Limit sets and the rate of convergence for one-dimensional cellular automata traffic models

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Abstract. A critical exponent of phase transition for one-dimensional cellular automata with conservative law is proposed and it is shown that the exponent represents the asymptotic behavior of the dynamics.

1 Introduction

For one-dimensional cellular automata with conservative law, ECA184 [9] and so on, there exists many rules exhibit phase transition in its dynamics according to the initial distribution shown in Fig. 1.

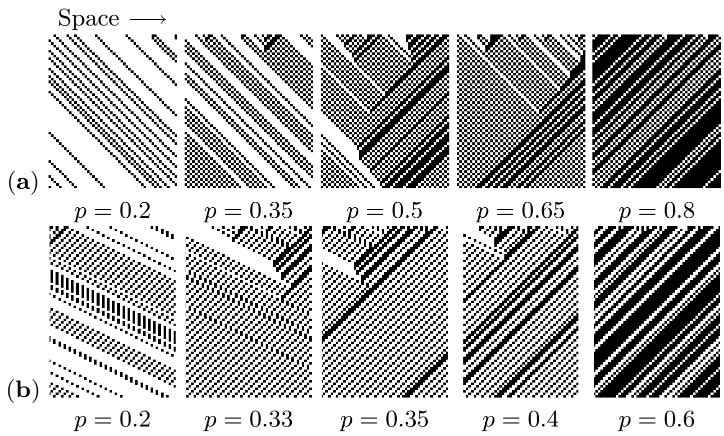


Fig. 1. Typical orbits of traffic models. Initial distribution of "1" is denoted by p. (**a**) ECA184. (**b**) EBCA1

The case of $p = 1/2$, the "speed" of convergence to limit set becomes significantly slow comparing to the case $p \neq 1/2$. Numerical studies for Fukui-Ishibashi model are in [3] and they observe that for $p = p_c$ the relaxation time diverges. We estimate such phenomena, that is the asymptotic behavior of one-dimensional cellular automata traffic model from initial distribution to the limit distribution with respect to Gibbs measure [1] on the limit set.

In [4] the equivalent condition that cellular automata have additive conserved quantity. Many cases of traffic models, shown in [2,7], the cellular automata satisfy its condition. With the condition we show the existence and the vanishing case of exponential rate in asymptotics.

2 Preliminary

Definition 1. Let S be a finite set and Λ be a finite subset of $\mathbb{Z}$. For a given map $f : S^{\Lambda} \to S$, we can define a map $\tau : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ for all $x = (x_i)_{i\in\mathbb{Z}} \in S^{\mathbb{Z}}$ by

$$(\tau x)_i = f(x_{i+\lambda}; \lambda \in \Lambda).$$

We call the dynamical system $(S^{\mathbb{Z}}, \tau)$ one-dimensional cellular automata with rule f.

It is clear that τ is shift commute continuous map with respect to the product topology. After here we denote the shift map by σ.

For one-dimensional cellular automata $\tau : S^{\mathbb{Z}} \mapsto S^{\mathbb{Z}}$, the limit set $X_\infty = \bigcap_{n\geq 0} \tau^n X_0$, $X_0 = S^{\mathbb{Z}}$ is always well defined. According to the convention in symbolic dynamics, we call $w \in S^* := \bigcup_{n\in\mathbb{N}} S^n$ a word and $|w|$ means the length of word w. For a countable word set W, $\langle W\rangle$ means the shift invariant closed set the words $w \in W$ generate by their concatenations. The shift map is denoted by σ.

Now we define the additive conservative quantity for cellular automata. Take a real-valued function U on $S^{\mathbb{Z}}$ depends only on finite number of indices, i.e. $U : S^m \to \mathbb{R}$, $U(x) = U(x_i, \cdots, x_{i+m-1})$.

Definition 2. We call U be additive conserved quantity for τ if $\rho_n(x) = \frac{1}{n}S_n U(x) := \frac{1}{n}\sum_{i=0}^{n-1} U(\sigma^i x)$ is invariant under τ for all $x \in \mathrm{Fix}(S^{\mathbb{Z}}, \sigma^n)$, where $\mathrm{Fix}(S^{\mathbb{Z}}, \sigma^n)$ is a set of all n-periodic configurations that is fixed under n-th iteration of σ. If τ has such U, we call τ is conservative.

Remark 1. For an additive conserved U, there exists finite range function q and a constant C such that

$$U(\tau x) = U(x) + q(x) - q(\sigma x) + C$$

for all $x \in S^{\mathbb{Z}}$. This fact is in [1,4]. From physical viewpoint as in [4,7] q corresponds to the flow.

We denote the average of flow as $Q_n(x) := \frac{1}{n}S_n q(x) = \frac{1}{n}\sum_{i=0}^{n-1} q(\sigma^i x)$ for all $x \in \mathrm{Fix}(S^{\mathbb{Z}}, \sigma^n)$. From ergodic theorems, there exist the limit $\rho := \lim_{n\to\infty} \rho_n(x) = \int_{S^{\mathbb{Z}}} U \mathrm{d}\mu$ and $Q := \lim_{n\to\infty} Q_n(x) = \int_{S^{\mathbb{Z}}} q \mathrm{d}\mu$ for almost every x with respect to any shift invariant probability ergodic measure μ on $S^{\mathbb{Z}}$.

The relation with ρ and Q on the limit set is called fundamental diagram like Fig. 2. From numerical experiments the critical density p_c corresponds to the critical points of the Fig., i.e. $\rho = 1/2$ in ECA184 and $\rho = 1/3$ in EBCA1.

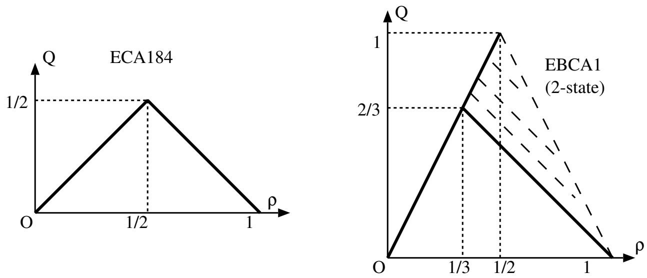


Fig. 2. Fundamental diagram for ECA184 and EBCA1[6,7]

Though it is well known that the conservative case contains rich and various phenomena, we concentrate the study to the phenomena of “phase transition” related to initial distribution described in Section 1.

3 Main result

After here we assume $S = \{1, 0\}$. At first we show the main result on the case of ECA184. We can see the essential situation here.

3.1 ECA184

The rule f of ECA184 is the following:

$$\Lambda = \{-1, 0, 1\} \text{ } and \text{ } f(x_{-1}x_0x_1) = \begin{cases} 1 \text{ if } x_{-1}x_0x_1 = 111, 101, 100, 011 \\ 0 \qquad\qquad \text{elsewhere.} \end{cases}$$

In this case, $X_0 = \{0, 1\}^{\mathbb{Z}}$,$X_\infty = \langle 0, 10\rangle \cup \langle 1, 10\rangle = X_- \cup X_+$. Set $x_c = (x_c)_{i\in\mathbb{Z}}$, if i is even $(x_c)_i = 1$ and otherwise $(x_c)_i = 0$. Note that $X_- \cap X_+ = \{x_c, \sigma x_c\}$.

This rule can be rewritten using boolean algebra as follows:

$$f(x_{-1}x_0x_1) = x_0 + (x_{-1}(x_1 + 1)) - (x_0(x_1 + 1)).$$

So the function $U(x) = x_0$ is a additive conservative quantity and the flow is $q(x_{-1}x_0) = x_{-1}(x_1 + 1)$.

Let μ^p be a Bernoulli measure on (X_0, σ) with respect to a probability vector $p = (1 - p, p)$, ν be a simple Markov measure on (X_-, σ) or (X_+, σ), and $a = (1 - a, a)$ be a left invariant probability vector for transition matrix of ν. Note that ν is a special case of Gibbs measure.

For $x \in X_\infty$ and set $S_n(x) = x_{-n} + \cdots + x_{n-1}$, we get the following result about the rate of convergence from X_0 to X_∞:

Proposition 1.

$$\begin{aligned} R(a,p,\nu) &:= \lim_{n\to\infty} -\frac{1}{n} \log \sum_{\frac{1}{2n} S_n(x)=a} \mu^p(\tau^{-n}[x_{-n}\cdots x_{n-1}])\nu[x_{-n}\cdots x_{n-1}] \\ &= H(a|p) + h_\nu(X_\infty, \sigma) \end{aligned}$$

where $[x_{-n}\cdots x_{n-1}] = \{y \in X_\infty;\ y_i = x_i \ \ -n \le i \le n-1\}$ *is called cylinder set,* $H(a|p)$ *is a relative entropy of two probability vectors and* $h_\nu(X_\infty, \sigma)$ *is the metrical entropy of* (X_∞, σ, ν)*. For* ν *is supported on* X_+ *or* X_-, $h_\nu(X_\infty, \sigma)$ *is well defined.*

Remark 2. The result shows that the rate is vanished if and only if $p = a = (1/2, 1/2)$, $\nu = \frac{1}{2}(\delta_{x_c} + \delta_{\sigma x_c})$ and it means that there are two cases, $R = 0$ and $R > 0$.

Remark 3. In [3] of Fig. 4 they study the mean convergence time from random initial configurations to stable states. From numerical experiment if $p \ne p_c$ the value remains finite and if $p = p_c$ it shows the divergence phenomena. Proposition 1 explains these numerical result.

If $R > 0$ we have exponential order of convergence, as appear in the fast dynamics of orbits. On the contrary, the case of $R = 0$ we have sub-exponential order, really polynomial that is appear in slow dynamics of orbits. This shows the existence of phase transition on $p = p_c$.

Proof (sketch). The essential idea of the proof is random walk representation of configurations.

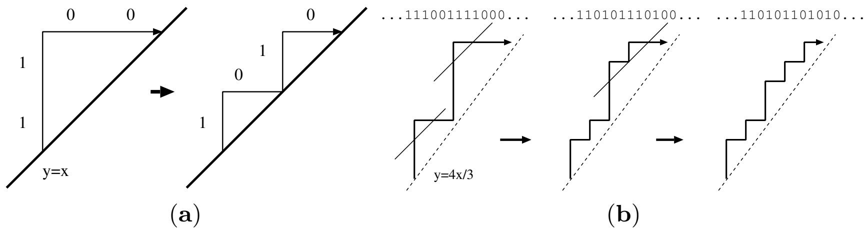


Fig. 3. ECA184: (**a**) The essential part of the rule in path space. (**b**) Its typical dynamics

Let $W_a = \{w \in S^{2n}\ ;\ \frac{1}{n} S_n(w^\infty) = a\}$. We count the initial configuration of length $4n$ which converges to W_a within n-steps. It is easy to see that the set of path of simple random walk never touch the line $y = ax$ of length $4n$ and return to the line at the last step is essential to the term.

For $S = \{0, 1\}$ and rational a, it is clear with certain amount of efforts to count up the number as in [5]

$$C_n \frac{2n!}{(2an)!(2(1-a)n)!} p^{2an}(1-p)^{2(1-a)n}$$

where $\lim_{n\to\infty} \log C_n/n = 0$. Using Stirling's formula, we have

$$\mathrm{e}^{2an\log a + (2(1-a)n)\log(1-a)}\mathrm{e}^{-2an\log p - 2(1-a)n\log(1-p)}.$$

From this we get the relative entropy term. The term h_ν comes from the uniformness of ν.

We complete for rational a. For irrational a, take the sequence $\{a_i\}$ which converges to a. By the continuity of $H(a|p)$ with respect to a, $H(a_i|p)$ also converges to $H(a|p)$ and ν is well defined whether a is rational or not. So we have the proposition.

Remark 4. More precisely we use harmonic analysis — generating function method. We get $\mathrm{e}^{-H(a|p)}$ as the convergence radius of generating function for first return time to the given line.

3.2 EBCA1

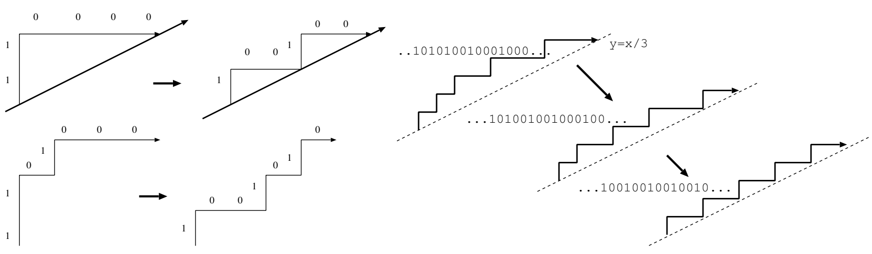
EBCA1 is a traffic model studied in [6,7]. The rule is described as following:

$$x_i^{t+1} = x_i^t + \min(b_{i-1}^t + b_{i-2}^t, 1 - x_i^t + b_i^t) - \min(b_i^t + b_{i-1}^t, 1 - x_{i+1}^t + b_{i+1}^t)$$

where $b_i = \min(x_i^t, 1 - x_{i+1}^t)$.

As ECA184, the limit set of EBCA1 has decomposition $X = X_- \bigcup X_+$ and essentially $X = X_- \bigcap X_+ = \{(100)^\infty\}$ where $X_- = \langle 0, 10\rangle$ and $X_+ = \langle 1, 10, 100\rangle$.

Its rule in path space and the dynamics is shown in Fig. 4.



(**a**) (**b**)

Fig. 4. EBCA1: (**a**) The essential part of the rule in path space. That is more complicated with ECA184. (**b**) Its typical dynamics

The problem for EBCA1 is existence of the metastable state as [6,7]. If the preimages of metastable state in X_0 has the positive measure with respect to μ^p, the assumption of our method is broken. Fortunately, we can show that the measure of metastable state is 0, so our method is still valid.

3.3 Generalization

We can generalize this more typical traffic models on $S = \{0, 1\}$ because our proof does not depend on the detail of its rule.

Our proof depends on the fact that the case of $p = p_c$ the limit set consists of finite number of configurations. That is the case that the intersection $X_+ \bigcap X_-$ has only finite number of points.

4 Discussion

The exponent R discussed the paper seems to describe well the phase transition of dynamics in one-dimensional cellular automata traffic models. This is the result not only for the "stable state" of the dynamics but for the "asymptoticity" of the dynamics. We think these point of view is important from the application to the real traffic or flow.

For our results, the essential part for a given rule is to determine the structure of attractor or stable states X_∞. Though it is difficult to set the general algorithms, there exists a class having "Lyapunov function $Q(x)$".

In [7] they proved that $Q(x)$ increases monotonously. The behavior corresponds to the "Lyapunov function" for the theory of ordinary differentiable equation. If we get the fact, the set $\{w \ ; \ q(w) = 1\}$ is a essential word set for attractor in $S = \{0, 1\}$. Though this point is discussed in [10] for steady state, the case of congestive state is remained yet.

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