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A Pincer Randomization Method for Valuing American Options*

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For European option we can obtain their exact values by using the so-called Black-Scholes formula, whereas there is no explicit exact solution for American counterparts despite that many researchers have attempted to obtain the solution. There have been many approximation methods developed for valuing American options, but they are not inclusive because they have both drawbacks and advantages with respect to their speed and accuracy. This paper develops an *interpolation* or *pincer* approximation based on the randomization methods due to Carr (1998) and Kimura (2004), using a pair of lower and upper bounds for option values derived by the Theta property. From numerical comparisons with other approximations, we see that our approximation has sufficient accuracy and efficiency for practical applications.

JEL Classification: G12, G13
Keywords: American options, Valuation, Pincer approximation, Randomization, Early exercise boundary

1. Introduction

Variety has come to options market nowadays since Black and Scholes (1973) and Merton (1973) published the seminal papers. In particular, the valuation of American options written on dividend-paying assets is an important issue in the market, because it can be widely applied to many other types of problems such as *real options*. Since McKean (1965) and Merton (1973) formulated the American option valuation as a free boundary problem, many researchers and practitioners have attempted to solve the analytical valuation problem of American options. However, no closed-form formulas have not yet been obtained. The difficulty is due to the possibility of early exercise whose unknown boundary must be determined as a part of the solution. Some researchers have made efforts toward developments of numerical approximation methods for pricing American options.

A simple and intuitive approximation was developed by Johnson (1983),

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which is based on analytical lower and/or upper bounds for the option value. Approximations are generated either by multiplying a coefficient to a single bound or by combining a pair of lower and upper bounds with weight coefficients. These coefficients are statistically estimated from a large set of the values of options actually traded in the market. In this sense, approximations based on bounds are of *experimental* or *implied* nature. Also, Broadie and Detemple (1996) have developed more sophisticated approximations in the same spirit. They obtained a lower bound of American call value by a capped American call option, while they numerically computed an upper bound through the integral representation of the early exercise premium ; see Carr et al. (1992) and Kim (1990). Using these bounds, they proposed two kinds of approximations called LBA (lower-bound approximation) and LUBA (lower-and-upper-bounds approximation). Numerical comparisons in Broadie and Detemple (1996) shows that both approximations can be quickly computed and that LUBA is more accurate than LBA.

Another fast and accurate approximation among existing methods is the randomization method proposed by Carr (1997), which is based on an American option with a random maturity. The random maturity follows the n-stage Erlangian distribution with mean equal to the original maturity. As $n \to \infty$, the Erlangian distribution converges to a point mass concentrated at the mean. Hence, for large n, it can be considered that the value of American option with random maturity approximates the true value. Although this idea is easy to understand, the Erlangian distribution is not suitable for obtaining a simple formula for the n-th approximation. In fact, Carr's formula for the American put value is given by a recursion of complex triple sums. To obtain a more tractable formula, an alternative randomization method has been recently developed by Kimura (2004), which used an order statistic for the random maturity. It has been shown that Kimura's randomization method is much simpler than Carr's method, providing almost the same accuracy. However, numerical solutions tend to behave unstably when we use high-precision computation. To remove this instability, we consider to refine Kimura's randomization by treating approximations with low precision as bounds of the true value. Interpolating these bounds, we propose approximations for the option value and the early exercise boundary. This hybrid scheme is called a *pincer randomization* in this paper.

The rest of this paper is organized as follows: In Section 2, we provide some preliminaries for the analysis. The primal focus is on the American put option because the call case can be analyzed by put-call symmetry relations. Section 3 provides an idea of the pincer randomization method. To examine the accuracy of our method, numerical comparisons with other approximations are shown in Section 4. Finally, we give a conclusion and some comments on future research in Section 5.

2. Preliminaries

2.1 Basic framework

Let $(S_t)_{t\geq 0}$ be the stock price in the capital market following the efficient market hypothesis. Assume that $(S_t)_{t\geq 0}$ is a risk-neutralized process governed by the stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}W_t, \qquad t \geq 0, \tag{2.1}$$

where $r > 0$ is the risk-free interest rate, $\delta \geq 0$ is a continuous dividend rate, $\sigma > 0$ is a volatility of the asset returns, and $(W_t)_{t\geq 0}$ is a standard Wiener process on a filtered probability space $(\Omega, (\mathbb{F}_t)_{t\geq 0}, \mathbb{F}, \mathbb{P})$, where $(\mathbb{F}_t)_{t\geq 0}$ is the natural filtration corresponding to W and the probability measure $\mathbb{P}$ is chosen so that the stock has mean rate of return r.

Let

$$P \equiv P(t,\ S_t) = P(t, S_t;\ K, r, \delta), \quad 0 \leq t \leq T$$

be the value of American put option written on $(S_t)_{t\geq 0}$ with maturity date $T > 0$ and exercise price $K > 0$. Similarly, let $C \equiv C(t,\ S_t) = C(t, S_t;\ K, r, \delta)$ $(0 \leq t \leq T)$ denote the value of the associated American call option with the same parameters as those in the put option. McDonald and Schroder (1998) proved that a *parity relation* holds between these two options, *i.e.*,

$$C(t, S;\ K, r, \delta) = P(t, K;\ S, \delta, r). \tag{2.2}$$

Because of the parity relation between P and C, we focus on the American put in this paper.

From the theory of arbitrage pricing, the fair value of the American put option at time t is given by solving an *optimal stopping problem*

$$P(t, S_t) = \operatorname*{ess\,sup}_{T_t \in [t,T]} \mathbb{E}\left[e^{-r(T_t - t)}(K - S_{T_t})^+ \mid \mathbb{F}_t\right], \qquad t \in [0, T], \tag{2.3}$$

where T_t is a stopping time of the filtration $(\mathbb{F}_t)_{t\geq 0}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$. Solving the optimal stopping problem (2.3) is equivalent to find the points $(t,\ S_t)$ for which early exercise is optimal. Let $\mathbb{S}$ and $\mathbb{C}$ denote the *stopping region* and *continuation region,* respectively. The stopping region $\mathbb{S}$ is defined by

$$\mathbb{S} = \left\{(u,\ x) \in [0,\ T] \times \mathbb{R}_+ \mid P(u,\ x) = (K - x)^+\right\}. \tag{2.4}$$

Of course, the continuation region $\mathbb{C}$ is the complement of $\mathbb{S}$ in $[0, T] \times \mathbb{R}_+$. The boundary that separates $\mathbb{S}$ from $\mathbb{C}$ is referred as the *early exercise boundary,* which is defined by

$$B_t = \sup\{x \in \mathbb{R}_+ \mid P(t,\ x) = (K - x)^+\}, \qquad t \in [0, T]. \tag{2.5}$$

The American put value P and the early exercise boundary $(B_t)_{t \in [0,T]}$ can be obtained by jointly solving a *free boundary problem*, which is specified by the Black-Scholes-Merton partial differential equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P(t, S)}{\partial S^2} + (r-\delta) S \frac{\partial P(t, S)}{\partial S} + \frac{\partial P(t, S)}{\partial t} - rP(t, S) = 0 \tag{2.6}$$

subject to the boundary conditions

$$\lim_{S \uparrow \infty} P(t, S) = 0, \tag{2.7}$$

$$\lim_{S \downarrow B_t} P(t, S) = K - B_t, \tag{2.8}$$

$$\lim_{S \downarrow B_t} \frac{\partial P(t, S)}{\partial S} = -1, \tag{2.9}$$

and the terminal condition

$$P(T, S) = (K - S)^+. \tag{2.10}$$

Equation (2.8) is usually called the *value matching condition* and Equation (2.9) is the *smooth pasting condition*. These conditions guarantee that premature exercise strategy on the early exercise boundary B_t will be optimal.

2.2 Randomization methods

2.2.1 Carr's randomization

Carr's randomization method consists of the following three steps:

1. Randomize the maturity T by an exponentially distributed random variable $\tilde{T}$ with mean $\mathbb{E}[\tilde{T}] = \lambda^{-1} = T$ in order to value the so-called Canadian option.

2. Extend the result to the case that $\tilde{T}$ is distributed as the n-stage Erlangian distribution with the same mean $\mathbb{E}[\tilde{T}] = \lambda^{-1} = T$.

3. Take the limit of the randomized option value by letting $n \to \infty$ to obtain the underlying American option value.

Let $h_n(t)$ $(t \geq 0)$ denote the pdf of the n-stage Erlangian distribution with mean $\lambda^{-1} = T$, *i.e.*,

$$h_n(t) = \frac{(nt/T)^{n-1}}{(n-1)!} \frac{n}{T} e^{-nt/T}, \quad t \geq 0.$$

Figure 1 illustrates that a sequence of $\{h_n(t)\}$ converges to Dirac's delta function concentrated at the mean T $(=1)$ as n gets large.

For a continuous function $g(t)$ $(t \geq 0)$, define

$$g_n^*(T) = \mathbb{E}[g(\tilde{T})] = \int_0^\infty g(t) h_n(t) \mathrm{d}t. \tag{2.11}$$

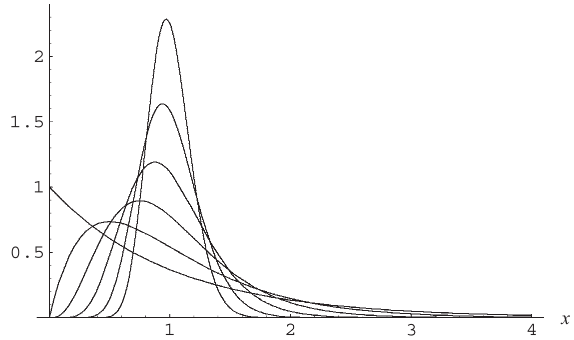


Figure 1. Convergence of the Erlangian pdf $h_n(\cdot)$ (n=1, 2, 4, 8, 16, 32)

Then, we have

$$\lim_{n\to\infty} g_n^*(T) = g(T) \tag{2.12}$$

that is the mathematical essence of Carr's randomization method.

2.2.2 Kimura's randomization

Instead of the n-stage Erlangian distribution, Kimura (2004) adopted an order static for the random maturity. In much the same way as in Carr's randomization, his method can be written by the following three steps:

1. Randomize the maturity T by an exponentially distributed random variable $\tilde{T}$ with mean $\mathbb{E}[\tilde{T}] = \lambda^{-1} = T$ in order to value the Canadian option.

2. Extend the result to the case that $\tilde{T}$ is distributed as an order statistic with the same mean $\mathbb{E}[\tilde{T}] = \lambda^{-1} = T$.

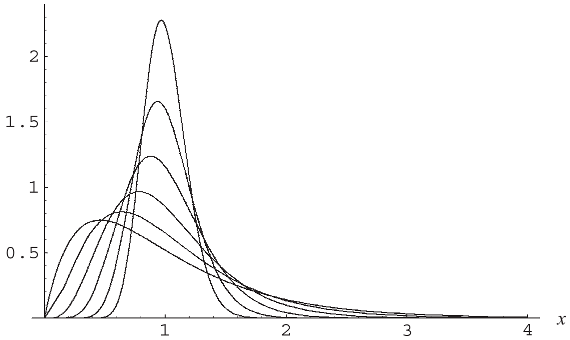
3. Take the limit of the randomized option value by letting $n, m \to \infty$ to obtain the underlying American option value.

The order static used by Kimura is defined as follows: Let $X_1, \cdots, X_{n+m}$ be iid random variables with parameter α (> 0), and let $X_{(i)}$ denote the i-th smallest of these random variables $(i = 1, \cdots, n+m)$. Then, the pdf of $X_{(n+1)}$ is given by

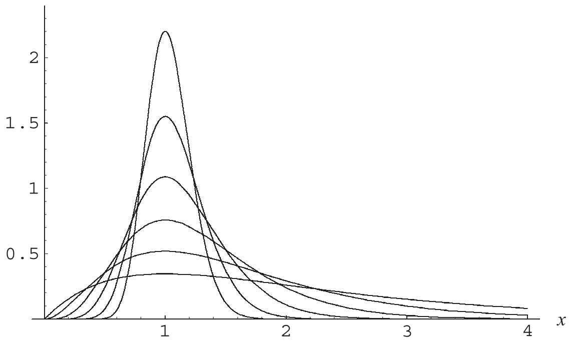
$$f_{n,m}(t) = \frac{(n+m)!}{n!(m-1)!}(1-e^{-\alpha t})^n \alpha e^{-m\alpha t}, \quad t \geq 0. \tag{2.13}$$

The mean and variance of $X_{(n+1)}$ are given by

$$\begin{aligned} \mathbb{E}[X_{(n+1)}] &= \frac{1}{\alpha}\sum_{i=0}^{n}\frac{1}{m+i}, \\ \mathbb{V}[X_{(n+1)}] &= \frac{1}{\alpha^2}\sum_{i=0}^{n}\frac{1}{(m+i)^2}. \end{aligned} \tag{2.14}$$



(a) mean matching: $\mathbb{E}[X_{(n+1)}] = 1$



(b) mode matching: $\mathbb{M}[X_{(n+1)}] = 1$

Figure 2. Convergence of the pdf $f_{n,m}(\cdot)$ ($n = m$=1, 2, 4, 8, 16, 32)

In addition, the modal value of $X_{(n+1)}$ is given by

$$\mathbb{M}[X_{(n+1)}] \equiv \arg\max_t f_{n,m}(t) = \frac{1}{\alpha}\log\frac{n+m}{m}. \tag{2.15}$$

Figure 2(a) and 2(b) show the convergence of the pdf as $n(=m) \to \infty$ for the cases that (a) $\mathbb{E}[X_{(n+1)}] = T \equiv 1$ and (b) $\mathbb{M}[X_{(n+1)}] = T \equiv 1$. From these figures we see that there is no difference between these two cases and the pdfs converge to Dirac's delta function concentrated at the mean $\mathbb{E}[X_{(n+1)}]$. By setting either $\mathbb{E}[X_{(n+1)}] = T$ or $\mathbb{M}[X_{(n+1)}] = T$, $X_{(n+1)}$ can be another candidate for the random maturity $\tilde{T}$, because $\lim_{n,m \to \infty} \mathbb{V}[X_{(n+1)}] = 0$. Kimura (2004) adopted the mode-matching $\mathbb{M}[X_{(n+1)}] = T$ in his randomization for computational convenience, because there is no significant difference between the two matchings for large numbers of n. For the mode-matching, α can be determined by

$$\alpha = \frac{1}{T}\log\frac{n+m}{m}. \tag{2.16}$$

For a continuous function $g(t)$ $(t \geq 0)$, define

$$g^*_{n,m}(T) = \mathbb{E}[g(\tilde{T})] = \int_0^\infty g(t) f_{n,m}(t)\,\mathrm{d}t. \tag{2.17}$$

Kimura (2004, Proposition 3) showed that the sequence $(g^*_{n,m})_{n,m \geq 1}$ can be efficiently computed by using the recursion

$$\begin{cases} g^*_{0,m} = \int_0^\infty m\alpha e^{-m\alpha t} g(t)\mathrm{d}t \\ g^*_{n,m} = \dfrac{n+m}{n} g^*_{n-1,m} - \dfrac{m}{n} g^*_{n-1,m+1}, \quad n \geq 1, \end{cases} \tag{2.18}$$

and that

$$\lim_{n,m\to\infty} g^*_{n,m}(T) = g(T). \tag{2.19}$$

2.3 Canadian options

It can be seen that the Canadian case is a common starting point in these randomization methods. Hence, we briefly summarize the fundamental results for the Canadian options.

For $\lambda > 0$ and $S_t = S$, let $p^* \equiv p^*(\lambda, S)$ denote the European-style Canadian put value at time $t \in [0, \tilde{T}]$. Note that $p^*(\lambda, S)$ does *not* depend on t by virtue of the memoryless property. Then, we have

Proposition 1 (Kimura (2004)) The value of the European-style Canadian put option is given by

$$p^*(\lambda, S) = \begin{cases} \xi(S) + \dfrac{\lambda}{\lambda+r}K - \dfrac{\lambda}{\lambda+\delta}S, & S < K \\ \eta(S), & S \geq K, \end{cases} \tag{2.20}$$

where

$$\xi(S) = \frac{1}{\theta_1-\theta_2}\frac{\lambda}{\lambda+\delta}\left(1-\frac{r-\delta}{\lambda+r}\theta_2\right)K\left(\frac{S}{K}\right)^{\theta_1}, \ S < K \tag{2.21}$$

$$\eta(S) = \frac{1}{\theta_1-\theta_2}\frac{\lambda}{\lambda+\delta}\left(1-\frac{r-\delta}{\lambda+r}\theta_1\right)K\left(\frac{S}{K}\right)^{\theta_2}, \ S \geq K \tag{2.22}$$

and the parameters $\theta_1 \equiv \theta_+ > 0$ are $\theta_2 \equiv \theta_- < 0$ are two roots of the quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 + \left(r-\delta-\frac{1}{2}\sigma^2\right)\theta - (\lambda+r) = 0,$$

i.e.

$$\theta_\pm = \frac{1}{\sigma^2}\left\{-\left(r-\delta-\frac{1}{2}\delta^2\right) \pm \sqrt{\left(r-\delta-\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\lambda+r)}\right\}. \tag{2.23}$$

For $\lambda > 0$ and $S_t = S$, let $P^* \equiv P^*(\lambda, S)$ denote the American-style Canadian put value. Then, we have

Proposition 2 (Kimura (2004)) The value of the European-style Canadian put option is given by

$$P^*(\lambda,S)=\begin{cases} K-S, & S \le B^* \le K \\ p^*(\lambda,S)+e^*(\lambda,S), & S > B^*, \end{cases} \tag{2.24}$$

where $B^* \equiv B^*(\lambda)$ is the Laplace-Carlson transform of the time-reversed early exercise boundary $\tilde{B}_\tau \equiv B_{T_{T-\tau}}(\tau \ge 0)$, *i.e.*,

$$B^*(\lambda)=\int_0^\infty \lambda e^{-\lambda\tau}\tilde{B}_\tau d\tau, \quad \lambda > 0 \tag{2.25}$$

and

$$e^*(\lambda,S)=-\frac{1}{\theta_2}\left\{\theta_1\xi(B^*)+\frac{\delta}{\lambda+\delta}B^*\right\}\left(\frac{S}{B^*}\right)^{\theta_2}, \qquad S > B^*. \tag{2.26}$$

Applying the value matching condition (2.8) to the option value $P^*(\lambda,S)$ in (2.24), we can obtain the following results that specify the early exercise boundary B^* of the American-style Canadian put option.

Proposition 3 (Kimura (2004))

(i) The early exercise boundary B^* of the American-style Canadian put option satisfies the equation

$$\lambda\left(\frac{B^*}{K}\right)^{\theta_1}=r(\theta_1-1)-\delta\theta_1\frac{B^*}{K}. \tag{2.27}$$

(ii) For the limiting case $\lambda \to 0$, we have

$$B^*(0)=\lim_{\tau\to\infty}\tilde{B}_\tau=\frac{r(\theta_1^o-1)}{\delta\theta_1^o}K=\frac{\theta_2^o}{\theta_2^o-1}K, \tag{2.28}$$

where $\theta_i^o=\lim_{\lambda\to 0}\theta_i (i=1,2)$. In particular, if $\delta=0$, then

$$B^*(0)=\lim_{\tau\to\infty}\tilde{B}_\tau=\frac{K}{1+\frac{\sigma^2}{2r}}. \tag{2.29}$$

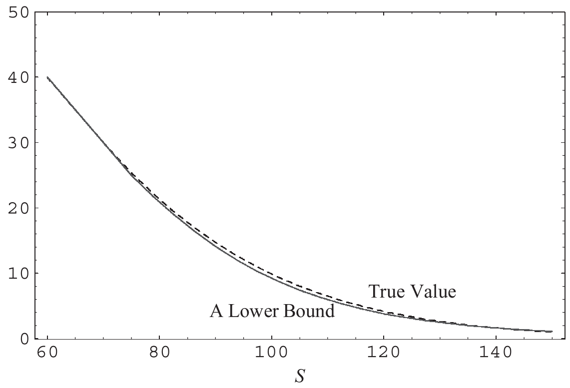
(iii) For the limiting case $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty}B^*(\lambda)=\tilde{B}_0=B_T=\min\left(\frac{r}{\delta},\ 1\right)K. \tag{2.30}$$

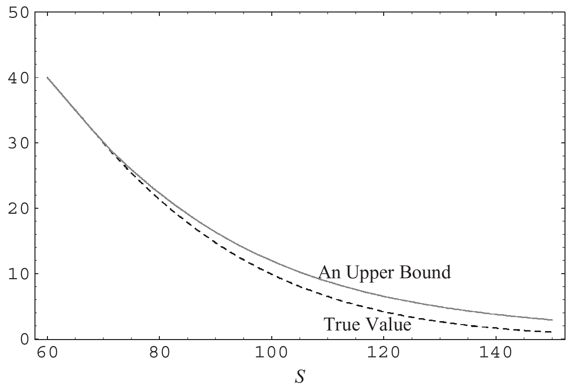
3. A Pincer Randomization Method

By virtue of the recursion (2.18), Kimura's randomization method is much simpler than Carr's method, providing almost the same accuracy. However, Kimura's method sometimes behaves unstably near the expiry in high-precision computation. The reasons for the instability are considered as

(i) the algorithm is fairly sensitive to the precision of the root B^* of the equation (2.27).



(a) A lower bound for the TRUE value



(b) An upper bound for the TRUE value

Figure 3. Lower and upper bounds (T=1.0, S=100, K=100, r=0.05, σ=0.3, δ=0)

(ii) the (n,m)-th approximation $g^*_{n,m}$ cannot appropriately satisfies the value matching condition in the recursive procedure.

In this section, we propose a refinement to overcome these difficulties, which is based on a pair of lower and upper bounds for a true value (say TRUE), and then TRUE is sandwiched between the bounds. This methods reflect some fundamental properties of the option Greek *Theta* and the order statistic. It is generally known that Theta indicates the ratio of the change in an American put value to decrease in time to expiration. Hence, the shorter the remaining time to expiration, the option value is cheaper.

3.1 Lower and upper bounds for the option value

First consider the mean-matching case. From Figure 2(a) and the Theta property of American put options that the mean-matching approximation for the option value always underestimates the true value when n and m are not large enough, giving a lower bound. Note that mean-matching approximation for the early exercise boundary provides an upper bound. Figure 3(a) shows that the lower bound is a tight one over the true value derived by the CRR bi-

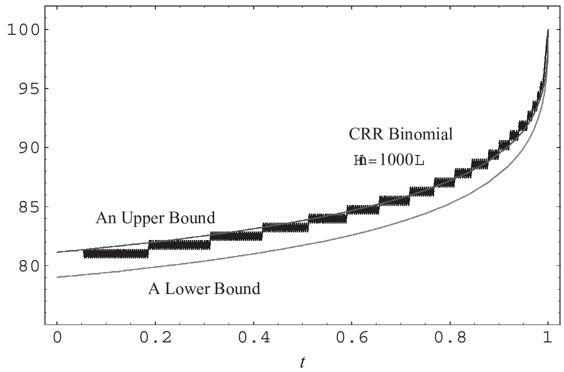


Figure 4. Lower and upper bounds for the early exercise boundary (K=100, r=0.05, σ=0.3, δ=0)

nomial method (Cox et al. (1979)) with $n = 1000$.

In the same manner as the mean-matching case, Figure 2(b) and the Theta property shows that the mode-matching approximation always overestimates the true value when n and m are small, *i.e*., it gives an upper bound. For the early exercise boundary, the mode-matching approximation provides a lower bound on the contrary. Figure 3(b) shows that the upper bound is less tight than the lower bound, where TRUE values are also computed by the CRR binomial method with $n = 1000$.

3.2 Interpolating lower and upper bounds

Figure 4 illustrates a relationship between the lower and upper bounds for the early exercise boundary. This figure shows that the the TRUE value is appropriately sandwiched between the bounds, and that the upper bound derived by the mean matching is a good approximation for the TRUE value. For the option value, the TRUE value is also in the bounds, and the lower bound is a good approximation for the TRUE one. From Figure 3, the mean matching provides more accurate approximations for the option value. From these observations, we employ the two interpolation methods below for valuing American put options, each of which does *not* have experimental nature as in LUBA (See Broadie and Detemple (1996)): Let $L(t, S_t)$ and $U(t, S_t)$ denote the lower and upper bound for the option value, respectively. Then, we define

- Arithmetic Average:

$$P_A(t,S_t) = \frac{L(t,S_t) + U(t,S_t)}{2}, \tag{3.1}$$

- Geometric Average:

$$P_G(t,S_t) = \sqrt{L(t,S_t) \times U(t,S_t)} \tag{3.2}$$

for approximations of the American put value. For the early exercise boundary, we define the arithmetic and geometric averages in a similar way. As de-

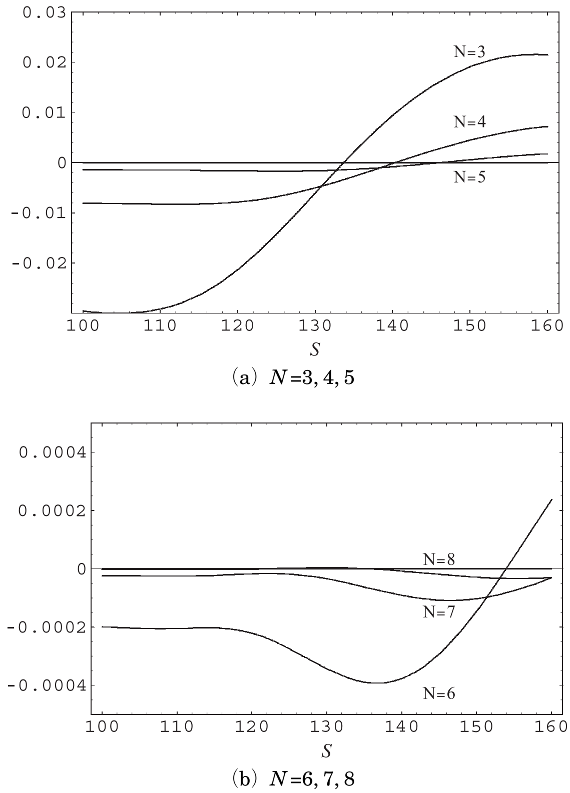


Figure 5. Relative percentage errors of the approximations for the vanilla European put value $p(0, S)$ (T=1.0, K=100, r=0.05, σ=0.3, δ=0.05)

scribed above, the upper bound of the early exercise boundary and the lower bound of the option value are good approximations for the TRUE values. Hence, we also add the upper-bound approximation for the early exercise boundary and the lower-bound approximation for the option value in comparisons.

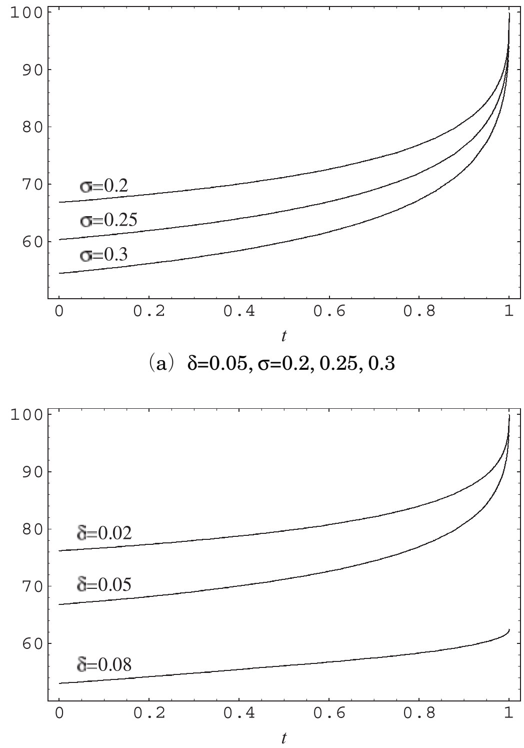
To determine an optimal level of (n, m) in the approximation $g^*_{n,m}$, we made a preliminary comparison between the European put value and its PR approximation. Let $p(t, S)$ denote the value of a vanilla European put option at time $t \in [0,T]$. Obviously, $p(t, S)$ can be computed by the Black-Scholes formula

$$p(t,S) = Ke^{-r(T-t)}\Phi\left(-d+\sigma\sqrt{T-t}\right) - Se^{-\delta(T-t)}\Phi(-d), \tag{3.3}$$

where

$$d = \frac{\log(S/K) + \left(r-\delta+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{3.4}$$

Figure 5 illustrates the relative percentage errors of approximations $g^*_{N,N}$ $(N = 3,\ldots,8)$ for $p(0, S)$ as functions of S. We see from Figure 5 that the approximations become better as N increases, being sufficiently accurate for



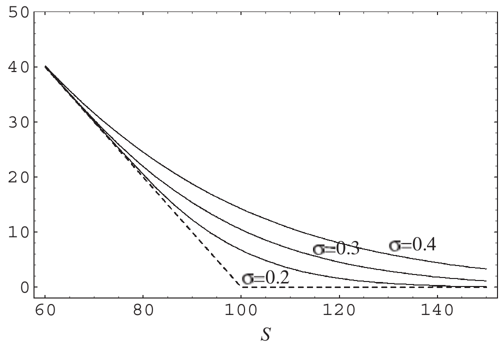
(b) δ=0.02, 0.05, 0.08, σ=0.2

Figure 6. Early exercise boundaries of put options (K=100, T=1.0, r=0.05)

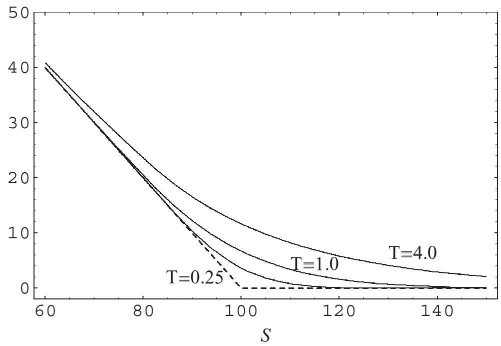
$N \geq 6$. Hence, we will employ $N = 8$ in our numerical experiments.

4. Computational Results

Figure 6(a)(6(b)) shows some relations between the early exercise boundary and the volatility (dividend rate). Also, Figure 7(a)(7(b)) shows some relations between the option values and the volatility (maturity). In order to check the performance of the PR method in detail, we compare them with other approximations for particular cases quoted from numerical experiments in AitSahlia and Carr (1997). Tables 1 and 2 summarize these results, in which we compute three approximations by the PR method with both the arithmetic and geometric averages and the lower-bound approximation named LB-Rondom. We employ the arithmetic average of the 1000- and 1001-steps binomial value as a bench mark of the TRUE value. For the methods of Kimura (2004), Carr (1997), and Geske and Johnson (1984), "N-pts" in these tables denote the number of steps of the N-point Richardson extrapolation. For the finite-difference results (Brennan and Schwartz (1977)), the parameters N and M denote the numbers of time and state steps, respectively. Also, the quadratic approximation (Barone-Adesi and Whaley (1987) and MacMillan (1886)), LBA and LUBA (Broadie and Detemple (1996)), approximations by Bunch and Johnson (1992) and Huang et al. (1996) are added for compari-



(a) T=1.0, σ=0.2, 0.3, 0.4



(b) T=0.25, 1.0, 4.0, σ=0.2

Figure 7. Values of put options (K=100, t=0, r=0.05, δ=0.02)

sons in these tables. See AitSahlia and Carr (1997) for details of their experiments.

The PR method performs very well and competes with the randomizations of Kimura and Carr. In addition, the PR method succeeds in the way that modified Kimura's randomization that always underestimates the TRUE value, because the PR method provides not only much more accurate approximations for valuing put options but also better approximation than Kimura. In addition, we see from these figures that the PR method is more accurate than LBA and LUBA, which are also the lower-bound and the lower-and-upper-bounds approximations, respectively.

Table 1 shows the impacts of the initial stock price S. The PR method with both of arithmetic and geometric average becomes accurate as S increases, because the early exercise premium relatively constitutes a smaller portion of the value for such cases. The fact is very well deserved from the viewpoints that the PR method can value European option values as accurate as the Black-Scholes formula and that we can decompose American option value into the early exercise premium plus European option value.

Table 2 demonstrates the impacts of the remaining time to maturity on the option value. For both cases of arithmetic and geometric averages, the PR method becomes accurate as the remaining time becomes long. For this tendency, we can give the same prospect from Table 1. In addition, from Tables 1

Table 1. A comparison of approximations for $P(0, S)$ ($T=3, K=100, r=0.06, \sigma=0.4, \delta=0.02$)

Method	$S=80$	$S=90$	$S=100$	$S=110$	$S=120$
Binomial	29.2601	24.8023	21.1294	18.0849	15.5428
PR method (Arithmetic Ave.)	28.8392	24.4533	20.8489	17.8681	15.3892
PR method (Geometric Ave.)	28.8373	24.4501	20.8445	17.8624	15.3823
LB-Random	28.5135	24.0614	20.4092	17.3971	14.9005
Kimura (8-pts)	28.7998	24.4246	20.7891	17.7971	15.2995
Carr (3-pts)	29.2323	24.7692	21.0835	18.0369	15.4873
Geske and Johnson	31.0305	26.1543	22.1114	18.7646	15.9911
Quadratic	29.4377	25.0614	21.4484	18.4418	15.9239
LBA	29.2105	24.7669	21.1039	18.0635	15.5252
LUBA	29.2540	24.7989	21.1306	18.0860	15.5437
Bunch and Johnson	29.9382	25.1566	21.3092	18.1558	15.5755
Huang et al.	29.7147	25.0136	21.2121	18.1173	15.5729
Finite Difference ($N=200, M=300$)	29.0584	24.4744	20.6330	14.5535	14.5535

Table 2. A comparison of approximations for $P(0, 100)$ ($K=100, r=0.06, \sigma=0.4, \delta=0.02$)

Method	$T=0.5$	$T=1.0$	$T=1.5$	$T=2.0$	$T=2.5$
Binomial	10.2741	13.8774	16.3682	18.2840	19.8349
PR method (Arithmetic Ave.)	10.3057	13.8392	16.2596	18.1109	19.6045
PR method (Geometric Ave.)	10.3016	13.8345	16.2547	18.1061	19.5998
LB-Random	10.0157	13.4765	15.8586	17.6884	19.1702
Kimura (8-pts)	10.1802	13.7083	16.1399	18.0090	19.5321
Carr (3-pts)	10.2759	13.8670	16.3469	18.2533	19.7960
Geske and Johnson	10.3159	14.0553	16.7200	18.8388	20.5970
Quadratic	10.2728	13.9142	16.4627	18.4476	20.0743
LBA	10.2697	13.8679	16.3545	18.4476	19.8134
LUBA	10.2750	13.8796	16.3712	18.2869	19.8371
Bunch and Johnson	10.2679	13.8904	16.4070	18.3487	19.9434
Huang et al.	10.2813	13.8756	16.3657	18.2948	19.8742
Finite Difference ($N=200, M=300$)	10.2614	13.8578	16.3158	18.1500	19.5442

and 2, we can see that the PR method with arithmetic average is accurate enough and is greater than the one with geometric average. Clearly, this reflects the fact that $P_A(t, S_t) \geq P_G(t, S_t)$ for all (t, S_t).

From the observations in Figures 3 and 4, it was considered that the lower-bound approximations for the option values would perform well. However, we see from Tables 1 and 2 that the lower-bound approximations are less accurate than other approximations. We also see from other numerical experiments that the randomization method with mean matching performs well if and only if dividend is zero for which the root B^* can be computed via

$$B^* = K\left(\frac{r(\theta_1 - 1)}{\lambda}\right)^{\frac{1}{\theta_1}} \tag{4.1}$$

without using Newton's method. These observations would imply that the accuracy of the lower-bound (or mean-matching) approximation is highly sensitive to the computational accuracy of the root B^*.

5. Conclusion

The previously established randomization methods have crucial problems such as (i) difficulty of implementation in Carr's randomization and (ii) unstable behavior near expiry in Kimura's randomization. To rectify these defects at the same time, we have adopted an interpolation approximation using a pair of lower and upper bounds obtained by Kimura's randomization. The idea is due to the Theta property of American put options.

The PR method generates accurate approximations when the initial stock price is in the out-of-money or the remaining time to maturity is long. It is straightforward to interpret these properties from the fact that American option value can decomposed into the early exercise premium and the associated European option value, the latter of which constitutes a greater portion of the whole value. However, the PR method still have a tendency of underestimation from the true value, which needs a further revision of the randomization.

Mathematical essence of randomization can be interpreted as an inversion of *Laplace* or *Fourier transforms*. This interpretation enables us to apply the randomization methods including the PR method to valuing other options, *e.g., exotic* or *path-dependent* options such as Asian, lookback, barrier options and so on. This is a future theme of extensive research. Another extension of the randomization method is the case that the stock return jumps accidentally, that is, the stock price process follows a jump-diffusion process or more generally a Lévy process. This remains as a future theme, too.

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