Finite Dimensional Semisimple $\mathbb{Q}$-Algebras

Takahiko Nakazi* and Takanori Yamamoto*

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education of Japan.

Abstract A $\mathbb{Q}$-algebra can be represented as an operator algebra on an infinite dimensional Hilbert space. However we don't know whether a finite $n$-dimensional $\mathbb{Q}$-algebra can be represented on a Hilbert space of dimension $n$ except $n = 1, 2$. It is known that a two dimensional $\mathbb{Q}$-algebra is just a two dimensional commutative operator algebra on a two dimensional Hilbert space. In this paper we study a finite $n$-dimensional semisimple $\mathbb{Q}$-algebra on a finite $n$-dimensional Hilbert space. In particular we describe a three dimensional $\mathbb{Q}$-algebra of the disc algebra on a three dimensional Hilbert space. Our studies are related to the Pick interpolation problem for a uniform algebra.

AMS Classification: 46J05; 46J10; 47A30; 47B38

Keywords: commutative Banach algebra; semisimple $\mathbb{Q}$-algebra; three dimension; norm; Pick interpolation

Takahiko Nakazi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: nakazi@math.sci.hokudai.ac.jp

Takanori Yamamoto
Department of Mathematics
Hokkai-Gakuen University
Sapporo 062-8605, Japan
E-mail: yamamoto@elsa.hokkai-s-u.ac.jp
1. Introduction

Let \( A \) be a uniform algebra on a compact Hausdorff space \( X \). If \( I \) is a closed ideal of \( A \), then the quotient algebra \( A/I \) is a commutative Banach algebra with unit. In this paper, if a Banach algebra \( B \) is isometrically isomorphic to \( A/I \), then \( B \) is called a \( Q \)-algebra. (F.Bonsall and J.Duncan called \( B \) an \( IQ \)-algebra.) B.Cole (cf. [1], p.272) showed that any \( Q \)-algebra is an operator algebra on a Hilbert space \( H \), that is, there exists an isometric isomorphism to an operator algebra on \( H \). Let \( \mu \) be a probability measure on \( X \) and \( H^2(\mu) \) the closure of \( A \) in \( L^2(\mu) \). \( H^2(\mu) \cap I^\perp \) denotes the annihilator of \( I \) in \( H^2(\mu) \). Let \( P \) be the orthogonal projection from \( H^2(\mu) \) onto \( H^2(\mu) \cap I^\perp \) and for any \( f \in A \) put

\[
S_{f}^\mu = P(f), \quad (f \in H^2(\mu) \cap I^\perp).
\]

Then \( S_{f+k}^\mu = S_{f}^\mu \) for \( k \) in \( I \) and \( \|S_{f}^\mu\| \leq \|f+I\| \). \( S^\mu \) is the map of \( A/I \) on operators on \( H^2(\mu) \cap I^\perp \) which sends \( f + I \to S_{f}^\mu \) for each \( f \) in \( A \). Hence \( S^\mu \) is a contractive homomorphism from \( A \) into \( B(H^2(\mu) \cap I^\perp) \) where \( B(H^2(\mu) \cap I^\perp) \) is the set of all bounded linear operators on \( H^2(\mu) \cap I^\perp \). The kernel of \( S^\mu \) contains \( I \). Then we say that \( S^\mu \) gives a contractive representation of \( A/I \) into \( B(H^2(\mu) \cap I^\perp) \). If \( \|S_{f}^\mu\| = \|f+I\|, (f \in A) \) then \( \ker S^\mu = I \) and we say that \( S^\mu \) gives an isometric representation of \( A/I \) on \( H^2(\mu) \cap I^\perp \).

Problem 1. Prove that any finite \( n \)-dimensional \( Q \)-algebra can be represented on a Hilbert space of finite dimension \( n \).

If \( S^\mu \) is isometric then we solve Problem 1. In fact, T.Nakazi and K.Takahashi (cf. [9]) solved Problem 1 for \( n = 2 \) in this way. It seems to be unknown for \( n \geq 3 \).

Problem 2. Describe a finite \( n \)-dimensional \( Q \)-algebra in finite \( n \)-dimensional commutative operator algebras with unit on a Hilbert space of finite dimension \( n \).

Problem 2 is clear for \( n = 1 \) and it was proved by S.W.Drury (cf. [4]) and T.Nakazi (cf. [8]) that a 2-dimensional commutative operator algebra with unit on a Hilbert space is just a \( Q \)-algebra. J.Holbrook (cf. [6]) proved that von Neumann’s inequality

\[
\|p(T)\| \leq \|p\|_{\infty}
\]

can fail for some polynomials \( p \) in 3 variables, where \( T = (T_1, T_2, T_3) \) is a triple of commuting contractions on \( C^4 \), and \( T_1, T_2, T_3 \) are simultaneously diagonalizable. Then we can construct a 4-dimensional commutative matrix algebra with unit on \( C^4 \), which is not a \( Q \)-algebra. If \( n \geq 4 \), then this implies that the set of all \( n \)-dimensional \( Q \)-algebra \( A/I \) is smaller than the set of all set of all \( n \)-dimensional commutative operator algebras with unit on a \( n \)-dimensional Hilbert space. If \( n = 3 \), then Problem 2 has not been solved yet. In this paper, we concentrate on a semisimple commutative Banach algebra and we study Problem 2. In Section 2, we will prove several general results of semisimple finite dimensional \( Q \)-algebras that will be used in the latter sections. In Section 3, we will study arbitrary semisimple \( n \) dimensional \( Q \)-algebras for \( n = 2, 3 \). In Section 4, we will study the isometric representation of \( A/I \). In Section 5, we will describe completely 3-dimensional semisimple \( Q \)-algebras of the disc algebra in 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space.
2. Semisimple and commutative matrix algebra

In this section, we study 3-dimensional commutative semisimple operator algebras on a 3-dimensional Hilbert space. In particular, we study when two such operator algebras are isometric or unitary equivalent. S.McCullough and V.Paulsen (cf. [7], Proposition 2.2) proved the similar result of Proposition 2.3. We use Lemma 2.1 to prove Proposition 2.2 and Proposition 2.3. In Example 2.6, we construct a 4-dimensional commutative matrix algebra with unit on $\mathbb{C}^4$ which is not a $Q$-algebra using the example of J.Holbrook (cf. [6]).

Lemma 2.1. Let $n \geq 2$ and let $H$ be an $n$-dimensional Hilbert space which is spanned by $k_1, k_2, \ldots, k_n$. Let

$$\psi_1 = \frac{k_1}{||k_1||}, \psi_j = \frac{k_j - \sum_{i=1}^{j-1} \langle k_j, \psi_i \rangle \psi_i}{||k_j - \sum_{i=1}^{j-1} \langle k_j, \psi_i \rangle \psi_i||} \quad (2 \leq j \leq n).$$

Then $\{\psi_1, \ldots, \psi_n\}$ is an orthonormal basis for $H$. Let $P_1, \ldots, P_n$ be the idempotent operators on $H$ such that $P_i k_i = k_i$, $P_i k_j = 0$ if $i \neq j$. For $1 \leq m \leq n$, let $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$, $(1 \leq i, j \leq n)$. Then $P_m = (a_{ij}^{(m)})_{1 \leq i, j \leq n}$ is an $n \times n$ matrix such that

$$P_1 = \begin{pmatrix} B_1 \\ O \end{pmatrix}, \ldots, P_m = \begin{pmatrix} O & B_m \\ O & O \end{pmatrix}, \ldots, P_n = \begin{pmatrix} O & B_n \end{pmatrix},$$

where $B_m$ is an $m \times (n - m + 1)$ matrix such that

$$B_1 = \begin{pmatrix} 1 & \ldots & a_{1n}^{(1)} \end{pmatrix}, \ldots, B_m = \begin{pmatrix} a_{1m}^{(m)} & \ldots & a_{1n}^{(m)} \\ \ldots & \ldots & \ldots \\ 1 & \ldots & a_{mn}^{(m)} \end{pmatrix}, \ldots, B_n = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Then $a_{mm}^{(m)} = 1$, and for $m \geq 2$,

$$a_{im}^{(m)} = \frac{\langle k_m, \psi_i \rangle}{\langle k_m, k_m \rangle - \sum_{h=1}^{m-1} \langle k_m, \psi_h \rangle \psi_h \rangle},$$

and for $m + 1 \leq j \leq n$,

$$a_{ij}^{(m)} = \frac{-\sum_{h=1}^{j-1} \langle k_j, \psi_h \rangle a_{ih}^{(m)}}{\langle k_j, \psi_j \rangle - \sum_{h=1}^{j-1} \langle k_j, \psi_h \rangle \psi_h \rangle}.$$

Since this lemma is proved by elementary calculations, the proof is omitted. It is well known that any $n$-dimensional commutative semisimple Banach algebra with unit $I$ is spanned by commuting idempotents $P_1, \ldots, P_n$ satisfying $P_1 + \ldots + P_n = I$. 
Proposition 2.2. In Lemma 2.1, for $1 \leq m \leq n$, $\text{rank} P_m = 1$, and $B = \text{span}\{P_1, \ldots, P_n\}$ is an $n$-dimensional semisimple commutative operator algebra with unit on $H$. Then $n \times n$ matrix $(a^{(m)}_{ij})$ for $P_m$ with respect to $\{\psi_1, \ldots, \psi_n\}$ is $a^{(m)}_{ij} = <P_m \psi_j, \psi_i>$, and

$P_1 = (a^{(1)}_{ij}) = \begin{pmatrix} 1 & a^{(1)}_{12} & \cdots & a^{(1)}_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, $P_2 = (a^{(2)}_{ij}) = \begin{pmatrix} 0 & a^{(2)}_{12} & a^{(2)}_{13} & \cdots & a^{(2)}_{1n} \\ 0 & 1 & a^{(2)}_{23} & \cdots & a^{(2)}_{2n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$,

$\ldots$, $P_n = (a^{(n)}_{ij}) = \begin{pmatrix} 0 & \cdots & 0 & a^{(n)}_{1n} \\ 0 & \cdots & 0 & a^{(n)}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a^{(n)}_{n-1n} \\ 0 & \cdots & 0 & 1 \end{pmatrix}$.

In Lemma 2.1, $a^{(m)}_{ij}$ is written using $k_1, \ldots, k_n$ and $\psi_1, \ldots, \psi_n$.

Proof. By the assumption of Lemma 2.1, $P_i k_i = k_i$ and $P_i k_j = 0$ if $i \neq j$. Hence $\text{rank} P_m = 1$. If $i \neq j$, then $P_i P_j k_m = \delta_{jm} P_i k_j = 0$, $(1 \leq m \leq n)$. Since $H = \text{span}\{k_1, \ldots, k_n\}$, this implies that $P_i P_j = 0$ if $i \neq j$. Hence $B$ is commutative. Since $P_i^2 k_m = \delta_{im} P_i k_m = P_i k_m$, $(1 \leq m \leq n)$, it follows that $P_i^2 = P_i$. Hence $B$ is semisimple and $n$-dimensional. Since $(P_1 + \ldots + P_n) k_m = P_m k_m = k_m$, $(1 \leq m \leq n)$, it follows that $P_1 + \ldots + P_n = I$. Hence $B$ has a unit $I$. This completes the proof.

Proposition 2.3. Let $H$ be a $3$-dimensional Hilbert space which is spanned by $k_1, k_2, k_3$. Let $\langle \cdot, \cdot \rangle$ denote the inner product, and let $\| \cdot \|$ denote the norm of $H$.

$\psi_1 = \frac{k_1}{\|k_1\|}$, $\psi_2 = \frac{k_2 - \langle k_2, \psi_1 \rangle \psi_1}{\|k_2 - \langle k_2, \psi_1 \rangle \psi_1\|}$, $\psi_3 = \frac{k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}$.

Then $\psi_1, \psi_2, \psi_3$ is an orthonormal basis in $H$. Let $P_i$ be the idempotent operator on $H$ such that $P_i k_i = k_i$, $P_i k_j = 0$ if $i \neq j$. For $m = 1, 2, 3$, the $3 \times 3$ matrix $(a^{(m)}_{ij})$ for $P_m$ with respect to $\{\psi_1, \psi_2, \psi_3\}$ is $a^{(m)}_{ij} = <P_m \psi_j, \psi_i>$. Then

$P_1 = (a^{(1)}_{ij}) = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2 = (a^{(2)}_{ij}) = \begin{pmatrix} 0 & -x & -yz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}$, $P_3 = (a^{(3)}_{ij}) = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix}$,

where

$x = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - \|\langle k_1, k_2 \rangle\|^2}}$, $y = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}$, $z = \frac{-\langle k_3, \psi_2 \rangle}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}$.
**Proof.** By Proposition 2.2, there exist \( x, y \) such that

\[
P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

By Lemma 2.1,

\[
x = a_{12}^{(1)} = \frac{-\langle k_2, k_1 \rangle}{\sqrt{||k_1||^2||k_2||^2 - |\langle k_1, k_2 \rangle|^2}},
\]

and

\[
y = a_{13}^{(1)} = \frac{-\sum_{h=1}^{2} \langle k_3, \psi_h \rangle a_{1h}^{(1)}}{||k_3 - \sum_{h=1}^{2} \langle k_3, \psi_h \rangle \psi_h||} = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{||k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2||}.
\]

By Proposition 2.2, there exist \( z, w \) such that

\[
P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & w \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & -w - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},
\]

because \( P_1 + P_2 + P_3 = I \). By Lemma 2.1,

\[
z = a_{23}^{(2)} = \frac{-\sum_{h=2}^{2} \langle k_3, \psi_h \rangle a_{2h}^{(2)}}{||k_3 - \sum_{h=1}^{2} \langle k_3, \psi_h \rangle \psi_h||} = \frac{-\langle k_3, \psi_2 \rangle}{||k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2||}.
\]

By Lemma 2.1,

\[
a_{12}^{(1)} = \frac{-\langle k_2, k_1 \rangle}{\sqrt{||k_1||^2||k_2||^2 - |\langle k_1, k_2 \rangle|^2}} = -a_{12}^{(2)}.
\]

Hence

\[
w = a_{13}^{(2)} = \frac{-\sum_{h=2}^{2} \langle k_3, \psi_h \rangle a_{1h}^{(2)}}{||k_3 - \sum_{h=1}^{2} \langle k_3, \psi_h \rangle \psi_h||} = \frac{-\langle k_3, \psi_2 \rangle a_{12}^{(2)}}{||k_3 - \sum_{h=1}^{2} \langle k_3, \psi_h \rangle \psi_h||} = za_{12}^{(2)} = -za_{12}^{(1)} = -xz.
\]

This completes the proof.

**Theorem 2.4.** Let \( P_1, P_2, P_3 \) be idempotent operators defined in Proposition 2.3. Let \( H' \) be a 3-dimensional Hilbert space. Let \( B' \) be a 3-dimensional semisimple commutative operator algebra on \( H' \). Then, there are idempotent operators \( Q_1, Q_2, Q_3 \) on \( H' \), an orthonormal basis \( \psi_1', \psi_2', \psi_3' \) in \( H' \) and complex numbers \( x_0, y_0, z_0 \) such that \( B' = \text{span}\{Q_1, Q_2, Q_3\} \) and, as matrices relative to \( \psi_1', \psi_2', \psi_3' \),

\[
Q_1 = \begin{pmatrix} 1 & x_0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -x_0 & -x_0z_0 \\ 0 & 1 & z_0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & x_0z_0 - y_0 \\ 0 & 0 & -z_0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Let \( \tau \) be the map of \( B \) on \( B' \) such that

\[
\tau (\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3) = \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3, \quad (\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C})�.
\]


(1) \( \tau \) is isometric if and only if
\[
|x|^2 + |y|^2 = |x_0|^2 + |y_0|^2,
\]
\[
(1 + |x|^2)(1 + |z|^2) = (1 + |x_0|^2)(1 + |z_0|^2),
\]
\[
|x|^2 + xz\bar{y} = |x_0|^2 + x_0z_0\bar{y_0}.
\]

(2) \( \tau \) is induced by a unitary map from \( H \) to \( H' \) if and only if there are complex numbers \( u_1, u_2, u_3 \) such that
\[
|u_1| = |u_2| = |u_3| = 1, \quad u_1 x = u_2 x_0, \quad u_1 y = u_3 y_0, \quad u_2 z = u_3 z_0.
\]
Then \( |x| = |x_0|, \quad |y| = |y_0|, \quad |z| = |z_0|, \quad xz\bar{y} = x_0z_0\bar{y_0} \).

**Proof.** (1) By the theorem of B.Cole and J.Wermer (cf. [3]), \( \tau \) is isometric if and only if, writing \( \text{tr} \) for trace,
\[
\text{tr}(P_i^*P_j) = \text{tr}(Q_i^*Q_j), \quad (1 \leq i, j \leq 3).
\]
If \( \tau \) is isometric, then
\[
1 + |x|^2 + |y|^2 = \text{tr}(P_1^*P_1) = \text{tr}(Q_1^*Q_1) = 1 + |x_0|^2 + |y_0|^2,
\]
\[
(1 + |x|^2)(1 + |z|^2) = \text{tr}(P_2^*P_2) = \text{tr}(Q_2^*Q_2) = (1 + |x_0|^2)(1 + |z_0|^2),
\]
\[
|x|^2 + xz\bar{y} = \text{tr}(P_1^*P_2) = \text{tr}(Q_1^*Q_2) = |x_0|^2 + x_0z_0\bar{y_0}.
\]
Conversely, if three equalities in (1) hold, then
\[
\text{tr}(P_1^*P_1) = 1 + |x|^2 + |y|^2 = 1 + |x_0|^2 + |y_0|^2 = \text{tr}(Q_1^*Q_1),
\]
\[
\text{tr}(P_2^*P_2) = (1 + |x|^2)(1 + |z|^2) = (1 + |x_0|^2)(1 + |z_0|^2) = \text{tr}(Q_2^*Q_2),
\]
\[
\text{tr}(P_1^*P_2) = |x|^2 + xz\bar{y} = |x_0|^2 + x_0z_0\bar{y_0} = \text{tr}(Q_1^*Q_2),
\]
\[
\text{tr}(P_2^*P_3) = xz(y - x) - |z|^2 = x_0z_0(y_0 - x_0y_0) - |z_0|^2 = \text{tr}(Q_2^*Q_3),
\]
\[
\text{tr}(P_3^*P_1) = y(xz - y) = y_0(x_0z_0 - y_0) = \text{tr}(Q_3^*Q_1),
\]
\[
\text{tr}(P_3^*P_3) = 1 + |z|^2 + |xz - y|^2 = 1 + |z_0|^2 + |x_0z_0 - y_0|^2 = \text{tr}(Q_3^*Q_3).
\]

(2) Suppose \( \tau \) is induced by a unitary map \( U = (u_{ij}), \ (1 \leq i, j \leq 3) \) from \( H \) to \( H' \). Since \( U P_1 = Q_1 U \), it follows that \( u_{11} = u_{31} = 0 \). Since \( U P_2 = Q_2 U \), it follows that \( u_{32} = 0 \). Hence \( U \) is an upper triangular matrix. Since the columns of \( U \) are pairwise orthogonal, \( U \) is a diagonal matrix. Hence there are complex numbers \( u_1, u_2, u_3 \) such that \( u_1, u_2, u_3 \) are diagonal element of \( U \), and \( |u_1| = |u_2| = |u_3| = 1 \). Since \( U P_1 = Q_1 U \), it follows that \( u_1 x = u_2 x_0, \ u_1 y = u_3 y_0 \). Since \( U P_2 = Q_2 U \), it follows that \( u_2 z = u_3 z_0 \). The converse is also true. This completes the proof.

**Example 2.5.** Let \( B_0 = \text{span}\{P_1, P_2, P_3\} \), where
\[
P_1 = \begin{pmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0 & -1 & -1 \\
0 & 1 & 1 \\
0 & 0 & 0 
\end{pmatrix}, \quad P_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 0 & 1 
\end{pmatrix}
\]
and let $B_1 = \text{span}\{P_1, P_2, P_3\}$. This is an example which is established by W.Wogen (cf. [3]). He proved that $B_0$ and $B_1$ are isometrically isomorphic, and not unitarily equivalent. There is another example as the following. Let $B_2 = \text{span}\{Q_1, Q_2, Q_3\}$, where

$$
Q_1 = \begin{pmatrix}
1 & \sqrt{2} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad Q_2 = \begin{pmatrix}
0 & -\sqrt{2} & -\sqrt{2}/3 \\
0 & 1 & 1/\sqrt{3} \\
0 & 0 & 0
\end{pmatrix}, \quad Q_3 = \begin{pmatrix}
0 & 0 & \sqrt{2}/3 \\
0 & 0 & -1/\sqrt{3} \\
0 & 0 & 1
\end{pmatrix}.
$$

Then $B_0$ and $B_2$ are 3-dimensional commutative operator algebras with unit. By the calculation,

$$
|x|^2 + |y|^2 = |x_0|^2 + |y_0|^2 = 2,
$$

$$
(1 + |x|^2)(1 + |z|^2) = (1 + |x_0|^2)(1 + |z_0|^2) = 4,
$$

$$
|x|^2 + x\bar{z}y = |x_0|^2 + x_0z_0\bar{y}_0 = 2.
$$

By (1) of Theorem 2.4, this implies that $B_0$ and $B_2$ are isometrically isomorphic. By (2) of Theorem 2.4, $B_0$ and $B_2$ are not unitarily equivalent.

3. One to one representation

In this section, we assume that $A/I$ is $n$-dimensional and semisimple. Hence there exist $\tau_1, \ldots, \tau_n$ in the maximal ideal space $M(A)$ of $A$ such that $\tau_i \neq \tau_j$ ($i \neq j$) and $I = \cap_{j=1}^n \ker \tau_j$. $S^\mu$ gives a contractive representation of $A/I$ into $B(H^2(\mu) \cap I^\perp)$ and $\dim H^2(\mu) \cap I^\perp \leq \dim A/I = n$. We study when $S^\mu$ is one to one from $A/I$ to $B(H^2(\mu) \cap I^\perp)$. It is clear that $S^\mu$ is one to one if and only if $\dim H^2(\mu) \cap I^\perp = \dim A/I$. For $1 \leq j \leq n$, there exist $f_j \in A$ such that $\tau_j(f_j) = \delta_{ij}$. Then $f_j + I$ is idempotent in $A/I$ and $A/I = \text{span}\{f_1 + I, \ldots, f_n + I\}$. The following two quantities are important to study $S^\mu$.

For $1 \leq j \leq n$,

$$
\rho_j = \sup\{ |\tau_j(f)| ; f \in \cap_{i \neq j} \ker \tau_i, ||f|| \leq 1 \}
$$

and

$$
\rho_j(\mu) = \sup\{ |\tau_j(f)| ; f \in \cap_{i \neq j} \ker \tau_i, ||f||_\mu \leq 1 \},
$$

where $||f||$ denotes the supnorm of $f$ in $A$ and $||f||_\mu = \langle f, f \rangle_\mu^{1/2} = (\int \langle f, f \rangle_\mu d\mu)^{1/2}$. Then it is easy to see that

$$
||f_j + I|| = \frac{1}{\rho_j}, \quad ||f_j + I||_\mu = \frac{1}{\rho_j(\mu)}.
$$

and

$$
||f_j + I|| \geq ||S_{f_j}^\mu|| \geq ||f_j + I||_\mu.
$$

If $S^\mu$ is one to one then $\tau_j$ has a bounded extension to $H^2(\mu)$. In fact, if $S^\mu$ is one to one then $\dim H^2(\mu) \cap I^\perp = n$ and so $\dim H^2(\mu) \cap (\ker \tau_j)^\perp = 1$ for $1 \leq j \leq n$. Then for $1 \leq j \leq n$, there exists $k_j \in H^2(\mu)$ such that

$$
\tau_j(f) = \langle f, k_j \rangle_\mu = \int X \bar{k}_j d\mu, \quad (f \in A).
$$
Proposition 3.1. There exists a one to one contractive representation $S^\mu$ of $A/I$.

Proof. Since $\tau_j \in M(A)$, there exists a positive representing measure $m_j$ of $\tau_j$ on $X$. Let $\mu = \sum_{j=1}^{n} m_j/n$. Then

$$|\tau_j(f)| = \int_X f dm_j \leq n(\int_X |f|^2 d\mu)^{1/2} = n\|f\|_\mu, \ (f \in A).$$

Hence $\tau_j$ has a unique bounded extension $\tilde{\tau}_j$ to $H^2(\mu)$ and $\tilde{\tau}_j \neq \tilde{\tau}_i \ (j \neq i)$. Since $H^2(\mu) \cap I^\perp = \bigcap_{j=1}^{n} \ker \tilde{\tau}_j$, dim $H^2(\mu) \cap I^\perp = n$. Hence $H^2(\mu) \cap I^\perp = \text{span}\{k_1,\ldots,k_n\}$. Suppose $S^\mu_j = 0$, then $\tilde{\tau}_j(f)k_j = (S^\mu_j)^*k_j = 0$. Hence $\tilde{\tau}_j(f) = 0, \ (1 \leq j \leq n)$. Hence $f \in \bigcap \ker \tau_j = I$. This implies that $S^\mu$ is one to one from $A/I$ to $B(H^2(\mu) \cap I^\perp)$. This completes the proof.

Theorem 3.2. Suppose that $S^\mu$ is a one to one contractive representation of $A/I$. Let $k_j$ be a function in $H^2(f_1)$ such that $\tau_j(f) = \langle f, k_j \rangle_\mu \ (f \in A)$, for $1 \leq j \leq n$. Then

1. $H^2(\mu) \cap I^\perp = \text{span}\{k_1,\ldots,k_n\}$ and $H^2(\mu) \cap I^\perp = \text{span}\{k_j\}$ where $I_j = \ker \tau_j$.
2. If $m_j = \|k_j\|^2_\mu \|k_j\|^2_\mu d\mu$ and $m = \sum_{j=1}^{n} m_j/n$, then $m_j$ is a representing measure for $\tau_j$ for each $1 \leq j \leq n$, and we may assume that $\mu$ is absolutely continuous with respect to $m$.
3. $\|S^\mu_{f_j}\| = \|k_j\|_\mu \|f_j + I\|_\mu$, for $1 \leq j \leq n$.

Proof. (1) Since $S^\mu$ is one to one, $\tau_j$ has a unique bounded extension $\tilde{\tau}_j$ to $H^2(\mu)$. In fact, if $S^\mu$ is one to one then dim $H^2(\mu) \cap I^\perp = n$ and so dim $H^2(\mu) \cap (\ker \tau_j)^\perp = 1$ for $1 \leq j \leq n$. Then there exists $k_j \in H^2(\mu)$ such that $\tau_j(f) = \langle f, k_j \rangle_\mu \ (f \in A)$. If $g \in I$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I^\perp$ for each $j$. Since $k_1,\ldots,k_n$ are linearly independent, $\{k_1,\ldots,k_n\}$ is a basis of $H^2(\mu) \cap I^\perp$. If $g \in I_j$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I_j^\perp$ for each $j$. Hence $k_j$ is a basis of $H^2(\mu) \cap I_j^\perp$.

(2) For $1 \leq j \leq n$,

$$\int_X f dm_j = \int_X f \frac{|k_j|^2_\mu}{\|k_j\|^2_\mu} d\mu = \frac{\langle f, k_j \rangle_\mu}{\|k_j\|^2_\mu} = \frac{\tilde{\tau}_j(f)k_j}{\|k_j\|^2_\mu} = \tau_j(f), \ (f \in A).$$

Hence $m_j$ is a representing measure for $\tau_j$. Let $m = \sum_{j=1}^{n} m_j/n$ and let $\mu = \mu^a + \mu^s$ be a Lebesgue decomposition by $m$. Then $H^2(\mu^a) \cap I^\perp = H^2(\mu) \cap I^\perp$ and so $H^2(\mu^s) \cap I^\perp = \{0\}$ where $\mu^a$ and $\mu^s$ are divided by their total masses. Hence $S^\mu_{f_j} = S^\mu_{f_j} + S^\mu_{f_j} = S^\mu_{f_j} + 0$ and so $\|S^\mu_{f_j}\| = \|S^\mu_{f_j}\|$ for $f \in A$.

(3) Since rank($S^\mu_{f_j}$) $= 1$, there exists $x_j \in H^2(\mu) \cap I^\perp$ such that $(S^\mu_{f_j})^*\phi = \langle \phi, x_j \rangle k_j = (k_j \otimes x_j)\phi$, ($\phi \in H^2(\mu) \cap I^\perp$). Then $\|S^\mu_{f_j}\| = \|S^\mu_{f_j}\|| = \|k_j \otimes x_j\| = \|k_j\|_\mu \|x_j\|_\mu$. Let $P$ be the orthogonal projection from $H^2(\mu)$ onto $H^2(\mu) \cap I^\perp$. Then

$$\langle P f_j, \phi \rangle = \langle S^\mu_{f_j} 1, \phi \rangle = \langle 1, (S^\mu_{f_j})^*\phi \rangle = \langle x_j, \phi \rangle \langle 1, k_j \rangle = \langle x_j, \phi \rangle, \ (\phi \in H^2(\mu) \cap I^\perp),$$

because $\langle 1, k_j \rangle = 1$. Hence $P f_j = x_j$. Hence

$$\|f_j + I\|_\mu = \|P f_j\|_\mu = \|x_j\|_\mu.$$ 

Hence $\|S^\mu_{f_j}\| = \|k_j\|_\mu \|f_j + I\|_\mu$. This completes the proof.
Let \( G(\tau) \) denote the Gleason part of \( \tau \). If \( G(\tau_i) = G(\tau_j) \), then we write \( \tau_i \sim \tau_j \).

**Proposition 3.3.** Suppose that \( \dim H^2(\mu) \cap I^\perp = n \), \( I^1 = \cap_{j \in N^1} \ker \tau_j \), \( I^2 = \cap_{j \in N^2} \ker \tau_j \), \( N^1 \cap N^2 = \emptyset \) and \( N^1 \cup N^2 = \{1, 2, ..., n\} \). Let \( \#N^j \) denote the number of elements in \( N^j \). If \( \tau_j \neq \tau_k \) whenever \( j \in N^1 \) and \( k \in N^2 \), then \( H^2(\mu) = H^2(\mu^1) \oplus H^2(\mu^2) \), \( H^2(\mu) \cap I^\perp = (H^2(\mu^1) \cap (I^1)^\perp) \oplus (H^2(\mu^2) \cap (I^2)^\perp) \), \( S^\mu = S^\mu_1 \oplus S^\mu_2 \) and \( \dim H^2(\mu) \cap (I^1)^\perp = \#N^j \), where \( \mu = \frac{\mu^1 + \mu^2}{2} \), \( \mu^1 \perp \mu^2 \) and \( \mu^j \) is a probability measure for \( j = 1, 2 \).

**Proof.** By (1) of Theorem 3.2, \( H^2(\mu) \cap I^\perp = \text{span}\{k_1, ..., k_n\} \). We may assume that \( N^1 = \{1, 2, ..., l\} \) and \( N^2 = \{l + 1, ..., n\} \). By (2) of Theorem 3.2, \( m_j = \|k_j\|^2 d\mu \) is a representing measure for \( \tau_j \) for each \( 1 \leq j \leq n \). Put \( \lambda^1 = \frac{1}{l} \sum_{j=1}^l m_j \) and \( \lambda^2 = \frac{1}{n-l} \sum_{j=l+1}^n m_j \) then \( \lambda^1 \perp \lambda^2 \) by definitions of \( N^1 \) and \( N^2 \). Let \( \mu = \mu_0^1 + \mu_0^2 \) be a Lebesgue decomposition with respect to \( \lambda^1 \) such that \( \mu_0^1 \ll \lambda^1 \) and \( \mu_0^2 \perp \lambda^1 \). Put \( \mu^1 = \mu_0^1/\|\mu_0^1\| \) and \( \mu^2 = \mu_0^2/\|\mu_0^2\| \). This completes the proof.

4. Isometric representation

In this section, we assume that \( A/I \) is \( n \)-dimensional and semisimple. Hence there exist \( \tau_1, ..., \tau_n \) in the maximal ideal space \( M(A) \) of \( A \) such that \( \tau_i \neq \tau_j \) \( (i \neq j) \) and \( I = \cap_{j=1}^n \ker \tau_j \). For \( 1 \leq j \leq n \), there exist \( f_j \in A \) such that \( \tau_i(f_j) = \delta_{ij} \). Then \( f_j + I \) is idempotent in \( A/I \) and \( A/I = \text{span}\{f_1 + I, ..., f_n + I\} \). If \( S^\mu \) is an isometric representation of \( A/I \), then \( \|S^\mu_{f_j}\| = \|f_j + I\| \) for \( 1 \leq j \leq n \). By (3) of Theorem 3.2, this implies that \( \|f_j + I\| = \|k_j\|_\mu \|f_j + I\|_\mu \). Hence, if \( S^\mu \) is an isometric representation of \( A/I \), then \( \|k_j\|_\mu = \|f_j + I\|/\|f_j + I\|_\mu \) for \( 1 \leq j \leq n \). Is the converse of this statement true? If \( n = 2 \), then the answer will be given in Proposition 4.4.

**Theorem 4.1.** Suppose that \( G(\tau_i) \cap G(\tau_j) \cap G(\tau_l) = \emptyset \) if \( i, j \) and \( l \) are different from each other. Then there exists an isometric representation \( S^\mu \) of \( A/I \).

**Proof.** By Proposition 3.3, if \( G(\tau_j) = \{\tau_j\} \), for all \( 1 \leq j \leq n \), then there exists an isometric representation \( S^\mu \) of \( A/I_j \) where \( I_j = \ker \tau_j \) and \( \mu^j \perp \mu \). If \( \mu = (\mu^1 + ... + \mu^n)/n \), then \( H^2(\mu) \cap I^\perp = (H^2(\mu^1) \cap I^\perp) \oplus ... \oplus (H^2(\mu^n) \cap I^\perp) \) and \( S^\mu_1 = S^\mu_1 \oplus ... \oplus S^\mu_n \) \( (f \in A) \). Therefore, the theorem is proved in the case when \( G(\tau_j) = \{\tau_j\} \), for all \( 1 \leq j \leq n \). It is sufficient to prove the theorem when \( \tau_i \sim \tau_j \) for some \( i, j (i \neq j) \). Suppose \( \tau_{2k-1} \sim \tau_{2k}, (1 \leq k \leq n_0) \) and \( G(\tau_l) = \{\tau_l\}, (2n_0 + 1 \leq l \leq n) \) for some \( n_0 \). Since \( G(\tau_l) \cap G(\tau_j) \cap G(\tau_l) = \emptyset \), it follows that \( \dim A/I_{ij} = 2 \) where \( I_{ij} = I_l \cap I_j = \ker \tau_i \cap \ker \tau_j \). By Corollary 1 in [9], there is a probability measure \( \mu_{ij} \) such that \( \|S^\mu_{f_j}\| = \|f + I_{ij}\| \) for all \( f \in A \). By Proposition 3.3, there are probability measures \( \mu_{2k-1,2k} \) \( (1 \leq k \leq n_0) \) and \( \mu_{l,2n_0 + 1} \leq l \leq n) \) such that \( \mu = (\mu^1 + \mu^2 + ... + \mu^{2n_0-1,2n_0} + \mu^{2n_0+1} + ... + \mu^n)/(n - n_0) \), \( H^2(\mu) \cap I^\perp = \emptyset \).
For example, we consider when \( n = 3 \) and \( \tau_1 \sim \tau_2 \not\sim \tau_3 \). Let \( I_{12} = I_1 \cap I_2 = \ker \tau_1 \cap \ker \tau_2 \). Then \( \dim A/I_{12} = 2 \). By Corollary 1 in [9], there is a probability measure \( \mu^{12} \) such that 
\[ \| S_f^{12} \| = \| f + I_{12} \| \] 
for all \( f \in A \). Let \( S'_{12} \) be the isometric representation of \( A/I_3 \) where \( I_3 = \ker \tau_3 \). Let \( \mu = (\mu^{12} + \mu^3)/2 \). Then \( \mu^{12} \perp \mu^3 \), \( H^2(\mu) \cap I^\perp_3 = (H^2(\mu^{12}) \cap I^\perp_{12}) \oplus (H^2(\mu^3) \cap I^\perp_3) \), 
\( S_f^{12} = S_f^1 \oplus S_f^3 \). \( (f \in A) \), \( (S_f^{12})^* k_j = \tau_j(f) k_j \), \( (j = 1, 2) \), and \( (S_f^{12})^* k_3 = \tau_3(f) k_3 \). Hence 
\[ \| S_f^{12} \| = \max(\| S_f^1 \|, \| S_f^3 \|) = \max(\| f + I_{12} \|, |\tau_3(f)|) = \sup_{\nu \in (A/I^*)^\perp, \| \nu \| \leq 1} \left| \int_X f \, d\nu \right| = \| f + I \|. \] 
Hence \( S'_{12} \) is an isometric representation of \( A/I \) where \( I = I_{12} \cap I_3 \). By the theorem of T. Nakazi (cf. [8]), \( \| f + I_{12} \| \) can be written using \( \rho_1 = \sup \{|\tau_1(f)| ; f \in \ker \tau_2, \| f \| \leq 1 \} \).

**Corollary 4.2.** Let \( A \) be a uniform algebra and \( I = \cap_{j=1}^n \ker \tau_j \) and \( \tau_i \not\sim \tau_j(i \neq j) \). Then there exists an isometric representation \( S'_{ij} \) of \( A/I \), and 
\[ \| f + I \| = \max(|\tau_1(f)|, ..., |\tau_n(f)|) \] 
Proof. Since \( \tau_i \not\sim \tau_j \) \( (i \neq j) \), there exist probability measures \( \mu^1, ... , \mu^n \) such that 
\[ \mu = (\mu^1 + ... + \mu^n)/n \], \( \mu^i \perp \mu^j \) \( (i \neq j) \), \( H^2(\mu) \cap I^\perp_3 = (H^2(\mu^1) \cap I^\perp_3) \oplus ... \oplus (H^2(\mu^n) \cap I^\perp_3) \), 
\( S_f^i = S_f^{mu} \oplus S_f^{nu} \). Since \( (S_f^{mu})^* k_j = \tau_j(f) k_j \), \( (S_f^{nu})^* \) is a rank 1 operator on \( H^2(\mu) \cap \ker \tau_j \) \( \perp \) \( \{ k_j \} \), it follows that 
\[ \| S_f^{12} \| = \| (S_f^{12})^* \| = |\tau_j(f)|. \] Then 
\[ \| S_f^{12} \| = \max(\| S_f^1 \|, ..., \| S_f^n \|) = \max(|\tau_1(f)|, ..., |\tau_n(f)|) = \sup_{\nu \in (A/I^*)^\perp, \| \nu \| \leq 1} \left| \int_X f \, d\nu \right| = \| f + I \|. \] 
This completes the proof.

**Corollary 4.3.** Let \( A \) be a uniform algebra and \( I = \cap_{j=1}^n \ker \tau_j \) and \( \tau_i \not\sim \tau_j(i \neq j) \). Suppose that \( S'_{ij} \) is an isometric representation of \( A/I \). Then,
1. \( \mu = \sum_{j=1}^n \mu^j \), \( \mu^i \perp \mu^j \) \( (i \neq j) \), \( \mu^j \ll m^j \) where \( \mu^j \) is a positive measure and \( m^j \) is some representing measure for \( \tau_j \).
2. \( S_f^i = S_f^{mu} \oplus S_f^{nu} \) \( (f \in A) \) where \( \mu^j \) is divided by its total variation and \( S_f^{nu} \) is an isometric representation of \( A/I_j \), where \( I_j = \ker \tau_j \).
3. \( S_f^i \) is an isometric representation of a diagonal \( n \times n \) matrix for any \( f \) in \( A \).

Proof. By the proof of (2) of Theorem 3.2 and Theorem 4.1, (1), (2) and (3) holds.

If \( A/I \) is 2-dimensional and semisimple, then there exist \( \tau_1, \tau_2 \) in \( M(A) \) such that \( \tau_1 \neq \tau_2 \) and \( I = \ker \tau_1 \cap \ker \tau_2 \). For \( j = 1, 2 \), there exists \( f_j \in A \) such that \( \tau_j(f_j) = \delta_{ij} \). Then \( f_j + I \) is idempotent in \( A/I \) and \( A/I = \mathrm{span}\{f_1 + I, f_2 + I\} \). If \( n = 2 \), then 
\[ \rho_1 = \sup \{|\tau_1(f)| ; f \in \ker \tau_2, \| f \| \leq 1 \}, \]
\[ \rho_1(\mu) = \sup \{ |\tau_1(f)| ; f \in \ker \tau_2, \|f\|_\mu \leq 1 \} \]

where \( \|f\| \) denotes the supnorm of \( f \) in \( A \) and \( \|f\|_\mu = \langle f, f \rangle_\mu = \langle f | f \rangle_\mu d\mu \|^{1/2} \). Then \( \rho_1 \) is a Gleason distance between \( \tau_1 \) and \( \tau_2 \), and \( \|f_1 + I\| = 1/\rho_1 \), \( \|f_1 + I\|_\mu = 1/\rho_1(\mu) \). The following proposition is essentially known (cf. Lemma 3 of [9]).

**Proposition 4.4.** If \( A/I \) is 2-dimensional and semisimple, then the following conditions are equivalent.

1. \( S^\mu \) is an isometric representation of \( A/I \).
2. \( \|k_1\|_\mu = \rho_1(\mu)/\rho_1 \).
3. \( \|k_1\|_\mu = \|f_1 + I\|/\|f_1 + I\|_\mu \).

**Proof.** By Theorem 3.2, (1) implies (3). By the above remark, (2) is equivalent to (3). It is sufficient to show that (3) implies (1). By Theorem 3.2, if (3) holds, then \( \|S^\mu_{f_1}\| = \|f_1 + I\| \).

By the above remark, this implies \( \|S^\mu_{f_1}\| = 1/\rho_1 \). By the theorem of T. Nakazi (cf. [8]), if \( I = \{ f \in A ; \tau_1(f) = \tau_2(f) = 0 \} \), then

\[
\|f + I\| = \sqrt{\frac{\tau_1(f) - \tau_2(f)}{2} \left( \frac{1}{\rho_1^2} - 1 \right) + \left( \frac{1}{\rho_1^2} - 1 \right)} \frac{\tau_1(f) - \tau_2(f)}{2} \left( \frac{1}{\rho_1^2} - 1 \right) + \left( \frac{1}{\rho_1^2} - 1 \right)}.
\]

Since \( \|S^\mu_{f_1}\| = 1/\rho_1 \), it follows from the theorem of I. Feldman, N. Krupnik and A. Markus (cf. [5]) that

\[
\|f + I\| = \|\tau_1(f)S^\mu_{f_1} + \tau_2(f)S^\mu_{f_2}\| = \|S^\mu\|.
\]

This completes the proof.

T. Nakazi and K. Takahashi [9] proved that there exists an isometric representation of \( A/I \) in the case when \( \dim A/I = 2 \). The following theorem gives a concrete matrix representation of \( A/I \).

**Theorem 4.5.** Suppose \( A/I \) is 3-dimensional and semisimple. If \( \tau_1 \sim \tau_2 \not\sim \tau_3 \) and \( S^\mu \) is an isometric representation of \( A/I \), then \( A/I \) is isometric to \( \{ S^\mu_f ; f \in A \} = \text{span}\{ S^\mu_{f_1}, S^\mu_{f_2}, S^\mu_{f_3} \} \), \( S^\mu_f = \tau_1(f)S^\mu_{f_1} + \tau_2(f)S^\mu_{f_2} + \tau_3(f)S^\mu_{f_3} \), and

\[
(S^\mu_{f_1})^* = \begin{pmatrix} 1 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S^\mu_{f_2})^* = \begin{pmatrix} 0 & -x & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S^\mu_{f_3})^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where

\[
x = \frac{-\langle k_2, k_1 \rangle_\mu}{\sqrt{\|k_1\|_\mu^2 \|k_2\|_\mu^2 - \|\langle k_1, k_2 \rangle_\mu \|^2}}.
\]

11
Proof. This follows from Lemma 2.1 and Theorem 4.1.

If \( B \subset B(H) \) and \( \dim H = 3 \), then

\[
P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},
\]

It follows from a 2-dimensional case that if \( y = z = 0 \), then \( B \) is a \( Q \)-algebra.

If the following condition (1) implies (2) for any distinct points \( \tau_1, \ldots, \tau_n \in M(A) \) and complex numbers \( w_1, \ldots, w_n \), then we say that \( A/I \) satisfies the Pick property.

1. \([1 - w_i \overline{w_j}]_{k_{ij}=1}^{n} \geq 0 \), where \( k_{ij} = \langle k_i, k_j \rangle_{\mu} \), and \( \tau_j(f) = \langle f, k_j \rangle_{\mu} \), \( f \in A \).
2. There exists \( f \in A \) such that \( \tau_j(f) = w_j \), \( (1 \leq j \leq n) \) and \( ||f + I|| \leq 1 \).

The following proposition is essentially known.

**Proposition 4.6.** Let \( A/I \) be an \( n \)-dimensional semisimple commutative Banach algebra. Then \( S^\mu : A/I \to B(H^2(\mu) \cap I^\perp) \) is isometric if and only if \( A/I \) satisfies the Pick property.

**Proof.** Suppose \( S^\mu \) is isometric. For any \( w_1, \ldots, w_n \in C \), there exists an \( f \in A \) such that \( \tau_j(f) = w_j \), \( (1 \leq j \leq n) \). Suppose \([1 - w_i \overline{w_j}]_{k_{ij}=1}^{n} \geq 0 \). For any complex numbers \( \alpha_1, \ldots, \alpha_n \), let \( k = \sum_{j=1}^{n} \alpha_j k_j \). Then \( ||k||^2_{\mu} = \sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j k_{ji} \). Since \( (S^\alpha_f)^{*} k_j = \tau_j(f) k_j \), \( (S^\mu_f)^{*} k = \sum_{j=1}^{n} \alpha_j \tau_j(f) k_j \). By (1),

\[
||k||^2_{\mu} - ||(S^\alpha_f)^{*} k||^2_{\mu} = \sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j (1 - w_i \overline{w_j}) k_{ji} \geq 0.
\]

Since \( H^2(\mu) \cap I^\perp \) is spanned by \( k_1, \ldots, k_n \), this implies that \( ||(S^\alpha_f)^{*}|| \leq 1 \). Since \( S^\mu \) is isometric, \( ||f + I|| = ||S^\alpha_f|| \leq 1 \). Therefore \( A/I \) satisfies the Pick property. Conversely, suppose \( A/I \) satisfies the Pick property and \( ||S^\mu_f|| = 1 \). Since \( (S^\alpha_f)^{*} k_j = \tau_j(f) k_j \) and \( ||(S^\alpha_f)^{*}|| = 1 \), it follows that

\[
\sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j (1 - \tau_i(f) \overline{\tau_j(f)}) k_{ji} = ||k||^2_{\mu} - ||(S^\alpha_f)^{*} k||^2_{\mu} \geq 0,
\]

and hence \([1 - \tau_i(f) \overline{\tau_j(f)}) k_{ji}]_{k_{ij}=1}^{n} \geq 0 \). By the Pick property, there exists \( g \in A \) such that \( ||g + I|| \leq 1 \) and \( \tau_j(g) = \tau_j(f) \), \( (1 \leq j \leq n) \). Therefore \( ||f + I|| = ||g + I|| \leq 1 = ||S^\mu_f|| \). Since the reverse inequality \( ||S^\mu_f|| \leq ||f + I|| \) is always holds, \( ||S^\mu_f|| = ||f + I|| \). This completes the proof.
In this section, we assume that $A$ is the disc algebra and $\dim A/I = 3$. For $f \in A$, let $\|f + I\| = \|f + I\|_{A/I}$. Since $M(A) = D = \{|z| \leq 1\}$, for each $1 \leq j \leq 3$, $\tau_j$ is just an evaluation functional at a point of $D$ and so we write that $\tau_1 = a$, $\tau_2 = b$ and $\tau_3 = c$, where $a$, $b$ and $c$ are in $D$. By Theorem 3.2, we may assume that $a$, $b$ and $c$ are in $D = \{|z| < 1\}$. Theorem 5.2 shows that the set of all 3-dimensional semisimple $Q$-algebras of the disc algebra is a proper subset in the set of all 3-dimensional semisimple commutative operator algebras with unit on a Hilbert space of dimension 3. However, Theorem 5.2 has not solved Problem 2 yet. We use Lemma 5.1 to prove Theorem 5.2. Let $a, b, c$ be the distinct points in the open unit disc $D$. Let $T(a, b, c)$ denote the subset of $C^3$ which consists of all $(x, y, z) \in C^3$ satisfying

$$1 + |x|^2 = \frac{|1 - \overline{ba}|}{a - b}^2, \quad 1 + |y|^2 = \frac{|1 - \overline{cb}|}{b - c}^2,$$

$$1 + |z|^2 = \frac{|1 - \overline{ac}|}{c - a}^2.$$

This implies that $x \neq 0, y \neq 0$, and $z \neq 0$. $T(a, b, c)$ is characterized by saying that the absolute values of $x, y, z$ are fixed and that their argument are arbitrary. In the following, we consider some inequalities of $x, y,$ and $z$. For $j = 1, 2, 3$, there exists $f_j \in A$ such that $\tau_j(f_j) = \delta_{ij}$. Hence, $f_1(a) = f_2(b) = f_3(c) = 1$, and $f_1(b) = f_1(c) = f_2(a) = f_2(c) = f_3(a) = f_3(b) = 0$.

**Lemma 5.1.** Let $a, b, c$ be the distinct points in $D$. Let $f \in A$. Let $I = \{g \in A : g(a) = g(b) = g(c) = 0\}$. Let $d\mu = \frac{d\theta}{2\pi}$.

1. $S_f = f(a)S_{f_1} + f(b)S_{f_2} + f(c)S_{f_3}$, and

$$\left(S_{f_1}^\mu\right)^* = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \left(S_{f_2}^\mu\right)^* = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad \left(S_{f_3}^\mu\right)^* = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(x, y, z) \in T(a, b, c)$.

2. $\|f + I\| = \|S_f^\mu\|$, $(f \in A)$. That is, $A/I$ is isometrically isomorphic to the 3-dimensional semisimple commutative operator algebra on $H^2(\mu) \cap I^\perp$ which is spanned by

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(x, y, z) \in T(a, b, c)$.

**Proof.** $H^2(\mu) \cap I^\perp$ is a 3-dimensional Hilbert space which is spanned by

$$k_1(z) = \frac{1}{1 - \overline{az}}, \quad k_2(z) = \frac{1}{1 - \overline{bz}}, \quad k_3(z) = \frac{1}{1 - \overline{cz}}.$$
For orthonormal basis $\psi_1, \psi_2, \psi_3$ defined in Proposition 2.3,

$$
\psi_1(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}, \quad \psi_2(z) = \frac{z-a \sqrt{1-|b|^2}}{1-\bar{a}z}, \quad \psi_3(z) = \frac{z-a - b \sqrt{1-|c|^2}}{1-\bar{a}z-1-\bar{b}z}.
$$

where

$$
\gamma_2 = -\left(\frac{a-b}{1-\bar{a}b}\right)^{-1}\left|\frac{a-b}{1-\bar{a}b}\right|, \quad \gamma_3 = \left(\frac{a-c}{1-\bar{a}c}\right)^{-1}\left|\frac{a-c}{1-\bar{a}c}\right|, \quad \gamma_3 = \left(\frac{a-c}{1-\bar{a}c}\right)^{-1}\left|\frac{b-c}{1-\bar{b}c}\right|.
$$

Since

$$
k_2 - (k_2, \psi_1)\psi_1 = \frac{(\bar{b}-\bar{a})(z-a)}{(1-\bar{b}a)(1-\bar{a}z)(1-\bar{b}z)},
$$

it follows that

$$
||k_2 - (k_2, \psi_1)\psi_1|| = \left|\frac{\bar{b}-\bar{a}}{1-\bar{b}a}\right| \frac{1}{\sqrt{1-|b|^2}}.
$$

Hence

$$
\psi_2 = \frac{k_2 - (k_2, \psi_1)\psi_1}{||k_2 - (k_2, \psi_1)\psi_1||} = \frac{z-a \sqrt{1-|b|^2}}{1-\bar{a}z} \frac{1}{1-\bar{b}z}.
$$

Since

$$
k_3 - (k_3, \psi_1)\psi_1 - (k_3, \psi_2)\psi_2 = \frac{(\bar{a}-\bar{c})(\bar{b}-\bar{c})(z-a)(z-b)}{(1-\bar{c}a)(1-\bar{c}b)(1-\bar{a}z)(1-\bar{b}z)(1-\bar{c}z)},
$$

it follows that

$$
\psi_3 = \frac{k_3 - (k_3, \psi_1)\psi_1 - (k_3, \psi_2)\psi_2}{||k_3 - (k_3, \psi_1)\psi_1 - (k_3, \psi_2)\psi_2||} = \frac{z-a}{1-\bar{a}z} \frac{z-b \sqrt{1-|c|^2}}{1-\bar{b}z} \frac{1}{1-\bar{c}z}.
$$

If we calculate $x, y, z$ using the formulas in Proposition 2.3, then it follows that $(x, y, z) \in T(a, b, c)$. Then

$$
x = \frac{-\langle k_2, k_1 \rangle}{\sqrt{||k_1||^2||k_2||^2 - ||\langle k_1, k_2 \rangle||^2}} = \frac{-\frac{1}{1-\bar{b}a}}{\sqrt{\frac{1}{(1-|a|^2)(1-|b|^2)} - \frac{1}{1-\bar{a}b}^2}} = \gamma_4 \frac{\sqrt{1-|a|^2} \sqrt{1-|b|^2}}{|a-b|},
$$

where

$$
\gamma_4 = -\frac{1-\bar{a}b}{|1-\bar{a}b|}.
$$

Hence

$$
1 + |x|^2 = \left|\frac{1-\bar{b}a}{a-b}\right|^2.
$$

Since

$$
-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x = \frac{\sqrt{1-|a|^2}}{1-\bar{c}a} \frac{1-\bar{a}b}{b-\bar{a}} \frac{\bar{c}-\bar{b}}{1-\bar{b}c},
$$

it follows that

$$
y = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{||k_3 - (k_3, \psi_1)\psi_1 - (k_3, \psi_2)\psi_2||} = \gamma_6 \frac{1-\bar{a}b \sqrt{1-|a|^2} \sqrt{1-|c|^2}}{\frac{a-b}{|a-c|}},
$$

14
where
\[ \gamma_6 = \left( \frac{a-b}{1-\overline{a}b} \right) \left( \frac{a-b}{1-\overline{a}b} \right)^{-1} \left( \frac{b-c}{1-\overline{b}c} \right)^{-1} \left( \frac{b-c}{1-\overline{b}c} \right) \left( \frac{1-\overline{c}a}{1-\overline{c}a} \right). \]

Since
\[ \langle k_3, \psi_2 \rangle = \frac{\overline{c}-\overline{a}}{\gamma_2} \sqrt{1-|b|^2} \frac{1}{1-\overline{a}c}, \]
it follows that
\[ z = \frac{-\langle k_3, \psi_2 \rangle}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|} = \frac{\gamma_6 \sqrt{1-|b|^2} \sqrt{1-|c|^2}}{|b-c|}. \]
where
\[ \gamma_6 = \left( \frac{a-b}{1-\overline{a}b} \right) \left( \frac{a-b}{1-\overline{a}b} \right)^{-1} \left( \frac{c-a}{1-\overline{a}c} \right)^{-1} \left( \frac{c-a}{1-\overline{a}c} \right) \left( \frac{1-\overline{b}c}{1-\overline{b}c} \right) \left( \frac{1-\overline{c}a}{1-\overline{c}a} \right). \]

Since \[ |\gamma_2| = |\gamma_3| = |\gamma_4| = |\gamma_5| = |\gamma_6| = 1, \]
it follows that
\[ 1 + |x|^2 \left( \frac{a-b}{1-\overline{a}b} \right)^2 = \left( \frac{1-\overline{a}c}{c-a} \right)^2, \quad 1 + |z|^2 = \left( \frac{1-\overline{b}c}{b-c} \right)^2. \]

Hence, (1) follows. It is sufficient to prove (2). By the theorem of D. Sarason (cf. [2], p.125, [10], Vol.1, p.231, [11]), \[ |f + I| = |S_{\mu}^f| \]. Then \( (S_{\mu}^f)^*k_1 = k_1, (S_{\mu}^f)^*k_2 = (S_{\mu}^f)^*k_3 = 0, (S_{\mu}^f)^*k_2 = k_2, (S_{\mu}^f)^*k_3 = (S_{\mu}^f)^*k_1 = 0, (S_{\mu}^f)^*k_3 = k_3, \) and \( (S_{\mu}^f)^*k_1 = (S_{\mu}^f)^*k_2 = 0 \). By Proposition 2.3,
\[ (S_{\mu}^f)^* = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{\mu}^f)^* = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{\mu}^f)^* = \begin{pmatrix} 0 & 0 & xz-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix}. \]

Since \( f - f(a)f_1 - f(b)f_2 - f(c)f_3 \in I \) and \( I(H^2(\mu) \cap I^1) \subset IH^2(\mu) \subset H^2(\mu) \cap I^1 \), it follows that
\[ (S_{\mu}^f - S_{\mu}^{f(a)f_1 + f(b)f_2 + f(c)f_3})\psi = S_{\mu}^{f-f(a)f_1 - f(b)f_2 - f(c)f_3} \psi = 0, \quad (\psi \in I^1_\mu). \]
Hence
\[ S_{\mu}^f = S_{\mu}^{f(a)f_1 + f(b)f_2 + f(c)f_3} = f(a)S_{\mu}^{f_1} + f(b)S_{\mu}^{f_2} + f(c)S_{\mu}^{f_3}. \]
This completes the proof.

For example, if \( (a,b,c) = (0, \frac{1}{2}, \frac{1}{3}) \) and \( (x,y,z) = (-\sqrt{3}, 4\sqrt{2}, -2\sqrt{6}) \), then the algebra \( \text{span}\{P_1, P_2, P_3\} \) is isometrically isomorphic to \( A/I \) which is a \( Q \)-algebra of a disc algebra.

**Theorem 5.2.** Let \( a, b, c \) be the distinct points in \( \mathbb{D} \). Let \( f \in A \). Let \( d\mu = \frac{du}{2\pi} \). Let \( I = \{ g \in A \mid g(a) = g(b) = g(c) = 0 \} \). If a 3-dimensional semisimple commutative operator algebra \( B \) on \( H^2(\mu) \cap I^1 \) is isometrically isomorphic to \( A/I \), then \( B \) is unitarily equivalent to the 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space \( H \) spanned by \( P_1, P_2, P_3 \) such that
\[ P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix}. \]
where \(x, y, z\) satisfy (1) \(\sim\) (3).

(1) \(\text{xyz} \neq 0,\)

(2) \[
\frac{1}{\sqrt{1 + |y|^2}} < \frac{1}{\sqrt{1 + |x|^2}} + \frac{1}{\sqrt{1 + |z|^2}},
\]

(3) \[
|y| > \frac{|xz|}{\sqrt{1 + |z|^2} + 1},
\]

**Proof.** By the theorem of B.Cole and J.Wermer (cf. [3]) and (2) of Theorem 2.4, we may assume that \(H\) is spanned by the orthonormal basis \(\psi_1, \psi_2, \psi_3\) which are calculated in the proof of Lemma 5.1. By Lemma 5.1, there are complex numbers \(x, y, z\) satisfying \((x, y, z) \in T(a, b, c)\). Since

\[
1 + |x|^2 = \left| \frac{1 - ba}{a - b} \right|^2 > 1, \quad 1 + |z|^2 = \left| \frac{1 - cb}{b - c} \right|^2 > 1,
\]

\(1 + |y|^2 = \left| \frac{a - b}{1 - ba} \right|^2 = \left| \frac{1 - ac}{c - a} \right|^2 > 1,
\]

(1) follows. Let

\[
\rho(z, w) = \frac{|z - w|}{|1 - \bar{w}z|}.
\]

Then

\[
\rho(a, b) = \frac{1}{\sqrt{1 + |x|^2}}, \quad \rho(b, c) = \frac{1}{\sqrt{1 + |z|^2}}, \quad \rho(c, a) = \sqrt{\frac{1 + |x|^2}{1 + |x|^2 + |y|^2}} > \frac{1}{\sqrt{1 + |y|^2}}.
\]

Since \(\rho(c, a) \leq \rho(a, b) + \rho(b, c)\), (2) follows. Let

\[
d(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.
\]

Since \(d(c, a) \leq d(a, b) + d(b, c)\),

\[
\frac{\sqrt{1 + |x|^2 + |y|^2} + \sqrt{1 + |x|^2}}{\sqrt{1 + |x|^2 + |y|^2} - \sqrt{1 + |x|^2}} \leq \sqrt{\frac{1 + |x|^2 + 1}{1 + |x|^2}} \cdot \sqrt{\frac{1 + |x|^2 + 1}{1 + |x|^2 - 1}}.
\]

Hence

\[
\frac{\sqrt{1 + |x|^2} + 1}{|y|} < \frac{\sqrt{1 + |x|^2} + |y|^2 + \sqrt{1 + |x|^2}}{|y|} \leq \frac{\sqrt{1 + |x|^2} + 1}{|z|} \cdot \frac{\sqrt{1 + |x|^2} + 1}{|x|}.
\]

This implies (3). This completes the proof.
Example 5.3. In Example 2.5, $\mathcal{B}_0$ is isometrically isomorphic to $\mathcal{B}_2$. Since $y_0 = 0$, it follows from Theorem 5.2 that $\mathcal{B}_2$ is not isometrically isomorphic to a 3-dimensional semisimple $Q$-algebra $A/I$ where $A$ is a disc algebra. Hence $\mathcal{B}_0$ is also not isometrically isomorphic to a $Q$-algebra $A/I$. Therefore $\mathcal{B}_0$ and $\mathcal{B}_2$ is the example to show that the set of all 3-dimensional semisimple $Q$-algebra $A/I$ where $A$ is a disc algebra is smaller than the set of all 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space.

References


