AN INSTABILITY CRITERION FOR ACTIVATOR INHIBITOR SYSTEMS IN A TWO-DIMENSIONAL BALL

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Abstract. Let B be a two-dimensional ball with radius R. Let $(u(x,y),\xi)$ be a non-constant steady state of the shadow system

$$u_t = D_u \Delta u + f(u,\xi) \ \text{ in } \ \mathrm{B}\times\mathbb{R}_+ \quad \text{and} \quad \tau\xi_t = \frac{1}{|B|}\iint_B g(u,\xi)dxdy \ \text{ in } \ \mathbb{R}_+,$$

$$\partial_\nu u = 0 \ \text{ on } \ \partial B\times\mathbb{R}_+,$$

where f and g satisfy the following: $f_\xi(u,\xi)<0$, $g_\xi(u,\xi)<0$ and there is a function $k(\xi)$ such that $g_u(u,\xi)=k(\xi)f_\xi(u,\xi)$. This system includes a special case of the Gierer-Meinhardt system and the FitzHugh-Nagumo system. We show that if $\mathcal{Z}\left[U_\theta(\,\cdot\,)\right]\geq 3$, then (u,ξ) is unstable for all $\tau>0$, where $U(\theta):=u(R\cos\theta,R\sin\theta)$ and $\mathcal{Z}\left[w(\,\cdot\,)\right]$ denotes the cardinal number of the zero level set of $w(\,\cdot\,)\in C^0(\mathbb{R}/2\pi\mathbb{Z})$. The contrapositive of this result is the following: if (u,ξ) is stable for some $\tau>0$, then $\mathcal{Z}\left[U_\theta(\,\cdot\,)\right]=2$. In the proof of these results, we use a strong continuation property of partial differential operators of second order on the boundary of the domain.

Keywords: instability; activator-inhibitor system; shadow system; reaction-diffusion system; nodal curve; nodal domain

1. Introduction and the main results

R. Casten and C. Holland [CH78] and H. Matano [Ma79] independently show that all the inhomogeneous steady states of single reaction diffusion equations with the Neumann boundary condition are unstable provided that the domain is convex. In the case of non-convex domains, the example of a stable inhomogeneous steady state is given in [Ma79]. Y. Nishiura [N94] shows that if the domain is a one-dimensional interval, then the so-called *shadow system* of a certain class of reaction diffusion systems does not have stable steady states which are non-constant and non-monotone (see also [NPY01]). In the case of high dimensional domains, it is known that there are several specific reaction diffusion systems, for example the Gierer-Meinhardt system [Mi05a, NTY01] and the FitzHugh-Nagumo system [O03], which admit stable inhomogeneous steady states even if the domain is convex, *e.g.*, a ball. If the system has the gradient structure or the skew-gradient structure [Y02b, Y02a], then there are instability results for inhomogeneous steady states. Specifically, all the inhomogeneous steady states are unstable provided that the system has the gradient structure and that the domain is convex (see [JM94, L96]). In the case of skew-gradient systems in a convex domain, several instability results for inhomogeneous steady states are obtained by [Y02a]. See [KY03, Y02b, K05] for other instability results of skew-gradient systems. However, it seems that the instability criterion of the steady states of a wide class of reaction diffusion systems,

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which clarify the relation between the stability and the profile, is not known except the one-dimensional case [N94, NPY01].

The aim of the present paper is to give a sufficient condition for the steady state of the shadow system of a wide class of reaction diffusion system, which is (SS) below with (N1) or (N2) below, to be *unstable* in the case that the domain is a two-dimensional ball (Theorem A). We also obtain a necessary condition for the steady state to be stable (Corollary B). Our sufficient condition requires only the information of the profile of the stationary solution *on the boundary* of the ball (see the assumption (A)). We also give a sufficient condition for the steady state to be radially symmetric (Lemma C) which is used to prove Theorem A. That sufficient condition is be proven by a new type of strong unique continuation properties on the boundary of the domain (Lemma D).

We will introduce notations and formulate our problem. Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{N}$ denote the set of the real numbers, the non-negative real numbers, the integer numbers and the nutural numbers respectively. In particular, the natural number shall not contain 0, hence $\mathbb{N} := \{1, 2, 3, \cdots\}$. Let $B \subset \mathbb{R}^2$ be a two-dimensional ball centered at the origin with radius R. In this paper, we consider the instability of the steady state of the following system:

$$\text{(SS)}\quad u_t = D_u\Delta u + f(u,\xi) \ \text{ in } \ B\times\mathbb{R}_+ \quad \text{and} \quad \tau\xi_t = \frac{1}{|B|}\iint_B g(u,\xi)dxdy \ \text{ in } \ \mathbb{R}_+,$$
$$\partial_\nu u = 0 \ \text{ on } \ \partial B\times\mathbb{R}_+,$$

where ∂_ν denotes the outer normal derivative at ∂B and τ is a positive real number. We call (SS) *the shadow system*, following [N82]. Without loss of generaliry, we can assume that $\iint_B dxdy = 1$.

The stationary solution of (SS) is given by the solution of

$$\text{(SE)}\qquad D_u\Delta u + f(u,\xi) = 0 \ \text{ in } \ B \quad \text{and} \quad \iint_B g(u,\xi)dxdy = 0,$$
$$\partial_\nu u = 0 \ \text{ on } \ \partial B,$$

where u is a function depending on space, and ξ is a spatially homogeneous function, *i.e.*, a constant. Note that ξ always satisfies the Neumann boundary condition.

Next, we state the assumption for the nonlinear terms f and g. In this paper, we assume that $f(\,\cdot\,,\,\cdot\,)$ and $g(\,\cdot\,,\,\cdot\,)$ are of class C^2, and that f and g satisfy the following (N1) or (N2):

$$\text{(N1)}\qquad \begin{array}{c} f_\xi(u,\xi) < 0,\ g_\xi(u,\xi) < 0 \ \text{ and } \ \text{there is a real-valued function } k(\xi) \\ \text{such that } g_u(u,\xi) = k(\xi)f_\xi(u,\xi). \end{array}$$

$$\text{(N2)}\qquad f_\xi(u,\xi) < 0,\ g_u(u,\xi) > 0 \ \text{ and } \ g_\xi(u,\xi) < 0.$$

The system satisfying (N2) is generally called *the activator inhibitor system*, and describes various natural phenomena.

Example 1.1. The shadow system of the Gierer-Meinhardt model [GM72] is the following:

$$\text{(GM)}\qquad u_t = \varepsilon^2\Delta u - u + \frac{u^p}{\xi^q} \quad \text{and} \quad \tau\xi_t = \frac{1}{|D|}\iint_D \left(-\xi + \frac{u^r}{\xi^s}\right)dxdy,$$

where (p,q,r,s) satisfy $0 < (p-1)/q < r/(s+1)$. This system is a model describing head formation of hydra which is a small creature. The functions u and ξ stand

for the concentration of biochemicals. Thus u and ξ are positive. The assumption on (p, q, r, s) comes from a biological reason. This system always satisfies (N2). Moreover if $p = r - 1$, then (N1) holds.

Example 1.2. The shadow system of the FitzHugh-Nagumo model [F61, NAY62] is the following:

$$u_t = \varepsilon^2 \Delta u + f_0(u) - \alpha\xi \quad \text{and} \quad \tau\xi_t = \frac{1}{|D|}\iint_D (\beta u - \gamma\xi)\, dxdy,$$

where α, β and γ are positive constants, and $f_0(u)$ is the so-called cubic like function. A typical example of $f_0(u)$ is $u(1-u)(u-\delta)$, $(0 < \delta < 1)$. This system is a model describing neuron pulse propagation. The conditions (N1) and (N2) are always satisfied.

Before we state conditions for the instability of steady states, we introduce notations about the profile of the steady state. Let (u, ξ) be a solution of (SE), and let

$$U(\theta) := u(R\cos\theta, R\sin\theta). \tag{1.1}$$

Note that U is of class C^1 provided that u is of class C^1. Let $w \in C^1(\mathbb{R}/2\pi\mathbb{Z})$. By $\mathcal{Z}[w(\,\cdot\,)]$ we denote the cardinal number of the zero level set of $w(\,\cdot\,)$, *i.e.*,

$$\mathcal{Z}[w(\,\cdot\,)] := \sharp\{\theta \in \mathbb{R}/2\pi\mathbb{Z};\ w(\theta) = 0,\ \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

For example, $\mathcal{Z}[\sin(\,\cdot\,)] = 2$.

We are in a position to state our instability condition, which is the following:

$$\mathcal{Z}[U_\theta(\,\cdot\,)] \geq 3 \ \text{ and } \ u \text{ is not constant in } B. \tag{A}$$

Our main concern is to prove the following theorem:

Theorem A. *Let* (u, ξ) *be a solution of* (SE)*. Suppose that* (A) *holds.*
(i) *If* (N1) *holds, then, for all* $\tau > 0$, (u, ξ) *is unstable.*
(ii) *If* (N2) *holds, then there is* $\tau_0 > 0$ *such that, for all* $\tau > \tau_0$, (u, ξ) *is unstable.*
Here the term unstable means that the linearized operator of (SS) *at* (u, ξ)*, which is* (EP) *in* Section 3*, has an eigenvalue with positive real part.*

The contrapositive of Theorem A (i) is the following:

Corollary B. *Suppose that* (N1) *holds. Let* (u, ξ) *be a solution of* (SE)*. If* (u, ξ) *is stable for some* $\tau > 0$*, then either* $\mathcal{Z}[U_\theta(\,\cdot\,)] = 2$ *or* u *is constant in* B*, where* U *is defined by* (1.1)*. Here the term stable means that the linearized operator of* (SS) *at* (u, ξ) *has no eigenvalue with positive real part.*

Remark 1.3. When $\varepsilon > 0$ in (GM) is small, (GM) has a steady state called boundary one-spike layer (see [NTY01, NT91, NT93]), where the boundary one-spike layer is the steady state satisfying the following: at exactly one point on the boundary, u attains the maximum and u almost vanishes outside a neighborhood of that point. In the case that the domain is a two-dimensional ball, if $p = r - 1$, then this boundary one-spike layer is stable for small $\tau > 0$ (see [NTY01]). Moreover $\mathcal{Z}[U_\theta(\,\cdot\,)] = 2$, where U is defined by (1.1) (see [LT01]).

Remark 1.4. If there exists a boundary k-spike layer ($k \geq 2$) of (SE) satisfying (N1), then it is unstable, because $\mathcal{Z}[U_\theta(\,\cdot\,)] \geq 2k$.

For later use, we divide (A) into the following two cases:

$$\mathcal{Z}\,[U_\theta(\,\cdot\,)] \geq 3 \ \text{ and } \ u \text{ is not radially symmetric,} \tag{A1}$$

$$\text{There is an open interval } \gamma \subset \mathbb{R}/2\pi\mathbb{Z} \text{ such that } U_\theta = 0 \text{ on } \gamma, \text{ and } u \text{ is not constant in } B. \tag{A2}$$

The following lemma is used to prove Theorem A. The lemma seems to be interesting itself.

Lemma C. *Let (u, ξ) be a solution of* (SE)*. If u satisfies* (A2)*, then u is radially symmetric. In particular, u is a constant on ∂B.*

Remark 1.5. If u is not radially symmetric, then $\{\theta \in \mathbb{R}/2\pi\mathbb{Z};\ U_\theta(\theta) = 0\}$ should consist of isolated points because of Lemma C and Lemma 4.3 below. Thus $\mathcal{Z}\,[U_\theta(\,\cdot\,)]$ is a finite integer and greater than or equal to 2. If u is radially symmetric, then $\mathcal{Z}\,[U_\theta(\,\cdot\,)] = \aleph_1$. Combining them, we see that the condition (A) is the disjoint union of the conditions (A1) and (A2). Moreover, we see

$$\mathcal{Z}\,[U_\theta(\,\cdot\,)] = \begin{cases} n \in \mathbb{N}\backslash\{1\} & \text{if } u \text{ is not radially symmetric;} \\ \aleph_1 & \text{if } u \text{ is radially symmetric.} \end{cases}$$

In order to prove Lemma C, we need the following strong unique continuation property:

Lemma D. *Let D be a bounded domain in $\mathbb{R}^2$ with boundary of class C^2. Let $\Phi \in C^1(D) \cap C^0(\overline{D})$ be a solution of*

$$L\Phi = 0 \ \ \textit{in} \ \ D \quad \text{and} \quad \partial_\nu \Phi = 0 \ \ \textit{on} \ \ \partial D, \tag{1.2}$$

where $L := d\Delta + V(x, y)$ and $V \in C^0(\overline{D})$. If there exist (infinitely many different) points $\{p_j\}_{j=1}^\infty$ such that each p_j satisfies (i) *or* (ii)*,*
(i) *$p_j \in \partial D$, and $\Phi(p_j) = 0$,*
(ii) *$p_j \in (\overline{D}\backslash\partial D)$, and $\Phi(p_j) = (\partial_{x_1}\Phi)(p_j) = (\partial_{x_2}\Phi)(p_j) = 0$,*
then $\Phi \equiv 0$ in $\overline{D}$.

If all p_j are on the boundary, then we do not need information of Φ on the interior of the domain. In this case, Lemma D seems to be new.

Remark 1.6. Let $m \geq 3$. Then there is a harmonic function u: $\mathbb{R}^{m-1} \times \mathbb{R}_+ \to \mathbb{R}$ which is C^1 up to the boundary and such that u and ∇u vanish on a common boundary set with positive measure (see [BW90, W95]).

This paper is organized as follows: in Section 2, we state known results which are used in the proof of the main results. In Section 3, we prove Theorem A and lemmas used in the proof of Theorem A. In Section 4, we prove Lemmas C and D.

2. Preliminaries

In this section we state several known results which are used in the proof of lemmas in Sections 3 and 4.

Let $h(\zeta)$ $(\zeta \in \mathbb{C})$ be a real-valued function satisfying the following: there is a constant $C > 0$ such that

$$|\Delta h| \leq C(|\nabla h| + |h|). \tag{2.1}$$

We can use the Carleman-Hartman-Wintner theory which is useful for studying the local behavior of the zero level set of the function h.

Proposition 2.1 ([HW53]). *Let $h(\zeta) = o(|\zeta|^n)$ as $|\zeta| \to 0$ for some $n \in \mathbb{N}$. Then either $h(\zeta) \equiv 0$ or there exists a $m \geq n$, $m \in \mathbb{N}$ such that*

$$\lim_{\substack{|\zeta|\to 0\\ \zeta\neq 0}} \frac{h_\zeta(\zeta)}{\zeta^m} \neq 0.$$

This proposition is due to Hartman-Wintner [HW53] and generalizes a result by Carleman [C33] (See [S90] for example).

Corollary 2.2. *If $h(\zeta) \not\equiv 0$, then $h(\zeta)$ has an asymptotic expansion of the form*

$$h(\zeta) = \mathrm{Re}(A\zeta^{m+1}) + o(|\zeta|^{m+1}), \quad \text{as} \quad |\zeta| \to 0, \tag{2.2}$$

where $\mathrm{Re}(A\zeta^{m+1})$ is the real part of the complex-valued function $A\zeta^{m+1}$, $m \geq n$ and

$$A = \frac{2}{m+1} \lim_{\substack{|\zeta|\to 0\\ \zeta\neq 0}} \frac{h_\zeta(\zeta)}{\zeta^m} \neq 0.$$

Remark 2.3. The real-valued function $\mathrm{Re}(A\zeta^m)$ has a particularly simple representation in polar coordinates (ρ, ω):

$$\mathrm{Re}(A\zeta^m) = \alpha\rho^m \cos(m\omega) + \beta\rho^m \sin(m\omega),$$

where α, $\beta \in \mathbb{R}$. Obviously the zero level set of $\mathrm{Re}(A\zeta^m)$ consists of m straight lines which meet at equal angle. See [HMN99] for details.

Corollary 2.4. *If the assumptions of* Corollary 2.2 *hold, then exactly $m+1$ nodal curves meet at equal angles, where m appears in* Corollary 2.2.

We identify the complex number $\zeta = \xi + \sqrt{-1}\eta$ $(\xi, \eta \in \mathbb{R})$ with the point $p \in D(\subset \mathbb{R}^2)$. Let Φ be a solution of (1.2). Then $d\,|\Delta\Phi| = |V\Phi|$. Therefore there exists a constant $C > 0$ such that Φ satisfies (2.1). Proposition 2.1 and Corollaries 2.2 and 2.4 can be applied to Φ.

Using Corollary 2.4, we obtain the following:

Corollary 2.5. *The zero level set of Φ, which is a solution of* (1.2)*, consists of either the whole domain D or C^1-curves and intersections among those curves.*

3. Proof of Theorem A

We prove Theorem A in this section.

Let D be a domain in $\mathbb{R}^N$. We write L^2 and H^1 as the usual Lebesgue space and Sobolev space of order one respectively. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\|\cdot\|_{L^2}$ denote the inner product in L^2 and the norm of L^2. Specifically,

$$\langle \psi_1, \psi_2 \rangle = \iint_D \psi_1(x,y)\psi_2(x,y)dxdy \quad \text{and} \quad \|\psi\|_{L^2} = \langle \psi, \psi \rangle^{\frac{1}{2}}.$$

Hereafter by (u, ξ) we denote the solution of (SE). In order to prove the instability of the steady state (u, ξ), we consider the eigenvalue problem

$$L_0 v + f_\xi(u,\xi)\eta = \lambda v \quad \text{and} \quad \langle v, g_u(u,\xi)\rangle + \langle 1, g_\xi(u,\xi)\rangle\,\eta = \lambda\tau\eta, \tag{EP}$$

$$\partial_\nu v = 0 \quad \text{on} \quad \partial B,$$

where $L_0 := D_u\Delta + f_u(u,\xi)$, and $(v,\eta) \in C^2 \times \mathbb{R}$. We want to prove the existence of the eigenvalue of (EP) with positive real part. It is important to prove the positivity of the second eigenvalue of the eigenvalue problem

$$L_0\phi = \mu\phi \ \text{ in } \ B \quad \text{ and } \quad \partial_\nu\phi = 0 \ \text{ on } \ \partial B. \tag{3.1}$$

From now on, let μ_n $(n \geq 1)$ denotes the n-th eigenvalue counting multiplicities, let ϕ_n denotes a corresponding eigenfunction satisfying that $\|\phi_n\|_{L^2} = 1$, and let $\mathrm{spec}(L_0) := \{\mu_n\}_{n=1}^{\infty}$.

Remark 3.1. If u is radially symmetric, then the first eigenfunction ϕ_1 is radially symmetric. In fact, differentiating $D_u\Delta\phi_1 + f_u(u,\xi)\phi_1 - \mu_1\phi_1 = 0$ with respect to θ, we have

$$\begin{aligned} 0 &= \partial_\theta D_u\Delta\phi_1 + f_{uu}(u,\xi)u_\theta\phi_1 + f_u(u,\xi)(\partial_\theta\phi_1) - \mu_1(\partial_\theta\phi_1) \\ &= D_u\Delta(\partial_\theta\phi_1) + f_u(u,\xi)(\partial_\theta\phi_1) - \mu_1(\partial_\theta\phi_1), \end{aligned}$$

where $\partial_\theta = -y\partial_x + x\partial_y$, we use the fact that $u_\theta \equiv 0$ and

$$\partial_\theta\Delta_{(x,y)} = \Delta_{(x,y)}\partial_\theta. \tag{3.2}$$

The above equality shows that $\partial_\theta\phi_1$ belongs to the first eigenspace. Since ϕ_1 is simple, there is $c \in \mathbb{R}$ such that $\partial_\theta\phi_1 = c\phi_1$. Since ϕ_1 is positive and $\partial_\theta\phi_1$ is periodic in θ, we see that $c = 0$. Thus $\partial_\theta\phi_1 = 0$, which indicates that ϕ_1 is radially symmetric.

At first, we assume that $\mu_2 > 0$ and prove the following:

Lemma 3.2. *Let (u,ξ) be a solution of* (SE). *Suppose that $\mu_2 > 0$ and that (u,ξ) satisfies* (A). *Here μ_2 is the second eigenvalue of* (3.1).
(i) *If* (N1) *holds, then, for all $\tau > 0$, (u,ξ) is unstable.*
(ii) *If* (N2) *holds, then there is $\tau_0 > 0$ such that, for any $\tau > \tau_0$, (u,ξ) is unstable.*

Proof. We consider (EP). From the second equation of (EP), we have

$$\eta = \frac{\langle v, g_u(u,\xi)\rangle}{\lambda\tau - \langle 1, g_\xi(u,\xi)\rangle}. \tag{3.3}$$

We consider the case that $\lambda > 0$. Thus $\lambda\tau > 0$. Owing to the assumptions (N1) and (N2), $\lambda\tau - \langle 1, g_\xi(u,\xi)\rangle > 0$. Thus the denominator of (3.3) does not vanish. Substituting (3.3) into (EP), we obtain the eigenvalue problem of v

$$(L_0 + A_{\lambda,\tau})v = \lambda v, \tag{3.4}$$

where $A_{\lambda,\tau}$ is a rank-one operator (*i.e.*, $\dim \mathrm{Ran} A_{\lambda,\tau} = 1$) and

$$A_{\lambda,\tau}v = \frac{\langle v, g_u(u,\xi)\rangle}{\lambda\tau - \langle 1, g_\xi(u,\xi)\rangle} f_\xi(u,\xi). \tag{3.5}$$

Note that λ appears in $A_{\lambda,\tau}$. Thus (3.4) is not a standard eigenvalue problem. We can see by the Sherman-Morrison formula that

$$(L_0 + A_{\lambda,\tau} - \lambda)^{-1} = \left(1 - \frac{(L_0-\lambda)^{-1}A_{\lambda,\tau}}{h(\lambda)}\right)(L_0-\lambda)^{-1}, \tag{3.6}$$

where

$$h(\lambda) = 1 + \frac{\left\langle (L_0-\lambda)^{-1}\left[f_\xi(u,\xi)\right], g_u(u,\xi)\right\rangle}{\lambda\tau - \langle 1, g_\xi(u,\xi)\rangle}. \tag{3.7}$$

Therefore under the condition that $\lambda \notin \mathrm{spec}(L_0)$, $L_0 + A_{\lambda,\tau} - \lambda$ is invertible if and only if

$$h(\lambda) \neq 0. \tag{3.8}$$

On the other hand, from the eigenfunction expansion we see

$$(L_0 - \lambda)^{-1}[\psi] = -\sum_{n\geq 1} \frac{\langle \psi, \phi_n \rangle}{\lambda - \mu_n} \phi_n. \tag{3.9}$$

Substituting (3.9) into $h(\lambda) = 0$, we obtain the following equation:

$$\lambda\tau - \langle 1, g_\xi(u,\xi)\rangle = h_0(\lambda),$$

where

$$h_0(\lambda) = \sum_{n\geq 1} \frac{a_n}{\lambda - \mu_n}$$

and

$$a_n = \langle f_\xi(u,\xi), \phi_n \rangle \langle \phi_n, g_u(u,\xi)\rangle .$$

We will prove the case (i).

Case (i):

We divide this case into three more cases.

Case $k(\xi) = 0$:

In this case, $a_1 = 0$. Thus $(L + A_{\mu_1,\tau} - \mu_1)[\phi_1] = 0$, which indicates that $\mu_1 (> \mu_2 > 0)$ is an eigenvalue of (EP). Thus (u,ξ) is unstable.

Case $k(\xi) < 0$:

In this case, $a_1 < 0$. If $a_2 \neq 0$, then $a_2 < 0$. Thus $\lim_{\lambda\uparrow\mu_1} h_0(\lambda) = +\infty$, $\lim_{\lambda\downarrow\mu_2} h_0(\lambda) = -\infty$ and $h(\lambda) \in C^0((\mu_2,\mu_1))$. Therefore, for any $\tau > 0$, there is $\tilde{\lambda} \in (\mu_2,\mu_1)$ such that $\tilde{\lambda}\tau - \langle 1, g_\xi\rangle = h_0(\tilde{\lambda})$, which indicates that (u,ξ) is unstable.

If $a_2 = 0$, then $(L + A_{\mu_2,\tau} - \mu_2)[\phi_2] = 0$. Thus $\mu_2 (> 0)$ is an eigenvalue of (EP). Thus (u,ξ) is unstable.

Case $k(\xi) > 0$:

Since $a_1 > 0$, $\lim_{\lambda\downarrow\mu_1} h_0(\lambda) = +\infty$, $\lim_{\lambda\uparrow+\infty} h_0(\lambda) = 0$ and $h(\lambda) \in C^0((\mu_1,+\infty))$. Hence, for any $\tau > 0$, there is $\tilde{\lambda} (> \mu_2 > 0)$ such that $\tilde{\lambda}\tau - \langle 1, g_\xi\rangle = h_0(\tilde{\lambda})$. Thus (u,ξ) is unstable.

Next we will prove the case (ii).

Case (ii):

Because of (N2), we see that $a_1 < 0$. Since $\lim_{\lambda\uparrow\mu_1} h_0(\lambda) = +\infty$, $\lim_{\lambda\downarrow\mu_2} h_0(\lambda) = +\infty$ and $h_0(\lambda) \in C^0((\mu_2,\mu_1))$, there is $\tau_0 > 0$ such that the following holds: for all $\tau > \tau_0$, there is $\tilde{\lambda} \in (\mu_2,\mu_1)$ such that $\tilde{\lambda}\tau - \langle 1, g_\xi\rangle = h_0(\tilde{\lambda})$. Thus (u,ξ) is unstable.

All the possibilities are verified. □ □

Owing to this lemma, it is sufficient to show that if (A) holds, then $\mu_2 > 0$. Firstly, we show that $\mu_2 > 0$ if (A1) holds. Before proving that, we state the Courant nodal theorem which is used in the proof of Lemma 3.4 below.

Proposition 3.3. *Let ζ_n $(n \geq 1)$ be the n-th eigenvalue, counting multiplicities, of the eigenvalue problem*

$$D_u \Delta\psi + V(x,y)\psi = \zeta\psi \quad \text{in} \quad B \qquad \text{and} \qquad \partial_\nu \psi = 0 \quad \text{on} \quad \partial B, \tag{3.10}$$

where $V = V(x,y) \in C^0$. Thus $\zeta_1 > \zeta_2 \geq \zeta_3 \geq \cdots \geq \zeta_n \geq \cdots$. Let ψ_n be an eigenfunction corresponding to ζ_n. If $\zeta_{n-1} > \zeta_n$, then the number of the connected component of $\{(x,y) \in B;\ \psi_n(x,y) \neq 0\}$ is less than or equal to n.

Lemma 3.4. *Let (u,ξ) be a solution of* (SE)*. If u satisfies* (A1)*, then $\mu_2 > 0$, where μ_2 is the second eigenvalue of* (3.1)*.*

Proof. Differentiating $D_u \Delta u + f(u,\xi) = 0$ with respect to θ, we obtain

$$D_u \Delta u_\theta + f_u(u,\xi) u_\theta = 0, \tag{3.11}$$

where $\partial_\theta = -y\partial_x + x\partial_y$ and we use (3.2). Since $\partial_\nu u_\theta = 0$ on ∂B, 0 is an eigenvalue of (3.1) and u_θ is a corresponding eigenfunction.

On the other hand, there is $C > 0$ such that

$$|\Delta u_\theta| \leq C(|\nabla u_\theta| + |u_\theta|), \tag{3.12}$$

because $D_u |\Delta u_\theta| = |f_u(u,\xi) u_\theta|$. Since (3.12) holds, we can use the Carleman-Hartman-Wintner theory and see that the zero level set of $u_\theta(\,\cdot\,,\,\cdot\,)$ consists of either the whole domain B or C^1-curves, which are called *nodal lines* or *nodal curves*, and finitely many intersections among those curves (see Corollary 2.5). Note that $u_\theta \not\equiv 0$ in B because of (A1). Now we assume that Lemma 4.3 which is proven later holds. Lemma 4.3 says that, for each isolated zero of u_θ on the boundary, there is a continuum that is a part of the zero level set of u_θ connecting to the zero of u_θ on the boundary. Moreover the lemma says that the continuum divides the domain B into at least two subdomains of B. Because of (A1), there are at least three zeros of u_θ on the boundary and three parts of the nodal curves connecting to zeros of u_θ on the boundary. These curves should divide the domain B into at least three subdomains. We call the subdomain *the nodal domain*.

Using the number of the connected component of the nodal domains, we will show that $\mu_2 > 0$. Recall that 0 is an eigenvalue of (3.1). Suppose that 0 is the first or the second eigenvalue. Then the number of the connected component of the nodal domain should be one or two (Proposition 3.3), which is a contradiction. Therefore there is $l \geq 3$ such that $\mu_2 > \mu_l = 0$. □ □

Secondly, we show that $\mu_2 > 0$ if (A2) holds. We used the Courant nodal theorem in the previous case. In the proof of Lemma 3.5 below, we use a variational characterization of μ_2 (see [RS78] for example). Let $\mathcal{H}[\psi]$ be a functional defined by

$$\mathcal{H}[\psi] := \iint_B \left(-D_u |\nabla \psi|^2 + f'(u,\xi)\psi^2 \right) dxdy.$$

Then

$$\mu_2 = \sup_{\substack{z \in \mathrm{span}\langle \phi_1 \rangle^\perp \cap H^1 \\ \|z\|_{L^2} = 1}} \mathcal{H}[z],$$

where ϕ_1 is the first eigenfunction of (3.1). Since $D_u \Delta u_x + f'(u,\xi) u_x = 0$, we see that

$$\begin{aligned} \mathcal{H}[u_x] &= \iint_B u_x \left(D_u \Delta u_x + f'(u,\xi) u_x \right) dxdy - D_u \int_{\partial B} u_x \partial_\nu u_x d\sigma \\ &= -D_u \int_{\partial B} u_x \partial_\nu u_x d\sigma, \end{aligned} \tag{3.13}$$

where we use Green's formula. This equality and the radial symmetry of u and ϕ_1 are keys in the proof of the following lemma:

Lemma 3.5. *Let* (u, ξ) *be a solution of* (SE). *If* u *satisfies* (A2), *then* $\mu_2 > 0$, *where* μ_2 *is the second eigenvalue of* (3.1).

Proof. If (A2) holds, then u is not constant in B and radially symmetric (see Lemma C). We see by Remark 3.1 that the first eigenfunction ϕ_1 of (3.1) is radially symmetric. In particular, $\phi_1(-x, y) = \phi_1(x, y)$ and $\phi_1(x, -y) = \phi_1(x, y)$. Since $u_x(x, y) = -u_x(-x, y)$, we see that

$$\langle \phi_1, u_x \rangle = 0. \tag{3.14}$$

We claim that $f(u, \xi) \neq 0$ at $r = R$, where $r = \sqrt{x^2 + y^2}$. Suppose the contrary, *i.e.*, that $f(u, \xi) = 0$ at $r = R$. Then the pair of constant functions (u, ξ) is a solution of (SS). Because of the unique continuation property on the boundary (see Lemma D), this constant solution is a unique solution and there is no other solution satisfying the boundary condition. We obtain a contradiction, because u is not constant (see (A2)).

On the other hand, we see by (3.13) that

$$\mathcal{H}[u_x] = -D_u \int_{\partial B} u_x \partial_\nu u_x d\sigma = 0,$$

where we use the fact that $u_x = u_r \cos\theta = 0$ on ∂B. Since $u_x = u_r \cos\theta$, we see that $\partial_\nu u_x = u_{rr} \cos\theta$ and that $D_u u_{rr} + f(u, \xi) = 0$ at $r = R$, where $r\partial_r = x\partial_x + y\partial_y$, $x = r\cos\theta$ and $y = r\sin\theta$. Here we use the fact that $\Delta = \partial_{rr} + \partial_r/r + \partial_{\theta\theta}/r^2$ and that $u_r = u_{\theta\theta} = 0$ at $r = R$. Since $D_u u_{rr} = -f(u, \xi) \neq 0$ at $r = R$, the function u_x does not satisfy the Neumann boundary condition, thus u_x is not a second eigenfunction. Owing to (3.14), $u_x \in \operatorname{span}\langle \phi_1 \rangle^\perp \cap H^1$. We can obtain

$$\mu_2 = \sup_{\substack{z \in \operatorname{span}\langle \phi_1 \rangle^\perp \cap H^1 \\ \|z\|_{L^2} = 1}} \mathcal{H}[z] > \mathcal{H}[u_x] = 0. \quad \square$$

□

Proof. Proof of Theorem A Combining Lemmas 3.4, 3.5 and 3.2, we obtain the desired conclusions. □ □

4. Proof of Lemmas C and D

In this section, we prove Lemmas C and D. To begin with, we prepare some notation. Let D be a two-dimensional bounded domain with boundary of class C^2. Let $\Phi = \Phi(x, y)$ be a solution of (1.2). Let $\Omega := \{\Omega_j\}$ denote the set of the connected components of $\{p \in D;\ \Phi(p) \neq 0\}$. Let $\Gamma := \{\Gamma_j\}$ denote the set of the connected components of the zero level set of Φ, which is denoted by $\{p \in D;\ \Phi(p) = 0\}$. In Lemma 4.2 we show that the cardinal number $\sharp\Gamma$ is finite.

Lemma 4.1. *The number of the connected components of* $\{p \in D;\ \Phi(p) \neq 0\}$, *which is denoted by* $\sharp\Omega$, *is finite.*

Proof. We will prove this lemma by contradiction. We assume that $\sharp\Omega = \infty$. We consider the eigenvalue problem

$$L\Phi = \rho\Phi \ \text{ in } \ D \qquad \text{and} \qquad \partial_\nu \Phi = 0 \ \text{ on } \ \partial D. \tag{EP2}$$

By $\{\rho_j\}_{j=1}^\infty$ we denote the eigenvalues of (EP2). From the standard theory of second order elliptic operators, we see that

$$\rho_1 > \rho_2 \geq \rho_3 \geq \cdots \geq \rho_j \geq \cdots, \qquad \lim_{j \to \infty} \rho_j = -\infty,$$

and that the number of the non-negative eigenvalues $\sharp\{\lambda_j \geq 0\}$ is finite.

We use the variational characterization of the eigenvalues of the form (see [RS78] for instance)

$$\rho_k = \sup_{\substack{X \subset H^1(D) \\ \dim X = k}} \inf_{\substack{v \in X \setminus \{0\} \\ \|v\|_{L^2(D)} = 1}} \iint_D \left(-d\,|\nabla v|^2 + V(x,y) v^2 \right) dxdy. \tag{4.1}$$

Here we consider the following l-dimensional subspace $X^{(l)}$ in $H^1(D)$:

$$X^{(l)} := \mathrm{span}\{v_1, v_2, \cdots, v_l\} = \left\{ v \in H^1(D);\ v = \sum_{j=1}^{l} c_j v_j, \quad c_1, c_2, \cdots, c_l \in \mathbb{R} \right\},$$

where

$$v_j(x,y) := \begin{cases} \Phi(x,y) & \text{if} \quad (x,y) \in \Omega_j; \\ 0 & \text{if} \quad (x,y) \notin \Omega_j. \end{cases}$$

Note that v is continuous on $\partial\Omega_1 \cup \cdots \cup \partial\Omega_l$ and that $v \in H^1(D)$. Then we see that, for any $v \in X^{(l)}$,

$$\iint_D \left(-d\,|\nabla v|^2 + V(x,y) v^2 \right) dxdy = \sum_{j=1}^{l} \iint_{\Omega_j} \left(-d\,|\nabla v_j|^2 + V(x,y) v_j^2 \right) dxdy$$

$$= \sum_{j=1}^{l} \left\{ \iint_{\Omega_j} \left(d\Delta v_j + V(x,y) v_j \right) v_j dxdy - d \int_{\partial\Omega_j} v_j \partial_\nu v_j d\sigma \right\} = 0,$$

where we use the fact that v_j or $\partial_\nu v_j$ vanishes for any point of $\partial\Omega_j$. Therefore by (4.1) we see that $\lambda_l \geq 0$. Any large number can be chosen for l, which indicates that $\sharp\{\lambda_j \geq 0\} = \infty$. This is a contradiction, because the number of the non-negative eigenvalues of (EP2) is finite. □ □

Lemma 4.2. *The number of the connected components of the zero level set of Φ, which is denoted by $\sharp\Gamma$, is finite.*

Proof. We prove the lemma by contradiction. Suppose that $\sharp\Gamma = \infty$. A connected component Γ_1 divides D into at least two parts. Because $\Gamma_2 \cap \Gamma_1 = \emptyset$, Γ_2 divides D into at least two parts. This process can be repeated infinitely many times. Thus $\sharp\Omega = \infty$, which contradicts to Lemma 4.1. □ □

Lemma 4.3. *If there is a point on the boundary $p \in \partial D$ satisfying that $\Phi(p) = 0$ and that p is an isolated point in $\{p \in \partial D;\ \Phi(p) = 0\}$, then there exists a connected component $\Gamma_j (\subset \Gamma)$ connecting to p. Moreover, Γ_j divides the domain into at least two subdomains.*

Proof. Because ∂D is of class C^2, there exists a family of open balls $\{B_j\}_{j=1}^{\infty}$ with radius r_j such that, for each $j \in \mathbb{N}$, B_j is in D and ∂B_j touches ∂D at only p, and that $r_j \to 0$ as $j \to \infty$.

We will show that, for each $j \in \mathbb{N}$, there is $q_j \in B_j$ such that $\Phi(q_j) = 0$. Suppose that $\Phi < 0$ in $\overline{B_j} \setminus \{p\}$ or that $\Phi > 0$ in $\overline{B_j} \setminus \{p\}$. Then only p attains the maximum or minimum of Φ in $\overline{B_j}$ respectively. Note that p satisfies an interior sphere condition (see [GT83] for details). Therefore the Hopf boundary point lemma tells us that $\partial_\nu \Phi > 0$ or $\partial_\nu \Phi < 0$ at p respectively (see [GT83]). This contradicts the Neumann boundary condition. There are only two cases. One case is that $\Phi \equiv 0$ in B_j. In

this case, we can clearly take $q_j(\in B_j)$ such that $\Phi(q_j)=0$. The other case is that there are $z_1,\ z_2\in B_j$ such that $\Phi(z_1)\geq 0$ and $\Phi(z_2)\leq 0$. Since B_j is convex, we see that $tz_1+(1-t)z_2\in B_j$ for $0\leq t\leq 1$. The mean value theorem says that there is $t_0\in[0,1]$ such that $\Phi(t_0z_1+(1-t_0)z_2)=0$. In this case, we choose $t_0z_1+(1-t_0)z_2$ as q_j. Taking a subsequence (if necessary), we can assume that $\{q_j\}_{j=1}^{\infty}$ are infinitely many different points satisfying that $q_j\to p$ as $j\to\infty$.

Because of Corollary 2.5, q_j is not an isolated point. Each q_j should be in one of Γ. Since $\sharp\Gamma<\infty$ (Lemma 4.2), there is $k\in\mathbb{N}$ such that Γ_k contains infinitely many points of $\{q_j\}_{j=1}^{\infty}$. We claim that Γ_k contains p. Suppose the contrary. There is a positive constant $\varepsilon>0$ such that $B_\varepsilon(p)\cap\Gamma_k=\emptyset$, where $B_\varepsilon(p)$ is the ball centered at p with radius ε. This is a contradiction, because there are only finitely many points in $D\backslash B_\varepsilon(p)$ and Γ_k cannot contain infinitely many points. Since Γ_k is the level set of the solution Φ, Γ_k is closed set in $\overline{D}$. Then Γ_k connects to p. It is clear that the two sides of ∂D with respect to p belong to different subdomains. Thus Γ_k divides the domain into at least two subdomains. □ □

Lemma 4.4. *If $\{q\in\partial D;\ \Phi(q)=0\}$ has a continuum γ, then $\Phi\equiv 0$ in D.*

Proof. We use a method similar to one used in the proof of Theorem 2 of [J93]. Let p be a point on γ except both the edges of γ, and let $U(\subset\mathbb{R}^2)$ be a neighborhood of p. We can transform the equation $D_u\Delta\Phi+V\Phi=0$ in $U\cap D$ into $\Delta\psi+b^{(1)}(x_1,x_2)\psi_{x_1}+b^{(2)}(x_1,x_2)\psi_{x_2}=0$ in $U\cap D$. In this transformation, we can assume that $\{(x_1,x_2)\in\partial D\cap B_\varepsilon(p);\ \Phi(x_1,x_2)=\Phi_\nu(x_1,x_2)=0\}=\{(x_1,x_2)\in\partial D\cap B_\varepsilon(p);\ \psi(x_1,x_2)=\psi_\nu(x_1,x_2)=0\}$, and that $\{(x_1,x_2)\in D\cap B_\varepsilon(p);\ \Phi(x_1,x_2)=\Phi_{x_1}(x_1,x_2)=\Phi_{x_2}(x_1,x_2)=0\}=\{(x_1,x_2)\in D\cap B_\varepsilon(p);\ \psi(x_1,x_2)=\psi_{x_1}(x_1,x_2)=\psi_{x_2}(x_1,x_2)=0\}$. See Lemmas 3.1, 3.2 and 3.3 of [J93]. We identify $\mathbb{R}^2$ with $\mathbb{C}$. By the Riemann mapping theorem, we see that there is an holomorphic mapping $\mathcal{F}:B_1(0)\to D\cap B_\varepsilon(p)$. Let $\xi+i\eta$ be a coodinate of $B_1(0)$, and $\tilde{\psi}(\xi,\eta):=\psi(x_1,x_2)$. Then $\tilde{\psi}$ should satisfy that $\Delta\tilde{\psi}+c^{(1)}(\xi,\eta)\tilde{\psi}_\xi+c^{(2)}(\xi,\eta)\tilde{\psi}_\eta=0$ and that $\tilde{\psi}=\tilde{\psi}_\xi=\tilde{\psi}_\eta=0$ on $\tilde{\gamma}$, where $\tilde{\gamma}:=\mathcal{F}(\gamma)$.

Now by a representation result of Bers-Nirenberg (see [BJS64]), we have

$$\tilde{\psi}_\xi-i\tilde{\psi}_\eta=e^{s(\zeta)}g(\zeta)\ \text{ for }\ \zeta=\xi+i\eta\in B_1(0)(\subset\mathbb{C}),$$

where $s(\zeta)$ is continuous on $\overline{B_1(0)}$ and real on $|\zeta|=1$, and $g(\zeta)$ is an analytic function on $B_1(0)$. Note that $e^{s(\zeta)}$ cannot vanish, hence $\{\tilde{\psi}_\xi-i\tilde{\psi}_\eta=0\}=\{g=0\}$. Because of the assumption of the lemma, there is a continuum $\tilde{\gamma}$ of $\partial B_1(0)$ such that $g=0$ on $\tilde{\gamma}$. Since the portion γ is (non-singular and) analytic and $g=0$ on $\tilde{\gamma}$, by analytic continuation we can extend the domain of g. Namely, there is a neighborhood $V(\subset\mathbb{R}^2)$ of a point on $\tilde{\gamma}$ except both the edges of $\tilde{\gamma}$ such that g is analytic in V. In particular, a portion of $\tilde{\gamma}$ is included by the interior of V. Since $g=0$ on $\tilde{\gamma}$, by the unique continuation property of complex analytic functions we see that $g\equiv 0$ on V. Therefore $\tilde{\psi}_\xi=\tilde{\psi}_\eta=0$ in $V\cap B_1(0)$. Because $\tilde{\psi}=0$ on $\tilde{\gamma}$, we see that $\tilde{\psi}\equiv 0$ in $V\cap B_1(0)$, hence $\psi\equiv 0$ in $D\cap B_\varepsilon(p)$. Using the usual strong continuation property at an interior point, we can obtain the conclusion. □ □

Proof. Proof of Lemma D Let $\{p_j\}_{j=1}^{\infty}$ be the set of points satisfying the assumption. We divide the possibilities into two cases.

Case 1: We assume that ∂D contains infinitely many points of $\{p_j\}_{j=1}^{\infty}$. Firstly, we assume that there are infinitely many p_j which is isolated in $\{p\in\partial D;\ \Phi(p)=0\}$. Since $\sharp\Gamma<\infty$ (Lemma 4.2), we see by Lemma 4.3 that there is $k\in\mathbb{N}$ such that

Γ_k contains infinitely many points of $\{p_j\}_{j=1}^{\infty}$. Using Corollary 2.5, we can see by an elementary topological argument of two-dimensional domains that Γ_k devides D into infinitely many connected components and that Φ does not vanishes in each connected component. Since $\sharp\Omega < \infty$ (Lemma 4.1), this is a contradiction. Secondly, we assume that there are at most finitely many p_j which is isolated in $\{q \in \partial D;\ \Phi(q) = 0\}$. In this case there should exist a closed continuum of $\{q \in \partial D;\ \Phi(q) = 0\}$. Because of Lemma 4.4, $\Phi \equiv 0$ in D.

Case 2: We assume that the interior of D contains infinitely many points of $\{p_j\}_{j=1}^{\infty}$. Since $\sharp\Gamma < \infty$ (Lemma 4.2), there is $k \in \mathbb{N}$ such that Γ_k contains infinitely many points of $\{p_j\}_{j=1}^{\infty}$ which is in the interior of D. Since Corollary 2.2 says that at least two nodal curves intersect transversely at each $p_j \in \Gamma_k$. We can show by an argument similar to one used in Case 1 that Γ_k divides D into infinitely many connected components, and that $\Phi \neq 0$ in each connected component. This is a contradiction, because $\sharp\Omega < \infty$ (Lemma 4.1). □ □

Proof. Proof of Lemma C Let (u, ξ) be a solution of (SS). Then u satisfies

$$D_u \Delta u_\theta + f'(u, \xi) u_\theta = 0,$$

where $\partial_\theta = -x_2 \partial_{x_1} + x_1 \partial_{x_2}$, and we use (3.2). Because of the assumption of this corollary, $u_\theta = 0$ on γ. Using Lemma D, we see that $u_\theta \equiv 0$ in B. We obtain the conclusion. □ □

Note added in proof. After this work was finished and submitted for publication, the author was informed that Lemma 3.1 (i) was obtained by Yanagida [Y02c] in the case that $k(\xi) = 1$ or $k(\xi) = -1$.

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