ON THE WORST CONDITIONAL EXPECTATION

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Abstract. We study continuous coherent risk measures on L^p, in particular, the worst conditional expectations. We show some representation theorems for them, extending the results of Artzner, Delbaen, Eber, Heath, and Kusuoka.

1. Introduction

The worst conditional expectations are important examples of coherent risk measures. Both concepts were introduced by Artzner et al. [1, 2]. There they *defined* risks (such as market risks) axiomatically and provided a unified framework for the analysis of them. In these two papers, the underlying probability space was supposed to be finite. Subsequently Delbaen [3] extended the theory of coherent risk measures to general probability spaces.

In [3], the space L^∞ of all bounded real random variables or the space L^0 of all real random variables was taken as the space of risks X to be measured. However, from the viewpoint of application, the assumption $X \in L^\infty$ seems to be inconvenient, while the space L^0 seems to be too large for a simple theory. In this paper, we take the intermediate spaces L^p, in particular L^1, as the spaces of risks. We can develop a simple theory at the cost of restricting ourselves to the *continuous* coherent risk measures on L^p.

Among such coherent risk measures, we are especially interested in the law invariant ones. The worst conditional expectations are again important examples. In fact, Kusuoka [4] proved that if the probability space was standard and nonatomic, then all the law invariant coherent risk measures on L^∞ with the "Fatou property" could be represented by the worst conditional expectations. We extend this result to the continuous, law invariant, coherent risk measures on L^p.

We refer to Nakano [5] where he also considers coherent risk measures on L^1. He uses them to measure the shortfall risks that appear in hedging contingent claims under constraints on the initial capital.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $1 \le p \le \infty$, we write L^p for $L^p(\Omega, \mathcal{F}, P)$ and $\|\cdot\|_p$ for its norm. A mapping $\rho : L^p \to \mathbf{R}$ is called a *coherent risk measure* if the following conditions are satisfied:

(1) if $X \ge 0$, then $\rho(X) \le 0$;

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(2) $\rho(X_1 + X_2) \le \rho(X_1) + \rho(X_2)$;
(3) $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \ge 0$;
(4) $\rho(a + X) = \rho(X) - a$ for $a \in \mathbf{R}$

(cf. [1]–[3]). We say that ρ is *continuous* if it is so in the norm $\|\cdot\|_p$, or, equivalently, there exists $C \in (0, \infty)$ such that $|\rho(X)| \le C\|X\|_p$ (see Lemma 2.1). From [3, Theorem 3.2] (see also [2, Proposition 4.1]), we obtain the following characterization.

Theorem 1.1. *Let $1 \le p < \infty$ and $(1/p) + (1/q) = 1$ ($q = \infty$ if $p = 1$). Then, for a mapping $\rho : L^p \to \mathbf{R}$, the following conditions are equivalent:*

(1) *The mapping ρ is a continuous coherent risk measure;*
(2) *There exists a set G of nonnegative random variables g with $E[g] = 1$ such that*

$$\sup_{g \in G} \|g\|_q < \infty, \tag{1.1}$$

$$\rho(X) = \sup_{g \in G} E[(-X)g] \quad (X \in L^p). \tag{1.2}$$

We refer to [5] for an analogue of this theorem for the lower semi-continuous coherent risk measures on L^1.

Let $\alpha \in (0, 1]$. There are several possible definitions of the worst conditional expectation ρ_α. In this paper, suggested by [3], we adopt the following one:

$$\rho_\alpha(X) := \sup_{g \in G(\alpha)} E[(-X)g] \qquad (X \in L^1), \tag{1.3}$$

where

$$G(\alpha) := \left\{ g : \begin{array}{l} g \text{ is a nonnegative random variable on } (\Omega, \mathcal{F}, P) \\ \text{such that } E[g] = 1,\ \|g\|_\infty \le 1/\alpha \end{array} \right\}. \tag{1.4}$$

Notice that

$$\rho_1(X) = E[-X] \qquad (X \in L^1).$$

We easily see that, for $1 \le p \le \infty$ and $\alpha \in (0, 1]$, the worst conditional expectation ρ_α defines a continuous coherent risk measure on L^p.

For $0 < \alpha < 1$, the worst conditional expectation WCE_α in [2] is defined by

$$\mathrm{WCE}_\alpha(X) := \sup \left\{ \frac{1}{P(A)} E[(-X)I_A] : A \in \mathcal{F},\ P(A) > \alpha \right\}.$$

We easily see that WCE_α also defines a continuous coherent risk measure on L^p with $1 \le p \le \infty$. In fact, if P is atomless, then, for $0 < \alpha < 1$, WCE_α and ρ_α coincide on L^∞ (see [3]), whence on L^1.

To give an explicit representation of $\rho_\alpha(X)$ for $X \in L^1$, we use an idea in statistical hypothesis testing. Let $X \in L^1$ and $\alpha \in (0, 1)$. We define two constants $k = k(X, \alpha) \in \mathbf{R}$ and $\gamma = \gamma(X, \alpha) \in [0, 1]$ by

$$k := \inf\{z \in \mathbf{R} : P(X > z) \le \alpha\}$$

and

$$\gamma := \begin{cases} \dfrac{\alpha - P(X > k)}{P(X = k)} & \text{if } P(X = k) > 0, \\ 0 & \text{if } P(X = k) = 0, \end{cases}$$

respectively. We define the random variable $\phi_{(X,\alpha)}$ (called the *most powerful test*) by

$$\phi_{(X,\alpha)} := I_{(X>k)} + \gamma I_{(X=k)}. \tag{1.5}$$

Notice that $E[\phi_{(X,\alpha)}] = \alpha$. Applying the Neyman–Pearson lemma, we prove the following theorem.

Theorem 1.2. *Let $\alpha \in (0,1)$. Then, for $X \in L^1$,*

$$\rho_\alpha(-X) = \frac{1}{\alpha} E\left[X\phi_{(X,\alpha)}\right]. \tag{1.6}$$

We say that a coherent risk measure ρ on L^p is *law invariant* if $\rho(X) = \rho(Y)$ for every pair (X,Y) of $X, Y \in L^p$ with the same distribution. Since the expectation on the right-hand side of (1.6) depends only on the distribution of X (and α), we immediately obtain the following corollary.

Corollary 1.3. *For $\alpha \in (0,1]$, the worst conditional expectation ρ_α is law invariant on L^1, hence on L^p with $1 < p \le \infty$.*

It should be noticed that, in Corollary 1.3, there is no restriction on $(\Omega, \mathcal{F}, P)$. In particular, it applies to those with atoms, such as discrete probability spaces. We may regard this as one advantage of definition (1.3) with (1.4).

For a random variable X, we write F_X for the distribution function of X,

$$F_X(x) := P(X \le x) \qquad (x \in \mathbf{R}).$$

We denote by Z_X the following Skorokhod representation of X:

$$Z_X(x) := \inf\{z \in \mathbf{R} : F_X(z) > x\} \qquad (0 < x < 1).$$

Then Z_X is a non-decreasing, right continuous function on $(0,1)$. We write Leb for the Lebesgue measure on $((0,1), \mathcal{B}(0,1))$. As is well known (see, e.g., [7, Section 3.12]), as a random variable on $((0,1), \mathcal{B}(0,1), \mathrm{Leb})$, Z_X has the same distribution as X. Using Theorem 1.2, we prove the following theorem.

Theorem 1.4. *Let $\alpha \in (0,1]$. Then, for $X \in L^1$,*

$$\rho_\alpha(-X) = \frac{1}{\alpha} \int_{1-\alpha}^{1} Z_X(x)dx. \tag{1.7}$$

Equality (1.7) is used as the definition of ρ_α on L^∞ in Kusuoka [4]. Following his method, we prove the following analogue of [4, Theorem 4].

Theorem 1.5. *Let $1 \le p < \infty$ and $(1/p) + (1/q) = 1$. We assume that*

$$(\Omega, \mathcal{F}, P) \text{ is a standard nonatomic probability space.} \tag{1.8}$$

Then, for a mapping $\rho : L^p \to \mathbf{R}$, the following conditions are equivalent:

(1) *The mapping ρ is a continuous, law invariant coherent risk measure;*
(2) *There exists a set M of probability measures on $((0,1], \mathcal{B}(0,1])$ such that*

$$\sup_{m \in M} \int_{(0,1]} \frac{1}{\alpha} m(d\alpha) < \infty \quad \text{if } p = 1, \tag{1.9}$$

$$\sup_{m \in M} \int_0^1 \left\{ \int_{[1-t,1]} \frac{1}{\alpha} m(d\alpha) \right\}^q dt < \infty \quad \text{if } 1 < p < \infty, \tag{1.10}$$

$$\rho(X) = \sup_{m \in M} \int_{(0,1]} \rho_\alpha(X) m(d\alpha) \quad (X \in L^p). \tag{1.11}$$

We prove Theorems 1.1, 1.2, and 1.4 in Section 2. In Section 3, we prove Theorem 1.5, and give an example showing that this assertion does not hold in general without the assumption (1.8).

2. Proofs of Theorems 1.2 and 1.4

Lemma 2.1. *Let $1 \le p \le \infty$. Then, for a coherent risk measure $\rho : L^p \to \mathbf{R}$, the following conditions are equivalent:*

(1) *The coherent risk measure ρ is continuous;*
(2) *There exists $C \in (0,\infty)$ such that $|\rho(X) - \rho(Y)| \le C\|X - Y\|_p$ for $X, Y \in L^p$;*
(3) *There exists $C \in (0,\infty)$ such that $\rho(X) \le C\|X\|_p$ for $X \in L^p$;*
(4) *There exists $C \in (0,\infty)$ such that $|\rho(X)| \le C\|X\|_p$ for $X \in L^p$.*

Proof. It is clear that (2) implies (1), and (4) implies (3) trivially. Since $\rho(0) = 0$, one can prove the implication (1) $\Rightarrow$ (4) in the same way as the standard proof of boundedness of continuous linear transformations. Suppose (3). Then $\rho(X) - \rho(Y) \le \rho(X - Y) \le C\|X - Y\|_p$, and similarly $\rho(Y) - \rho(X) \le C\|X - Y\|_p$. Hence (2). □

Proof of Theorem 1.1. (2) $\Rightarrow$ (1) By Hölder's inequality, we have

$$\rho(X) \le \sup\{\|g\|_q : g \in G\} \cdot \|X\|_p.$$

Using this, we can easily prove (1).

(1) $\Rightarrow$ (2) Let $X \in L^\infty$ and let X_n be a uniformly bounded sequence that decreases to X a.s. Then $X_n \to X$ in L^p, and so, by the continuity of ρ, we have $\rho(X_n) \to \rho(X)$. This implies that the restriction of ρ on L^∞ has the Fatou property (see [3]). Therefore, by [3, Theorem 3.2], there exists a set G of nonnegative random variables g with $E[g] = 1$ such that $\rho(X) = \mu(X)$ for $X \in L^\infty$, where

$$\mu(X) := \sup_{g \in G} E[(-X)g]. \tag{2.1}$$

From this and Lemma 2.1, we see that there exists $C \in (0,\infty)$ such that

$$E[|X|g] \le C\|X\|_p \qquad (g \in G,\ X \in L^\infty).$$

Following the standard method, we obtain (1.1) from this. Equality (2.1) now defines a continuous coherent risk measure μ on L^1, whence ρ and μ coincide on L^1. □

Proof of Theorem 1.2. First we assume that $X \ge C$ for some $C \in \mathbf{R}$. Clearly we may assume $P(X > C) > 0$. Set $Y := X - C$. Then $\rho_\alpha(-X) = \rho_\alpha(-Y) + C$. We introduce the following class of "randomized tests" ϕ:

$$\Phi(\alpha) := \left\{ \phi : \begin{array}{l} \phi \text{ is a random variable on } (\Omega, \mathcal{F}, P) \text{ such that} \\ 0 \le \phi \le 1 \ (P \text{ a.s.}), \quad E[\phi] = \alpha \end{array} \right\}.$$

Then from the definition of ρ_α it follows that

$$\rho_\alpha(-Y) = \frac{E[Y]}{\alpha} \sup\left\{E^{P_1}[\phi] : \phi \in \Phi(\alpha)\right\},$$

where E^{P_1} stands for expectation with respect to the probability measure P_1 given by

$$\frac{dP_1}{dP} = \frac{Y}{E[Y]}.$$

The Neyman-Pearson lemma (cf. [6, Chapter III, Section 3]) now gives

$$\sup\left\{E^{P_1}[\phi] : \phi \in \Phi(\alpha)\right\} = E^{P_1}\left[\phi_{(Y,\alpha)}\right]$$

or

$$\rho_\alpha(-Y) = \frac{1}{\alpha} E\left[\phi_{(Y,\alpha)} Y\right].$$

We have

$$\begin{aligned} k(Y,\alpha) &= \inf\{z : P(Y > z) \le \alpha\} \\ &= \inf\{u : P(X > u) \le \alpha\} - C \\ &= k(X,\alpha) - C. \end{aligned}$$

So

$$\begin{aligned} \{X > k(X,\alpha)\} &= \{Y > k(Y,\alpha)\}, \\ \{X = k(X,\alpha)\} &= \{Y = k(Y,\alpha)\}, \end{aligned}$$

whence $\phi_{(X,\alpha)} = \phi_{(Y,\alpha)}$. Combining,

$$\begin{aligned} \rho_\alpha(-X) &= \frac{1}{\alpha} E\left[Y\phi_{(Y,\alpha)}\right] + C = \frac{1}{\alpha} E\left[Y\phi_{(X,\alpha)}\right] + \frac{C}{\alpha} E\left[\phi_{(X,\alpha)}\right] \\ &= \frac{1}{\alpha} E\left[X\phi_{(X,\alpha)}\right]. \end{aligned}$$

We now assume that X is an arbitrary element of L^1. Write

$$X_n := XI_{(X \ge -n)} - nI_{(X < -n)} \qquad (n = 1, 2, \dots).$$

If $a > -n$, then

$$\{X_n > a\} = \{X \ge -n\} \cap \{X > a\} = \{X > a\}.$$

Thus if $n > -k(X,\alpha)$, then $\phi_{(X,\alpha)} = \phi_{(X_n,\alpha)}$, and so

$$\rho_\alpha(-X_n) = \frac{1}{\alpha} E\left[X_n\phi_{(X,\alpha)}\right].$$

Since $X_n \to X$ as $n \to \infty$ in L^1 and $0 \le \phi_{(X,\alpha)} \le 1$, we have

$$E\left[X_n\phi_{(X,\alpha)}\right] \to E\left[X\phi_{(X,\alpha)}\right] \qquad (n \to \infty);$$

also the continuity of ρ_α implies that $\rho_\alpha(-X_n)$ tends to $\rho_\alpha(-X)$ as $n \to \infty$. Thus (1.6) follows. □

Proof of Theorem 1.4. We may (and shall) assume $0 < \alpha < 1$. For $X \in L^1$, write

$$Y_X(x) := \inf\{z \in \mathbf{R} : F_X(z) \ge x\} \qquad (0 < x < 1).$$

Then $\mathrm{Leb}(Z_X = Y_X) = 1$ (see [7, p. 35]). So, instead of (1.7), we may prove

$$\rho_\alpha(-X) = \frac{1}{\alpha}\int_{1-\alpha}^{1} Y_X(x)dx. \tag{2.2}$$

Since, as a random variable on $((0,1), \mathcal{B}(0,1), \mathrm{Leb})$, Y_X has the same distribution as X, Theorem 1.2 yields

$$\rho_\alpha(-X) = \frac{1}{\alpha}\int_0^1 Y_X(x)\phi_{(Y_X,\alpha)}(x)dx.$$

Now

$$k(Y_X,\alpha) = k(X,\alpha) = \inf\{z \in \mathbf{R} : F_X(z) \ge 1-\alpha\} = Y_X(1-\alpha).$$

Define $c \in [1-\alpha, 1]$ by

$$c := \inf\{x \in (0,1) : Y_X(x) > Y_X(1-\alpha)\} \quad (:= 1 \text{ if the set is empty}).$$

Then since Y_X is nondecreasing on $(0,1)$, we have

$$\{x \in (0,1) : Y_X(x) > Y_X(1-\alpha)\} = \begin{cases} (c,1) \text{ or } [c,1) & \text{if } c < 1, \\ \emptyset & \text{if } c = 1, \end{cases}$$

whence

$$\alpha - \mathrm{Leb}\{x \in (0,1) : Y_X(x) > Y_X(1-\alpha)\} = c - (1-\alpha).$$

We also have

$$Y_X(x) = Y_X(1-\alpha) \qquad (1-\alpha \le x < c);$$

in particular, $c = 1-\alpha$ if $\mathrm{Leb}\{x \in (0,1) : Y_X(x) = Y_X(1-\alpha)\} = 0$. Thus

$$\begin{aligned}
&\int_0^1 Y_X(x)\phi_{(Y_X,\alpha)}(x)dx \\
&= Y_X(1-\alpha)\gamma(Y_X,\alpha)\mathrm{Leb}\{x \in (0,1) : Y_X(x) = Y_X(1-\alpha)\} + \int_c^1 Y_X(x)dx \\
&= Y_X(1-\alpha)\left\{c - (1-\alpha)\right\} + \int_c^1 Y_X(x)dx = \int_{1-\alpha}^1 Y_X(x)dx,
\end{aligned}$$

whence (2.2). □

3. Proof of Theorems 1.5

Proposition 3.1. *Let $1 \le p < \infty$ and $X \in L^p$. Write*

$$X_n := -nI_{(X<-n)} + XI_{(-n\le X\le n)} + nI_{(X>n)} \qquad (n = 1, 2, \dots). \tag{3.1}$$

Then

$$\int_0^1 |Z_X(t) - Z_{X_n}(t)|^p \, dt \to 0 \qquad (n \to \infty).$$

Proof. We have

$$P(X_n \le z) = \begin{cases} 0 & (-\infty < z < -n), \\ P(X \le z) & (-n \le z < n), \\ 1 & (n \le z < \infty). \end{cases}$$

Suppose $0 < x < P(X \le -n)$. Then $Z_X(x) \le -n$. On the other hand, since $x < P(X \le -n) = P(X_n \le -n)$, we have $Z_{X_n}(x) \le -n$, hence $= -n$. Thus

$$|Z_X(x) - Z_{X_n}(x)| \le |Z_X(x)| \qquad (0 < x < P(X \le -n)).$$

Next we suppose $P(X \le -n) \le x < P(X < n)$. Then since

$$\lim_{z\uparrow n} P(X \le z) = P(X < n),$$

we see that $P(X \le n - \epsilon) > x$ for $\epsilon > 0$ small enough. So

$$Z_X(x) = \inf\{-n < z < n : P(X \le z) > x\}.$$

On the other hand,

$$P(X_n \le -n) = P(X \le -n) \le x < P(X < n) = P(X_n < n),$$

so that, similarly

$$Z_{X_n}(x) = \inf\{-n < z < n : P(X_n \le z) > x\}.$$

Since $P(X \le z) = P(X_n \le z)$ for $-n < z < n$, it follows that

$$Z_X(x) = Z_{X_n}(x) \qquad (P(X \le -n) \le x < P(X < n)).$$

Finally we suppose $P(X < n) \le x < 1$. Then, for any $\epsilon > 0$, we have

$$x \ge P(X < n) = P(X_n < n) \ge P(X_n \le n - \epsilon),$$

so that $Z_{X_n}(x) \geq n - \epsilon$. Since ϵ is arbitrary, we see that $Z_{X_n}(x) \geq n$, hence $= n$. Similarly, we have $Z_X(x) \geq n$. Thus

$$|Z_X(x) - Z_{X_n}(x)| \leq |Z_X(x)| \qquad (P(X < n) \leq x < 1).$$

Combining,

$$\begin{aligned}&\int_0^1 |Z_X(x) - Z_{X_n}(x)|^p \, dx \\ &\leq \int_0^{P(X \leq -n)} |Z_X(x)|^p dx + \int_{P(X<n)}^1 |Z_X(x)|^p dx \quad \to \quad 0 \qquad (n \to \infty).\end{aligned}$$

Thus we obtain the proposition. □

By assumption (1.8), we may (and shall) assume that

$$(\Omega, \mathcal{F}, P) \text{ is equal to the Lebesgue space } ((0,1), \mathcal{B}(0,1), \text{Leb}). \tag{3.2}$$

Then, for a random variable X on $(\Omega, \mathcal{F}, P)$, the Skorokhod representation Z_X is again a random variable on $(\Omega, \mathcal{F}, P)$. Now if X is nondecreasing and right continuous, then

$$X(t) = Z_X(t) \qquad (0 < t < 1). \tag{3.3}$$

Indeed, from the definition of Z_X, we find that

$$P(X \leq Z_X(t)) \geq t, \quad P(X < Z_X(t)) \leq t \qquad (0 < t < 1).$$

The latter implies $X(s) \geq Z_X(t)$ for $t < s < 1$, whence $X(t) \geq Z_X(t)$. Similarly, the former implies $X(s) \leq Z_X(t)$ for $0 < s < t$, whence $X(t-) \leq Z_X(t)$. Since $X(t-) = X(t)$ a.s., we have $X(t) = Z_X(t)$ a.s. However, both X and Z_X are right continuous, whence (3.3).

For $1 \leq q \leq \infty$, we define

$$G^q := \{g \in L^q : g \geq 0 \ (P \text{ a.s.}), \quad E[g] = 1\}.$$

Recall that, for a random variable X, we denote by F_X its distribution function.

Proposition 3.2. *Let $1 \leq p < \infty$ and $(1/p) + (1/q) = 1$. Then, for $X \in L^p$ and $g \in G^q$,*

$$E[Z_X Z_g] = \sup\{E[Xf] : f \in G^q, \ F_f = F_g\}.$$

Proof. Write

$$\mu(X) := \sup\{E[(-X)f] : f \in G^q, \ F_f = F_g\} \qquad (X \in L^p).$$

Then since $\|f\|_q = \|g\|_q$ if $F_f = F_g$, the mapping $\mu : L^p \to \mathbf{R}$ defines a continuous coherent risk measure on L^p. Define X_n by (3.1). Then $X_n \in L^\infty$, and so [4, Proposition 14] implies $E[Z_{X_n} Z_g] = \mu(-X_n)$. Now $\|X - X_n\|_p \to 0$, so that the continuity of μ implies $\mu(-X_n) \to \mu(-X)$. On the other hand, by Proposition 3.1, we have

$$\lim_{n\to\infty} E[Z_{X_n} Z_g] = E[Z_X Z_g].$$

Thus the proposition follows. □

Proposition 3.3. *Let $1 \leq p < \infty$ and $(1/p) + (1/q) = 1$. Then, for a mapping $\rho : L^p \to \mathbf{R}$, the following conditions are equivalent:*

(1) *The mapping ρ is a continuous, law invariant coherent risk measure;*

(2) *There exists a set H of nondecreasing, right continuous probability density functions on* $(0,1)$ *such that*

$$\sup_{h\in H} \|h\|_q < \infty, \tag{3.4}$$

$$\rho(-X) = \sup_{h\in H} E[Z_X h] \quad (X \in L^p). \tag{3.5}$$

Proof. (2) $\Rightarrow$ (1) Clearly ρ is law invariant. Define G by

$$G := \{g \in G^q : Z_g \in H\}.$$

Then since $Z_h = h$ for $h \in H$, we see that $\{Z_g : g \in G\} = H$. Thus

$$\rho(-X) = \sup_{g\in G} E[Z_X Z_g].$$

Now if $g \in G$ and $F_f = F_g$ for $f \in G^q$, then $f \in G$. So by Proposition 3.2 we have

$$\sup_{g\in G} E[Z_X Z_g] = \sup_{g\in G} \sup\{E[Xf] : f \in G^q,\ F_f = F_g\}. = \sup_{g\in G} E[Xg].$$

Moreover,

$$\sup_{g\in G} \|g\|_q = \sup_{g\in G} \|Z_g\|_q = \sup_{h\in H} \|h\|_q < \infty.$$

Thus ρ is a continuous coherent risk measure on L^p.

(1) $\Rightarrow$ (2) The restriction of ρ on L^∞ defines a law invariant, coherent risk measure on L^∞ with the Fatou property (see the proof of Theorem 1.1). So, by [4, Lemma 10], there exists a set H of nondecreasing, right continuous probability density functions on $(0,1)$ such that

$$\rho(-X) = \sup_{h\in H} E[Z_X h] \qquad (X \in L^\infty). \tag{3.6}$$

By Lemma 2.1, we can take $C > 0$ such that $\rho(-X) \le C\|X\|_p$ for $X \in L^p$.

Let $h \in H$. Then since $Z_h = h$, it follows from [4, Proposition 14] that, for any nonnegative $X \in L^\infty$,

$$E[Xh] \le E[Z_X Z_h] = E[Z_X h] \le \rho(-X) \le C\|X\|_p.$$

This implies

$$E[|X|h] \le C\|X\|_p \qquad (h \in H,\ X \in L^\infty).$$

From this, we obtain (3.4). So if we write

$$\mu(X) := \sup_{h\in H} E[(-X)h] \qquad (X \in L^p),$$

then μ defines a continuous coherent risk measure on L^p. For $X \in L^p$, we define X_n by (3.1). Then (3.6) implies $\rho(-X_n) = \mu(-Z_{X_n})$ for $n = 1, 2, \ldots$. If we let $n \to \infty$, then by Proposition 3.1 we obtain $\rho(-X) = \mu(Z_X)$ or (3.5). □

Proof of Theorem 1.5. We prove the implication (1) $\Rightarrow$ (2). Let H be as in Proposition 3.3. For $h \in H$, we define

$$f(t) = \begin{cases} 0 & (-\infty < t < 0), \\ h(0+) & (t = 0), \\ h(t) & (0 < t < 1). \end{cases} \tag{3.7}$$

Then, as in the proof of [4, Theorem 4], it follows from Theorem 1.4 that, for $X \in L^p$,

$$E[Z_X h] = \int_{[0,1)} \left\{ \int_x^1 Z_X(t)dt \right\} df(x) = \int_{[0,1)} \rho_{1-x}(-X)(1-x)df(x)$$
$$= \int_{(0,1]} \rho_\alpha(-X)m(d\alpha),$$

where m is the probability measure on $(0,1]$ defined by

$$m(d\alpha) := \alpha df \circ \phi \tag{3.8}$$

with $\phi(t) := 1-t$. Now if $1 < p < \infty$, then

$$\sup_{m \in M} \int_0^1 \left\{ \int_{[1-t,1]} \frac{1}{\alpha} m(d\alpha) \right\}^q dt = \int_0^1 h(t)^q dt = \|h\|_q^q,$$

while if $p = 1$, then

$$\int_{(0,1]} \frac{1}{\alpha} m(d\alpha) = h(1-) = \|h\|_\infty.$$

Thus (2) follows from Proposition 3.3.

The proof of the implication (2) $\Rightarrow$ (1) is similar. For $m \in M$, we define the nondecreasing, right continuous function h on $(0,1)$ so that (3.7) and (3.8) hold. We prove representation (3.5) with (3.4) from (2), and apply Proposition 3.3 to obtain (1). □

Example 3.4. In this example, we show that Theorem 1.5 does not hold in general without (1.8). We set $\Omega = \{\omega_1, \omega_2\}$ and $\mathcal{F} = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2\}\}$. Choose p_1, p_2 so that $0 < p_1 < p_2 < 1$, $p_1 + p_2 = 1$, and define the probability measure P on $(\Omega, \mathcal{F})$ by $P\{\omega_1\} = p_1$, $P\{\omega_2\} = p_2$. Then two random variables on Ω have the same distribution if and only if they are identical. Thus, on this probability space, all the coherent risk measures are law invariant. Take $g_1 > 0$ and $g_2 > 0$ so that $g_1 p_1 + g_2 p_2 = 1$, $g_1 p_1 > g_2 p_2$. We define ρ by

$$\rho(-X) := E[Xg] = X_1 g_1 p_1 + X_2 g_2 p_2,$$

where $X(\omega_i) = X_i$, $g(\omega_i) = g_i$ for $i = 1, 2$. Now suppose Theorem 1.5 (2) holds. Then as in the proof of (2) $\Rightarrow$ (1) in Theorem 1.5, we get representation (3.5). If we take $(0,1)$ and $(1,0)$ as (X_1, X_2), then we obtain

$$g_2 p_2 = \sup_{h \in H} \int_{p_1}^1 h(t)dt, \quad g_1 p_1 = \sup_{h \in H} \int_{p_2}^1 h(t)dt.$$

However this contradicts the assumption $g_1 p_1 > g_2 p_2$ since $p_1 < p_2$ and $h \geq 0$ for $h \in H$.

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