Local Synchronization Rate in Oscillator Lattices

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Oscillator lattices with random frequency distribution are numerically studied. We measured the local synchronization rate R in various oscillator lattices and found that on the lattices with more connections the oscillators are not always easier to synchronize. We also found strange behavior in which R is decreased as the coupling strength K is increased.

§1. Introduction

Synchronization of coupled oscillators in various types of networks has recently attracted much attention. One of the type which has been studied well is the system of globally coupling oscillators. In this paper, we study oscillator lattices with nearest-neighbor coupling. Specifically, we use phase oscillators with sine-coupling:

$$\frac{d\phi_i}{dt} = \omega_i - K \sum_{j \in n.n.\ of\ i} \sin(\phi_i - \phi_j) \ , \tag{1·1}$$

where ω_i obeys Gaussian distribution with mean 0 and variance 1. Global frequency order of such systems was originally studied in Refs. 1) and 2), inspired by the discovery of entrainment transition in globally coupled oscillators.[3)] Hong et al.[4)] recently analyzed global phase order in high-dimensional oscillator lattices.

Different from the previous studies on global order, we study local frequency order in the present paper, measuring the *local synchronization rate* R

$$R \equiv \left\langle \frac{\text{Number of synchronized nearest-neighbor pairs}}{\text{Number of nearest-neighbor pairs}} \right\rangle , \tag{1·2}$$

where $\langle \cdots \rangle$ implies random frequency average and we define synchronization between oscillators i and j when their mean frequencies $\tilde{\omega}_i$ and $\tilde{\omega}_j$ satisfy the condition

$$|\tilde{\omega}_i - \tilde{\omega}_j| < \frac{\pi}{T} \tag{1·3}$$

in the numerical simulation of sufficiently long time T.[1)] If various lattices with a common K are compared, it may be considered that a lattice with more neighboring oscillators generally tends to give a larger R, because strong coupling to neighbors implies easy synchronization. In the next section, however, the results of our numerical simulations show that this naive scenario is not always true.

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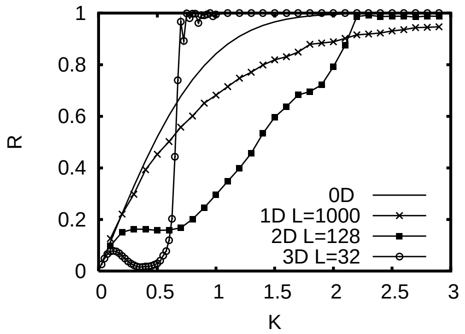

Fig. 1. Local synchronization rate in 1D, 2D and 3D square lattices with the length L on a side. 0D means the analytical result of coupled two oscillators, shown for comparison.

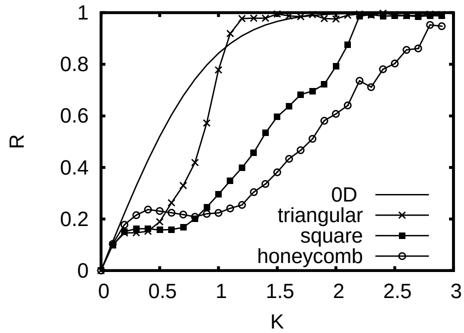

Fig. 2. Local synchronization rate in 2D triangular, square and honeycomb lattices with the length $L = 128$ on a side. The meaning of 0D is the same as in Fig. 1.

§2. Numerical simulations

Numerical simulations for Eq. (1·1) were carried out using the fourth-order Runge-Kutta method with the time step 0.01. Figure 1 shows the local synchronization rate R as a function of K in the 1D, 2D and 3D square lattices with the length L on a side under the periodic boundary condition. The mean frequency of each oscillator was measured for the time $T = 10000$. Random frequency average of Eq. (1·1) was omitted since we considered that the system size is sufficiently large. In this figure, for small K, R decreases with the dimension of the square lattice. We also notice strange behavior in 2D and 3D lines: There is a region that R decreases as K is increased.

Figure 2 shows another example of strange behavior of R. In this figure, we used various 2D lattices. The number of nearest-neighbor oscillators is 6, 4 and 3 for the triangular, square and honeycomb lattice respectively. For small K, R decreases as the number of nearest-neighbor oscillators is increased. The tendency for R to decrease as K is increased is clear in the honeycomb lattice.

§3. Size dependency of R

In the previous section, we found that on the lattices with more connections the oscillators are not always easier to synchronize. However, is this result unusual? We may consider that neighboring oscillators with random frequency disturb synchronization. To show that neighboring oscillators work in favor of synchronization, we consider globally coupled oscillators as an extreme case:

$$\frac{d\phi_i}{dt} = \omega_i - K \sum_{j=1}^{N} \sin(\phi_i - \phi_j) \ . \tag{3·1}$$

Note that the coupling strength K is not divided by N unlike the usual case.

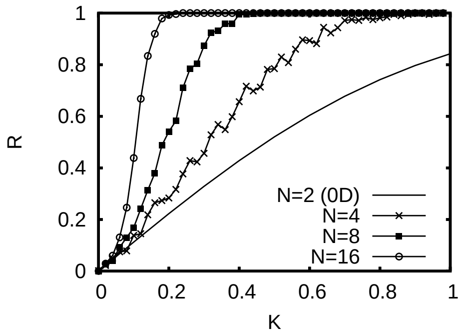

Fig. 3. Size dependency of R in global coupled oscillators.

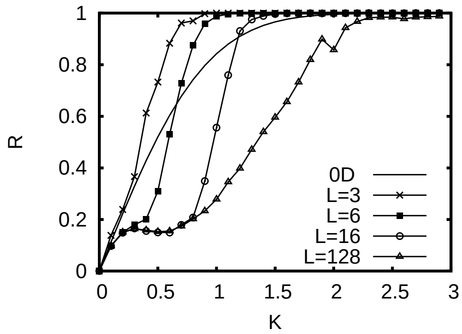

Fig. 4. Size dependency of R in 2D square lattices.

Figure 3 shows size dependency of R in globally coupled oscillators. From this figure, we see that as the number of neighboring oscillators is increased, R is surely increased. Figure 4 shows size dependency of R in 2D square lattices. From this figure, we see that as the system size becomes larger, R becomes smaller. These results implies that in order to emerge strange behavior as seen in the previous section, the extendedness of the system is essential.

§4. A problem on numerical simulations

In our numerical simulations, we have used $T = 10000$ as the simulation time so far. Is this T sufficiently long? In Figs. 5 and 6, we show T dependency of R in 1D and 2D lattices. From these figures, we see that the simulations of 1D lattice have no problem, but R in 2D (and 3D, not shown) lattices does not converge yet.

Let us consider the possibility of giving the correct R values numerically. Consider probability distribution $P(|\Delta\tilde{\omega}|)$ of the absolute value of the difference between the mean frequencies of the neighboring pairs: $\Delta\tilde{\omega} = \tilde{\omega}_i - \tilde{\omega}_j$. If we assume the form

$$P(|\Delta\tilde{\omega}|) = R\delta(|\Delta\tilde{\omega}|) + f(|\Delta\tilde{\omega}|) \ , \tag{4·1}$$

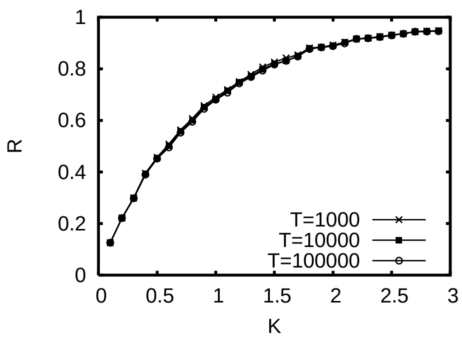

Fig. 5. T dependency of R in 1D lattice (L=1000).

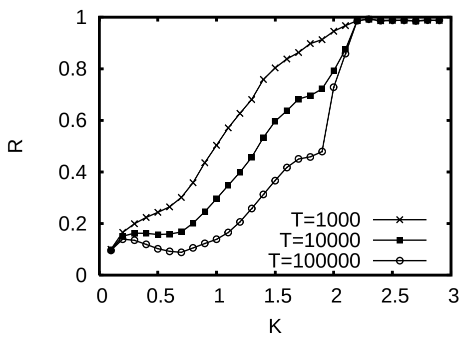

Fig. 6. T dependency of R in 2D square lattice (L=128).

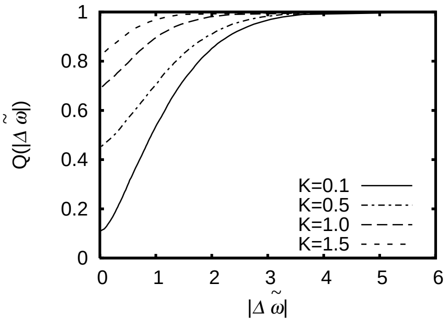

Fig. 7. Cumulative probability distribution $Q(|\Delta\tilde{\omega}|)$ in 1D lattice ($L = 10000$).

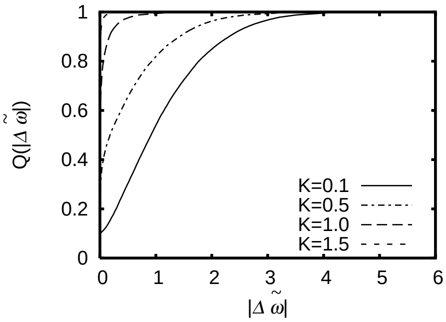

Fig. 8. Cumulative probability distribution $Q(|\Delta\tilde{\omega}|)$ in 2D square lattice ($L = 128$).

where f is non-singular function, the height of the delta peak at $|\Delta\tilde{\omega}| = 0$ is the local synchronization rate R.

Though P itself seems hard to define numerically, cumulative probability distribution $Q(|\Delta\tilde{\omega}|) \equiv \int_0^{|\Delta\tilde{\omega}|} P(x)dx$ is easier to calculate. Figures 7 and 8 show $Q(|\Delta\tilde{\omega}|)$. From these figures, we see that since $R = Q(0)$, it appears that though R is well-defined in 1D lattice, R cannot be determined from some lines of 2D lattice, because $Q'(0)$ seems to diverge.

§5. Summary and discussion

We numerically studied the local synchronization rate R in oscillator lattices and found that on the lattices with more connections the oscillators are not always easier to synchronize. We also found strange behavior in which R is decreased as the coupling strength K is increased. From the results of the system size dependency of R, we understood that in order to emerge such behavior of R, the extendedness of the system is essential. We believe that chaotic behavior which comes from the extendedness is important, but the analysis is not yet done. From the results of the simulation time dependency of R, we could not say that we obtain reliable results in 2D (and 3D) systems. It seems that 2D systems have essential difficulty in calculating R. Clarifying the difference between 1D and 2D systems is one of the remaining problems. Fukuda et al.[5] carried out an experiment on synchronization in BZ-reaction beads arranged on 2D lattices. Though the main purpose of their experiment was the study on noise effects on synchronization, in the noiseless case, they obtained the results similar to our numerical results. Therefore, we expect that our results obtained in this study is generic.

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