_{\mathrm{I}} \otimes \Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}, \\
V_{\mathrm{I}} &:= \kappa_{\mathrm{I}}(v_{\mathrm{I}}) \otimes \mathbf{1}_{\mathrm{I_c}}, \\
\hat{\kappa}_{\mathrm{I}}(A) &:= \kappa_{\mathrm{I}}(A) \otimes \mathbf{1}_{\mathrm{I_c}} \text{ for } A \in \mathcal{A}_{\mathrm{I}}, \\
\hat{\pi}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A) &:= \mathbf{1}_{\mathrm{I}} \otimes \pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A) \text{ for } A \in \mathcal{A}_{\mathrm{I_c}},
\end{aligned} \tag{24}$$

where $\mathbf{1}_{\mathrm{I}}$ and $\mathbf{1}_{\mathrm{I_c}}$ are the identity operators on $\mathcal{K}_{\mathrm{I}}$ and $\mathcal{H}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}$, v_{I} is given by (16). Noting $\mathrm{Ad}(v_{\mathrm{I}}) = \Theta|_{\mathcal{A}_{\mathrm{I}}}$, we have a unique representation κ of the total system $\mathcal{A}$ on $\mathcal{K}$ satisfying

$$\kappa(A) = \hat{\kappa}_{\mathrm{I}}(A) \quad \text{for} \quad A \in \mathcal{A}_{\mathrm{I}}, \tag{25}$$

and

$$\kappa(B_+) = \hat{\pi}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(B_+) \quad \text{for} \quad B_+ \in \mathcal{A}^{e}_{\mathrm{I_c}}, \quad \kappa(B_-) = V_{\mathrm{I}} \hat{\pi}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(B_-) \quad \text{for} \quad B_- \in \mathcal{A}^{o}_{\mathrm{I_c}}. \tag{26}$$

By (14), i.e., the product property of $\varphi^{\beta H(\mathrm{I})}$ for $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_{\mathrm{I_c}}$, we verify that this $\left(\mathcal{K},\ \kappa,\ \Psi\right)$ gives a GNS triplet of $\varphi^{\beta H(\mathrm{I})}$. We have also

$$\begin{aligned}
\mathfrak{M}_{\kappa} &:= \kappa(\mathcal{A})'' = \Big(\kappa_{\mathrm{I}}(\mathcal{A}_{\mathrm{I}}) \otimes \pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(\mathcal{A}_{\mathrm{I_c}})\Big)'' \\
&= (\kappa_{\mathrm{I}}(\mathcal{A}_{\mathrm{I}}))'' \otimes \mathfrak{M}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}.
\end{aligned} \tag{27}$$

Since ${\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}$ is even, and is $\Theta|_{\mathcal{A}_{\mathrm{I_c}}}$-invariant, we have a unitary operator $U_{\mathrm{I_c}}$ of $\mathcal{H}_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}$ which implements $\Theta|_{\mathcal{A}_{\mathrm{I_c}}}$ in its GNS space $\big(\mathcal{H}_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}},\ \pi_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}},\ \Omega_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}\big)$. As (17), $\mathrm{Ad}(U_{\mathrm{I_c}})$ determines the even and odd parts of $\mathfrak{M}_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}$. Accordingly by (18), the grading is induced on the center $\mathfrak{Z}_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}$ and it is decomposed into $\mathfrak{Z}^e_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}$ and $\mathfrak{Z}^o_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}$.

For $\mathcal{A}_{\mathrm{I}}$, $\kappa_{\mathrm{I}}(v_{\mathrm{I}})$ gives a unitary operator implementing $\Theta|_{\mathcal{A}_{\mathrm{I}}}$. By the construction of $\big(\mathcal{K},\ \kappa,\ \Psi\big)$,

$$U := \kappa_{\mathrm{I}}(v_{\mathrm{I}}) \otimes U_{\mathrm{I_c}} \in \mathfrak{B}(\mathcal{K}) \tag{28}$$

gives a unitary operator which implements Θ for $\varphi^{\beta H(\mathrm{I})}$. This U gives a grading for $\mathfrak{M}_\kappa$ and it is split into $\mathfrak{M}^e_\kappa$ and $\mathfrak{M}^o_\kappa$. Also by this grading the center $\mathfrak{Z}_\kappa := \mathfrak{M}_\kappa \cap \mathfrak{M}'_\kappa$ is decomposed into $\mathfrak{Z}^e_\kappa$ and $\mathfrak{Z}^o_\kappa$.

Note that the center of the tensor product of a pair of von Neumann algebras is equal to the tensor product of their centers by the commutant theorem (Corollary 5.11 in I.V. of [T]). Since $\mathcal{A}_{\mathrm{I}}$ is a full matrix algebra, and the center of any state on it is trivial, by (27) we have

$$\mathfrak{Z}_\kappa = \mathbf{1}_{\mathrm{I}} \otimes \mathfrak{Z}_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}. \tag{29}$$

Moreover from (28) and (29) it follows that

$$\mathfrak{Z}^e_\kappa = \mathbf{1}_{\mathrm{I}} \otimes \mathfrak{Z}^e_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}, \quad \mathfrak{Z}^o_\kappa = \mathbf{1}_{\mathrm{I}} \otimes \mathfrak{Z}^o_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}, \tag{30}$$

where we have noted that the grading of $\mathfrak{Z}_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}$ is determined by the unitary $U_{\mathrm{I_c}}$. The equalities (29) and (30) give

$$\mathfrak{Z}_\kappa \sim^{\Theta} \mathfrak{Z}_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}}. \tag{31}$$

Combining (31) with (23) we have

$$\mathfrak{Z}_{\varphi_{\mathrm{I_c}}} \sim^{\Theta} \mathfrak{Z}_{{\varphi^{\beta H(\mathrm{I})}}_{\mathrm{I_c}}} \sim^{\Theta} \mathfrak{Z}_\kappa. \tag{32}$$

Since $\big(\mathcal{K},\ \kappa,\ \Psi\big)$ and $\big(\mathcal{H}_\varphi,\ \pi_\varphi,\ \Omega^{\beta H(\mathrm{I})}_\varphi\big)$ are both GNS representations of the same state $\varphi^{\beta H(\mathrm{I})}$ on $\mathcal{A}$, they are apparently unitary equivalent. The representation $\big(\mathcal{H}_\varphi,\ \pi_\varphi,\ \Omega^{\beta H(\mathrm{I})}_\varphi\big)$ obviously induces the same von Neumann algebra for $\big(\mathcal{H}_\varphi,\ \pi_\varphi,\ \Omega_\varphi\big)$, namely $\mathfrak{M}_\varphi$. Hence $\big(\mathcal{K},\ \kappa,\ \Psi\big)$ and $\big(\mathcal{H}_\varphi,\ \pi_\varphi,\ \Omega_\varphi\big)$ are unitary equivalent. Taking the restriction of the unitary map which conjugates those equivalent representations of $\mathcal{A}$ onto the center, we have

$$\mathfrak{Z}_\kappa \sim^{\Theta} \mathfrak{Z}_\varphi. \tag{33}$$

From (32) and (33), it follows that

$$\mathfrak{Z}_{\varphi_{\mathrm{I_c}}} \sim^{\Theta} \mathfrak{Z}_{\varphi}, \tag{34}$$

which is what we would like to have. □

Remark 1. We note that the identification of two von Neumann algebras in (31) and in (34) does not imply that the underlying $\mathbf{C}^*$-systems $\mathcal{A}$ and $\mathcal{A}_{\mathrm{I_c}}$ are conjugated to each other in those representations.

Remark 2. We shall explain that the formula (34) does not hold in general by an example. Take one-dimensional lattice $\mathbb{Z}$ and a site of it, say the origin 0. We prepare a non-factor quasi-free state ρ [MaV] on $\mathcal{A}_{\{0\}_{\mathrm{c}}}$, where $\{0\}_{\mathrm{c}}$ denote the complementary region of $\{0\}$. The factor decomposition of ρ is given by $\rho = 1/2(\psi + \psi\Theta)$, where ψ is a noneven factor state of $\mathcal{A}_{\{0\}_{\mathrm{c}}}$. Take a (unique) product state extension of the tracial state $\mathrm{tr}_{\{0\}}$ of $\mathcal{A}_{\{0\}}$ and ψ to the total system $\mathcal{A}$, which is denoted $\tilde{\psi}$. We see that the state $\tilde{\psi}\Theta$ on $\mathcal{A}$ is equal to the state extension of $\mathrm{tr}_{\{0\}}$ and $\psi\Theta$ to $\mathcal{A}$. Let $\big(\mathcal{H}_{\tilde{\psi}},\ \pi_{\tilde{\psi}},\ \Omega_{\tilde{\psi}}\big)$ be a GNS triplet of $\tilde{\psi}$. Take an odd unitary u of $\mathcal{A}_{\{0\}}$, say, $1/\sqrt{2}(a_0 + a_0^*)$. Define $\xi := 1/\sqrt{2}(\Omega_{\tilde{\psi}} + \pi_{\tilde{\psi}}(u)\Omega_{\tilde{\psi}})$, which is a unit vector of $\mathcal{H}_{\tilde{\psi}}$. Let φ_ξ denote the state determined by $\varphi_\xi(A) := \Big(\pi_{\tilde{\psi}}(A)\xi, \xi\Big)$ for $A \in \mathcal{A}$. It is clear that this φ_ξ is a factor state of $\mathcal{A}$ by its construction. By direct computation, its restriction onto $\mathcal{A}_{\{0\}_{\mathrm{c}}}$ is equal to ρ. Hence φ_ξ is a factor state whose restriction to the subsystem $\mathcal{A}_{\{0\}_{\mathrm{c}}}$ is non-factor.

5 Violation of the local thermal stability for noneven KMS states

For some technical reason we shall work with KMS states [HHuWi] (not directly with Gibbs states). Let α_t ($t \in \mathbb{R}$) be a one-parameter group of $*$-automorphisms of $\mathcal{A}$. A state φ is called an (α_t, β)-KMS state if it satisfies

$$\varphi\big(A\alpha_{i\beta}(B)\big) = \varphi(BA)$$

for every $A \in \mathcal{A}$ and $B \in \mathcal{A}_{ent}$, where $\mathcal{A}_{ent}$ denotes the set of all $B \in \mathcal{A}$ for which $\alpha_t(B)$ has an analytic extension to $\mathcal{A}$-valued entire function $\alpha_z(B)$ as a function of $z \in \mathbb{C}$.

Our dynamics α_t is assumed to be even, namely $\alpha_t\,\Theta = \Theta\,\alpha_t$ for each $t \in \mathbb{R}$. We also put the following assumptions in order to relate α_t with

some $\delta \in D(\mathcal{A}_{\mathrm{loc}})$.
(I) The domain of the generator δ_α of α_t includes $\mathcal{A}_{\mathrm{loc}}$.
(II) $\mathcal{A}_{\mathrm{loc}}$ is a core of δ_α.

The next proposition asserts the equivalence of the KMS and Gibbs conditions under (I, II). The proof was given for the lattice fermion systems in Theorem 7.5 (the implication from KMS to Gibbs under the assumption (I) and Theorem 7.6 (the converse direction under the assumption (I, II)) of [AM2]. The proof for the graded lattice systems can be done in much the same way and we shall omit it. We emphasize that this equivalence does not require the evenness of states, which becomes essential in the proof of Proposition 5.

Proposition 4. *Let α_t be an even dynamics satisfying the conditions* $(\mathrm{I}, \mathrm{II})$. *Let $\delta(\in D(\mathcal{A}_{\mathrm{loc}}))$ be the restriction of its generator δ_α to $\mathcal{A}_{\mathrm{loc}}$. Then a state φ of $\mathcal{A}$ satisfies (α_t, β)-KMS condition if and only if it satisfies (δ, β)-Gibbs condition.*

One would ask whether fermion grading symmetry is perfectly preserved or not for non-zero temperature states. (It is plausible that we can derive more stronger statement about the unbroken symmetry of fermion grading for KMS states than Proposition 1.) We leave this question for future study. Here we show the following rather weak statement. Suppose that there is a nonzero odd element in the center of some even KMS state for even dynamics satisfying (I, II), then there always exist noneven KMS states that do not satisfy the local thermal stability (LTS). This LTS refers to LTS-P in the terminology of [AM1] (not LTS-M there). The content of the local thermal stability condition is summarized in Appendix.

We shall give some preparation. Let φ be an arbitrary even (α_t, β)-KMS state. For $\mathrm{I} \in \mathfrak{F}_{\mathrm{loc}}$, which is now fixed, $\varphi^{\beta H(\mathrm{I})}$ denotes the perturbed state of φ by $\beta H(\mathrm{I})$. From the given $\delta \in D(\mathcal{A}_{\mathrm{loc}})$ and $\mathrm{I} \in \mathfrak{F}_{\mathrm{loc}}$, a new $*$-derivation $\tilde{\delta} \in D(\mathcal{A}_{\mathrm{loc}})$ is given as follows. Let $\Phi \in \mathcal{P}$ denote the potential corresponding to δ. Define a new potential $\tilde{\Phi} \in \mathcal{P}$ by

$$\tilde{\Phi}(\mathrm{J}) := 0, \text{ if } \mathrm{J} \cap \mathrm{I} \neq \emptyset, \quad \text{and} \quad \tilde{\Phi}(\mathrm{J}) := \Phi(\mathrm{J}), \text{ otherwise.} \tag{35}$$

We denote the $*$-derivation corresponding to $\tilde{\Phi}$ by $\tilde{\delta} \in D(\mathcal{A}_{\mathrm{loc}})$. By definition, $\tilde{\delta}$ acts trivially on $\mathcal{A}_{\mathrm{I}}$. The one-parameter group of $*$-automorphisms of $\mathcal{A}$ generated by $\tilde{\delta}$ is equal to the perturbation of α_t by $H(\mathrm{I})$ given in terms of the Dyson-Schwinger expansion series and denoted $\tilde{\alpha}_t$. By Proposition 4 and its proof found in [AM2], $\varphi^{\beta H(\mathrm{I})}$ satisfies $(\tilde{\alpha}_t, \beta)$-KMS condition and $(\tilde{\delta}, \beta)$-Gibbs condition.

We recall the GNS representation $\big(\mathcal{K},\ \kappa,\ \Psi\big)$ of $\varphi^{\beta H(\mathrm{I})}$ previously defined in (24) (25) (26). Let p be a nonzero projection in $\mathfrak{Z}_\kappa$ which has a unique even-odd decomposition $p = p_+ + p_-$, $p_+ \in \mathfrak{Z}_\kappa^e$ and $p_- \in \mathfrak{Z}_\kappa^o$. By (29) we can write $p = \mathbf{1}_\mathrm{I} \otimes q$ with some $q \in \mathfrak{Z}_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}$. Furthermore by (30), we have $p_+ = \mathbf{1}_\mathrm{I} \otimes q_+$ with $q_+ \in \mathfrak{Z}^e_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}$ and $p_- = \mathbf{1}_\mathrm{I} \otimes q_-$ with $q_- \in \mathfrak{Z}^o_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}$. We define a positive linear functional on $\mathcal{A}$ by

$$\varphi^{\beta H(\mathrm{I})}_{<p>}(A) := (\kappa(A)\Psi, p\Psi) \quad \text{for } A \in \mathcal{A}. \tag{36}$$

We take its restriction onto $\mathcal{A}_{\mathrm{I_c}}$. For $A_+ \in \mathcal{A}^e_{\mathrm{I_c}}$, we have

$$\begin{aligned}
\varphi^{\beta H(\mathrm{I})}_{<p>}(A_+) &= (\kappa(A_+)\Psi, p\Psi) \\
&= \Big(\Big(\mathbf{1}_\mathrm{I} \otimes \pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A_+)\Big)\ \Omega_\mathrm{I} \otimes \Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}},\ \Omega_\mathrm{I} \otimes q\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}\Big) \\
&= \Big(\pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A_+)\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}},\ q\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}\Big) \\
&= \Big(\pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A_+)\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}},\ q_+\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}\Big),
\end{aligned} \tag{37}$$

where in the last equality we have used the evenness of $\varphi^{\beta H(\mathrm{I})}$. For $A_- \in \mathcal{A}^o_{\mathrm{I_c}}$,

$$\begin{aligned}
\varphi^{\beta H(\mathrm{I})}_{<p>}(A_-) &= (\kappa(A_-)\Psi, p\Psi) \\
&= \Big(\Big(\kappa_\mathrm{I}(v_\mathrm{I}) \otimes \pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A_-)\Big)\ \Omega_\mathrm{I} \otimes \Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}, \Omega_\mathrm{I} \otimes q\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}\Big) \\
&= \mathrm{tr}_\mathrm{I}(v_\mathrm{I}) \Big(\pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A_-)\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}, q\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}\Big) = 0,
\end{aligned} \tag{38}$$

where we have used $\mathrm{tr}_\mathrm{I}(v_\mathrm{I}) = 0$.

If p is even, i.e. $p = p_+ = \mathbf{1}_\mathrm{I} \otimes q_+$ with $q_+ \in \mathfrak{Z}^e_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}$, then from (37), (38), and $\varphi^{\beta H(\mathrm{I})}(A_-) = 0$ for any $A_- \in \mathcal{A}^o$, it follows that

$$\varphi^{\beta H(\mathrm{I})}_{<p>}(A) = \Big(\pi_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}(A)\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}, q_+\Omega_{\varphi^{\beta H(\mathrm{I})}{}_{\mathrm{I_c}}}\Big) \tag{39}$$

for any $A \in \mathcal{A}_{\mathrm{I_c}}$.

Suppose that $\mathfrak{Z}^o_\kappa$ is not empty. Take any nonzero $f \in \mathfrak{Z}^o_\kappa$. Then $f + f^*$ and $if + (if)^*$ are self-adjoint elements in $\mathfrak{Z}^o_\kappa$. Since at least one of them is nonzero, we can take a self-adjoint element in $\mathfrak{Z}^o_\kappa$ whose operator norm is less than 1 and shall denote such element by f. Let $p_f := 1/2(1+f)$, which is a positive operator. Define a noneven state

$$\psi := 2\varphi^{\beta H(\mathrm{I})}_{<p_f>} \tag{40}$$

by substituting this p_f into p of (36). We easily see that $\psi\Theta$ is equal to $2\varphi^{\beta H(\mathrm{I})}_{<p_{-f}>}$ for $p_{-f} := 1/2(1-f)$. Their averaged state $1/2(\psi+\psi\Theta)$ is obviously equal to $\varphi^{\beta H(\mathrm{I})}$.

Proposition 5. *Let α_t be an even dynamics satisfying* (I, II) *and let φ be an arbitrary even (α_t,β)-KMS state. For $\mathrm{I}\in\mathfrak{F}_{\mathrm{loc}}$, let $\tilde{\alpha}_t$ denote the perturbed dynamics of α_t by the local Hamiltonian $H(\mathrm{I})$. Let $\tilde{\Phi}$ denote the potential for $\tilde{\alpha}_t$ given as (35). If the odd part of the center of the perturbed state $\varphi^{\beta H(\mathrm{I})}$ is not empty, then the noneven $(\tilde{\alpha}_t,\beta)$-KMS states ψ and $\psi\Theta$ given as (40) violate $(\tilde{\Phi},\,\beta)$-LTS condition.*

Proof. Since $\varphi^{\beta H(\mathrm{I})}$ is an $(\tilde{\alpha}_t,\beta)$-KMS state, ψ and $\psi\Theta$ are also $(\tilde{\alpha}_t,\beta)$-KMS states by Theorem 5.3.30 [BR]. Accordingly $\varphi^{\beta H(\mathrm{I})}$, ψ and $\psi\Theta$ are all $(\tilde{\delta},\beta)$-Gibbs states by Proposition 4.

We consider the state restrictions of $\varphi^{\beta H(\mathrm{I})}$, ψ, and $\psi\Theta$ onto $\mathcal{A}_{\mathrm{I_c}}$. Since the even parts of p_f and p_{-f} are both scalar, it follows from (37) that

$$\varphi^{\beta H(\mathrm{I})}|_{\mathcal{A}^e_{\mathrm{I_c}}} = \psi|_{\mathcal{A}^e_{\mathrm{I_c}}} = \psi\Theta|_{\mathcal{A}^e_{\mathrm{I_c}}}.$$

Due to to (38) all of them are even when restricted to $\mathcal{A}_{\mathrm{I_c}}$. Hence we have

$$\varphi^{\beta H(\mathrm{I})}|_{\mathcal{A}_{\mathrm{I_c}}} = \psi|_{\mathcal{A}_{\mathrm{I_c}}} = \psi\Theta|_{\mathcal{A}_{\mathrm{I_c}}}. \tag{41}$$

Denote the local Hamiltonians for the new potential $\tilde{\Phi}$ determined by the formula (Φ-e) by $\{\tilde{H}(\mathrm{J})\}_{\mathrm{J}\in\mathfrak{F}_{\mathrm{loc}}}$. From (35) it follows that

$$\tilde{H}(\mathrm{I}) = 0,$$

and hence

$$\varphi^{\beta H(\mathrm{I})}(\tilde{H}(\mathrm{I})) = \psi(\tilde{H}(\mathrm{I})) = \psi\Theta(\tilde{H}(\mathrm{I})) = 0. \tag{42}$$

We compute conditional entropy of $\varphi^{\beta H(\mathrm{I})}$, ψ and $\psi\Theta$ for the finite region I. The definition of conditional entropy is given in (47). Noting (14) we have

$$\widetilde{S}_{\mathrm{I}}(\varphi^{\beta H(\mathrm{I})}) = -S(\mathrm{tr}_{\mathrm{I}}\circ\varphi^{\beta H(\mathrm{I})}|_{\mathcal{A}_{\mathrm{I_c}}},\,\varphi^{\beta H(\mathrm{I})}) = -S(\mathrm{tr}_{\mathrm{I}}\circ\varphi^{\beta H(\mathrm{I})}|_{\mathcal{A}_{\mathrm{I_c}}},\,\mathrm{tr}_{\mathrm{I}}\circ\varphi^{\beta H(\mathrm{I})}|_{\mathcal{A}_{\mathrm{I_c}}}) = 0, \tag{43}$$

which is the maximum value of $\widetilde{S}_{\mathrm{I}}(\cdot)$.

For ψ, using (41) and then (14) we have

$$\begin{aligned}\widetilde{S}_{\mathrm{I}}(\psi) &= -S(\mathrm{tr}_{\mathrm{I}}\circ\psi|_{\mathcal{A}_{\mathrm{I_c}}},\,\psi)\\ &= -S(\mathrm{tr}_{\mathrm{I}}\circ\varphi^{\beta H(\mathrm{I})}|_{\mathcal{A}_{\mathrm{I_c}}},\,\psi)\\ &= -S(\varphi^{\beta H(\mathrm{I})},\,\psi).\end{aligned} \tag{44}$$

Since $\varphi^{\beta H(\mathrm{I})} \neq \psi$, the former is even and the latter is noneven, it follows from this equality and the strictly positivity of relative entropy (see [A4]) that

$$\widetilde{S}_{\mathrm{I}}(\psi) < 0.$$

By the automorphism invariance (acting two states in the argument) of relative entropy, we have

$$\widetilde{S}_{\mathrm{I}}(\psi\Theta) = \widetilde{S}_{\mathrm{I}}(\psi) < 0. \tag{45}$$

Substituting (42), (43), and (45) into (48), we obtain

$$F^{\tilde{\Phi}}_{\mathrm{I},\beta}(\psi) = F^{\tilde{\Phi}}_{\mathrm{I},\beta}(\psi\Theta) < F^{\tilde{\Phi}}_{\mathrm{I},\beta}(\varphi^{\beta H(\mathrm{I})}) = 0. \tag{46}$$

This strict inequality with (41) shows that ψ and $\psi\Theta$ do not satisfy $(\tilde{\Phi}, \beta)$-LTS condition (49), although both of them satisfy $(\tilde{\delta}, \beta)$-Gibbs condition. $\square$

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A Appendix

A.1 Local thermal stability (LTS) condition

Let $(\mathcal{A},\ \{\mathcal{A}_{\mathrm{I}}\}_{\mathrm{I}\in\mathfrak{F}_{\mathrm{loc}}})$ be a lattice system considered in § 4. In [AM1] the local thermal stability (LTS) is studied for the lattice fermion systems. It is easy to see that the same formulation is available for the graded lattice systems under consideration.

Let ω be a state of $(\mathcal{A},\ \{\mathcal{A}_{\mathrm{I}}\}_{\mathrm{I}\in\mathfrak{F}_{\mathrm{loc}}})$. For $\mathrm{I}\in\mathfrak{F}_{\mathrm{loc}}$, the conditional entropy of ω is defined in terms of the relative entropy (19) by

$$\widetilde{S}_{\mathrm{I}}(\omega) := -S(\mathrm{tr}_{\mathrm{I}}\circ\omega|_{\mathcal{A}_{\mathrm{I_c}}},\ \omega) = -S(\omega\cdot E_{\mathrm{I_c}},\ \omega) \le 0, \tag{47}$$

where $E_{\mathrm{I_c}}$ is the conditional expectation onto $\mathcal{A}_{\mathrm{I_c}}$ with respect to the tracial state and $\omega\cdot E_{\mathrm{I_c}}(A) := \omega(E_{\mathrm{I_c}}(A))$ for $A\in\mathcal{A}$.

Let $\varPhi\in\mathcal{P}$. The conditional free energy of ω for $\mathrm{I}\in\mathfrak{F}_{\mathrm{loc}}$ is given by

$$F^{\varPhi}_{\mathrm{I},\beta}(\omega) := \widetilde{S}_{\mathrm{I}}(\omega) - \beta\omega(H(\mathrm{I})), \tag{48}$$

where $H(\mathrm{I})$ is a local Hamiltonian for I with respect to $\varPhi$.

Definition 4. *Let $\varPhi$ be a potential in $\mathcal{P}$. A state φ of $\mathcal{A}$ is said to satisfy the local thermal stability condition for $\varPhi$ at inverse temperature β or $(\varPhi,\ \beta)$-LTS condition if for each* $\mathrm{I}\in\mathfrak{F}_{\mathrm{loc}}$

$$F^{\varPhi}_{\mathrm{I},\beta}(\varphi) \ge F^{\varPhi}_{\mathrm{I},\beta}(\omega) \tag{49}$$

for any state ω satisfying $\omega|_{\mathcal{A}_{\mathrm{I_c}}} = \varphi|_{\mathcal{A}_{\mathrm{I_c}}}$.

There is the other definition of local thermal stability in [AM1] that has the same variational principle formula as above but takes the commutant algebra $\mathcal{A}'_{\mathrm{I}}$ as the complementary outside system of a local region I instead of $\mathcal{A}_{\mathrm{I_c}}$. We shall call this alternative local thermal stability condition LTS$'$ condition, where the superscript '$'$' stands for the commutant. (Also by '$'$' we mean that this formalism is not so natural compared to Definition 4 if we respect the given quasi-local structure. Nevertheless, there are some *mathematically* good points with LTS$'$ as will be noted in the next paragraph.)

The equivalence of KMS and LTS$'$ conditions holds for the lattice fermion systems without assuming the evenness on states. For our LTS, on the contrary, such evenness assumption is required in deriving its equivalence to the KMS condition. (The formalism of LTS$'$ using commutants for complementary outside systems makes it possible to exploit the known arguments for quantum spin lattice systems [AS].)