1. INTRODUCTION

Let \( T \in (0, \infty) \). We consider the stock price process \((S_t)_{0 \leq t \leq T}\) that is governed by the stochastic differential equation

\[
dS_t = S_t(bdt + \sigma dY_t), \quad 0 \leq t \leq T, \tag{1.1}
\]

where \( \sigma \) and the initial value \( S_0 \) are positive constants, and \( b \in \mathbb{R} \). In the classical Black-Scholes model, Brownian motion is used as the driving noise process \( Y \), and the resulting price process \( S \) becomes Markovian. In Anh and Inoue (2005), Anh et al. (2005) and Inoue et al. (2006), the Gaussian process \( Y_t = B_t - \int_0^t \{ \frac{1}{\sqrt{2 \pi \sigma^2 s}} e^{-\frac{(y-p)^2}{2 \sigma^2 s}} dB_u \} ds, \quad 0 \leq t \leq T \), which has stationary increments, is used instead as the driving noise process \( Y \) in (1.1), where \( p \) and \( q \) are real constants such that \( 0 < q < \infty \), \( -q < p < \infty \), and \((B_t)_{t \in \mathbb{R}}\) is a one-dimensional Brownian motion defined on a probability space \((\Omega, \mathcal{F}, P)\) satisfying \( B_0 = 0 \). The parameters \( p \) and \( q \) describe the memory of \( Y \), and the resulting stock price process \( S \) becomes non-Markovian.

We write \((\mathcal{F}_t)_{0 \leq t \leq T}\) for the \( P \)-augmentation of the filtration generated by the process \((Y_t)_{0 \leq t \leq T}\). The theory of innovation processes as described in Liptser and Shiryaev (2001) tells us that \( Y \) is an \((\mathcal{F}_t)\)-semimartingale (cf. Anh and Inoue, 2005, Theorem 3.1) though the above representation itself is not a semimartingale representation of \( Y \) since \((B_t)\) is not \((\mathcal{F}_t)\)-adapted. In fact, using the prediction theory for \( Y \) which is developed in Anh et al. (2005), the following explicit semimartingale representation of \( Y \) is obtained (Inoue et al., 2006, Theorem 2.1):

\[
Y_t = W_t - \int_0^t \left\{ \int_0^s l(s, u) dW_u \right\} ds, \quad 0 \leq t \in T, \tag{1.2}
\]

where \((W_t)_{0 \leq t \leq T}\) is a one-dimensional Brownian motion, called the innovation process, satisfying \( \sigma(W_s : 0 \leq s \leq t) = \sigma(Y_s : 0 \leq s \leq t) \) for \( 0 \leq t \leq T \) and \( l(t, s) \) is a bounded Volterra kernel given explicitly by

\[
l(t, s) = pe^{-(p+q)(t-s)} \left\{ 1 - \frac{2pq}{(2q + p)^2 e^{2qs} - p^2} \right\}, \quad 0 \leq s \leq t \leq T. \tag{1.3}
\]
Thus the process $Y$ has the virtue that it simultaneously possesses the property of a stationary increments process and the simple semimartingale representation (1.2) with (1.3). The market model driven by $Y$ is arbitrage-free and complete since the process $Y$ becomes a Brownian motion under a suitable probability measure (see Anh and Inoue, 2005, Section 3).

As is well known, binary approximation of the Black-Scholes model plays a very important role for the model in many ways. Sottinen (2001) constructed a binary market model that approximates the market model driven by fractional Brownian motion, and investigated the arbitrage opportunities in the binary model. In this paper, we construct a binary market model with memory that approximates the continuous-time market model driven by $Y$ in (1.2). However, rather than considering the special kernel $l(t, s)$ in (1.3), we take a general bounded measurable Volterra kernel $l(t, s)$. We remark that any centered Gaussian process $Y = (Y_t)_{0 \leq t \leq T}$ that is equivalent to a Brownian motion has a canonical representation of the form (1.2) with $l(t, s)$ satisfying square integrability (see Hida and Hitsuda, 1991, Chapter VI). Thus, in this paper, we consider a subclass consisting of $Y$ for which $l(t, s)$ is bounded. As in Sottinen (2001), the key feature to the construction of the approximating binary market is to prove a Donsker-type theorem for the process $Y$ (Theorem 2.1).

As stated above, the market driven by $Y$ in (1.2) with (1.3) is arbitrage-free unlike that driven by a fractional Brownian motion. However, the approximating binary market model may admit arbitrage opportunities. We consider conditions for their existence or non-existence. We also study the rate at which the arbitrage probability tends to zero.

2. A Donsker-type theorem

Let $T \in (0, \infty)$. In what follows, we write $C = C_T$ for positive constants, depending on $T$, which may not be necessarily equal to each other. Let $n \in \mathbb{N}$. In Sections 2 and 3, we write $\sum_{s \leq t} X_s = \sum_{i=1}^{[nt]} X_{(i/n)}$ and $\prod_{s \leq t} X_s = \prod_{i=1}^{[nt]} X_{(i/n)}$.

Let $l(t, s)$ be a bounded measurable function on $[0, T] \times [0, T]$ that vanishes whenever $s > t$. Let $W = (W_t)_{0 \leq t \leq T}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. We define the process $Y = (Y_t)_{0 \leq t \leq T}$ by (1.2).

We put, for $t, u \in [0, T]$,

$$z(t, u) := \int_u^t l(s, u) \, ds, \quad y(t, u) := 1 - z(t, u).$$

Then both $z(t, u)$ and $y(t, u)$ are bounded and continuous on $[0, T] \times [0, T]$, and it holds that $Y_t = \int_0^t y(t, u) \, dW_u$ for $0 \leq t \leq T$. Let $C$ be a positive constant satisfying, for $(t_1, u), (t_2, u) \in [0, T] \times [0, T]$,

$$|z(t_1, u) - z(t_2, u)| = |y(t_1, u) - y(t_2, u)| \leq C|t_1 - t_2|. \quad (2.1)$$

Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of i.i.d. random variables with $E[\xi_i] = 0$, $E[(\xi_i)^2] = 1$ and $E[(\xi_i)^4] < \infty$. We define the process $W^{(n)} = (W^{(n)}_t)_{0 \leq t \leq T}$ by

$$W^{(n)}_t := \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i, \quad 0 \leq t \leq T,$$

where $[x]$ denotes the greatest integer not exceeding $x$. The process $W^{(n)}$ converges weakly to $W$ in the Skorohod space by Donsker’s theorem (see, e.g., Billingsley,
We define the process $Y^{(n)} = (Y_t^{(n)})_{0 \leq t \leq T}$ by

$$Y_t^{(n)} := \int_0^t y\left(\frac{|nt|}{n}, \frac{t}{n}\right) dW_t^{(n)}, \quad 0 \leq t \leq T.$$ 

Then it follows that $Y_t^{(n)} = n^{-1/2} \sum_{i=1}^{|nt|} y\left(\frac{|nt|}{n}, i/n\right) \xi_i$ for $0 \leq t \leq T$.

Here is the Donsker-type theorem for $Y$.

**Theorem 2.1.** The process $Y^{(n)}$ converges weakly to $Y$ as $n \to \infty$.

**Proof.** We first show that the finite-dimensional distributions of $Y^{(n)}$ converge to those of $Y$ as $n \to \infty$. Thus, for $a_1, \ldots, a_d \in \mathbb{R}$ and $t_1, \ldots, t_d \in [0, T]$, we show that $X^{(n)}$ converges to a normal distribution with variance $\text{Var}(X)$, where $X^{(n)} := \sum_{k=1}^d a_k Y_{t_k}^{(n)}$ and $X := \sum_{k=1}^d a_k Y_{t_k}$. We have

$$\text{Var}(X^{(n)}) = \sum_{k,l=1}^d a_k a_l \frac{1}{n} \sum_{i=1}^{|nt_k \wedge t_l|} y\left(\frac{|nt_k|}{n}, i/n\right) y\left(\frac{|nt_l|}{n}, i/n\right)$$

$$= \sum_{k,l=1}^d a_k a_l \int_0^{|nt_k \wedge t_l|} y\left(\frac{|nt_k|}{n}, \frac{|nt_l|+1}{n}\right) y\left(\frac{|nt_k|}{n}, \frac{|nt_l|}{n}\right) ds,$$

where $t \wedge s := \min(t, s)$. The function $(t_1, t_2, u) \mapsto y(t_1, u) y(t_2, u)$ is continuous, whence uniformly continuous, on the compact set $[0, T]^3$. From this and the fact that $0 \leq t - (|nt|/n) < 1/n$, we see that

$$\lim_{n \to \infty} \text{Var}(X^{(n)}) = \sum_{k,l=1}^d a_k a_l \int_0^{t_k \wedge t_l} y(t_k, s) y(t_l, s) ds = \text{Var}(X).$$ (2.2)

We may assume $\text{Var}(X) > 0$. For, otherwise, (2.2) implies that $X^{(n)}$ converges to $X = 0$ in law. We put $b_i^{(n)} := \sum_{k=1}^d a_k y\left(\frac{|nt_k|}{n}, i/n\right)$ and $X_i^{(n)} := n^{-1/2} b_i^{(n)} \xi_i$ for $n, i = 1, 2, \ldots$. Then we have $X^{(n)} = \sum_{i=1}^{|nt|} X_i^{(n)}$ for $n = 1, 2, \ldots$. We need to show the following Lindeberg’s condition: for every $\epsilon > 0$,

$$\lim_{n \to \infty} \sum_{i=1}^{|nt|} E \left[ (X_i^{(n)})^2 1_{\{|X_i^{(n)}| > \epsilon \sigma^{(n)}\}} \right] = 0,$$ (2.3)

where $\sigma^{(n)} := \sqrt{\text{Var}(X^{(n)})}$. Choose a positive constant $M$ satisfying $|b_i^{(n)}| \leq M$ for $n, i = 1, 2, \ldots$. Then since $|X_i^{(n)}| \leq M n^{-1/2} |\xi|$, we have

$$\sum_{i=1}^{|nt|} E \left[ (X_i^{(n)})^2 1_{\{|X_i^{(n)}| > \epsilon \sigma^{(n)}\}} \right] \leq \sum_{i=1}^{|nt|} E \left[ (M n^{-1/2} \xi)^2 1_{\{|M n^{-1/2} \xi| > \epsilon \sigma^{(n)}\}} \right]$$

$$= \sum_{i=1}^{|nt|} M^2 n^{-1} E \left[ \xi^2 1_{\{|\xi| \geq M^{-1} \sigma^{(n)} \sqrt{n}\}} \right] \leq M^2 TE \left[ (\xi)^2 1_{\{|\xi| \geq M^{-1} \sigma^{(n)} \sqrt{n}\}} \right].$$

We obtain (2.3) from this. By (2.2) and (2.3), we can apply the central limit theorem (cf. Billingsley, 1968, Theorem 7.2), so that $X^{(n)}$ converges to $X$ in law, as desired.

Next we show that, for $0 \leq t_1 \leq t \leq t_2 \leq T$ and $n = 1, 2, \ldots$,

$$E \left[ |Y_t^{(n)} - Y_{t_1}^{(n)}|^2 |Y_{t_2}^{(n)} - Y_{t_1}^{(n)}|^2 \right] \leq C |t_2 - t_1|^2.$$ (2.4)
The theorem follows from this and Theorem 15.6 of Billingsley (1968). However, if \( t_2 - t_1 < 1/n \), then either \( t_1 \) and \( t \) or \( t \) and \( t_2 \) lie in the same subinterval \([m/n, m+1/n)\) for some \( m \), whence the left hand side of (2.4) is zero. Therefore we may assume that \( t_2 - t_1 \geq 1/n \).

We show that

\[
E \left[ |Y_t^{(n)} - Y_s^{(n)}|^4 \right] \leq C|t - s|^2
\]

for \( t, s \) and \( n \) satisfying

\[
0 \leq s < t \leq T, \quad t - s \geq 1/n.
\]

This implies (2.4) under the condition \( t_2 - t_1 \geq 1/n \) since

\[
E \left[ |Y_t^{(n)} - Y_{t_1}^{(n)}|^2 |Y_{t_2}^{(n)} - Y_t^{(n)}|^2 \right] \leq E \left[ |Y_t^{(n)} - Y_{t_1}^{(n)}|^4 \right]^{1/2} E \left[ |Y_{t_2}^{(n)} - Y_t^{(n)}|^4 \right]^{1/2}
\]

\[
\leq C|t - t_1||t_2 - t| \leq C|t_2 - t_1|^2.
\]

For distinct \( i, j, k \) and \( l \), \( E[(\xi_i)^3 \xi_j] = E[(\xi_i)^2 \xi_k \xi_l] = E[\xi_i \xi_j \xi_k \xi_l] = 0 \). Hence, \( E[|Y_t^{(n)} - Y_s^{(n)}|^4] = n^{-2} E[\sum_{i=1}^{nt}(y(\frac{nt}{n}, s) - y(\frac{nt}{n}, t))4] \) is

\[
E[(\xi_i)^4]n^{-2} \sum_{i=1}^{nt} (y(\frac{nt}{n}, s) - y(\frac{nt}{n}, t))^4 + \frac{6}{n^2}E[(\xi_i)^2]^2 \sum_{1 \leq i < j \leq nt} (y(\frac{nt}{n}, s) - y(\frac{nt}{n}, t))^2 (y(\frac{nt}{n}, s) - y(\frac{nt}{n}, t))^2
\]

\[
= (I_1 + I_2)E[(\xi_i)^4] + 6(J_1 + J_2 + J_3)E[(\xi_i)^2]^2
\]

for \( t, s \) and \( n \) satisfying (2.6), where

\[
I_1 := n^{-2} \sum_{i=1}^{nt} (y(\frac{nt}{n}, s) - y(\frac{nt}{n}, t))^4, \quad I_2 := n^{-2} \sum_{i=nt+1}^{nt} y(\frac{nt}{n}, t)^4
\]

and

\[
J_1 := n^{-2} \sum_{(i,j) \in \Lambda_1} (y(\frac{nt}{n}, s) - y(\frac{nt}{n}, t))^2 (y(\frac{nt}{n}, s) - y(\frac{nt}{n}, t))^2,
\]

\[
J_2 := n^{-2} \sum_{(i,j) \in \Lambda_2} (y(\frac{nt}{n}, s) - y(\frac{nt}{n}, t))^2 y(\frac{nt}{n}, t)^2,
\]

\[
J_3 := n^{-2} \sum_{(i,j) \in \Lambda_3} y(\frac{nt}{n}, s)^2 y(\frac{nt}{n}, t)^2
\]

with \( \Lambda_1 = \{(i,j) : 1 \leq i < j \leq \left[ nt \right] \} \), \( \Lambda_2 = \{(i,j) : 1 \leq i \leq \left[ nt \right], \left[ ns \right] < j \leq \left[ nt \right] \} \), and \( \Lambda_2 = \{(i,j) : \left[ ns \right] < i < j \leq \left[ nt \right] \} \). By (2.6), \( \left[ nt \right] - \left[ ns \right] \leq nt - ns + 1 = n(t - s + \frac{1}{n}) \leq 2n(t - s) \), so that \( \# \Lambda_1 \leq Cn^2 \), \( \# \Lambda_2 \leq Cn^2(t - s) \), and \( \# \Lambda_3 \leq Cn^2(t - s)^2 \). Therefore, using (2.1), we have \( |J_l| \leq C(t - s)^2 \) for \( l = 1, 2, 3 \) and \( t, s \) and \( n \) satisfying (2.6). Similarly, \( |I_l| \leq C(t - s)^2 \) for \( l = 1, 2, 3 \). Thus (2.5) follows.

Denote by \( \Delta X \) and \( [X] \) the jump and quadratic variation processes of a process \( X \), respectively, i.e., \( \Delta X_t := X_t - \lim_{s \uparrow t} X_s \), \( [X]_t := \sum_{s \leq t} (\Delta X_s)^2 \).

**Theorem 2.2.** The process \( \Delta Y^{(n)} \) converges to zero in probability, while \( [Y^{(n)}] \) converges to the deterministic process \((t)_{0 \leq t \leq T} \) in probability.
Proof. By (2.5) with (2.6), \( E[(Y_t^{(n)})^4] \leq E[(Y_t^{(n)})^4] \leq Cn^{-2} \), so that,

\[
E \left[ \sup_{0 \leq s \leq T} (\Delta Y_s^{(n)})^4 \right] \leq E \left[ \sum_{t \leq T} (\Delta Y_t^{(n)})^4 \right] = \sum_{t \leq T} E \left[ (\Delta Y_t^{(n)})^4 \right] \leq C \frac{nT}{n^2} \to 0
\]

as \( n \to \infty \). Thus \( Y_t^{(n)} \) converges to zero in probability.

We put \( Z_t^{(n)} := \int_0^t z(\frac{\lfloor nt \rfloor}{n}, s) dW_s^{(n)} \) for \( 0 \leq t \leq T \). Then we have \( Y_t^{(n)} = W_t^{(n)} - Z_t^{(n)} \), whence \( [Y_t^{(n)}]_t = [W_t^{(n)}]_t - 2 \sum_{s \leq t} (\Delta W_s^{(n)})(\Delta Z_s^{(n)}) + [Z_t^{(n)}]_t \).

Since \( z(n, u) = 0 \), we have

\[
Z_t^{(n)} = z^2 \frac{nT}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left\{ z(\frac{\lfloor nt \rfloor}{n}, \frac{i}{n}) - z(\frac{\lfloor nt \rfloor - 1}{n}, \frac{i}{n}) \right\} \xi_i \quad (= 0 \text{ if } \lfloor nt \rfloor = 1).
\]

From this and (2.1), \( E[(\Delta Z_t^{(n)})^2] \) is at most

\[
E \left[ (\Delta Z_t^{(n)})^2 \right] = \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} \left( z(\frac{\lfloor nt \rfloor}{n}, \frac{i}{n}) - z(\frac{\lfloor nt \rfloor - 1}{n}, \frac{i}{n}) \right)^2 \leq \frac{nT}{n} \frac{C^2}{n^2} = \frac{C}{n}.
\]

Since \( Z_t^{(n)} \) is increasing, we see that

\[
E \left[ \sum_{0 \leq t \leq T} \left| Z_t^{(n)} \right| \right] = \sum_{t \leq T} E \left[ (\Delta Z_t^{(n)})^2 \right] \leq nT \frac{C}{n^2} = \frac{C}{n}.
\]

Thus \( Z_t^{(n)} \) converges to zero in probability.

We have \( [W_t^{(n)}]_t - t = (\lfloor nt \rfloor / n) - t + (1/n) \sum_{i=1}^{\lfloor nT \rfloor} \{ (\xi_i)^2 - 1 \} \). Let \( \epsilon > 0 \). Then, by Kolmogorov’s inequality (see, e.g., Williams, 1991, Section 14.6),

\[
P \left( \sup_{0 \leq t \leq T} \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} \{ (\xi_i)^2 - 1 \} \geq \epsilon \right) = P \left( \sup_{0 \leq t \leq T} \left| \sum_{i=1}^{\lfloor nT \rfloor} \{ (\xi_i)^2 - 1 \} \right| \geq n\epsilon \right) \leq \frac{1}{\epsilon^2 n^2} \sum_{i=1}^{\lfloor nT \rfloor} E \left[ (\xi_i^2 - 1)^2 \right] \leq \frac{nT}{\epsilon^2 n^2} E \left[ (\xi_i^2 - 1)^2 \right] \to 0
\]

as \( n \to \infty \). From this and the fact that \( 0 \leq t - (\lfloor nt \rfloor / n) < 1/n \), we see that \( [W_t^{(n)}]_t \) converges to the deterministic process \( (t) \) in probability.

By Schwarz’s inequality, we have

\[
\left| \sum_{s \leq t} \left( \Delta W_s^{(n)} \right)(\Delta Z_s^{(n)}) \right| \leq [W_t^{(n)}]_{t=1/2}^{1/2} [Z_t^{(n)}]_{t=1/2}^{1/2} \leq [W_t^{(n)}]_{t=1/2}^{1/2} [Z_t^{(n)}]_{t=1/2}^{1/2},
\]

whence, by (2.7),

\[
E \left[ \sup_{0 \leq s \leq T} \left| \sum_{s \leq t} \left( \Delta W_s^{(n)} \right)(\Delta Z_s^{(n)}) \right| \right] \leq E \left[ [W_t^{(n)}]_{t=1/2}^{1/2} [Z_t^{(n)}]_{t=1/2}^{1/2} \right]
\]

\[
\leq E \left[ [W_t^{(n)}]_{t=1/2}^{1/2} \right] E \left[ [Z_t^{(n)}]_{t=1/2}^{1/2} \right] \leq T^{1/2} \cdot (Cn^{-1})^{1/2} = Cn^{-1/2}.
\]

Thus the process \( \left( \sum_{s \leq t} \left( \Delta W_s^{(n)} \right)(\Delta Z_s^{(n)}) \right) \) also converges to zero in probability. Combining, we see that \( [Y_t^{(n)}]_t \) converges to \( (t) \) in probability. \( \square \)
3. APPROXIMATING BINARY MARKET

Let $Y$ be as defined in Section 2. For $T, \sigma \in (0, \infty)$ and a deterministic continuous function $b(\cdot)$ on $[0, T]$, we consider the stock price process $S$ that is governed by the following more general stochastic differential equation than (1.1):

$$dS_t = S_t \{b(t) dt + \sigma dY_t\}, \quad 0 \leq t \leq T,$$

where the initial value $S_0$ is a positive constant. By (1.2) and the Itô formula, the solution $S$ is given by $S_t = S_0 \exp \left\{ \sigma Y_t + \int_0^t b(s) ds - \frac{1}{2} \sigma^2 t \right\}$. For $n = 1, 2, \ldots$, let $Y^{(n)}$ be as in Section 2. We consider the process $S^{(n)} = (S^{(n)}_t)_{0 \leq t \leq T}$ defined by

$$S^{(n)}_t := \prod_{s \leq t} \left\{ 1 + \sigma \Delta Y^{(n)}_{s} + \frac{1}{n} b\left(\left\lfloor \frac{[ns]}{n} \right\rfloor \right) \right\}, \quad 0 \leq t \leq T.$$

The aim of this section is to prove that $S^{(n)}$ converges weakly to the process $S$.

As in Eq. (10) and (11) of Sottinen (2001), we put

$$Y^{(1,n)}_t := \sum_{s \leq t} \Delta Y^{(n)}_{s} 1_{\{|\Delta Y^{(n)}_{s}| < \frac{1}{2} \sigma^{-1}\}}, \quad Y^{(2,n)}_t := \sum_{s \leq t} \Delta Y^{(n)}_{s} 1_{\{|\Delta Y^{(n)}_{s}| \geq \frac{1}{2} \sigma^{-1}\}}.$$

Then we have

$$Y^{(1,n)}_t = Y^{(1,n)}_1 + Y^{(2,n)}_t, \quad (3.1)$$
$$[Y^{(1,n)}]_t = \sum_{s \leq t} (\Delta Y^{(n)}_{s})^2 1_{\{|\Delta Y^{(n)}_{s}| < \frac{1}{2} \sigma^{-1}\}}, \quad (3.2)$$
$$[Y^{(2,n)}]_t = \sum_{s \leq t} (\Delta Y^{(n)}_{s})^2 1_{\{|\Delta Y^{(n)}_{s}| \geq \frac{1}{2} \sigma^{-1}\}}, \quad (3.3)$$
$$[Y^{(n)}]_t = [Y^{(1,n)}]_t + [Y^{(2,n)}]_t. \quad (3.4)$$

**Lemma 3.1.** The process $[Y^{(2,n)}]$ converges to zero in probability, whence $[Y^{(1,n)}]$ converges to the deterministic process $(t)$ in probability. The process $Y^{(2,n)}$ converges to zero in probability, whence $Y^{(1,n)}$ converges weakly to $Y$.

**Proof.** By Theorem 2.2, the process $\Delta Y^{(n)}$ converges to zero in probability. Since (3.2) implies that $P(\sup_{0 \leq s \leq T} |\Delta Y^{(n)}_t| > \varepsilon)$ is at most $P(\sup_{0 \leq s \leq T} |\Delta Y^{(2,n)}_t| > \varepsilon)$, $\Delta Y^{(n)}$ converges to zero in probability. Therefore, by Theorem 2.2 and (3.4), $[Y^{(1,n)}]$ converges to zero in probability.

In the same way, since $P(\sup_{0 \leq s \leq T} |Y^{(2,n)}_t|) \leq P(\sup_{0 \leq s \leq T} |\Delta Y^{(n)}_t|) \leq 2 \sigma^{-1}$, it follows from Theorem 2.2 that $Y^{(2,n)}$ converges to zero in probability. Therefore, by Theorem 2.1, (3.1) and Theorem 4.1 of Billingsley (1968), $Y^{(1,n)}$ converges weakly to $Y$. \ 

**Theorem 3.2.** The process $S^{(n)}$ converges weakly to $S$.

**Proof.** Write $S^{(n)}_t = S^{(1,n)}_t S^{(2,n)}_t$, where $S^{(2,n)}_t := \prod_{s \leq t} \{1 + \sigma \Delta Y^{(2,n)}_{s}\}, \ S^{(1,n)}_t := \prod_{s \leq t} \{1 + \sigma \Delta Y^{(1,n)}_{s} + (1/n) b\left(\left\lfloor \frac{[ns]}{n} \right\rfloor \right)\}$, and the processes $S^{(n)}$ are as above. We claim the following: (i) $S^{(1,n)}$ converges weakly to $S$; (ii) $S^{(2,n)}$ converges to one in probability.

The claim (ii) implies that $S^{(1,n)}(S^{(2,n)} - 1)$ converges to zero in probability (see Problem 1 in Billingsley, 1968, p. 28). Since $S^{(n)}_t = S^{(1,n)}_t (S^{(2,n)}_t - 1) + S^{(1,n)}_t$, we
see from (i) and Theorem 4.1 of Billingsley (1968) that $S^{(n)}$ converges weakly to $S$, as desired.  

For $\epsilon > 0$, we have $P(\sup_{0 \leq t \leq T} |S^{(2,n)}_t - 1| \geq \epsilon) \leq P(\sup_{0 \leq t \leq T} |\Delta Y^{(n)}_t| > \frac{1}{n}\sigma^{-1})$. Since the process $\Delta Y^{(n)}$ converges to zero in probability by Theorem 2.2, $S^{(2,n)}$ converges to one in probability. Thus (ii) follows.

We prove (i). Since the exponential is a continuous functional in the Skorohod topology, it is enough to prove that $\log S^{(1,n)}$ converges weakly to the process $(\sigma Y_t + \int_0^t b(s) ds - \frac{1}{2}\sigma^2 t)$. Notice that $|\sigma \Delta Y^{(1,n)}_t| + \frac{1}{n}b(|n|/n)| < \frac{1}{4}$ for sufficiently large $n$ and $t \in [0, T]$, whence the logarithm $\log S^{(1,n)}$ is well defined for such $n$.

We have $\log(1 + x) = x - \frac{1}{2}x^2 + r(x)x^3$ for $|x| < 1$, where $r(x)$ is a bounded function on $|x| \leq \frac{1}{4}$. Hence

$$\log S^{(1,n)}_t = \sum_{s \leq t} \left\{ \sigma \Delta Y^{(1,n)}_s + \frac{1}{n}b \left( \frac{|ns|}{n} \right) - \frac{1}{2} \left( \sigma \Delta Y^{(1,n)}_s + \frac{1}{n}b \left( \frac{|ns|}{n} \right) \right)^2 + r \left( \sigma \Delta Y^{(1,n)}_s + \frac{1}{n}b \left( \frac{|ns|}{n} \right) \right) \cdot \left( \sigma \Delta Y^{(1,n)}_s + \frac{1}{n}b \left( \frac{|ns|}{n} \right) \right)^3 \right\}$$

$$= \sigma Y^{(1,n)}_t + \sum_{s \leq t} \frac{1}{n}b \left( \frac{|ns|}{n} \right) - \frac{1}{2} \Phi^{(n)}_t + \Psi^{(n)}_t,$$

where $\Phi^{(n)}_t := \sum_{s \leq t} \left\{ (1/n)b(\lfloor ns/n \rfloor/n) + \sigma \Delta Y^{(1,n)}_s \right\}^2$ and

$$\Psi^{(n)}_t := \sum_{s \leq t} r \left( \sigma \Delta Y^{(1,n)}_s + \frac{1}{n}b \left( \frac{|ns|}{n} \right) \right) \cdot \left( \sigma \Delta Y^{(1,n)}_s + \frac{1}{n}b \left( \frac{|ns|}{n} \right) \right)^3.$$

We put $\Gamma^{(n)}_t := \sum_{s \leq t} \frac{1}{n}b \left( \frac{|ns|}{n} \right)\Delta Y^{(1,n)}_s$. Then

$$\Phi^{(n)}_t = n^{-2} \sum_{s \leq t} b \left( \frac{|ns|}{n} \right)^2 + 2\sigma \Gamma^{(N)}_t + \sigma^2 [Y^{(1,n)}]_t.$$

Since $b(\cdot)$ is bounded, the first term $n^{-2} \sum_{s \leq t} b \left( \frac{|ns|}{n} \right)^2$ goes to 0 as $n \to \infty$. By Lemma 3.1, the third term $\sigma^2 [Y^{(1,n)}]$ converges to $(\sigma^2 t)$ in probability. As for the second term, it holds that

$$\sup_{0 \leq t \leq T} |\Gamma^{(n)}_t| \leq C \sup_{s \leq T} |\Delta Y^{(1,n)}_s| \leq C \leq |\Delta Y^{(n)}_t|.$$ 

Since $\Delta Y^{(n)}$ converges to zero in probability by Theorem 2.2, so does $\Gamma^{(n)}$. Thus the process $(\Phi_t)$ converges to $(\sigma^2 t)$. Since $\sup_{0 \leq t \leq T} \Psi_t \leq C \left( \frac{1}{n} + \sup_{s \leq T} |\Delta Y^{(1,n)}_s| \right)\Phi_T$, we see that the process $(\Psi_t)$ converges to zero in probability. Using these facts, Lemma 3.1, and Theorem 4.1 of Billingsley (1968), we see that $\log S^{(1,n)}$ converges weakly to $(\sigma Y_t + \int_0^t b(s) ds - \frac{1}{2}\sigma^2 t)$. \qed

4. Arbitrage Opportunities in the Binary Market

In this section, we study the arbitrage opportunities in the approximating binary market model with memory constructed in Section 3. For simplicity, we assume that the function $b(\cdot)$ is a real constant as in (1.1).

Let $N \in \mathbb{N}, r, b \in \mathbb{R}$, and $\sigma \in (0, \infty)$. The number $N$ corresponds to $n$ in Sections 2 and 3. Let $g(t, u)$ be as in Section 2. We define $r^{(N)} := r/N$, $b^{(N)} := b/N$. The $[NT]$-period market $\mathcal{M}^{(N)}$ consists of a share of the money market with price
process \((B_n^{(N)})_{n=0,1,...,|NT|}\) and a stock with price process \((S_n^{(N)})_{n=0,1,...,|NT|}\). The prices are governed respectively by

\[
B_0^{(N)} = 1, \quad B_n^{(N)} = B_{n-1}^{(N)}(1 + r^{(N)}), \quad n = 1, \ldots, |NT|, \\
S_0^{(N)} = s_0, \quad S_n^{(N)} = S_{n-1}^{(N)}(1 + b^{(N)} + X_n^{(N)}), \quad n = 1, \ldots, |NT|,
\]

where \(s_0\) is a positive constant,

\[X_n^{(N)} := \sigma \Delta Y_n^{(N)} = \frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n} \{y\left(\frac{i}{N}, \frac{i}{N}\right) - y\left(\frac{n-1}{N}, \frac{n-1}{N}\right)\} \xi_i\]

and \(\{\xi_i\}\) are i.i.d. random variables such that \(P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2\). By Theorem 3.2, the binary market model \(M^{(N)}\) approximates the continuous-time market model with bond price process \((e^{rt})\) and stock price process \(S\) in (1.1).

Given the values of \(\xi_1, \ldots, \xi_{n-1}\), the random variable \(X_n^{(N)}\) takes the following two possible values \(u_n\) and \(d_n\): 

\[d_n = d_n(\xi_1, \ldots, \xi_{n-1}) = \frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n-1} \{y\left(\frac{i}{N}, \frac{i}{N}\right) - y\left(\frac{n-1}{N}, \frac{n-1}{N}\right)\} \xi_i - \frac{\sigma}{\sqrt{N}}, \]

\[u_n = u_n(\xi_1, \ldots, \xi_{n-1}) = \frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n-1} \{y\left(\frac{i}{N}, \frac{i}{N}\right) - y\left(\frac{n-1}{N}, \frac{n-1}{N}\right)\} \xi_i + \frac{\sigma}{\sqrt{N}}.
\]

We investigate the arbitrage opportunities in \(M^{(N)}\). Choose \(C \in \mathbb{R}\) so that

\[|y(t, u) - y(s, u)| \leq C|t - s|, \quad 0 \leq t, s, u \leq T. \quad (4.1)\]

**Theorem 4.1.** Suppose that \(T < 1/C\). Then there exists an integer \(N_0\) such that for each \(N \geq N_0\), the market \(M^{(N)}\) is arbitrage-free.

**Proof.** From the condition \(TC < 1\), we have an integer \(N_0\) such that

\[\frac{b}{N} < \frac{\sigma}{\sqrt{N}} (TC + 1) > -1, \quad |r - b| < \sqrt{N}(1 - TC)\sigma \quad (4.2)\]

if \(N \geq N_0\). Let \(n \in \{1, \ldots, [NT]\}\). Then, by (4.1), \(\min_{\xi \in \{-1, 1\}^{n-1}} d_n(\xi) = -\frac{\sigma}{\sqrt{N}} \sum_{i=1}^{n-1} \{y\left(\frac{i}{N}, \frac{i}{N}\right) - y\left(\frac{n-1}{N}, \frac{n-1}{N}\right)\} \xi_i\) is at least

\[-\sigma N^{-1/2} \left[\left((n-1)/N\right) + 1\right] \geq -\sigma N^{-1/2} (TC + 1).
\]

This and (4.2) yield \(b^{(N)} + X_n^{(N)} \geq (b/N) + \min_{\xi \in \{-1, 1\}^{n-1}} d_n(\xi) > -1\) for \(N \geq N_0\) and \(n = 1, \ldots, [NT]\), whence \(S_n > 0\).

We show that \(M^{(N)}\) is arbitrage-free for \(N \geq N_0\). By Proposition 6.1.2 of Dzhaparidze (1996), \(M^{(N)}\) is free from arbitrage opportunities if and only if

\[d_n < r^{(N)} - b^{(N)} < u_n, \quad n = 1, \ldots, [NT]. \quad (4.3)\]

However, \(\max_{\xi \in \{-1, 1\}^{n-1}} d_n(\xi) = \sigma N^{-1/2} \sum_{i=1}^{n-1} \{y\left(\frac{i}{N}, \frac{i}{N}\right) - y\left(\frac{n-1}{N}, \frac{n-1}{N}\right)\} - \sigma N^{-1/2}\) is at most \(-\sigma N^{-1/2}[1 - ((n-1)/N)] \leq -\sigma N^{-1/2}(1 - TC)\). Similarly, we see that \(\min_{\xi \in \{-1, 1\}^{n-1}} u_n(\xi) = -\sigma N^{-1/2} \sum_{i=1}^{n-1} \{y\left(\frac{i}{N}, \frac{i}{N}\right) - y\left(\frac{n-1}{N}, \frac{n-1}{N}\right)\} + \sigma N^{-1/2}\) is at least \(\sigma N^{-1/2}[1 - ((n-1)/N)] \geq \sigma N^{-1/2}(1 - TC)\). Thus, by (4.2), (4.3) holds for \(N \geq N_0\). \(\square\)
By Theorem 4.1, the market $\mathcal{M}^{(N)}$ is arbitrage-free for $T$ small enough and $N$ large enough. However, in general, the market $\mathcal{M}^{(N)}$ may admit arbitrage opportunities, as we see below.

Suppose that there exists a positive constant $C$ such that $l(s, u) \geq C$ for $0 \leq u < s \leq T$. Let $T > 1/C$. We assume that $r \leq b$. Then, $d_{[NT]}(-1, \ldots, -1)$ is

$$
\frac{\sigma}{\sqrt{N}} \sum_{i=1}^{[NT]-1} \frac{l(s, \xi_i^T)}{\sqrt{N}} ds - \frac{\sigma}{\sqrt{N}} \left( \frac{C([NT] - 1)}{N} - 1 \right).
$$

Since $TC > 1$, we have $d_{[NT]}(-1, \ldots, -1) > r_N - b_N$ or $S_{[NT]} > (1 + r_N)S_{[NT] - 1}$ for $N$ large enough. Therefore, if the value of $(\xi_1, \ldots, \xi_{[NT] - 1})$ turns out to be $(-1, \ldots, -1)$, then we have an arbitrage opportunity: we may buy stocks at time $[NT] - 1$ using money obtained by short-selling bonds. In a similar fashion, we can show that if $T > 1/C, r < b$ and $N$ is large enough, then the value $(1, \ldots, 1)$ of $(\xi_1, \ldots, \xi_{[NT] - 1})$ gives an arbitrage opportunity.

Put $P_N = \mathbb{P}\left( \bigcup_{n=1}^{[NT]} \{ d_n < r(N) - b(N) < u_n \} \right)$. As we see in the proof of Theorem 4.1, the binary market $\mathcal{M}^{(N)}$ is arbitrage-free if and only if $P_N = 0$. The next theorem gives the rate at which the arbitrage probability $P_N$ tends to zero.

**Theorem 4.2.** There exists a positive constant $C = C_T$ such that, for each $\alpha \in (0, 1)$, we have $N(\alpha) \in N$ satisfying $P_N \leq C^{N^{-\alpha}}$ for $N \geq N(\alpha)$.

**Proof.** Set $\beta = (\alpha + 1)/2$, and choose $N(\alpha) \in N$ so large that

$$
N^{\beta/2} C \sqrt{T} < \sqrt{N} - \frac{|r - b|}{\sigma}, \quad N^{\beta/2} > 4 \quad (4.4)
$$

if $N \geq N(\alpha)$. Then $d_1 < r(N) - b(N) < u_1$. For $N \geq N(\alpha)$ and $n = 2, \ldots, [NT]$, we put $\lambda := N^{\beta/2}$, $M_n := \max_{1 \leq m \leq n-1} \sum_{i=1}^{m} \eta_i$ and

$$
s_{n-1} := \left[ N \sum_{i=1}^{n-1} \left\{ \frac{y(\xi_i, \xi_i)}{\sqrt{N}} - y(\frac{\xi_{i-1}}{N}, \xi_i) \right\} \right]^{1/2}.
$$

where $\eta_i := \sqrt{N} \left\{ y(\xi_i, \xi_i) - y(\frac{\xi_{i-1}}{N}, \xi_i) \right\}$ for $i = 1, 2, \ldots$. By (4.1), we have $s_{n-1} \leq C \sqrt{T}$. This and (4.4) imply that $P((r - b)/N \leq d_n)$ is at most

$$
P\left( \frac{\sigma^{-1}(r - b) + \sqrt{N}}{M_{n+1}} \right) \leq P(M_{n+1} \geq \lambda C \sqrt{T}) \leq P(M_{n+1} \geq \lambda s_{n-1}).
$$

Similarly we have $P(u_n \leq (r - b)/N) \leq P(M_{n+1} \geq \lambda s_{n-1})$. Since $\frac{1}{\lambda} \lambda > 1$ and

$$
\max_{1 \leq i \leq n-1} |\eta_i| = \max_{1 \leq i \leq n-1} |\sqrt{N} \{ y(\xi_i, \xi_i) - y(\frac{\xi_{i-1}}{N}, \xi_i) \}| \leq s_{n-1},
$$

it follows from Eq. (12.16) in Billingsley (1968, p. 89) that $P(M_{n+1} \geq \lambda s_{n-1}) \leq C_0 \lambda^{-1}$ for some constant $C_0 > 0$ independent of $N$ and $n$; notice that $\eta_i$ here corresponds to $\xi_i$ in Eq. (12.16) in Billingsley (1968, p. 89). Hence, $P_N$ is at most

$$
\sum_{n=2}^{[NT]} \left\{ P\left( \frac{r - b}{N} \leq d_n \right) + P\left( u_n \leq \frac{r - b}{N} \right) \right\} \leq \frac{2[NT]}{N^{2\beta}} C_0 \leq \frac{2TC_0}{N^{\alpha}}.
$$

Thus the theorem follows. \qed
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