Bifurcation Study of Kuramoto Transition of Random Oscillator Network

Takashi Ichinomiya

Laboratory of Nonlinear Studies and Computation, Research Institute for Electronic Science, Hokkaido University, Sapporo 060-0812, Japan

(Received June 17, 2005)

We study the Kuramoto transition of oscillators in random network and Barabási-Albert network model. In both cases, the results of numerical simulation show good coincidence with the mean-field analysis.

Synchronization is a widely spreading behaviour in nature. It is seen in the firing of neurons in our brain, flushing of fireflies, and some chemical reactions. One of the great progress of the study of synchronization was made by Kuramoto.[1] He showed that the dynamics of many kinds of oscillators can be reduced to the dynamics of phase oscillator, and analyzed the synchronization of it. One of the important results of his works is the discovery of the Kuramoto transition, the synchronization of globally coupled oscillators.

However, global coupling is seldom found in nature. For example, a neuron in the brain is not connected to all other neurons. It is connected to a finite number of neurons, and the global coupling is far from real network of neurons. Recently, the study of complex network has been developed.[2] In the complex network, each node links to a finite number of other nodes. In this decade, many real networks such as the internet, metabolic networks, neurons in the brain, food web, and co-authorship of papers have been investigated. We have realized that many real networks have common structure, such as scale-free degree distribution $P(k) \propto k^{-\gamma}$, where $P(k)$ is the distribution function of degree of nodes.

Therefore, the study of synchronization in complex network is important problem to study the synchronization in real systems. The study in this regard had been mainly carried out by numerical simulations. After the simulation by Watts,[3] Hong et al. carried out detailed investigation of the Kuramoto transition in Watts-Strogatz model and obtained the phase diagram.[4] Moreno and Pacheco studied the synchronization in the Barabási-Albert(BA) network and concluded that the critical coupling constant K_c is not zero in the BA network.[5]

Recently, we developed the analytical theory of the Kuramoto transition in random networks.[6),7] Using the mean-field approximation, we concluded that as the size of network approaches infinity, K_c approaches 0 in the scale-free random network if $\gamma \leq 3$.

This result seems to contradict to that of Moreno and Pacheco. It is usually believed that the property of the BA network is similar to the $\gamma = 3$ random scale-free network. If K_c is not 0 in the BA network, it suggests that the BA network gives different dynamics of oscillators from that given by the random scale-free network.

However, this discrepancy may be due to the difference of the order parameter. Moreno and Pacheco used $\langle e^{i\theta} \rangle$, where $\langle \cdots \rangle$ means the average over all nodes, as the order parameter. On the other hand, our paper used $\langle ke^{i\theta} \rangle / \langle k \rangle$ as the order parameter, where k is the degree of each node. The difference of the order parameter will lead to the inconsistence of the critical coupling. Moreover, we note the coupling dependence of the order parameter may be different from that in globally coupled oscillators. As we showed,[7] the critical coupling is not proportional to $\sqrt{K - K_c}$ in the scale-free network at $N \to \infty$, since the higher-order term in the amplitude equation is not negligible. The coefficients of the higher-order term in the amplitude equation diverges at $N \to \infty$, which makes the determination of the critical coupling difficult.

Of course, there is a possibility that K_c for random network is different from that for the BA model. Though many numerical results suggest that the BA network seems similar to the random scale-free network, we should be careful to state that the synchronization in the BA network is essentially same as that in the random network. In this study, we carry out the simulation for scale-free random network model and the BA model, and investigate the difference of the synchronization in each model.

The model we study is coupled phase oscillators, whose dynamics are described by

$$\frac{d\theta_i}{dt} = \omega_i + K \sum_j a_{i,j} \sin(\theta_j - \theta_i), \tag{1}$$

where θ_i and ω_i are the phase and velocity of oscillator at node i, respectively. $a_{i,j}$ is the adjacent matrix that describes the network. If node i and j are connected, $a_{i,j} = 1$ and otherwise $a_{i,j} = 0$. In this paper, we examine the synchronization on two network models. First one is the scale-free network produced by the BA model. In our simulation, we use the network whose mean degree is $\langle k \rangle = 10.0$. Such a network can be produced by repeatedly adding new nodes which have 5 edges. The other network is the random networks that have the same degree distribution as the BA model. From our previous studies on random networks, the synchronization of this network will be well described by the mean-field approximation. We assume that the distribution of ω_i is given by $g(\omega) = \frac{1}{\sqrt{\pi}} \exp(-\omega^2)$.

Before presenting the results of simulations, we summarize the results of the mean-field approximation.[6),7)] The mean-field analysis of random network models shows that the order parameter is given by $r = \langle k_i e^{-i\theta_i} \rangle / \langle k_i \rangle$, where k_i is the degree of node i. We note that this order parameter coincides with that for globally coupled oscillator, $\langle e^{-i\theta_i} \rangle$, if we take $P(k_i) = \delta(k_i - N + 1)$ at $N \to \infty$. Using this order parameter, the phase distribution ρ is given by

$$\rho(\theta, \omega, k) = \begin{cases} \delta(\theta - \arcsin(\frac{\omega}{Kkr})) & \text{if } \frac{|\omega|}{Kkr} \leq 1, \\ \frac{C(k,\omega)}{|\omega - Kkr \sin\theta|} & \text{otherwise,} \end{cases} \tag{2}$$

where $C(k, \omega)$ is the normalization factor that makes $\int \rho(\theta, \omega, k) d\theta = 1$. We note that ρ depends on not only ω but also k, the degree of the node. This dependence does

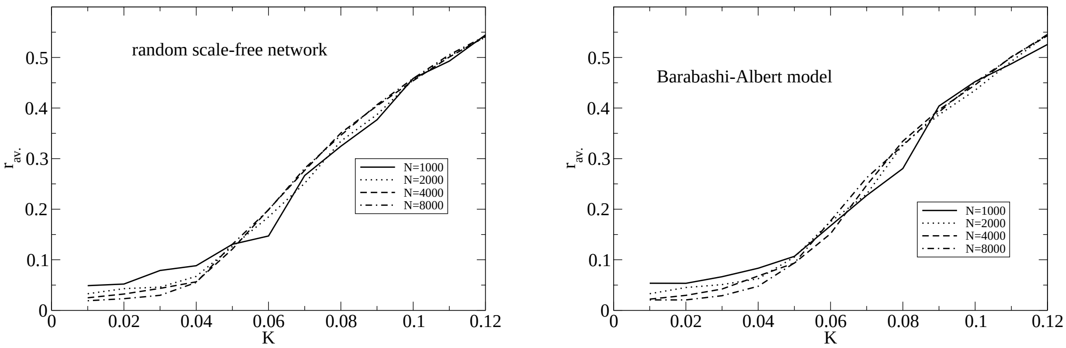

Fig. 1. The size and coupling dependence of the order parameter in the random scale-free network model and the Barabási-Albert model.

not appear in globally coupled oscillators. In addition, the center-manifold reduction of the Kuramoto transition suggests that the singularity of the order parameter at $K = K_c$ is strongly suppressed in random scale-free networks.

First we show the coupling dependence of the order parameter. In Fig. 1, we plot the mean value of order parameter $\langle k_i e^{i\theta}\rangle/\langle k_i\rangle$ for the both model over 50 samples.

In the case of $N = 1000$, the order parameter strongly fluctuates. This suggests that at small N the fluctuation is so strong that we cannot obtain reliable order parameter from 50 samples. However, at $N = 8000$ the order parameter smoothly depends on the coupling. There is no significant difference of the order parameter between two network models at $N = 8000$. This result suggests that the Kuramoto transition in the BA network is similar to that in the random scale-free network model. We also note that the coupling dependence of the order parameter seems different from the globally coupled model, $r \propto \sqrt{K - K_c}$. As K decreases, the order parameter smoothly goes to 0. This result coincides with the analysis of our previous paper.[7] In the previous paper, we investigated the amplitude equation of the order parameter r, $\frac{dr}{dt} = p_0 r + \sum p_n r^{2n+1}$, and found that p_n diverges in random scale-free network at $N \to \infty$. Since the square-root dependence of r on $K - K_c$ is observed only when higher-order term of the amplitude equation can be neglected, it is not observed in scale-free networks.

One of the important results obtained from mean-field theory is the correlation between distribution of phase and degree, given by Eq. (2). In Fig. 2, we plot the distribution of phase for the random and BA network at $k = 5$ and 12, $K = 0.10$, $N = 4000$. The distribution of phase ρ shows good coincidence between these two models. In both models, the distribution of θ is well described by Eq. (2). This result strongly suggests that the Kuramoto transition in the BA model can be approximated by that in the random scale-free network model. Therefore we conclude that the Kuramoto transition in the BA model is the same as that in random scale-free network model, both qualitatively and quantitatively.

In conclusion, we study the Kuramoto transition in the scale-free random network model and the BA model. The result of the scale-free random network model coincides with the mean-field analysis, and the result of the BA model shows no inconsistency with that of the random network model. These results suggest that the

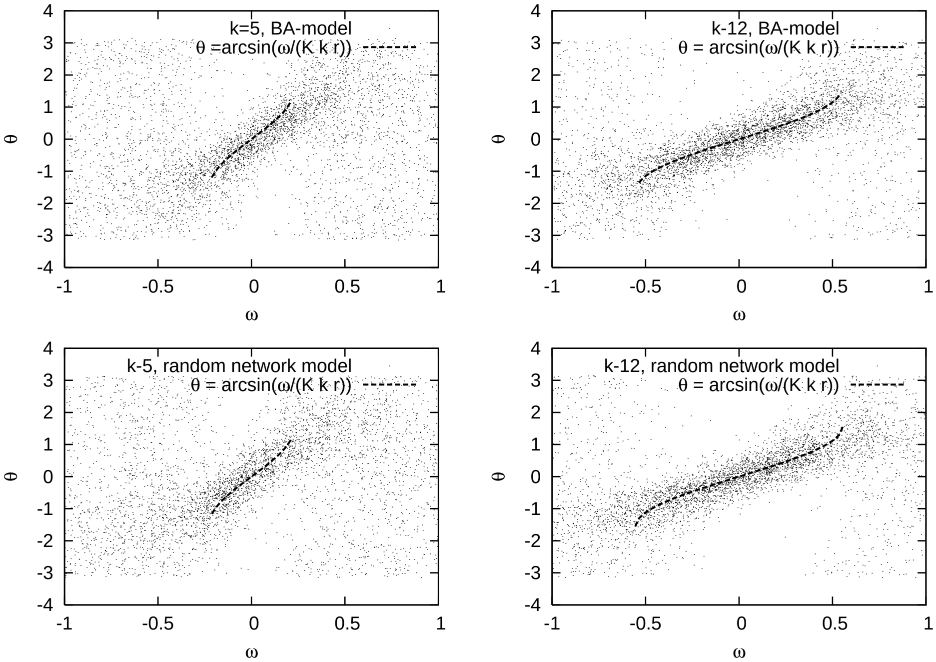

Fig. 2. (ω, θ) distribution of oscillators with degree 5 (left) and 12 (right) in the BA (upper) and random scale-free (lower) networks.

Kuramoto transition in the BA model is qualitatively and quantitatively the same as that in the random network model. We also note that the relaxation time estimation by Moreno and Pacheco also suggests the validity of the mean-field analysis in the BA model. They estimated the relaxation time $\langle\tau\rangle$, the time for synchronization of the node whose degree is k from numerical simulation. They concluded $\langle\tau\rangle$ is proportional to $k^{-\nu}$, where $\nu = 0.96$. On the other hand, the mean-field analysis predicts $\nu = -1$, because each oscillator couples to the mean-field with strength proportional to Kk. The value obtained by the mean-field approximation is very close to the one obtained by their simulation, and this result strongly suggests that the mean-field theory works well for the synchronization in the BA network.

We acknowledge Y. Nishiura, T. Yanagita, M. Iima and Y. Nakao for fruitful discussion.

References

1) Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
2) R. Albert and A-L. Barabási, Rev. Mod. Phys. **74** (2002), 47.
3) D. Watts, *Small Worlds* (Princeton University Press, Princeton, NJ, 1999).
4) H. Hong, M. Y. Choi and B. J. Kim, Phys. Rev. E **65** (2002), 026139.
5) Y. Moreno and A. F. Pacheco, Europhys. Lett. **68** (2004), 603.
6) T. Ichinomiya, Phys. Rev. E **70** (2004), 026116.
7) T. Ichinomiya, Prog. Theor. Phys. **113** (2005), 1.