Properties of the Dirac–Weyl operator with a strongly singular gauge potential[a)]

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Considered is a quantum system of a charged particle moving in the plane $\mathbf{R}^2$ under the influence of a perpendicular magnetic field concentrated on some fixed isolated points in $\mathbf{R}^2$. Such a magnetic field is represented as a finite linear combination of the two-dimensional Dirac delta distributions and their derivatives, so that the gauge potential of the magnetic field also may be strongly singular at those isolated points. Properties of the Dirac–Weyl operator with such a singular gauge potential are investigated. It is seen that some of them depend on whether the magnetic flux is locally quantized or not. Particular attention is paid to the zero-energy state. For each of the self-adjoint realizations of the Dirac–Weyl operator, the number of the zero-energy states is computed. It is shown that, in the present case, a theorem of Aharonov and Casher [Phys. Rev. A **19**, 2461 (1979)], which relates the total magnetic flux to the number of zero-energy states, does not hold. It is also proven that the spectrum of every self-adjoint extension of the minimal Dirac–Weyl operator is equal to $\mathbf{R}$.

I. INTRODUCTION

In a previous paper,[1] the author considered, from an operator-theoretical point of view, a quantum system of a charged particle with charge $q \in \mathbf{R} \setminus \{0\}$ moving in the plane $\mathbf{R}^2$ under the influence of a perpendicular magnetic field, where the gauge potential of the magnetic field is allowed to be strongly singular at some fixed isolated points $\mathbf{a}_\nu \in \mathbf{R}^2$, $\nu = 1,\ldots,n$. A basic result in Ref. 1 is that the momentum operators of the system commute in the strong sense if and only if the magnetic flux is locally quantized, i.e., the magnetic flux of every rectangle not intersecting $\mathbf{a}_\nu$ ($\nu = 1,\ldots,n$) is an integer multiple of $2\pi/q$. This result was applied to show that there is a class of non-Schrödinger representations of the canonical commutation relations associated with the physical situation in which the magnetic flux is not locally quantized, which corresponds to the occurence of the Aharonov–Bohm effect (cf. also Ref. 2). It was also shown in Ref. 1 that, if the magnetic flux is locally quantized, then the magnetic field must be concentrated on the points $\mathbf{a}_\nu$, $\nu = 1,\ldots,n$, and hence it is represented as a finite linear combination of the two-dimensional Dirac distributions and their derivatives (but the converse is not true).

In connection with these results, it is interesting to ask how other properties of the quantum system depend on whether the magnetic flux is locally quantized or not. This is a motivation of the present work. Thus, in this article, restricting ourselves to the case where the magnetic field is concentrated on some isolated points, we investigate properties of the quantum system.

Another motivation of this work comes from an interesting paper by Aharonov and Casher,[3] who showed that the Dirac–Weyl operator with a regular gauge potential (or equivalently the corresponding Pauli Hamiltonian) has exactly N zero-energy states with N being the largest integer strictly less than $(q/2\pi) \times$ (the total magnetic flux) > 0 (cf. also Refs. 4, 5, and references therein). Our question is: Does this result also hold for the quantum system

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with such a singular magnetic field as described above? We shall give in this article a negative answer to this question, deriving new formulas on the number of the zero-energy states.

The outline of the present article is as follows. In Sec. II, we describe the quantum system we are going to study and present preliminary results, which include some of the results obtained in Ref. 1. We are primarily interested in properties of the Dirac–Weyl operator associated with the quantum system, but, for a comparison, we also consider the Schrödinger Hamiltonian defined via a quadratic form. We show that the Schrödinger Hamiltonian has no zero-energy states (Theorem 2.5). In Sec. III we discuss some of the operator theoretical aspects of the Dirac–Weyl operator Q with a "natural" domain. It is proven that, if the magnetic flux is locally quantized, then Q is self-adjoint and its square is equal to the Schrödinger Hamiltonian (Theorem 3.1). In the case where the magnetic flux is not locally quantized, we show by an explicit construction that there exist at least two self-adjoint extensions of Q. Section IV is concerned with zero-energy states. We first show that, if the magnetic flux is locally quantized, then Q has no zero-energy states (Theorem 4.2). Hence the Aharonov–Casher theorem does not hold for the Dirac–Weyl Hamiltonian Q. To examine other possibilties that the quantum system under consideration has zero-energy states, we consider the minimal version $Q_{\min}$ of Q with domain $D(Q_{\min})=C_0^\infty(\mathbf{R}^2\setminus\{\mathbf{a}_1,\ldots,\mathbf{a}_n\})$. It is shown that $\bar{Q}_{\min}$, the closure of $Q_{\min}$, has no zero-energy states (Lemma 4.3). We construct two self-adjoint extensions $Q_{\min}^{(j)}$, $j=1,2$, of $Q_{\min}$. Identifying explicitly their kernel, we prove that, under certain conditions, they have degenerate zero-energy states determined by the magnetic flux at each point $\mathbf{a}_\nu$ and the number of the zero-energy states increases, tending to infinity, as $n\to\infty$ (Theorem 4.7 and Proposition 4.9). This may be a remarkable phenomenon. We also discuss the relevance of these results to the index theory as well as supersymmetry. In the last section, we identify the spectrum of the Dirac–Weyl operators introduced in the previous sections. We show that the spectrum of every self-adjoint extension of $Q_{\min}$ is equal to $\mathbf{R}$ (Theorem 5.1). Thus the spectrum of the Dirac–Weyl operators that are self-adjoint extensions of $Q_{\min}$ does not depend on whether the magnetic flux is locally quantized or not.

II. PRELIMINARY RESULTS

We consider a quantum system of a charged particle with charge $q\in\mathbf{R}\setminus\{0\}$ moving in the plane $\mathbf{R}^2$ under the influence of a perpendicular magnetic field B concentrated on some fixed isolated points $\mathbf{a}_\nu=(a_{\nu1},a_{\nu2})\in\mathbf{R}^2$, $\nu=1,\ldots,n$. Such a field B is given by a real distribution of the form

$$B(\mathbf{r})=\sum_{\nu=1}^{n}\sum_{0<\alpha+\beta<m}C_{\alpha,\beta}^{(\nu)}D_x^\alpha D_y^\beta\delta(\mathbf{r}-\mathbf{a}_\nu),\quad \mathbf{r}=(x,y)\in\mathbf{R}^2, \tag{2.1}$$

with a non-negative integer m and real constants $C_{\alpha,\beta}^{(\nu)}$, where D_x and D_y denote the distributional partial differential operators in x and y, respectively, and $\delta(\mathbf{r})$ is the Dirac delta distribution on $\mathbf{R}^2$ (e.g., Ref. 6, Chap. II, Sec. 4.5). A gauge potential $\mathbf{A}(\mathbf{r})$ of the magnetic field B is defined to be an $\mathbf{R}^2$-valued function $\mathbf{A}=(A_1,A_2)$ on the domain

$$M=\mathbf{R}^2\setminus\{\mathbf{a}_1,\ldots,\mathbf{a}_n\} \tag{2.2}$$

such that

$$B=D_xA_2-D_yA_1 \tag{2.3}$$

in the sense of distributions on $\mathbf{R}^2$.

We denote by Δ the two-dimensional Laplacian

$$\Delta=D_x^2+D_y^2. \tag{2.4}$$

Using the well-known formula

$$\Delta \log|\mathbf{r}-\mathbf{a}_\nu| = 2\pi\delta(\mathbf{r}-\mathbf{a}_\nu),$$

we see that the distribution

$$\phi(\mathbf{r}) = \sum_{\nu=1}^{n} \sum_{0<\alpha+\beta<m} \frac{C_{\alpha,\beta}^{(\nu)}}{2\pi} D_x^\alpha D_y^\beta \log|\mathbf{r}-\mathbf{a}_\nu| \tag{2.5}$$

satisfies

$$\Delta\phi(\mathbf{r}) = B(\mathbf{r}). \tag{2.6}$$

This allows us to take as a gauge potential of the magnetic field

$$\mathbf{A} = (A_1, A_2) = (-D_y\phi, D_x\phi). \tag{2.7}$$

Explicitly we have

$$A_1(\mathbf{r}) = -\sum_{\nu=1}^{n} \sum_{0<\alpha+\beta<m} \frac{C_{\alpha,\beta}^{(\nu)}}{2\pi} D_x^\alpha D_y^\beta \left(\frac{y-a_{\nu 2}}{|\mathbf{r}-\mathbf{a}_\nu|^2}\right), \quad \mathbf{r}\in M, \tag{2.8}$$

$$A_2(\mathbf{r}) = \sum_{\nu=1}^{n} \sum_{0<\alpha+\beta<m} \frac{C_{\alpha,\beta}^{(\nu)}}{2\pi} D_x^\alpha D_y^\beta \left(\frac{x-a_{\nu 1}}{|\mathbf{r}-\mathbf{a}_\nu|^2}\right), \quad \mathbf{r}\in M. \tag{2.9}$$

Note that A_j $(j=1,2)$ can have strong singularity at each point $\mathbf{r}=\mathbf{a}_\nu$, $\nu=1,\ldots,n$, with order more than one.

Definition (2.7) implies also that $\mathbf{A}$ is divergence-free on M. It follows that

$$\tilde{A}(z) = A_2(x,y) + iA_1(x,y), \quad z = x+iy, \tag{2.10}$$

must be a holomorphic function on the domain

$$D_M = \mathbf{C}\setminus\{a_\nu\}_{\nu=1}^{n}, \tag{2.11}$$

where $a_\nu = a_{\nu 1} + ia_{\nu 2}$. In fact, we can show that

$$\tilde{A}(z) = \frac{1}{2\pi} \sum_{\nu=1}^{n} \sum_{k=0}^{m} \frac{C_k^{(\nu)}}{(z-a_\nu)^{k+1}}, \quad z\in D_M, \tag{2.12}$$

with

$$C_k^{(\nu)} = (-1)^k k! \sum_{\alpha=0}^{k} C_{\alpha,k-\alpha}^{(\nu)} i^{k-\alpha}. \tag{2.13}$$

We use a system of units where the light speed c and the Planck constant $\hbar$ are equal to 1. Let

$$p_1 = -iD_x, \quad p_2 = -iD_y, \tag{2.14}$$

in $L^2(\mathbf{R}^2)$. The momentum operator $\mathbf{P} = (P_1, P_2)$ with the gauge potential $\mathbf{A}$ is defined by

$$P_j = p_j - qA_j, \quad j=1,2, \tag{2.15}$$

in $L^2(\mathbf{R}^2)$ with domain $D(P_j)=D(p_j)\cap D(A_j)$. Let

$$M_1=\{(x,y)\in\mathbf{R}^2 \mid y\neq a_{\nu 2},\quad \nu=1,2,\ldots,n\},$$

$$M_2=\{(x,y)\in\mathbf{R}^2 \mid x\neq a_{\nu 1},\quad \nu=1,\ldots,n\}.$$

The following theorem is a special case of Theorem 3.2 in Ref. 1.

Theorem 2.1: Each P_j $(j=1,2)$ is essentially self-adjoint on $C_0^\infty(M_j)$.

We denote the closure of P_j by $\bar{P}_j$.

By Eqs. (2.1) and (2.3), we have

$$D_xA_2(\mathbf{r})-D_yA_1(\mathbf{x})=0,\quad \mathbf{r}\in M. \tag{2.16}$$

Hence

$$[P_1,P_2]=0 \quad \text{on } C_0^\infty(M). \tag{2.17}$$

This suggests that $\bar{P}_1$ and $\bar{P}_2$ may have a chance to commute in a proper sense. We say that two self-adjoint operators S and T STRONGLY COMMUTE if their spectral projections commute (Ref. 7, Sec. VIII 5). It is shown that *S and T strongly commute if and only if for all $a,b\in\mathbf{R}$*

$$e^{iaS}e^{ibT}=e^{ibT}e^{iaS}.$$

To state a result on the strong commutativity of $\bar{P}_1$ and $\bar{P}_2$, we recall a concept concerning the magnetic flux.[1] Let $a,b\in\mathbf{R}$ and $C(x,y;a,b)$ be the rectangular closed curve: $(x,y)\to(x+a,y)\to(x+a,y+b)\to(x,y+b)\to(x,y)$ in M and $D(x,y;a,b)$ be its interior domain. Then the magnetic flux passing through $D(x,y;a,b)$ is given by

$$\Phi_{a,b}(x,y)=\int_{C(x,y;a,b)}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'. \tag{2.18}$$

For each $a,b\in\mathbf{R}$, the function $\Phi_{a,b}$ is defined on the set $D_{a,b}=(\mathbf{R}\setminus\{a_{\nu 1},a_{\nu 1}-a\}_{\nu=1}^n\})\times(\mathbf{R}\setminus\{a_{\nu 2},a_{\nu 2}-b\}_{\nu=1}^n\})$. Let $\mathbf{Z}$ be the set of integers. We say that THE MAGNETIC FLUX IS LOCALLY QUANTIZED if $\Phi_{a,b}$ is a $2\pi\mathbf{Z}/q$-valued function for all $a,b\in\mathbf{R}$. Using Eqs. (2.10) and (2.12), we can easily show that

$$\Phi_{a,b}(x,y)=\sum_{\mathbf{a}_\nu\in D(x,y;a,b)}\gamma_\nu, \tag{2.19}$$

with

$$\gamma_\nu=C_{0,0}^{(\nu)}=C_0^{(\nu)}. \tag{2.20}$$

In particular, the total magnetic flux Φ is given by

$$\Phi=\sum_{\nu=1}^{n}\gamma_\nu. \tag{2.21}$$

Thus it follows that the *the magnetic flux is locally quantized if and only if γ_ν is an integer multiple of $2\pi/q$ for all $\nu=1,\ldots,n$.* Note that the quantization of the total magnetic flux (i.e., the case where Φ is an integer multiple of $2\pi/q$, the "global" quantization of the magnetic flux) does not imply the local quantization of the magnetic flux.

The Lebesgue measure of the set $\mathbf{R}^2 \setminus D_{a,b}$ is zero. Hence $\Phi_{a,b}$ defines a unique self-adjoint multiplication operator in $L^2(\mathbf{R}^2)$. We denote it by the same symbol $\Phi_{a,b}$. The following theorem has been proven in Ref. 1.

Theorem 2.2: For all $a,b \in \mathbf{R}$,

$$e^{ia\bar{P}_1} e^{ib\bar{P}_2} = \exp(-iq\Phi_{a,b}) e^{ib\bar{P}_2} e^{ia\bar{P}_1}. \tag{2.22}$$

In particular, $\bar{P}_1$ and $\bar{P}_2$ strongly commute if and only if the magnetic flux is locally quantized.

Remark 2.3: Physically, $(\exp(ia\bar{P}_1)\exp(ib\bar{P}_2)\Psi)(\mathbf{r})$ and $(\exp(ib\bar{P}_2)\exp(ia\bar{P}_1)\Psi)(\mathbf{r})(a,b \in \mathbf{R}, \mathbf{r} = (x,y) \in \mathbf{R}^2, \Psi \in L^2(\mathbf{R}^2))$ mean the parallel transport, along the curves $(x,y) \to (x,y+b) \to (x+a,y+b)$, and $(x,y) \to (x+a,y) \to (x+a,y+b)$, respectively, of the wave function Ψ under the influence of the gauge potential $\mathbf{A}$. Formula (2.22) shows that the function $-q\Phi_{a,b}$ gives the phase shift between these two parallel transports. Hence Theorem 2.2 tells us that, in the present idealized system, *the Aharonov–Bohm effect occurs if and only if the magnetic flux is not locally quantized.*

For later use, we prove the following fact.

Lemma 2.4: $\ker \bar{P}_j = \{0\}$, $j = 1,2$.

Proof: Let

$$\psi_1(x,y) = -\sum_{\nu=1}^{n} \sum_{0<\alpha+\beta<m} \frac{C_{\alpha,\beta}^{(\nu)}}{2\pi} D_x^\alpha D_y^\beta \arctan\left(\frac{x-a_{\nu 1}}{y-a_{\nu 2}}\right), \quad (x,y) \in M_1,$$

$$\psi_2(x,y) = \sum_{\nu=1}^{n} \sum_{0<\alpha+\beta<m} \frac{C_{\alpha,\beta}^{(\nu)}}{2\pi} D_x^\alpha D_y^\beta \arctan\left(\frac{y-a_{\nu 2}}{x-a_{\nu 1}}\right), \quad (x,y) \in M_2.$$

Then we have

$$A_1 = D_x \psi_1 \quad \text{on } M_1,$$

$$A_2 = D_y \psi_2 \quad \text{on } M_2.$$

Hence

$$P_j = e^{iq\psi_j} p_j e^{-iq\psi_j} \quad \text{on } C_0^\infty(M_j). \tag{2.23}$$

Let $j = 1,2$, be fixed and $f \in \ker \bar{P}_j$. Then, since $C_0^\infty(M_j)$ is a core for $\bar{P}_j$ (Theorem 2.1), there exists a sequence $\{f_n\}_{n=1}^\infty$ in $C_0^\infty(M_j)$ such that $f_n \to f$, $P_j f_n \to 0$ in $L^2(\mathbf{R}^2)$ as $n \to \infty$. By Eq. (2.23) we have

$$p_j e^{-iq\psi_j} f_n \to 0,$$

which, together with the closedness of p_j, implies that $e^{-iq\psi_j} f \in D(p_j)$ and

$$p_j e^{-iq\psi_j} f = 0.$$

It is well-known that $\ker p_j = \{0\}$. Hence $\exp(-iq\psi_j) f = 0$, so that $f = 0$. Thus the desired result follows. ∎

If the charged particle is nonrelativistic with mass $m_0 > 0$, then the Hamiltonian of the quantum system under consideration may be given as the Schrödinger Hamiltonian $H(\mathbf{A})$ defined as the self-adjoint operator associated with the non-negative, closed, quadratic form

$$s(f,g)=\frac{(\bar{P}_1 f,\bar{P}_1 g)+(\bar{P}_2 f,\bar{P}_2 g)}{2m_0}, \quad f,g\in D(\bar{P}_1)\cap D(\bar{P}_2), \tag{2.24}$$

so that

$$D(H(\mathbf{A})^{1/2})=D(\bar{P}_1)\cap D(\bar{P}_2)$$

and

$$(H(\mathbf{A})^{1/2}f,H(\mathbf{A})^{1/2}g)=s(f,g), \quad f,g\in D(H(\mathbf{A})^{1/2}). \tag{2.25}$$

(For a representation theorem for closed semibounded quadratic forms, see, e.g., Ref. 7, Sec. VIII 6.) As a corollary of Lemma 2.4, we have the following.

Theorem 2.5: The Hamiltonian $H(\mathbf{A})$ has no zero-energy states

$$\ker H(\mathbf{A})=\{0\}. \tag{2.26}$$

Proof: By Eq. (2.25), we can show that

$$\ker H(\mathbf{A})=\ker \bar{P}_1\cap\ker \bar{P}_2,$$

hence Lemma 2.4 gives Eq. (2.26). ■

III. THE DIRAC–WEYL OPERATOR

In what follows, the domain $D(S+T)$ of the sum $S+T$ of two linear operators S and T from a Hilbert space to another is always taken to be $D(S)\cap D(T)$ unless otherwise stated.

Let σ_j, $j=1,2,3$, be the Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The Dirac–Weyl operator is given by

$$Q=\sigma_1\bar{P}_1+\sigma_2\bar{P}_2 \tag{3.1}$$

acting in $L^2(\mathbf{R}^2;\mathbf{C}^2)$, which describes a Hamiltonian for a quantum system of a spin-1/2 massless Dirac particle with charge q under the influence of the gauge potential $\mathbf{A}$. In this section, we discuss the problem of the self-adjoint realization of Q. In the present case, this problem is not so trivial, because the gauge potential $\mathbf{A}(\mathbf{r})$ can be strongly singular at $\mathbf{r}=\mathbf{a}_\nu$, $\nu=1,\ldots,n$. We remark that, as for Dirac operators with singular potentials in three dimensions, there have been a number of studies, see, e.g., Refs. 8, 9, and references therein. The singularities treated in these studies, however, are the Coulomb-type. In our case, as is seen from Eqs. (2.8) and (2.9), the singularity of the gauge potential can be much more singular than that.

The first of our results is the following.

Theorem 3.1: Suppose that the magnetic flux is locally quantized. Then Q is self-adjoint and Q^2 is given by

$$Q^2=\bar{P}_1^2+\bar{P}_2^2. \tag{3.2}$$

Moreover, there exists a unitary operator U such that

$$UQU^{-1}=(\operatorname{sgn}\bar{P}_2)(\bar{P}_1^2+\bar{P}_2^2)^{1/2}\sigma_3, \tag{3.3}$$

where $\operatorname{sgn} x=1$ if $x\geqslant 0$ and $\operatorname{sgn} x=-1$ if $x<0$.

To prove this theorem, we recall some definitions and facts in the theory of anticommuting self-adjoint operators.[10–12] We say that two self-adjoint operators S and T in a Hilbert space STRONGLY ANTICOMMUTE if for all $a\in\mathbf{R}$, $\exp(iaS)$ leaves $D(T)$ invariant and

$$e^{iaS}Tf=Te^{-iaS}f, \quad f\in D(T).$$

It is shown that the definition is symmetric in S and T. Definitions equivalent to this form can be found in Refs. 10–12. The following is a basic result on strongly anticommuting self-adjoint operators.

Lemma 3.2 (Ref. 10): Let S and T be strongly anticommuting self-adjoint operators in a Hilbert space. Then $S+T$ is self-adjoint and

$$(S+T)^2=S^2+T^2.$$

For our purpose, we need the following lemma.

Lemma 3.3: Let S_1 and S_2 be strongly commuting self-adjoint operators in a Hilbert space $\mathscr{H}$. Then $S_1\otimes\sigma_1$ and $S_2\otimes\sigma_2$ are strongly anticommuting self-adjoint operators in the Hilbert space $\mathscr{H}\otimes\mathbf{C}^2$.

Proof: Since σ_j, $j=1,2$, are Hermitian matrices, it follows from a general theory of tensor products of self-adjoint operators (e.g., Ref. 7, Sec. VIII 10) that $T_j=S_j\otimes\sigma_j$, $j=1,2$, are self-adjoint in $\mathscr{H}\otimes\mathbf{C}^2$. Let E_j be the spectral measure of S_j. Then, for all Borel sets G in $\mathbf{R}$, $E_1(G)$ and $E_2(G)$ commute. Hence

$$\mathscr{D}=\cup_{a,b>0}R(E_1([-a,a])E_2([-b,b]))$$

is dense in $\mathscr{H}$, where $R(S)$ denotes the range of the operator S, and forms a set of entire analytic vectors for both S_1 and S_2. Let $f\in\mathscr{D}\hat{\otimes}\mathbf{C}^2$ (algebraic tensor product). Then

$$T_1^{2n}f=(S_1^{2n}\otimes I)f,$$

$$T_1^{2n+1}f=(S_1^{2n+1}\otimes\sigma_1)f, \quad n\geqslant 0,$$

where I denotes identity. Hence, for all $t\in\mathbf{R}$,

$$e^{itT_1}f=[(\cos tS_1)\otimes I+i(\sin tS_1)\otimes\sigma_1]f.$$

Using the fact that $\sigma_1\sigma_2=-\sigma_2\sigma_1$, we have

$$(I\otimes\sigma_2)e^{itT_1}f=[(\cos tS_1)\otimes I-i(\sin tS_1)\otimes\sigma_1](I\otimes\sigma_2)f. \tag{3.4}$$

The operators $\cos tS_1$ and $\sin tS_1$ leave $\mathscr{D}$ invariant. Hence the left-hand side of Eq. (3.4) is in $D(S_2\otimes I)$. Applying $S_2\otimes I$ to both sides of Eq. (3.4) and using the strong commutativity of S_1 and S_2, we obtain

$$T_2e^{itT_1}f=e^{-itT_1}T_2f.$$

By a simple limiting argument using the fact that $\mathscr{D}\hat{\otimes}\mathbf{C}^2$ is a core of T_2, we conclude that T_1 and T_2 strongly anticommute. ■

Remark 3.4: We can also prove the converse of the above lemma: *Let S_1 and S_2 be self-adjoint operators in a Hilbert space such that $S_1\otimes\sigma_1$ and $S_2\otimes\sigma_2$ strongly anticommute. Then S_1 and S_2 strongly commute.* For the proof of this fact, see Ref. 13.

Lemma 3.5: Let S_1 and S_2 be as in Lemma 3.3. Then $S_1 \otimes \sigma_1 + S_2 \otimes \sigma_2$ is self-adjoint in $\mathscr{H} \otimes \mathbf{C}^2$ and

$$(S_1 \otimes \sigma_1 + S_2 \otimes \sigma_2)^2 = S_1^2 \otimes I + S_2^2 \otimes I.$$

Proof: This follows from Lemmas 3.3, 3.2, and the fact that $(S_j \otimes \sigma_j)^2 = S_j^2 \otimes I$. ■

We still need a lemma.

Lemma 3.6 (Ref. 14, Theorem 4.4): Let S and T be strongly anticommuting self-adjoint operators in a Hilbert space $\mathscr{H}$. Suppose that T is injective. Let $T = U_T |T|$ be the polar decomposition of T. Then there exists a unitary operator V on $\mathscr{H}$ such that

$$V(S+T)V^{-1} = U_T (S^2 + T^2)^{1/2}.$$

Remark 3.7: Lemma 3.6 is an abstract and *nonperturbative* version of the so-called *Foldy–Wouthuysen–Tani transformation* of the usual Dirac operator (e.g., Ref. 15, Chap. 4).

Proof of Theorem 3.1: Under the assumption, $\bar{P}_1$ and $\bar{P}_2$ strongly commute (Theorem 2.2). Hence we can apply Lemma 3.5 with $S_j = \bar{P}_j$ to obtain the first half of the theorem.

To prove the second half, let

$$W = \frac{1 - i\sigma_1}{\sqrt{2}}.$$

Then W is unitary and $W\sigma_2 W^{-1} = \sigma_3$. Hence

$$WQW^{-1} = \sigma_1 \bar{P}_1 + \sigma_3 \bar{P}_2.$$

It is easy to see that the polar decomposition of $\sigma_3 \bar{P}_2$ is given by $\sigma_3 \bar{P}_2 = \sigma_3 (\operatorname{sgn} \bar{P}_2) |\bar{P}_2|$. By Lemma 2.4, $\sigma_3 \bar{P}_2$ is injective. Hence we can apply Lemma 3.6 with $S = \sigma_1 \bar{P}_1$ and $T = \sigma_3 \bar{P}_2$ to obtain Eq. (3.3). ■

It is natural to ask what if the magnetic flux is not locally quantized. Unfortunately we have not been able to give a definite answer to this question. In the present article, we content ourselves with showing that, if Q is not essentially self-adjoint, then Q has at least two different self-adjoint extensions (see also Sec. IV B). To this end, we first note that Q is written

$$Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix}, \tag{3.5}$$

with

$$Q_\pm = \bar{P}_1 \pm i\bar{P}_2. \tag{3.6}$$

Theorem 3.1 implies that $Q_\pm$ is closed if the magnetic flux is locally quantized. In the case where the magnetic flux is not locally quantized, this is not obvious, but it is easily shown that $Q_\pm$ is closable. We denote their closure by $\bar{Q}_\pm$ and define

$$Q_1 = \begin{pmatrix} 0 & Q_+^* \\ \bar{Q}_+ & 0 \end{pmatrix} \tag{3.7}$$

and

$$Q_2 = \begin{pmatrix} 0 & \bar{Q}_- \\ Q_-^* & 0 \end{pmatrix}. \tag{3.8}$$

Proposition 3.8: Each Q_j $(j=1,2)$ is a self-adjoint extension of Q. Moreover, σ_3 leaves $D(Q_j)$ invariant and

$$\sigma_3 Q_j + Q_j \sigma_3 = 0, \quad j=1,2.$$

Proof: The self-adjointness of Q_j is easily proven (note that $Q_\pm^* = (\bar{Q}_\pm)^*$). It follows from Eq. (3.6) that

$$Q_- \subset Q_+^*, \quad Q_+ \subset Q_-^*,$$

which imply that each Q_j is an extension of Q. The second half of the proposition is easily checked. ■

Remark 3.9: If the magnetic flux is locally quantized, then, by Theorem 3.1, $Q=Q_j$, $j=1,2$.

Remark 3.10: The idea of the above construction of self-adjoint extensions of Q has also been used in Ref. 16 (cf. also Ref. 17).

Remark 3.11: Let $\hat{Q}$ be any self-adjoint extension of Q such that $\sigma_3\hat{Q}+\hat{Q}\sigma_3=0$ on $D(\hat{Q})$ and define $H=\hat{Q}^2$. Then the quadruple $\{L^2(\mathbf{R}^2;\mathbf{C}^2),\hat{Q},H,\sigma_3\}$ is a model of supersymmetric quantum mechanics (SSQM).[18–21,5] In this context, $\hat{Q}$ and H are called the *supercharge* and the *supersymmetric Hamiltonian*, respectively. Hence, from this point of view too, it is interesting to analyze properties of $\hat{Q}$.

IV. ZERO-ENERGY STATE

A. The case where the magnetic flux is locally quantized

Lemma 4.1: If the magnetic flux is locally quantized, then

$$H(\mathbf{A}) = Q^2. \tag{4.1}$$

Proof: Under this assumption, Eq. (3.2) implies that $Q^2 \subset H(\mathbf{A})$. But both of these operators are self-adjoint. Hence Eq. (4.1) follows. ■

Theorem 4.2: If the magnetic flux is locally quantized, then $\ker Q=\{0\}$, i.e., Q has no zero-energy states.

Proof: By Eq. (4.1), we have $\ker H(\mathbf{A}) = \ker Q$, which, together with Theorem 2.5, gives the desired result. ■

Theorem 4.2 shows that the Aharonov–Casher theorem in Ref. 3 does not hold on zero-energy states of the Dirac–Weyl operator Q if the magnetic flux is locally quantized. We also remark that the case where the magnetic flux is locally quantized corresponds to the nonoccurrence of the Aharonov–Bohm effect (see Remark 2.3).

B. The case where the magnetic flux is not necessarily locally quantized

We first consider the minimal version of the Dirac–Weyl operator Q

$$Q_{\min} = Q \restriction C_0^\infty(M), \tag{4.2}$$

which is symmetric and hence closable. We denote its closure by $\bar{Q}_{\min}$.

Lemma 4.3:

$$\ker \bar{Q}_{\min} = \{0\}.$$

Proof: Let $\Psi \in \ker \bar{Q}_{\min}$. Then there exists a sequence $\Psi_n \in C_0^\infty(M)$ such that $\Psi_n \to \Psi$ and $Q_{\min}\Psi_n \to 0$ in $L^2(\mathbf{R}^2)$ as $n\to\infty$. Let $\|\cdot\|$ denote the norm of $L^2(\mathbf{R}^2;\mathbf{C}^2)$. By Eq. (2.17) and the fact $\sigma_1\sigma_2 = -\sigma_2\sigma_1$, we have

$$\|Q_{\min}\Psi_n\|^2=\|P_1\Psi_n\|^2+\|P_2\Psi_n\|^2.$$

Hence $P_j\Psi_n\to 0$, $j=1,2$, as $n\to\infty$, which imply that $\Psi\in D(\bar{P}_1)\cap D(\bar{P}_2)$ and $\bar{P}_j\Psi=0$, $j=1,2$. Hence, by Lemma 2.4, $\Psi=0$. ■

Although Lemma 4.3 shows that $\bar{Q}_{\min}$ has no zero-energy states, self-adjoint extensions of $\bar{Q}_{\min}$ may have zero-energy states. In fact this is true, as is shown below.

Let

$$Q_{\pm,\min}=Q_{\pm}\restriction C_0^\infty(M). \tag{4.3}$$

Note that

$$Q_{+,\min}=-2i\bar{\partial}-iq\tilde{A}(z)^*,\quad Q_{-,\min}=-2i\partial+iq\tilde{A}(z),$$

on $C_0^\infty(M)$, where $\partial=\partial/\partial z$ and $\bar{\partial}=\partial/\partial\bar{z}$. We have

$$\bar{Q}_{\min}=\begin{pmatrix}0 & \bar{Q}_{-,\min}\\ \bar{Q}_{+,\min} & 0.\end{pmatrix}.$$

Lemma 4.3 implies that

$$\ker\bar{Q}_{\pm,\min}=\{0\}. \tag{4.4}$$

In the same way as in the proof of Proposition 3.8, we can show that the operators

$$Q_{\min}^{(1)}=\begin{pmatrix}0 & Q_{+,\min}^*\\ \bar{Q}_{+,\min} & 0\end{pmatrix} \tag{4.5}$$

and

$$Q_{\min}^{(2)}=\begin{pmatrix}0 & \bar{Q}_{-,\min}\\ Q_{-,\min}^* & 0\end{pmatrix} \tag{4.6}$$

are self-adjoint extensions of $Q_{\min}$.

We want to determine the zero-energy states of $Q_{\min}^{(j)}$, $j=1,2$. In order to do that, it is convenient to consider generalized zero-energy states of Q. Let $\mathscr{D}'(M)$ be the space of distributions on M. We say that $\Psi=(\Psi_+,\Psi_-)$ is a GENERALIZED ZERO-ENERGY STATE of Q if $\Psi_\pm\in\mathscr{D}'(M)$ satisfying

$$\langle\Psi,(Qu)^*\rangle=0,\quad u=\begin{pmatrix}u_+\\ u_-\end{pmatrix}\in C_0^\infty(M;\mathbf{C}^2), \tag{4.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical bilinear form on $\mathscr{D}'(M;\mathbf{C}^2)\times C_0^\infty(M;\mathbf{C}^2)$. We denote by $\mathscr{S}_0(Q)$ the space of all the generalized zero-energy states of Q. Let

$$H_\phi(M)=\{e^{-q\phi}f\,|\,f \text{ is holomorphic on } D_M\}, \tag{4.8}$$

$$H_\phi^a(M)=\{e^{q\phi}g\,|\,g \text{ is antiholomorphic on } D_M\}. \tag{4.9}$$

Lemma 4.4:

$$\mathscr{S}_0(Q)=\left\{\Psi=\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}\middle|\Psi_+\in H_\phi(M),\Psi_-\in H_\phi^a(M)\right\}. \tag{4.10}$$

Proof: The idea of the proof of this lemma is similar to that of the Aharonov–Casher theorem.[3] For the sake of completeness, we give the proof. Let $\mathscr{V}$ be the set on the right-hand side of Eq. (4.10) and $\Psi=(\Psi_+,\Psi_-)\in\mathscr{V}$. Then there exist a holomorphic function f and an antiholomorphic function g on D_M such that $\Psi_+=e^{-q\phi}f$ and $\Psi_-=e^{q\phi}g$ on M. Using Eqs. (3.6) and (2.7), we have

$$Q_+=-2ie^{-q\phi}\bar{\partial}e^{q\phi},\quad \text{on}\quad C^\infty(M), \tag{4.11}$$

$$Q_-=-2ie^{q\phi}\partial e^{-q\phi},\quad \text{on}\quad C^\infty(M). \tag{4.12}$$

Using these formulas, we can show that Eq. (4.7) holds. Hence $\Psi\in\mathscr{S}_0(Q)$. Thus $\mathscr{S}_0(Q)\supset\mathscr{V}$.

To prove the converse inclusion relation, let $\Psi=(\Psi_+,\Psi_-)\in\mathscr{S}_0(Q)$, so that Eq. (4.7) holds. Noting that $\exp(-q\phi)$ is a one-to-one mapping from $C_0^\infty(M)$ onto itself, we have

$$\langle e^{q\phi}\Psi_+,\bar{\partial}u\rangle=0$$

for all $u\in C_0^\infty(M)$. In particular, taking $u=\partial v$ with $v\in C_0^\infty(M)$ arbitrary, we obtain

$$\langle e^{q\phi}\Psi_+,\Delta v\rangle=0.$$

Hence, by the elliptic regularity of the Laplacian Δ, Ψ_+ is in $C^\infty(M)$. Similarly we have $\Psi_-\in C^\infty(M)$. Thus we obtain

$$Q_\pm\Psi_\pm=0$$

as partial differential equations in M. It follows from Eqs. (4.11) and (4.12) that

$$\bar{\partial}e^{q\phi}\Psi_+=0,\quad \partial e^{-q\phi}\Psi_-=0\quad \text{on}\quad M,$$

which imply that $\Psi_+\in H_\phi(M)$ and $\Psi_-\in H_\phi^a(M)$. Hence $\Psi\in\mathscr{V}$. Thus $\mathscr{S}_0(Q)\subset\mathscr{V}$. ■

Remark 4.5: Lemma 4.4 shows that Q has infinitely many generalized zero-energy states, a phenomenon which we have already encountered in the case where B and $\mathbf{A}$ are "regular."[3] Moreover, such a phenomenon also occurs in some of other Dirac operators appearing in supersymmetric quantum mechanics[16,22,23] (in Ref. 23, the existence of infinitely many *normalizable zero-energy states* of a Dirac operator is shown).

Let $\mathbf{Z}_+$ be the set of non-negative integers. We introduce

$$\mathbf{W}_\pm=\left\{(p,k_1,\ldots,k_n)\in\mathbf{Z}_+\times\mathbf{Z}^n\,|\,p+\sum_{\nu=1}^n k_\nu<\pm\frac{q\Phi}{2\pi}-1,\quad k_\nu>\pm\frac{q\gamma_\nu}{2\pi}-1,\quad \nu=1,\ldots,n\right\}, \tag{4.13}$$

and set

$$N_\pm(n;q)=\#\mathbf{W}_\pm, \tag{4.14}$$

the number of the elements of $\mathbf{W}_\pm$. We have

$$N_-(n;q)=N_+(n;-q). \tag{4.15}$$

Let

$$F(z)=-\frac{1}{4\pi}\sum_{\nu=1}^{n}\sum_{k=1}^{m}\frac{C_k^{(\nu)}}{k(z-a_\nu)^k},$$

and, for $(p_\pm,k_1,\ldots,k_n)\in\mathbf{Z}_+\times\mathbf{Z}^n$, define

$$\Omega^+_{p_+,k_1,\ldots,k_n}(\mathbf{r})=\left(\prod_{\nu=1}^{n}|z-a_\nu|^{-q\gamma_\nu/2\pi}(z-a_\nu)^{k_\nu}\right)P_+(z)e^{iq\,\mathrm{Im}\,F(z)},\tag{4.16}$$

$$\Omega^-_{p_-,k_1,\ldots,k_n}(\mathbf{r})=\left(\prod_{\nu=1}^{n}|z-a_\nu|^{q\gamma_\nu/2\pi}(\bar{z}-\bar{a}_\nu)^{k_\nu}\right)P_-(\bar{z})e^{-iq\,\mathrm{Im}\,\overline{F(z)}},\tag{4.17}$$

with $P_\pm$ a polynomial of order $p_\pm$ such that $P_+(a_\nu)\neq 0$, $P_-(\bar{a}_\nu)\neq 0$, $\nu=1,\ldots,n$.

Lemma 4.6: (i) The function $\Omega^\pm_{p_\pm,k_1,\ldots,k_n}$ satisfies the partial differential equation

$$Q_\pm\Psi^\pm_{p_\pm,k_1,\ldots,k_n}(\mathbf{r})=0\quad\text{on}\quad M.\tag{4.18}$$

(ii) The function $\Omega^\pm_{p_\pm,k_1,\ldots,k_n}$ is in $L^2(\mathbf{R}^2)$ if and only if $(p_\pm,k_1,\ldots,k_n)\in\mathbf{W}_\pm$.

Proof: (i) By Eqs. (2.7), (2.10), and (2.12), we can show that ϕ can be written

$$\phi(\mathbf{r})=\mathrm{Re}\,F(z)+\sum_{\nu=1}^{n}\frac{\gamma_\nu}{2\pi}\log|\mathbf{r}-\mathbf{a}_\nu|.\tag{4.19}$$

Using this equation, Eqs. (4.11) and (4.12), we see that Eq. (4.18) holds.

(ii) We have

$$|\Omega^+_{p_+,k_1,\ldots,k_n}(\mathbf{r})|\sim\mathrm{const}\,|\mathbf{r}|^{-(q\Phi/2\pi)+p_++\Sigma_{\nu=1}^{n}k_\nu}$$

as $|\mathbf{r}|\to\infty$ and

$$|\Omega^+_{p_+,k_1,\ldots,k_n}(\mathbf{r})|\sim\mathrm{const}\,|\mathbf{r}-\mathbf{a}_\nu|^{-(q\gamma_\nu/2\pi)+k_\nu}$$

as $\mathbf{r}\to\mathbf{a}_\nu$. Hence the desired assertion about $\Omega^+_{p_+,k_1,\ldots,k_n}$ follows. Similarly we can prove the assertion about $\Omega^-_{p_-,k_1,\ldots,k_n}$. ■

We now come to the main result in this section.

Theorem 4.7:

(i) We have

$$\ker Q^{(1)}_{\min}=\left\{\begin{pmatrix}0\\ \Omega^-_{p,k_1,\ldots,k_n}\end{pmatrix}\middle|\,(p,k_1,\ldots,k_n)\in\mathbf{W}_-\right\},\tag{4.20}$$

$$\ker Q^{(2)}_{\min}=\left\{\begin{pmatrix}\Omega^+_{p,k_1,\ldots,k_n}\\ 0\end{pmatrix}\middle|\,(p,k_1,\ldots,k_n)\in\mathbf{W}_+\right\},\tag{4.21}$$

where $\ker Q^{(1)}_{\min}=\{0\}$ if $\mathbf{W}_-=\emptyset$, and $\ker Q^{(2)}_{\min}=\{0\}$ if $\mathbf{W}_+=\emptyset$. In particular,

$$\dim\ker Q^{(1)}_{\min}=N_-(n;q),\tag{4.22}$$

$$\dim\ker Q^{(2)}_{\min}=N_+(n;q).\tag{4.23}$$

(ii) If (a) $n=1$ or (b) $n\geqslant 2$ and the magnetic flux is locally quantized, then

$$\ker Q^{(j)}_{\min}=\{0\},\quad j=1,2. \tag{4.24}$$

Proof: (i) By Eq. (4.4), we have

$$\ker Q^{(1)}_{\min}=\left\{\begin{pmatrix}0\\ \Psi\end{pmatrix}\middle|\,\Psi\in\ker Q^{*}_{+,\min}\right\},\quad \ker Q^{(2)}_{\min}=\left\{\begin{pmatrix}\Psi\\ 0\end{pmatrix}\middle|\,\Psi\in\ker Q^{*}_{-,\min}\right\}.$$

Hence, by Lemma 4.6, the sets on the right-hand sides of Eqs. (4.20) and (4.21) are included in $\ker Q^{(1)}_{\min}$ and $\ker Q^{(2)}_{\min}$, respectively.

To prove the converse inclusion relations, let

$$\Psi=(\Psi_+,\Psi_-)\in\ker Q^{*}_{-,\min}\oplus\ker Q^{*}_{+,\min}.$$

Note that

$$\ker Q^{*}_{-,\min}\oplus\ker Q^{*}_{+,\min}=\ker Q^{*}_{\min}=L^2(\mathbf{R}^2;\mathbf{C}^2)\cap\mathscr{S}_0(Q).$$

Hence $\Psi\in\mathscr{S}_0(Q)$ and $\Psi_\pm\in L^2(\mathbf{R}^2)$. Therefore, by Lemma 4.4, there exist a holomorphic function f and an antiholomorphic function g on D_M such that

$$\Psi_+=e^{-q\phi}f,\quad \Psi_-=e^{q\phi}g.$$

Taking Eq. (4.19) into account together with the condition $\Psi_+\in L^2(\mathbf{R}^2)$, we see that f must be of the form

$$f(z)=e^{qF(z)}h(z)$$

with a meromorphic function h on $\mathbf{C}\cup\{\infty\}$ with possible poles at $z=a_\nu$, $\nu=1,\ldots,n$. Thus Ψ_+ has to take the form

$$\Psi_+=\Omega^{+}_{p,k_1,\ldots,k_n},$$

with some $(p,k_1,\ldots,k_n)\in\mathbf{Z}_+\times\mathbf{Z}^n$ and a polynomial P_+ of degree p such that $P_+(a_\nu)\neq 0$, $\nu=1,\ldots,n$. By Lemma 4.6(ii), $(p,k_1,\ldots,k_n)$ must be in $\mathbf{W}_+$. Similarly we can show that Ψ_- must be of the form

$$\Psi_-=\Omega^{-}_{p,k_1,\ldots,k_n},$$

with some $(p,k_1,\ldots,k_n)\in\mathbf{W}_-$. Thus $(0,\Psi_-)$ and $(\Psi_+,0)$ are in the set on the right-hand sides of Eqs. (4.20) and (4.21), respectively.

We write $(p,k_1,\ldots,k_n)=(p,\mathbf{k})$. Then $\{\Omega^{\pm}_{p_j\mathbf{k}_j}\}_{j=1}^{l}$ are linearly independent if and only if $(p_j\mathbf{k}_j)\neq(p_i,\mathbf{k}_i)$, $i\neq j,i,j=1,\ldots,l$. Thus Eqs. (4.22) and (4.23) follow.

(ii) If $n=1$, then it is easy to see that $\mathbf{W}_\pm=\emptyset$. Hence $N_\pm(1;q)=0$. Thus Eq. (4.24) follows. Suppose that $n\geqslant 2$ and the magnetic flux is locally quantized, so that $l_\nu=q\gamma_\nu/2\pi\in\mathbf{Z}$, $\nu=1,\ldots,n$. Let $(p,k_1,\ldots,k_n)\in\mathbf{W}_+$. Then

$$k_\nu \geqslant l_\nu, \quad \nu=1,\ldots,n,$$

$$p+\sum_{\nu=1}^{n} k_\nu < \sum_{\nu=1} l_\nu - 1,$$

which implies that $p<-1$. Hence $\mathbf{W}_+=\emptyset$. Similarly we can show that $\mathbf{W}_-=\emptyset$. Thus Eq. (4.24) follows. ■

As a corollary of Theorem 4.7(i), we have the following.

Corollary 4.8: Let $n\geqslant 2$ and suppose that $N_+(n;q)+N_-(n;q)\geqslant 1$. Then $Q_{\min}$ is not essentially self-adjoint.

Proof: If $Q_{\min}$ were essentially self-adjoint, then $\bar{Q}_{\min} = Q^*_{\min}$. By Lemma 4.3, we have $\ker \bar{Q}_{\min}=\{0\}$. On the other hand, under the present assumption, Theorem 4.7(i) gives $\ker Q^*_{\min}\neq\{0\}$. Thus we are led to a contradiction. ■

Finally, we show that, under some conditions, $N_\pm(n;q)\geqslant 1$. For a positive number x, we denote by $[x]$ (resp. $\{x\}$) the largest integer less than or equal to x (resp. the largest integer less than x).

Proposition 4.9: Let $n\geqslant 2$ and

$$\epsilon_\nu(q)=\frac{q\gamma_\nu}{2\pi}-\left[\frac{q\gamma_\nu}{2\pi}\right], \quad \nu=1,\ldots,n. \tag{4.25}$$

Then the following holds.

(i) Suppose that

$$\sum_{\nu=1}^{n} \epsilon_\nu(q) > 1. \tag{4.26}$$

Then

$$N_+(n;q)\geqslant \left\{ \sum_{\nu=1}^{n} \epsilon_\nu(q) \right\}. \tag{4.27}$$

In particular, if

$$\sum_{\nu=1}^{n} \epsilon_\nu(q) \to \infty \quad \text{as} \quad n\to\infty,$$

then

$$N_+(n;q)\to\infty \quad \text{as} \quad n\to\infty. \tag{4.28}$$

(ii) Let

$$m_+(n;q)=\#\{\nu\in\{1,\ldots,n\}\,|\,\epsilon_\nu(q)>0\}$$

and suppose that

$$m_+(n;q) > 1+\sum_{\nu=1}^{n} \epsilon_\nu(q). \tag{4.29}$$

Then

$$N_-(n;q) \geqslant m_+(n;q) - 1 - \left[\sum_{\nu=1}^{n} \epsilon_\nu(q) \right]. \tag{4.30}$$

In particular, if

$$m_+(n;q) - 1 - \sum_{\nu=1}^{n} \epsilon_\nu(q) \to \infty \quad \text{as} \quad n \to \infty,$$

then

$$N_-(n;q) \to \infty \quad \text{as} \quad n \to \infty. \tag{4.31}$$

Proof: (i) Let

$$l_\nu = \left[\frac{q\gamma_\nu}{2\pi} \right].$$

If $p \in \mathbf{Z}_+$ satisfies

$$p < \sum_{\nu=1}^{n} \epsilon_\nu(q) - 1,$$

then $(p, l_1, \ldots, l_n) \in \mathbf{W}_+$. Hence

$$N_+(n;q) \geqslant \# \left\{ p \in \mathbf{Z}_+ \,|\, 0 \leqslant p < \sum_{\nu=1}^{n} \epsilon_\nu(q) - 1 \right\} = \left\{ \sum_{\nu=1}^{n} \epsilon_\nu(q) \right\}.$$

Thus Eq. (4.27) follows. Formula (4.28) is a direct consequence of Eq. (4.27).

(ii) Let

$$k_\nu = \begin{cases} -l_\nu; & \text{if} \quad \epsilon_\nu(q) = 0 \\ -l_\nu - 1; & \text{if} \quad \epsilon_\nu(q) > 0. \end{cases}$$

If $p \in \mathbf{Z}_+$ satisfies

$$p < m_+(n;q) - 1 - \sum_{\nu=1}^{n} \epsilon_\nu(q),$$

then $(p, k_1, \ldots, k_n) \in \mathbf{W}_-$. Hence

$$N_-(n;q) \geqslant \# \left\{ p \in \mathbf{Z}_+ \,|\, 0 \leqslant p < m_+(n;q) - 1 - \sum_{\nu=1}^{n} \epsilon_\nu(q) \right\} = m_+(n;q) - 1 - \left[\sum_{\nu=1}^{n} \epsilon_\nu(q) \right].$$

Thus we obtain Eq. (4.30). Formula (4.31) follows from Eq. (4.30). ■

Remark 4.10: By Eq. (4.15), part (i) also gives an estimate for $N_-(n;q)$. If Eqs. (4.26) and (4.29) hold, then $N_\pm(n;q) \geqslant 1$. Formula (4.28) physically means that, under the condition given there, the number of zero-energy states of $Q_{\min}^{(2)}$ increases, tending to infinity, as the number of the points at which the magnetic field passes through increases. This is an interesting phenomenon to be noted. A similar consideration applies to the zero-energy states of $Q_{\min}^{(1)}$.

Remark 4.11: Let

$$H_j = (Q_{\min}^{(j)})^2.$$

Then, as already mentioned in Remark 3.11, each $\mathbb{M}_j = \{L^2(\mathbf{R}^2;\mathbf{C}^2), Q^{(j)}_{\min}, H_j\sigma_3\}$ is a model of SSQM. In SSQM, supersymmetry is said to be broken if the supersymmetric Hamiltonian has no zero-energy states.[18–21] Theorem 4.7 and Proposition 4.9 imply the following: (i) If $n=1$ or the magnetic flux is locally quantized, then the supersymmetry of the model $\mathbb{M}_j$ is broken. (ii) If $n\geqslant 2$ and Eq. (4.26) (resp. with q replaced by $-q$) is satisfied (hence the magnetic flux is not locally quantized), then the supersymmetry of the model $\mathbb{M}_1$ (resp. $\mathbb{M}_2$) is not broken. These are interesting phenomena.

Remark 4.12: The operator H_j is a self-adjoint extension of $H_{\min} = Q^2_{\min} = (\bar{P}_1^2 + \bar{P}_2^2) \restriction C_0^\infty(M)$. *Under the assumption of Corollary 4.8, $H_{\min}$ is not essentially self-adjoint.* For, if it were essentially self-adjoint, then $\bar{H}_{\min} = \bar{Q}^2_{\min} = H_j$. Hence $\ker \bar{Q}_{\min} = \ker Q^{(j)}_{\min}$. But, $\ker \bar{Q}_{\min} = \{0\}$ (Lemma 4.3) and $\ker Q^{(j)}_{\min} \neq 0$. Thus we are led to a contradiction.

C. Connection with index theory

The results in Theorem 4.7 can be rephrased in terms of the index theory. We recall some definitions in the index theory (e.g., Ref. 24 Chap. IV, Sec. 5). Let T be a densely defined closed linear operator from a Hilbert space to another. The index $\operatorname{ind}(T)$ of T is defined by

$$\operatorname{ind}(T) = \dim\ker T - \dim\ker T^*,$$

provided that at least one of $\dim\ker T$ and $\dim\ker T^*$ is finite. If both (resp. at least one) of $\ker T$ and $\ker T^*$ are (resp. is) finite-dimensional and $R(T)$ is closed, then T is said to be (resp. semi-) Fredholm. It is known that, if T is semi-Fredholm, then $\operatorname{ind}(T)$ is invariant under compact perturbations relative to T, which is called the *stability* or the *topological invariance of the index.*

Theorem 4.7 is translated into the following.

Theorem 4.13: For all $n\geqslant 1$,

$$\operatorname{ind}(\bar{Q}_{+,\min}) = -N_-(n;q), \tag{4.32}$$

$$\operatorname{ind}(\bar{Q}_{-,\min}) = -N_+(n;q). \tag{4.33}$$

Remark 4.14: Index formulas (4.32) and (4.33), which are determined by the magnetic flux at *each point* $\mathbf{a}_\nu$, $\nu=1,\ldots,n$, not by the total magnetic flux only, are essentially different from those of $Q_\pm$ with "regular" gauge potentials (see Refs. 3–5).

From the point of view of the topological invariance of the index, it is interesting to examine whether $\bar{Q}_{\pm,\min}$ are Fredholm or not. We shall do it in the next section. Unfortunately, the result is negative.

V. SPECTRAL PROPERTY

In this section we investigate the spectral property of self-adjoint extensions of $Q_{\min}$. As the following theorem shows, their spectrum is independent of whether the magnetic flux is locally quantized or not.

Theorem 5.1: Every self-adjoint extension $\hat{Q}$ of $Q_{\min}$ satisfies

$$\sigma(\hat{Q}) = \mathbf{R}. \tag{5.1}$$

In particular,

$$\sigma(Q_j) = \sigma(Q^{(j)}_{\min}) = \mathbf{R}, \quad j=1,2. \tag{5.2}$$

This theorem is a special case of a more general one. To state it, we recall a definition. A measurable function V on $\mathbf{R}^d$ $(d\geqslant 1)$ is said to be almost locally in $L^2(\mathbf{R}^d)$ if for each $\epsilon>0$ and each bounded set Ω in $\mathbf{R}^d$, there is a closed set $F\subset\Omega$ such that the Lebesgue measure of F is smaller than ϵ and $V\in L^2(\Omega\setminus F)$. It is shown that, if V is almost locally in $L^2(\mathbf{R}^2)$, then $C_0^\infty(\mathbf{R}^d)\cap D(V)$ is dense in $L^2(\mathbf{R}^d)$ (Ref. 25, Chap. 4, Sec. 6).

Let B_j, $j=1,2$, be real-valued measurable functions on $\mathbf{R}^2$ which are almost locally in $L^2(\mathbf{R}^2)$. Then it is easy to see that the Dirac–Weyl operator

$$\boldsymbol{D}=\sigma_1(p_1-B_1)+\sigma_2(p_2-B_2),$$

with $D(\boldsymbol{D})=C_0^\infty(\mathbf{R}^2;\mathbf{C}^2)\cap D(B_1)\cap D(B_2)$ is symmetric in $L^2(\mathbf{R}^2;\mathbf{C}^2)$.

Theorem 5.2: Let B_j be as above. Suppose that there are sequences $\{\mathbf{c}_n\}_{n=1}^\infty\subset\mathbf{R}^2$ and $\{t_n\,|\,t_n>0,n\geqslant 1\}$ with $t_n\to\infty$ as $n\to\infty$ such that

$$\int_{|\mathbf{r}-\mathbf{c}_n|<t_n}|B_j(\mathbf{r})|^2d\mathbf{r}<\infty,\quad n\geqslant 1,$$

$$\frac{1}{t_n^2}\int_{|\mathbf{r}-\mathbf{c}_n|<t_n}|B_j(\mathbf{r})|^2d\mathbf{r}\to 0\quad\text{as}\quad n\to\infty,\quad j=1,2.$$

Then every self-adjoint extension $\hat{\boldsymbol{D}}$ of $\boldsymbol{D}$ satisfies

$$\sigma(\hat{\boldsymbol{D}})=\mathbf{R}. \tag{5.3}$$

We prove this theorem by extending the method given in Ref. 25, Chap. 4, Sec. 6. (Note that $\boldsymbol{D}$ is a matrix whose entries are linear operators. Hence results in the cited literature are not immediately applicable.) To do that, we need a lemma.

Lemma 5.3: Let $\mathbf{k}\in\mathbf{R}^2$ and

$$U(\mathbf{k})=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -(k_1-ik_2)/|\mathbf{k}|\\ (k_1+ik_2)/|\mathbf{k}| & 1\end{pmatrix}.$$

Then $U(\mathbf{k})$ is unitary and

$$U(\mathbf{k})^*(\sigma_1k_1+\sigma_2k_2)U(\mathbf{k})=|\mathbf{k}|\sigma_3\,.$$

The proof of this lemma is straightforward. Hence it is omitted.

Let

$$\mathbf{v}_+=U(\mathbf{k})\begin{pmatrix}1\\0\end{pmatrix},\quad \mathbf{v}_-=U(\mathbf{k})\begin{pmatrix}0\\1\end{pmatrix}.$$

Then

$$|\mathbf{v}_\pm|=1$$

and, by Lemma 5.3, we have

$$\left(\sum_{j=1}^2\sigma_jk_j\right)\mathbf{v}_\pm=\pm|\mathbf{k}|\mathbf{v}_\pm\,. \tag{5.4}$$

Proof of Theorem 5.2: Let $\mathbf{k}\in\mathbf{R}^2$ and $\rho\in C_0^\infty(\mathbf{R}^2)$ such that $\rho(\mathbf{r})=0$ for $|\mathbf{r}|\geqslant 1$ and $\int|\rho(\mathbf{r})|^2d\mathbf{r}=1$. Define

$$\rho_n(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{t_n}\rho\left(\frac{\mathbf{r}-\mathbf{c}_n}{t_n}\right).$$

Then one can easily show that $\rho_n\in C_0^\infty(\mathbf{R}^2)\cap D(B_j)$, $j=1,2$, with

$$\|\rho_n\|=1$$

and

$$\|(p_j-k_j)\rho_n\|\to 0(n\to\infty),\quad j=1,2. \tag{5.5}$$

Moreover, setting $K=\sup_{\mathbf{r}\in\mathbf{R}^2}|\rho(\mathbf{r})|$, we have

$$\|B_j\rho_n\|^2\leqslant\frac{K^2}{t_n^2}\int_{|\mathbf{r}-\mathbf{c}_n|<t_n}|B_j(\mathbf{r})|^2d\mathbf{r}.$$

Hence, we obtain

$$\|B_j\rho_n\|\to 0(n\to\infty),\quad j=1,2,$$

which, together with Eq. (5.5), imply that

$$\|(p_j-B_j-k_j)\rho_n\|\to 0(n\to\infty),\quad j=1,2. \tag{5.6}$$

Let

$$\Psi_n^{(\pm)}=\rho_n\mathbf{v}_\pm\,.$$

Then $\Psi_n^{(\pm)}\in C_0^\infty(\mathbf{R}^2;\mathbf{C}^2)\cap D(B_j)$, $j=1,2$, with

$$\|\Psi_n^{(\pm)}\|=1.$$

Using Eq. (5.4), we have

$$\|(\hat{\boldsymbol{D}}\mp|\mathbf{k}|)\Psi_n^{(\pm)}\|=\left\|\sum_{j=1}^{2}\sigma_j\mathbf{v}_\pm(p_j-B_j-k_j)\rho_n\right\|\leqslant\sum_{j=1}^{2}\|(p_j-B_j-k_j)\rho_n\|.$$

Hence, by Eq. (5.6), we obtain

$$\|(\hat{\boldsymbol{D}}\mp|\mathbf{k}|)\Psi_n^{(\pm)}\|\to 0\quad(n\to\infty),$$

which imply that $\pm|\mathbf{k}|\in\sigma(\hat{\boldsymbol{D}})$. Since $\mathbf{k}\in\mathbf{R}^2$ is arbitrary, Eq. (5.3) follows. ∎

Remark 5.4: The method of the proof of Theorem 5.2 also works in the case of Dirac operators in d-dimensions of the form

$$\boldsymbol{D}=\sum_{\mu=1}^{d}\gamma^\mu\left(-i\frac{\partial}{\partial x_\mu}-qA_\mu(x)\right),\quad x=(x_1,\ldots,x_d)\in\mathbf{R}^d,$$

where $\{\gamma^\mu\}_{\mu=1}^d$ is a set of the gamma matrices satisfying $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\delta_{\mu\nu}$ and $\mathbf{A}(x)=(A_1(x),\ldots,A_d(x))$ is a gauge potential. Under the d-dimensional version of the assumption of Theorem 5.2, we can show that the spectrum of every self-adjoint extension of the operator

$$\boldsymbol{D}\restriction C_0^\infty(\mathbf{R}^d)\cap[\cap_{j=1}^d D(A_j)]$$

is equal to $\mathbf{R}$.

Proof of Theorem 5.1: We need only to show that $B_j = A_j$ satisfies the assumption of Theorem 5.2. It is easy to see that A_j is almost locally in $L^2(\mathbf{R}^2)$. Let

$$\alpha = \max_{\nu=1,\ldots,n} |\mathbf{a}_\nu|$$

and $\mathbf{c}_m \in \mathbf{R}^2$ and $t_m > 0$ be such that $t_m \to \infty$ as $m \to \infty$ and

$$|\mathbf{c}_m| = t_m + m + 4\alpha + 1.$$

We set

$$K_m = \{\mathbf{r} \in \mathbf{R}^2 \mid |\mathbf{r} - \mathbf{c}_m| \leqslant t_m\}.$$

Then, for every $\mathbf{r} \in K_m$, we have

$$|\mathbf{r} - \mathbf{a}_\nu| \geqslant |\mathbf{c}_m - \mathbf{a}_\nu| - |\mathbf{r} - \mathbf{c}_m| \geqslant |\mathbf{c}_m| - \alpha - t_m = m + 3\alpha + 1, \tag{5.7}$$

hence, for all m,

$$K_m \cap \{\mathbf{a}_\nu\}_{\nu=1}^n = \emptyset.$$

By Eq. (5.7), we have for all $\mathbf{r} \in K_m$

$$|\mathbf{r}| \geqslant |\mathbf{r} - \mathbf{a}_\nu| - |\mathbf{a}_\nu| \geqslant m + 2\alpha + 1.$$

Moreover, for all $\mathbf{r} \in K_m$,

$$\frac{|\mathbf{r}|}{|\mathbf{r} - \mathbf{a}_\nu|} \leqslant \frac{|\mathbf{r} - \mathbf{a}_\nu| + |\mathbf{a}_\nu|}{|\mathbf{r} - \mathbf{a}_\nu|} \leqslant 1 + \frac{\alpha}{m + 3\alpha + 1} < \frac{4}{3}.$$

Hence

$$\frac{1}{|\mathbf{r} - \mathbf{a}_\nu|} < \frac{4}{3} \frac{1}{|\mathbf{r}|}, \quad \mathbf{r} \in K_m.$$

Using this inequality and Eq. (2.12), we have

$$|A_j(\mathbf{r})| \leqslant \frac{C}{|\mathbf{r}|^l}, \quad \mathbf{r} \in K_m,$$

with an integer $l \geqslant 1$ and a constant $C > 0$. Hence

$$\frac{1}{t_m^2} \int_{K_m} |A_j(\mathbf{r})|^2 d\mathbf{r} \leqslant \frac{\text{const}}{(m + 2\alpha + 1)^{2l}} \to 0 \, (m \to \infty).$$

Thus the desired result follows. ■

Theorem 5.1 implies the following.

Theorem 5.5: The operators $\bar{Q}_{\pm,\min}$ are not semi-Fredholm (hence not Fredholm).

This theorem is proven by employing the following lemma.

Lemma 5.6: Let T be a densely defined closed linear operator from a Hilbert space $\mathscr{H}_1$ to another one $\mathscr{H}_2$ such that $R(T)$ and $R(T^*)$ are closed. Let

$$Q_T=\begin{pmatrix}0 & T^* \\ T & 0\end{pmatrix},$$

which is self-adjoint in $\mathscr{H}_1 \oplus \mathscr{H}_2$. Then

$$\inf \sigma(|Q_T|)\setminus\{0\}>0. \tag{5.8}$$

Proof: By the present assumption and a general theorem (e.g., Ref. 24, Chap. IV, Sec. 5, Theorem 5.2), there is a constant $c>0$ such that

$$\|Tf\|\geqslant c\|f\|, \quad f\in(\ker T)^{\perp}\cap D(T),$$

$$\|T^*g\|\geqslant c\|g\|, \quad g\in(\ker T^*)^{\perp}\cap D(T^*).$$

Let $\psi=(f,g)\in(\ker Q_T)^{\perp}\cap D(Q_T)$. Then, using the fact that

$$\ker Q_T=\ker T\oplus\ker T^*,$$

we have $f\in(\ker T)^{\perp}\cap D(T)$ and $g\in(\ker T^*)^{\perp}\cap D(T^*)$. Hence

$$\|Q_T\psi\|^2=\|T^*g\|^2+\|Tf\|^2\geqslant c^2\|g\|^2+c^2\|f\|^2=c^2\|\psi\|^2. \tag{5.9}$$

Since Q_T is self-adjoint and ψ is an arbitrary elememt in $(\ker Q_T)^{\perp}\cap D(Q_T)$, Eq. (5.9) implies Eq. (5.8) [note that, for any self-adjoint operator A, we have $D(A)=D(|A|)$, $\ker A=\ker|A|$]. ■

Proof of Theorem 5.5: Suppose that $\bar{Q}_{+,\min}$ were semi-Fredholm. Then $R(\bar{Q}_{+,\min})$ is closed. Hence $R(Q^*_{+,\min})$ is also closed (e.g., Ref. 24, Chap. IV, Sec. 5, Theorem 5.13). Therefore, by Lemma 5.6, $\inf\sigma(|Q^{(1)}_{\min}|)\setminus\{0\}>0$. But this contradicts Eq. (5.2). Thus $\bar{Q}_{+,\min}$ is not semi-Fredholm. Similarly we can show that $\bar{Q}_{-,\min}$ is not semi-Fredholm. ■

Remark 5.7: By Theorem 5.5, $\operatorname{ind}(\bar{Q}_{\pm,\min})$ may be unstable even under relatively compact perturbations. It would be desirable to find an example of such perturbations.

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