ON THE EQUATIONS OF STATIONARY PROCESSES WITH DIVERGENT DIFFUSION COEFFICIENTS

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ABSTRACT. We investigate a class of Langevin equations with delay. The random noises in the equations are adopted so that they are in accordance with linear response theory in statistical physics. We prove that every purely nondetermistic, stationary Gaussian process with divergent diffusion coefficients as well as reflection positivity is characterized as the unique solution of one of such equations. This extends the results of Okabe to processes with divergent diffusion coefficients. A correspondence between the decays of the delay coefficient of the equation and the correlation function of the solution is obtained. We see that it is of different type from the case of finite diffusion coefficients.

1. INTRODUCTION

This paper aims to generalize Okabe's theory on KMO-Langevin equations to processes with divergent diffusion coefficients. KMO-Langevin equations are stochastic differential equations with delay which describe the time evolution of purely nondeterministic, stationary Gaussian processes with reflection positivity. A purely nondeterministic, weakly stationary process is called *reflection positive* if the correlation function $R(t) := E(X(0)X(t))$ of X is in the form

$$R(t) = \int_0^\infty e^{-|t|\lambda}\sigma(d\lambda) \qquad (t \in \mathbf{R}) \tag{1.1}$$

with some bounded Borel measure σ on $(0,\infty)$. We note that in this paper all stationary processes and stationary random distributions are assumed to have expectation zero. *The diffusion coefficient* D of X is a finite or infinite positive number defined by

$$D = \int_0^\infty R(t)dt. \tag{1.2}$$

Let $\alpha > 0$, and ρ be a Borel measure on $(0,\infty)$ such that

$$\int_0^\infty \frac{1}{1+\lambda}\rho(d\lambda) < \infty, \qquad \int_0^\infty \frac{1}{\lambda}\rho(d\lambda) = \infty. \tag{1.3}$$

We put

$$\gamma(t) = \chi_{(0,\infty)}(t)\int_0^\infty e^{-t\lambda}\rho(d\lambda) \qquad (t \in \mathbf{R}), \tag{1.4}$$

and

$$\Delta_W(\xi) = 2\sqrt{2\pi}\alpha \mathrm{Re}\left[-i\xi - \frac{i\xi}{2\pi}\int_0^{\infty-} e^{i\xi t}\gamma(t)dt\right] \qquad (\xi \in \mathbf{R}\setminus\{0\}). \tag{1.5}$$

This paper, which introduces Langevin equations describing *long-memory* processes, was published in J. Fac. Sci. Univ. Tokyo Sect. IA **40** (1993), 307–336.

Let W be a stationary Gaussian random distribution, defined on a probability space $(\Omega, \mathcal{F}, P)$, with spectral density Δ_W. We are concerned with a stationary solution of the following stochastic differential equation

$$\dot{X} = -\gamma * \dot{X} + W \tag{1.6}$$

with the causality condition

$$M_t(X) = M_t(W) \qquad \text{for any } t \in \mathbf{R}, \tag{1.7}$$

where, for any stationary random distribution Y, $M_t(Y)$ denotes the closed linear hull of $\{Y(\phi) : \phi \in \mathcal{S}(\mathbf{R}),\ \operatorname{supp} \phi \subset (-\infty, t]\}$ in $L^2(\Omega)$. We call (1.6) *the second KMO-Langevin equation*. *The first KMO-Langevin equation* is an equation with white noise as random force, and will be treated in our forthcoming paper.

We state our main results.

Theorem 1.1. *There exists a unique stationary random distribution X which satisfies* (1.6)–(1.7). *The solution X is a purely nondeterministic, stationary Gaussian process with reflection positivity.*

Theorem 1.2. *Let X be the stationary Gaussian process which satisfies* (1.6)–(1.7), *and let R be the correlation function of X. Then*

(i) $R(0) = \sqrt{2\pi}\alpha$.

(ii) $\displaystyle \frac{1}{R(0)} \int_0^\infty e^{i\zeta t} R(t) dt = \left[-i\zeta - i\zeta \int_0^\infty e^{i\zeta t} \gamma(t) dt\right]^{-1} \qquad (\mathrm{Im}\ \zeta > 0).$

(iii) $\displaystyle \int_0^\infty R(t) dt = \infty.$

(iv) $\displaystyle X = \lim_{M \to \infty} \frac{1}{R(0)} (\chi_{(0,M)} R) * W$ *as random distributions.*

Theorem 1.3. *Let R be the correlation function of the stationary process X which satisfies* (1.6)–(1.7). *Let $0 < p < 1$, and let l be a slowly varying function. Then the following are equivalent:*

$$\gamma(t) \sim t^{-p} l(t) \quad (t \to \infty), \tag{1.8}$$

$$R(t) \sim \frac{R(0) \sin(p\pi)}{\pi} \cdot \frac{t^{-(1-p)}}{l(t)} \quad (t \to \infty). \tag{1.9}$$

We explain a physical background following Kubo–Toda–Hashitsume [9], pp.35-36. Suppose that the x-component $X(t)$ of velocity of a particle with mass m obeys the equation:

$$m \frac{dX(t)}{dt} = -\beta X(t) - \int_{-\infty}^t \gamma(t-s) \dot{X}(s) ds + W(t). \tag{1.10}$$

Here $W(t)$ is the thermal noise. The positive constants k and T denote the Boltzmann constant and the absolute temperature of the fluid, respectively. Then, the following two formulas must hold:

$$\frac{1}{m(-i\omega + \gamma[\omega])} = \frac{1}{kT} \int_0^\infty E(X(0)X(t)) e^{i\omega t} dt, \tag{1.11}$$

$$m\gamma[\omega] = \frac{1}{kT} \int_0^\infty E(W(0)W(t)) e^{i\omega t} dt, \tag{1.12}$$

where $\gamma[\omega] := [\beta - i\omega \int_0^\infty \gamma(t) e^{i\omega t} dt]/m$. The equality (1.11) is called *the fluctuation-dissipation theorem of the first kind*, and (1.12) *the fluctuation-dissipation theorem of the second kind*. The second is a corollary of the first, and the first can be derived from the linear response theory. The first is a generalization of the Einstein relation, while the second is a generalization of the Nyquist theorem. We note that the equation (1.6) can be formally rewritten as (1.10) with $m = 1$ and $\beta = 0$. If put $m = 1$, and $R(0) = kT$, according to the equipartition law in statistical physics, in (1.10), then we see that (ii) corresponds to (1.11); in view of (i), (1.5) corresponds to (1.12); since the equality (iii) can be seen as $D = R(0)/0+$, it corresponds to the Einstein relation.

The case $\beta > 0$ in (1.10) was first studied mathematically by Okabe [11], and subsequently by Okabe [12], [15], Inoue [5], and Okabe-Inoue [16]. Discrete parameter versions of (1.10) were studied by Okabe [13], [14], Inoue [6] and Okabe–Inoue [16]. Especially in Okabe [15] and Inoue [5], it was shown that if $\gamma(t) \sim t^{-p} l(t)$ as $t \to \infty$ with $0 < p < \infty$ and l slowly varying, then

$$R(t) \sim \frac{R(0)p}{\beta^2} t^{-(p+1)} l(t) \qquad (t \to \infty). \tag{1.13}$$

This slowly decaying tail corresponds to the Alder–Wainwright effects discovered by Alder and Wainwright [1] through molecular dynamic computations. See also Oobayashi–Kohno–Utiyama [17]. We see from Theorem 1.3 that long-time behaviors of different type occur if $\beta = 0$.

Since our case is more singular than the previous case $\beta > 0$, we need some different techniques from the previous works. Among them, the awareness that the random noise W in (1.6) is purely nondeterministic is most important. From this we are naturally led to the idea that general theory of purely nondeterministic, stationary random distributions can be applied. This is shown in sections 4 and 5.

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2. Reflection positivity

In this paper when we say *stationary process*, it always means a real, mean continuous, weakly stationary process with expectation zero, defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $X = (X(t) : t \in \mathbf{R})$ be a purely nondeterministic stationary process. Let R be the correlation function of X: $R(t) = E(X(t)X(0))$ $(t \in \mathbf{R})$. Let Δ be *the spectral density* of X:

$$R(t) = \int_{-\infty}^{\infty} e^{-it\xi} \Delta(\xi) d\xi \qquad (t \in \mathbf{R}). \tag{2.1}$$

The spectral density Δ is a positive, even, and integrable function on $\mathbf{R}$ which satisfies $(1 + \lambda^2)^{-1} \log \Delta(\lambda) \in L^1(\mathbf{R})$. *The outer function* h of X is an outer function in the Hardy class H^2 defined by

$$h(\zeta) = \exp\left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1 + \lambda\zeta}{\lambda - \zeta} \frac{\log \Delta(\lambda)}{1 + \lambda^2} d\lambda\right] \qquad (\mathrm{Im}\ \zeta > 0). \tag{2.2}$$

Since h belongs to H^2, there exists a $\mathrm{l.i.m.}_{\eta \downarrow 0} h(\xi + i\eta)$ in $L^2(\mathbf{R})$, which we also denote by $h(\xi)$. It holds that

$$|h(\xi)|^2 = \Delta(\xi) \qquad (\text{a.e. } \xi \in \mathbf{R}). \tag{2.3}$$

The canonical representation kernel E of X is defined by $E(t) = \hat{h}(t)$, where $\hat{h}$ denotes the Fourier transform of h:

$$\hat{h}(t) = \text{l.i.m.} \int_{-M}^{M} e^{-it\xi} h(\xi) d\xi \qquad (t \in \mathbf{R}). \tag{2.4}$$

It holds that $E(t) = 0$ in $(-\infty, 0)$. Furthermore we have the following equalities:

$$h(\zeta) = \frac{1}{2\pi} \int_{0}^{\infty} e^{i\zeta t} E(t) dt \qquad (\operatorname{Im} \zeta > 0), \tag{2.5}$$

$$R(t) = \frac{1}{2\pi} \int_{0}^{\infty} E(|t| + s) E(s) ds \qquad (t \in \mathbf{R}). \tag{2.6}$$

For the details, we refer to Dym–McKean [3].

We introduce three classes of measures:

$$\Sigma_0 = \{\sigma : \ \sigma \text{ is a non-zero bounded Borel measure on } (0, \infty)\}, \tag{2.7}$$

$$N_0 = \{\nu : \nu \text{ is a non-zero Borel measure on } (0, \infty) \text{ such that } \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{\lambda + \lambda'} \nu(d\lambda) \nu(d\lambda') < \infty\}, \tag{2.8}$$

$$M_0 = \{\mu : \mu \text{ is a non-zero Borel measure on } (0, \infty) \text{ such that } \int_{0}^{\infty} \frac{1}{1 + \lambda} \mu(d\lambda) < \infty\}. \tag{2.9}$$

Note that $\Sigma_0 \subset M_0$ and $N_0 \subset M_0$. For $\sigma \in \Sigma_0$, we define a positive definite function R_σ by

$$R_\sigma(t) = \int_{0}^{\infty} e^{-|t|\lambda} \sigma(d\lambda) \qquad (t \in \mathbf{R}). \tag{2.10}$$

Example 2.1. Let $0 < p < \infty$. If we define a Borel measure σ on $(0, \infty)$ by $\sigma(d\lambda) = \lambda^{p-1} e^{-\lambda} d\lambda / \Gamma(p)$, then we see that σ is in Σ_0 and that $R_\sigma(t) = (1 + |t|)^{-p}$.

For $\mu \in M_0$, we define a function $K_\mu(t)$ in $\mathbf{R}$ by

$$K_\mu(t) = \chi_{(0,\infty)}(t) \int_{0}^{\infty} e^{-\lambda t} \mu(d\lambda) \qquad (t \in \mathbf{R}). \tag{2.11}$$

Note that if ν is in N_0, then K_ν is square integrable on $\mathbf{R}$.

A purely non-deterministic, stationary process X is called *reflection positive* if the correlation function R of X is in the form $R = R_\sigma$ with some $\sigma \in \Sigma_0$. *The diffusion coefficient* D of X is a finite or infinite positive number defined by

$$D = \int_{0}^{\infty} R(t) dt. \tag{2.12}$$

The class $D < \infty$ was studied in Okabe [11], and it was shown that the canonical representation kernel E must be in the form $E = K_\nu$, where ν is an element of N_0 with some regularity conditions. In this section we generalize this fact so as to contain the class $D = \infty$.

For $\sigma \in \Sigma_0$, we define a non-negative function Δ_σ by

$$\Delta_\sigma(\xi) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\lambda}{\lambda^2 + \xi^2} \sigma(d\lambda) \qquad (\xi \in \mathbf{R} \setminus \{0\}). \tag{2.13}$$

Then $\Delta_\sigma(\xi)$ and $(1+\xi^2)^{-1}\log\Delta_\sigma(\xi)$ are both integrable on $\mathbf{R}$ because Δ_σ coincides with the spectral density of a purely nondeterministic stationary process with correlation function R_σ (Theorem 2.1 and Corollary 2.1 in Okabe [10]). For $\sigma \in \Sigma_0$, define an outer function h_σ in H^2 by

$$h_\sigma(\zeta) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+\xi\zeta}{\xi-\zeta}\frac{\log\Delta_\sigma(\xi)}{1+\xi^2}d\xi\right] \quad (\mathrm{Im}\,\zeta > 0). \tag{2.14}$$

Then h_σ coincides with the outer function of a stationary process with correlation function R_σ.

For $\mu \in M_0$ we define a function F_μ by

$$F_\mu(\zeta) = \frac{1}{2\pi}\int_0^\infty \frac{1}{\lambda - i\zeta}\mu(d\lambda) \quad (\mathrm{Im}\,\zeta \geq 0, \zeta \neq 0). \tag{2.15}$$

Note that

$$F_\mu(\zeta) = \frac{1}{2\pi}\int_0^\infty e^{i\zeta t}K_\mu(t)dt \qquad (\mathrm{Im}\,\zeta > 0). \tag{2.16}$$

The following generalization of Theorem A in Okabe [14] plays an important role in our argument.

Theorem 2.2. *For any $\mu \in M_0$, $(1+\lambda^2)^{-1}\log|F_\mu(\lambda)|$ is integrable on $\mathbf{R}$, and it holds that*

$$F_\mu(\zeta) = \exp\left[\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+\lambda\zeta}{\lambda-\zeta}\frac{\log|F_\mu(\lambda)|}{1+\lambda^2}d\lambda\right] \quad (\mathrm{Im}\,\zeta > 0). \tag{2.17}$$

Proof. The proof of this theorem is similar to the proof of Theorem A in Okabe [14] except for its inequality (v). We substitute for (v) the following inequality: for any $R \geq 3$ and $0 \leq \theta \leq \pi$,

$$\left|\mathrm{Log}\ F_\mu(Re^{i\theta})\right| \leq \pi + \left|\log\frac{1}{\pi}\int_0^\infty \frac{1}{1+\lambda}\mu(d\lambda)\right| + \left|\log\frac{\epsilon\mu([\epsilon, M])}{2\pi(M+R)^2}\right|. \tag{2.18}$$

Here Log denotes the principal branch, and positive numbers ϵ and M are chosen so that $\mu([\epsilon, M]) > 0$. The inequality (2.18) follows easily from the following two estimates : for any $R \geq 3$ and $0 \leq \theta \leq \pi$,

$$\begin{aligned}|F_\mu(Re^{i\theta})| &\leq \frac{1}{2\pi}\int_{(0,1)}\frac{1}{R-\lambda}\mu(d\lambda) + \frac{1}{2\pi}\int_{[1,\infty)}\frac{1}{\lambda}\mu(d\lambda)\\ &\leq \frac{1}{\pi}\int_0^\infty \frac{1}{1+\lambda}\mu(d\lambda),\end{aligned}$$

$$\begin{aligned}|F_\mu(Re^{i\theta})| &= \left|\frac{1}{2\pi}\int_0^\infty \frac{(\lambda + R\sin\theta) + iR\cos\theta}{\lambda^2 + 2R\lambda\sin\theta + R^2}\mu(d\lambda)\right|\\ &\geq \frac{1}{2\pi}\int_\epsilon^M \frac{\lambda}{(\lambda+R)^2}\mu(d\lambda) \geq \frac{\epsilon\mu([\epsilon, M])}{2\pi(M+R)^2}.\end{aligned}$$

□

The following proposition can be proved in the same as the proof of Theorem A in Okabe [14].

Proposition 2.3. *Let μ be a Borel measure in M_0. Then*

$$|F_\mu(\xi)|^2 = \frac{1}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+\xi^2}\tau(d\lambda) \qquad (\xi \in \mathbf{R}\setminus\{0\}),$$

where τ is a Borel measure on $(0,\infty)$ defined by

$$\tau(d\lambda) = \frac{1}{2\pi}\left(\int_0^\infty \frac{1}{\lambda+\lambda'}\mu(d\lambda')\right)\mu(d\lambda).$$

Now we establish a one-to-one correspondence between Σ_0 and N_0 in the following two theorems. Define a map $S : N_0 \ni \nu \mapsto \sigma = S(\nu) \in \Sigma_0$ by

$$\sigma(d\lambda) = \frac{1}{2\pi}\left(\int_0^\infty \frac{1}{\lambda+\lambda'}\nu(d\lambda')\right)\nu(d\lambda). \tag{2.19}$$

Theorem 2.4. *Let $\nu \in N_0$ and $\sigma \in \Sigma_0$. Then $\sigma = S(\nu)$ if and only if $h_\sigma(\zeta) = F_\nu(\zeta)$ for any* Im $\zeta > 0$.

Proof. Let $\nu \in N_0$. We put $\sigma = f(\nu)$. It follows from Proposition 2.3 that $|F_\nu(\xi)|^2 = \Delta_\sigma(\xi)$ for any $\xi \in \mathbf{R}\setminus\{0\}$. Then by Theorem 2.2 we see that $F_\nu(\zeta) = h_\sigma(\zeta)$ for any Im $\zeta > 0$.

Conversely, let $\nu \in N_0$ and $\sigma \in \Sigma_0$, and assume that $F_\nu(\zeta) = h_\sigma(\zeta)$ for any Im $\zeta > 0$. Letting $\eta \downarrow$ in $\zeta = \xi + i\eta$, we have $F_\nu(\xi) = h_\sigma(\xi)$ for any $\xi \in \mathbf{R}\setminus\{0\}$. We put $\sigma' = S(\nu)$. Then, for almost every $\xi \in \mathbf{R}$,

$$\Delta_\sigma(\xi) = |h_\sigma(\xi)|^2 = |F_\nu(\xi)|^2 = \Delta_{\sigma'}(\xi),$$

so that the Laplace transforms of σ and σ' coincide with each other because they are the Fourier transforms of Δ_σ and $\Delta_{\sigma'}$, respectively. By the uniqueness of the Laplace transform we see that $\sigma = \sigma' = S(\nu)$. □

Theorem 2.5. *The map S is a bijection from N_0 onto Σ_0.*

Proof. In view of Theorem 2.4, the injectivity of S follows from the uniqueness of Stieltjes transform.

Conversely, let σ be a Borel measure in Σ_0. Then the same argument as the proof of Theorem 5.1 in Okabe [11] shows the existence of a Borel measure ν on $[0,\infty)$ such that $\int_{[0,\infty)}(1+\lambda)^{-1}\nu(d\lambda) < \infty$ and

$$\frac{1}{2\pi}\int_{[0,\infty)} \frac{1}{\lambda - i\zeta}\nu(d\lambda) = h_\sigma(\zeta) \qquad (\mathrm{Im}\zeta > 0).$$

In view of Theorem 2.4, to obtain the result, it is enough to show that $\nu(\{0\}) = 0$ and

$$\int_0^\infty\int_0^\infty \frac{1}{\lambda+\lambda'}\nu(d\lambda)\nu'(d\lambda) < \infty. \tag{2.20}$$

Letting $\eta \downarrow$ in $\zeta = \eta + i\xi$, we obtain

$$\left|\frac{1}{2\pi}\int_{[0,\infty)} \frac{1}{\lambda - i\xi}\nu(d\lambda)\right|^2 = \Delta_\sigma(\xi) \qquad (\xi \in \mathbf{R}\setminus\{0\}). \tag{2.21}$$

Since

$$\left|\int_{[0,\infty)} \frac{1}{\lambda - i\xi}\nu(d\lambda)\right| \geq \int_{[0,\infty)} \frac{|\xi|}{\lambda^2+\xi^2}\nu(d\lambda) \geq \frac{\nu(\{0\})}{|\xi|},$$

it follows from the integrability of Δ_σ that $\nu(\{0\}) = 0$. Now if we define a Borel measure σ' on $(0, \infty)$ by the right hand side of (2.19), then it follows from Proposition 2.3 and (2.21) that

$$\Delta_\sigma(\xi) = \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 + \xi^2} \sigma'(d\lambda) \qquad (\xi \in \mathbf{R} \setminus \{0\}),$$

and so

$$\sigma'((0,\infty)) = \int_0^\infty \left(\frac{1}{\pi} \int_{-\infty}^\infty \frac{\lambda}{\lambda^2 + \xi^2} d\xi \right) \sigma'(d\lambda) = \int_{-\infty}^\infty \Delta_\sigma(\xi) d\xi < \infty.$$

Thus we obtain (2.20). This completes the proof. □

Theorem 2.6. *A purely nondeterministic stationary process X is reflection positive if and only if its canonical representation kernel E of X is in the form $E = K_\nu$ with some $\nu \in N_0$. Furthermore if the correlation function of X is equal to R_σ with $\sigma \in \Sigma_0$, then the canonical representation kernel E is equal to K_ν with $\nu = S^{-1}(\sigma)$.*

Proof. Let X be a purely non-deterministic stationary process.

Let $\nu \in N_0$, and assume that E_ν is the canonical representation kernel of X. Let R be the correlation function of X. Then by (2.6) we have $R = R_\sigma$ with $\sigma = f(\nu)$; thus X is reflection positive.

Conversely, assume that the correlation function R of X is in the form $R = R_\sigma$ with $\sigma \in \Sigma_0$. If we put $\nu = f^{-1}(\sigma)$, then by Theorem 2.4 we have

$$h_\sigma(\zeta) = F_\nu(\zeta) = \frac{1}{2\pi} \int_0^\infty e^{i\zeta t} K_\nu(t) dt \qquad (\mathrm{Im}\ \zeta > 0).$$

This, together with the uniqueness of the Laplace transform, shows that $E_\nu(t)$ is the canonical representation kernel of X. This completes the proof. □

3. KMO-Langevin data

We define a subset Σ_∞ of Σ_0 by

$$\Sigma_\infty = \{\sigma \in \Sigma_0 : \int_0^\infty \frac{1}{\lambda} \sigma(d\lambda) = \infty\}. \tag{3.1}$$

Since the condition $\int_0^\infty \lambda^{-1} \sigma(d\lambda) = \infty$ implies $\int_0^\infty R_\sigma(t) dt = \infty$, this subset Σ_∞ corresponds to reflection positive, purely nondeterministic stationary processes with divergent diffusion coefficients. We introduce a set $\mathcal{L}_\infty$ consisting of pairs (α, ρ) with a positive number α and a Borel measure ρ on $(0, \infty)$ such that $\int_0^\infty K_\rho(t) dt = \infty$:

$$\mathcal{L}_\infty = \left\{ (\alpha, \rho) \in \mathbf{R}^+ \times M_0 : \int_0^\infty \frac{1}{\lambda} \rho(d\lambda) = \infty \right\}. \tag{3.2}$$

Theorem 3.1. *The relation*

$$\int_0^\infty \frac{1}{\lambda - i\zeta} \sigma(d\lambda) = \sqrt{2\pi} \alpha \left[-i\zeta - i\zeta \int_0^\infty \frac{1}{\lambda - i\zeta} \rho(d\lambda) \right]^{-1} \qquad (\mathrm{Im}\ \zeta > 0) \tag{3.3}$$

determines a bijection $\sigma \to L(\sigma) = (\alpha, \rho)$ from Σ_∞ onto $\mathcal{L}_\infty$.

Proof. If we assume that a bounded Borel measure σ on $[0,\infty)$ satisfies (3.3) together with a positive number α and a Borel measure ρ on $[0,\infty)$, then we easily get the following two correspondences between σ and ρ:

$$\int_{[0,\infty)} \frac{1}{\lambda}\sigma(d\lambda) = \infty \Leftrightarrow \rho(\{0\}) = 0, \qquad \sigma(\{0\}) = 0 \Leftrightarrow \int_{[0,\infty)} \frac{1}{\lambda}\rho(d\lambda) = \infty. \tag{3.4}$$

Here the integrand λ^{-1} is assumed to be $+\infty$ at $\lambda = 0$.

Let σ be a Borel measure in Σ_∞. If we put $\sigma_n(d\lambda) = \chi_{[1/n,\infty)}(\lambda)\sigma(d\lambda)$ for $n = 1, 2, \cdots$, then, by Theorem 8.5 in Okabe [11], there exists for each n a triplet $(\alpha_n, \beta_n, \rho_n)$ of two positive numbers α_n, β_n and a Borel measure $\rho_n \in M_0$ such that

$$F_{\sigma_n}(\zeta) \cdot (\beta_n - i\zeta - 2\pi i\zeta F_{\rho_n}(\zeta)) = \alpha_n/\sqrt{2\pi}. \tag{3.5}$$

In view of Theorem 8.2 in Okabe [11], α_n tends to the limit $\alpha := \sigma([0,\infty))/\sqrt{2\pi}$ as $n \to \infty$ and β_n tends to zero as $n \to \infty$, so that, by taking $\zeta = i$ in (3.5), we get the estimate

$$\sup_n \int_0^\infty 1/(1+\lambda)\rho_n(d\lambda) < \infty. \tag{3.6}$$

By (3.6), there exist a subsequence $n_1 < n_2 < \cdots$ and a bounded Borel measure $\tilde{\rho}$ on $[0,\infty]$ such that $(1+\lambda)^{-1}\rho_{n_k}(d\lambda)$ weakly converges to $\tilde{\rho}$ on $[0,\infty]$ as $k \to \infty$. Transition to the limit $k \to \infty$ yields

$$F_\sigma(\zeta) \cdot (c_1 - ic_2\zeta - 2\pi i\zeta F_\rho(\zeta)) = \alpha/\sqrt{2\pi},$$

where $c_1 = \tilde{\rho}(\{0\})$, $c_2 = 1 + \tilde{\rho}(\{\infty\})$ and $\rho(d\lambda) = \chi_{(0,\infty)}(1+\lambda)\tilde{\rho}(d\lambda)$. Since we have by definition,

$$\int_0^\infty \frac{1}{\lambda}\sigma(d\lambda) = \infty, \qquad \alpha = \sigma([0,\infty))/\sqrt{2\pi},$$

it follows that $c_1 = 0$, $c_2 = 1$. Thus we obtain (3.3) . Combining this with (3.4), we see that there exists a pair (α, ρ) in $\mathcal{L}_\infty$ which satisfies (3.3) with σ. Furthermore, it follows from the uniqueness of the Stieltjes transform that such a pair (α, ρ) is uniquely determined by σ; the map L in the theorem is thus well-defined. The uniqueness of Stieltjes transform proves also that L is one-to-one.

Finally we prove the onto property of L. Let (α, ρ) be a pair in $\mathcal{L}_\infty$, and choose $\beta_n > 0$ so that β_n decreases to 0 as $n \to \infty$. By Theorem 8.5 in Okabe [11], there exists for each n a bounded Borel measure σ_n on $[0,\infty)$ which satisfies (3.5) with $\alpha_n = \alpha$, $\rho_n = \rho$. Theorem 8.2 in Okabe [11] implies the equality $\sigma_n([0,\infty)) = \sqrt{2\pi}\alpha$, so that there exist a subsequence $n_1 < n_2 < \cdots$ and a bounded Borel measure σ' on $[0,\infty]$ such that σ_{n_k} converges to σ' weakly on $[0,\infty]$ as $k \to \infty$. If we put $\sigma(A) = \sigma'(A \cap [0,\infty))$, then the triplet of σ, α and ρ satisfies (3.3), and so, by (3.4), σ is in Σ_∞. We have thus shown the existence of an element σ of Σ_∞ which satisfies (3.3) with (α, ρ), and so, the map L is onto. This completes the proof. □

Theorem 3.2. *Let $\sigma \in \Sigma_\infty$, and $(\alpha, \rho) = L(\sigma)$. Then $\sigma((0,\infty)) = \sqrt{2\pi}\alpha$.*

Proof. Take $\zeta = i\eta$ in (3.3). Then

$$\int_0^\infty \frac{\eta}{\lambda+\eta}\sigma(d\lambda) = \sqrt{2\pi}\alpha\left(1+\int_0^\infty \frac{1}{\lambda+\eta}\rho(d\lambda)\right)^{-1} \qquad (\eta > 0),$$

so letting $\eta \uparrow \infty$, we obtain the result. $\square$

According to Definition 8.3 in Okabe [11], we give the following definition.

Definition 3.3. Let σ be a Borel measure in Σ_∞. Then we call the pair $(\alpha, \rho) = L(\sigma) \in \mathcal{L}_\infty$ *the second KMO-Langevin data* associated with σ.

4. Kubo noise

Let $\mathcal{D}(\mathbf{R})$ be the space of all functions of class C^∞ on $\mathbf{R}$ with compact support. We introduce the same topology on $\mathcal{D}(\mathbf{R})$ as in the theory of distributions. In this paper when we say *stationary random distribution*, it always means a real, weakly stationary random distribution with expectation zero defined on a probability space $(\Omega, \mathcal{F}, P)$, i.e., a system of random variables $\{Y(\phi) : \phi \in \mathcal{D}(\mathbf{R})\}$ such that

(i) the mapping $\mathcal{D}(\mathbf{R}) \ni \phi \mapsto Y(\phi) \in L^2(\Omega)$ is linear, and continuous in the L^2-sense;
(ii) $Y(\phi) = \overline{Y(\overline{\phi})}$ for any $\phi \in \mathcal{D}(\mathbf{R})$;
(iii) $E(Y(\phi)) = 0$ and $E(Y(\tau_h\phi)\overline{Y(\tau_h\psi)}) = E(Y(\phi)\overline{Y(\psi)})$ for any $\phi, \psi \in \mathcal{D}(\mathbf{R})$ and $h \in \mathbf{R}$.

Here $\overline{a}$ denotes the complex conjugate of a, and τ_h denotes the shift transformation:

$$\tau_h\phi(t) = \phi(t+h). \tag{4.1}$$

We refer to Itô [7] for the details. For any stationary random distribution Y and $t \in \mathbf{R}$, $M_t(Y)$ denotes the closed linear hull of

$$\{Y(\phi) : \phi \in \mathcal{D}(\mathbf{R}),\ \mathrm{supp}\ \phi \subset (-\infty, t]\}$$

in $L^2(\Omega)$. A stationary random distribution Y is called *purely nondeterministic* if it holds that

$$\bigcap_{t\in\mathbf{R}} M_t(Y) = \{0\}. \tag{4.2}$$

See Rozanov [18].

Let X be a purely nondeterministic, reflection positive, stationary Gaussian process with divergent diffusion coefficient. In this section, we introduce for X a purely nondeterministic, stationary Gaussian random distribution I which corresponds to the Kubo noise constructed by Okabe [11] when the diffusion coefficient of the process is finite. Let R, E and h be the correlation function, canonical representation kernel and outer function of X, respectively. We put

$$h_I(\zeta) = \frac{1}{\sqrt{2\pi}}\frac{h(\zeta)}{R[\zeta]} \qquad (\mathrm{Im}\ \zeta > 0), \tag{4.3}$$

where $R[\zeta]$ denotes the Fourier–Laplace transform of R:

$$R[\zeta] = \frac{1}{2\pi}\int_0^\infty e^{i\zeta t}R(t)dt \qquad (\mathrm{Im}\ \zeta > 0). \tag{4.4}$$

Let σ be a Borel measure in Σ_∞ such that $R = R_\sigma$. We put $\nu = S^{-1}(\sigma) \in N_0$, where S is the map introduced in section 2. Then, by Theorem 2.6, E is equal to K_ν. Therefore we have

$$h_I(\zeta) = \frac{1}{\sqrt{2\pi}}\frac{F_\nu(\zeta)}{F_\sigma(\zeta)} \qquad (\mathrm{Im}\ \zeta > 0). \tag{4.5}$$

By (4.5), the boundary value $h_I(\xi) := \lim_{\eta\downarrow 0} h(\xi + i\eta)$ exists for any $\xi \neq 0$ and it holds that

$$h_I(\xi) = \frac{1}{\sqrt{2\pi}}\frac{F_\nu(\xi)}{F_\sigma(\xi)} \qquad (\xi \in \mathbf{R} \setminus \{0\}). \tag{4.6}$$

Set $(\alpha, \rho) = L(\sigma) \in \mathcal{L}_\infty$. We define a function Δ_I by

$$\Delta_I(\xi) = \frac{\sqrt{2\pi}}{\pi\alpha}\int_0^\infty \frac{\xi^2}{\lambda^2+\xi^2}\rho(d\lambda) \qquad (\xi \in \mathbf{R},\ \xi \neq 0). \tag{4.7}$$

Proposition 4.1. *It holds that*

(i) $\dfrac{\Delta_I(\xi)}{1+\xi^2} \in L^1(\mathbf{R})$,

(ii) $\dfrac{\log \Delta_I(\xi)}{1+\xi^2} \in L^1(\mathbf{R})$,

(iii) $|h_I(\xi)|^2 = \Delta_I(\xi) \qquad (\xi \in \mathbf{R},\ \xi \neq 0)$.

Proof. A simple calculation shows that

$$\int_{-\infty}^\infty \frac{\Delta_I(\xi)}{1+\xi^2}d\xi = \frac{\sqrt{2\pi}}{\alpha}\int_0^\infty \frac{1}{1+\lambda}\rho(d\lambda) < \infty.$$

Thus (i) follows. The proof of (iii) is almost identical to the one of Lemma 8.3 in Okabe [11]. By (iii) and (4.6), we have

$$\Delta_I(\xi) = \frac{1}{2\pi}\frac{|F_\nu(\xi)|^2}{|F_\sigma(\xi)|^2} \qquad (\xi \in \mathbf{R}\setminus\{0\}),$$

so that (ii) follows from Theorem 2.2. □

Theorem 4.2. *It holds that*

$$h_I(\zeta) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{1+\lambda\zeta}{\lambda-\zeta}\frac{\log\Delta_I(\lambda)}{1+\lambda^2}d\lambda\right] \qquad (\mathrm{Im}\ \zeta > 0). \tag{4.8}$$

Proof. Since both σ and ν are in M_0, Theorem 2.2 can be applied to F_ν and F_σ. Therefore by (4.5) and (4.6) we have

$$h_I(\zeta) = \exp\left[\frac{1}{\pi i}\int_{-\infty}^\infty \frac{1+\lambda\zeta}{\lambda-\zeta}\frac{\log|h_I(\xi)|}{1+\lambda^2}d\lambda\right] \qquad (\mathrm{Im}\ \zeta > 0),$$

so the theorem follows from Proposition 4.1 (iii). □

Once we obtain (4.8) with Proposition 4.1 (i), (ii), then we are immediately led to a purely nondeterministic, stationary Gaussian random distribution I which has spectral density Δ_I. Though this procedure seems standard, we explain it for reader's convenience, following the survey Hida–Maruyama–Nishio [4] from which the author learned the theory. For any purely nondeterministic, stationary

Gaussian process Y with canonical representation kernel E_Y, there exists a unique Brownian motion $(B(t) : t \in \mathbf{R})$ such that $B(0) = 0$ and

$$M_t(Y) = M_t(\dot{B}) \qquad (t \in \mathbf{R}), \tag{4.9}$$

$$Y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} E_Y(t-s)dB(s) \qquad (t \in \mathbf{R}). \tag{4.10}$$

This $(B(t))$ is called *the canonical Brownian motion* of Y (cf. Karhunen [8]). Now we note that

$$\frac{1}{1-i\zeta} = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1+\lambda\zeta}{\lambda-\zeta}\frac{\log\{1/(1+\lambda^2)\}}{1+\lambda^2}d\lambda\right] \qquad (\mathrm{Im}\,\zeta > 0); \tag{4.11}$$

a simple proof of this equality is to take a Dirac measure δ_1 with unit mass at 1 in Theorem 2.2. By (4.8) and (4.11), we have

$$\frac{h_I(\zeta)}{1-i\zeta} = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1+\lambda\zeta}{\lambda-\zeta}\frac{\log\{\Delta_I(\lambda)/(1+\lambda^2)\}}{1+\lambda^2}d\lambda\right] \qquad (\mathrm{Im}\,\zeta > 0), \tag{4.12}$$

so that $h_I(\zeta)/(1-i\zeta)$ is an outer function in the Hardy class H^2 (cf. Dym–McKean [3]). In particular, the boundary value $h_I(\xi)/(1-i\xi)$ is square integrable on $\mathbf{R}$. Therefore if we are given a one-dimensional Brownian motion $(B(t) : t \in \mathbf{R})$, then we can define a stationary Gaussian random distribution I by the following equality: for any $\phi \in \mathcal{D}(\mathbf{R})$,

(4.13)
$$I(\phi) = \sqrt{2\pi}\int_{-\infty}^{\infty} \left(\frac{h_I(\xi)}{1-i\xi}\cdot(1-i\xi)\hat{\phi}(\xi)\right)\check{}(t)dB(t) = \sqrt{2\pi}\int_{-\infty}^{\infty}(h_I\hat{\phi})\check{}(t)dB(t).$$

It follows from (4.13) and Proposition 4.1 (iii) that

$$E(I(\phi)\overline{I(\psi)}) = \int_{\mathbf{R}} \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\Delta_I(\xi)d\xi \qquad (\phi, \psi \in \mathcal{D}(\mathbf{R})), \tag{4.14}$$

which implies that Δ_I is the spectral density of I. Of particular importance is the following causal property:

$$M_t(I) = M_t(\dot{B}) \qquad (t \in \mathbf{R}). \tag{4.15}$$

In particular, we see from (4.15) that I is purely nondeterministic. We can prove (4.15) as follows. Set $e(t) = \exp(t)$ $(t \le 0)$, $= 0$ $(t > 0)$, and set $Y(\phi) = I(e * \phi)$ for any $\phi \in \mathcal{D}(\mathbf{R})$. Then we see that $\{Y(\phi)\}$ is equivalent to a stationary process with $h_I(\zeta)/(1-i\zeta)$ as outer function and $(B(t) - B(0))$ as canonical Brownian motion. Therefore $M_t(Y) = M_t(\dot{B})$ for any $t \in \mathbf{R}$. On the other hand, since $I(\phi) = Y(\phi - d\phi/dt)$ for any $\phi \in \mathcal{D}(\mathbf{R})$, it holds that $M_t(I) = M_t(Y)$ $(t \in \mathbf{R})$. Hence we obtain (4.15). We finish quoting from Hida–Maruyama–Nishio [4] with this.

Definition 4.3. Let σ be a Borel measure in Σ_∞, and let X be a stationary Gaussian process with correlation function R_σ. If $B = (B(t) : t \in \mathbf{R})$ is the canonical Brownian motion of X, then we call the purely nondeterministic, stationary Gaussian random distribution I defined by (4.13) the *Kubo noise* of X.

Theorem 4.4. *Let X be a purely nondeterministic, reflection positive,stationary Gaussian process with divergent diffusion coefficient, and let I be the Kubo noise of X. Then for any $t \in \mathbf{R}$, $M_t(X) = M_t(I)$.*

Proof. The theorem follows immediately from (4.15) and (4.9). $\square$

In the following we are concerned with the representation of X in terms of the Kubo noise I of X. If the diffusion coefficient of X is finite, this is carried out in Theorem 8.3 in Okabe [11]; this representation reads formally as

$$X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} R(t-s) I(s) ds. \tag{4.16}$$

We will see below that a similar representation holds even if the diffusion coefficient is infinite, that is, even if R is not integrable; in so doing we need to interpret the integral in the right hand side of (4.16) as an improper integral.

Let k be an integrable function on $\mathbf{R}$, and let Y be a stationary random distribution. Then we have

$$E\left[\left(\int_{-\infty}^{\infty} |k(t) Y(\tau_t \phi)| dt\right)^2\right] \le \|k\|_1^2 \cdot \|\phi\|_2^2 < \infty \qquad (\phi \in \mathcal{D}(\mathbf{R})).$$

Therefore we can define another stationary random distribution $k * Y$ by

$$(k * Y)(\phi) = \int_{-\infty}^{\infty} k(s) Y(\tau_s \phi) ds \qquad (\phi \in \mathcal{D}(\mathbf{R})). \tag{4.17}$$

We call $k * Y$ *the convolution* of k and Y. If Y is in the form

$$Y(\phi) = \sqrt{2\pi} \int_{-\infty}^{\infty} (g \cdot \hat{\phi})\tilde{}(t) dB(t) \tag{4.18}$$

with some g which satisfies

$$\int_{-\infty}^{\infty} \frac{|g(\xi)|^2}{(1+\xi^2)^k} d\xi < \infty \tag{4.19}$$

for some $k \in \mathbf{N} \cup \{0\}$, then we have a similar representation for $k * Y$ as follows.

Lemma 4.5. *Let $k \in L^1(\mathbf{R})$, and $f \in L^2(\mathbf{R})$. Then*

$$\int_{-\infty}^{\infty} k(s) \left\{\int_{-\infty}^{\infty} f(t+s) dB(t)\right\} ds = \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} k(s) f(t+s) ds\right\} dB(t). \tag{4.20}$$

Proof. If we put

$$Z(s) := \int_{-\infty}^{\infty} f(t+s) dB(t), \qquad h(t) := \int_{-\infty}^{\infty} k(s) f(t+s) ds,$$

then we have

$$E[Z(s)^2] = \|f\|_2^2, \qquad \|h\|_2 = \|k\|_1 \cdot \|f\|_2.$$

In particular, both sides of (4.20) are well-defined. Now

$$E\left[\left|\int_{-\infty}^{\infty} k(s) Z(s) ds - \int_{-\infty}^{\infty} h(t) dB(t)\right|^2\right] = I_1 - 2\mathrm{Re}(I_2) + I_3, \tag{4.21}$$

where

$$I_1 = E\left[\left|\int_{-\infty}^{\infty} k(s)Z(s)ds\right|^2\right],$$

$$I_2 = E\left[\left\{\int_{-\infty}^{\infty} k(s)Z(s)ds\right\}\left\{\int_{-\infty}^{\infty} \overline{h(t)}dB(t)\right\}\right],$$

$$I_3 = E\left[\left|\int_{-\infty}^{\infty} h(t)dB(t)\right|^2\right].$$

Since

$$\int_{-\infty}^{\infty} |k(s)| \cdot E[|Z(s)\int_{-\infty}^{\infty} h(t)dB(t)|]ds \le \|f\|_2 \cdot \|h\|_2 \cdot \|k\|_1 < \infty,$$

Fubini–Tonelli theorem shows that

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} k(s) \cdot E[Z(s)\int_{-\infty}^{\infty} \overline{h(t)}dB(t)]ds \\ &= \iiint_{\mathbf{R}^3} k(s_1)\overline{k(s_2)}f(t+s_1)\overline{f(t+s_2)}ds_1ds_2dt. \end{aligned}$$

Similarly we have $I_1 = I_3 = I_2$. Thus it follows from (4.21) that

$$E\left[\left|\int_{-\infty}^{\infty} k(s)Z(s)ds - \int_{-\infty}^{\infty} h(t)dB(t)\right|^2\right] = 0.$$

This completes the proof. □

For any function $f \in L^2(\mathbf{R})$, $\widetilde{f}$ denotes the inverse Fourier transform of f:

$$\widetilde{f}(t) = \underset{M\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-M}^{M} e^{it\xi}f(\xi)d\xi. \tag{4.22}$$

Proposition 4.6. *Let $k \in L^1(\mathbf{R})$, and let Y be a stationary random distribution of the form* (4.18) *with some g which satisfies* (4.19) *for some $k \in \mathbf{N} \cup \{0\}$. Then*

$$(k * Y)(\phi) = (2\pi)^{3/2}\int_{-\infty}^{\infty} (\widetilde{k} \cdot g \cdot \hat{\phi})\tilde{}(t)dB(t) \qquad (\phi \in \mathcal{D}(\mathbf{R})). \tag{4.23}$$

Proof. By (4.19), the function $g(\xi)\hat{\phi}(\xi)$ is in $L^2(\mathbf{R})$. Since

$$(g \cdot \widehat{\tau_s\phi})\tilde{}(t) = (g \cdot \hat{\phi})\tilde{}(t+s),$$

it follows from Lemma 4.5 that

$$(k * Y)(\phi) = \sqrt{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} k(s)(g\cdot\hat{\phi})\tilde{}(t+s)ds\right\}dB(t).$$

A simple calculation shows that

$$\int_{-\infty}^{\infty} k(s)(g\cdot\hat{\phi})\tilde{}(t+s)ds = (2\pi)(\widetilde{k}\cdot g\cdot\widehat{\phi})\tilde{}(t).$$

Thus we obtain (4.23). □

Theorem 4.7. *Let X be a stationary process whose correlation function R is in the form $R = R_\sigma$ with $\sigma \in \Sigma_\infty$. Let I be the Kubo noise of X. Then as random distributions,*

$$X = \lim_{M\to\infty} \frac{1}{\sqrt{2\pi}}(\chi_{(0,M)}R) * I. \tag{4.24}$$

Proof. The inverse Fourier transform of $\chi_{(0,M)}(t)R(t)$ is equal to $F_\sigma(\xi)(1 - q_M(\xi))$, where

$$q_M(\xi) = \frac{e^{i\xi M}}{2\pi F_\sigma(\xi)} \int_0^\infty \frac{e^{-\lambda M}}{\lambda - i\xi}\sigma(d\lambda) \qquad (\xi \neq 0).$$

We put $\nu = S^{-1}(\sigma)$. Then by (4.6), (4.13) and Proposition 4.6, for any $\phi \in \mathcal{D}(\mathbf{R})$,

$$E[|X(\phi) - (2\pi)^{-1/2}\{(\chi_{(0,M)}R) * I\}(\phi)|^2] = (2\pi)^{-1}\int_{-\infty}^\infty |q_M(\xi)|^2 |F_\nu(\xi)\hat{\phi}(\xi)|^2 d\xi.$$

First we have $\lim_{M\to\infty} q_M(\xi) = 0$ and $|F_\nu(\xi)\hat{\phi}(\xi)|^2 \in L^1(\mathbf{R})$. Since we have for any $\xi \neq 0$,

$$\begin{aligned} \left|\int_0^\infty \frac{e^{-\lambda M}}{\lambda - i\xi}\sigma(d\lambda)\right| &= \left|\int_0^\infty \frac{\lambda e^{-\lambda M}}{\lambda^2 + \xi^2}\sigma(d\lambda) + i\int_0^\infty \frac{\xi e^{-\lambda M}}{\lambda^2+\xi^2}\sigma(d\lambda)\right| \\ &\leq \left|\int_0^\infty \frac{\lambda}{\lambda^2+\xi^2}\sigma(d\lambda) + i\int_0^\infty \frac{\xi}{\lambda^2+\xi^2}\sigma(d\lambda)\right| \\ &= |2\pi F_\sigma(\xi)|, \end{aligned} \tag{4.25}$$

it holds that $|q_M(\xi)| \leq 1$. Therefore Lebesgue's dominated convergence theorem yields

$$\lim_{M\to\infty} E[|X(\phi) - (2\pi)^{-1/2}\{(\chi_{(0,M)}R) * I\}(\phi)|^2] = 0,$$

which implies (4.24). This completes the proof. $\square$

5. KMO-Langevin equations

In this section, we are concerned with an equation which describes the time evolution of a stationary process with correlation function R_σ for some $\sigma \in \Sigma_\infty$. We shall adopt the Kubo noise as random force in the equation; the equation with white noise as random force will be discussed in our forthcoming paper.

Proposition 5.1. *Let μ be a Borel measure in M_0, and let Y be a stationary random distribution of the form* (4.18) *with some g which satisfies* (4.19) *for some $k \in \mathbf{N} \cup \{0\}$. Then for any $\phi \in \mathcal{D}(\mathbf{R})$,*

$$\text{l.i.m.}_{M\to\infty}\, \{(\chi_{(0,M)}K_\mu) * \dot{Y}\}(\phi) = -(2\pi)^{3/2}\int_{-\infty}^\infty \{i\xi F_\mu(\xi)g(\xi)\hat{\phi}(\xi)\}\tilde{\ }(t)dB(t). \tag{5.1}$$

Proof. First we see that $(1+\xi^2)|g(\xi)\hat{\phi}(\xi)|^2 \in L^1(\mathbf{R})$. For any $\xi \neq 0$,

$$\begin{aligned} |\xi F_\mu(\xi)| &\leq \int_{(0,1)} \frac{|\xi|}{|\lambda - i\xi|}\mu(d\lambda) + \int_{[1,\infty)} \frac{|\xi|}{|\lambda - i\xi|}\mu(d\lambda) \\ &\leq \int_{(0,1)} \mu(d\lambda) + |\xi|\int_{[1,\infty)} \frac{1}{\lambda}\mu(d\lambda) \leq (1+\xi^2)^{1/2}\int_0^\infty \frac{2}{1+\lambda}\mu(d\lambda), \end{aligned}$$

so $|\xi F_\mu(\xi)|^2/(1+\xi^2)$ is bounded. In particular, the right-hand side of (5.1) is well-defined. The inverse Fourier transform of $\chi_{(0,M)}K_\mu$ is equal to $F_\mu(\xi) - q_M(\xi)$, where

$$q_M(\xi) = \frac{e^{i\xi M}}{2\pi}\int_0^\infty \frac{e^{-\lambda M}}{\lambda - i\xi}\mu(d\lambda) \qquad (\xi \neq 0),$$

and hence by Proposition 4.6

$$\begin{aligned}E[|\{(\chi_{(0,M)}K_\mu) * \dot{X}\}(\phi) + (2\pi)^{3/2}\int_{-\infty}^\infty \{i\xi F_\mu(\xi)g(\xi)\hat{\phi}(\xi)\}\tilde{}(t)dB(t)|^2] \\ = 2\pi\int_{-\infty}^\infty |\xi q_M(\xi)g(\xi)\hat{\phi}(\xi)|^2 d\xi.\end{aligned}$$

We see that $q_M(\xi) \to 0$ as $M \to \infty$. The same estimate as (4.25) yields $|q_M(\xi)| \leq |F_\mu(\xi)|$ $(\xi \neq 0)$, and so $|\xi q_M(\xi)|^2/(1+\xi^2)$ is bounded. Therefore, by virtue of Lebesgue's dominated convergence theorem, we obtain (5.1). $\square$

Let μ and Y be as in Proposition 5.1. We define a stationary random distribution $K_\mu * \dot{Y}$ by

$$K_\mu * \dot{Y} = \lim_{M\to\infty}(\chi_{(0,M)}K_\mu) * \dot{Y}, \tag{5.2}$$

and we call it the *convolution* of K_μ and $\dot{Y}$.

Remark 5.2. We can show that $\lim_{M\to\infty}(\chi_{(0,M)}K_\mu) * \dot{Y}$ exists for any stationary random distribution Y by using the spectral representation of Y (see Itô [7]) instead of (4.18). The proof is similar to the one of Proposition 5.1.

Let σ be a Borel measure in Σ_∞, and let X be a stationary Gaussian process with correlation function $R = R_\sigma$. Let $(\alpha, \rho) = L(\sigma)$ be the KMO-Langevin data associated with σ, and let I be the Kubo noise of X. We put $\gamma(t) = K_\rho(t)$ $(t \in \mathbf{R})$.

Theorem 5.3. *As random distributions,*

$$\dot{X} = -\gamma * \dot{X} + \alpha I. \tag{5.3}$$

Proof. We put $\nu = S^{-1}(\sigma)$. Then by Proposition 5.1 we have for any $\phi \in \mathcal{D}(\mathbf{R})$,

$$\begin{aligned}\dot{X}(\phi) + \underset{M\to\infty}{\text{l.i.m.}}\ \{(\chi_{(0,M)}\gamma) * \dot{X}\}(\phi) - \alpha I(\phi) \\ = -\sqrt{2\pi}\int_{-\infty}^\infty \left\{\left(i\xi + 2\pi i\xi F_\rho(\xi) + \frac{\alpha}{\sqrt{2\pi}F_\sigma(\xi)}\right)F_\nu(\xi)\hat{\phi}(\xi)\right\}\tilde{}(t)dB(t).\end{aligned}$$

Since $(\alpha, \rho) = L(\sigma)$ says

$$i\xi + 2\pi i\xi F_\rho(\xi) + \frac{\alpha}{\sqrt{2\pi}F_\sigma(\xi)} = 0 \qquad (\xi \in \mathbf{R}\setminus\{0\}), \tag{5.4}$$

we obtain (5.3). $\square$

According to Definition 8.2 in Okabe [11], we give the following definition.

Definition 5.4. We call (5.3) *the second KMO-Langevin equation.*

So far we have started from a stationary Gaussian process with correlation function R_σ for some $\sigma \in \Sigma_\infty$. Now we change our starting point; we start from a pair $(\alpha, \rho) \in \mathcal{L}_\infty$. We put $\gamma(t) = K_\rho(t)$ $(t \in \mathbf{R})$, $\sigma = L^{-1}(\alpha, \rho) \in \Sigma_\infty$ and $\nu = S^{-1}(\sigma) \in N_0$. Define a function Δ_W by

$$\Delta_W(\xi) = 2\sqrt{2\pi}\alpha \mathrm{Re}\left[-i\xi - \frac{i\xi}{2\pi}\int_0^{\infty-} e^{i\xi t}\gamma(t)dt\right] \qquad (\xi \in \mathbf{R} \setminus \{0\}), \tag{5.5}$$

where $\int^{\infty-}$ denotes the improper integral;

$$\int^{\infty-} f(t)dt = \lim_{M\to\infty}\int^{M} f(t)dt.$$

Since γ is non-increasing and in $L^1_{loc}[0,\infty)$, $\int_0^{\infty-} e^{i\xi t}\gamma(t)dt$ exists for any $\xi \in \mathbf{R}\setminus\{0\}$ (cf. Titchmarsh [19], Theorem 6). We see that Δ_W has the following two kinds of representations: for $\xi \in \mathbf{R} \setminus \{0\}$,

$$\Delta_W(\xi) = \frac{\sqrt{2\pi}\alpha\xi}{\pi}\int_0^{\infty-}\gamma(t)\sin\xi t dt = \frac{\sqrt{2\pi}\alpha}{\pi}\int_0^{\infty}\frac{\xi^2}{\lambda^2+\xi^2}\rho(d\lambda). \tag{5.6}$$

Proposition 5.5. *It holds that*

(i) $\dfrac{\Delta_W(\xi)}{1+\xi^2} \in L^1(\mathbf{R})$,

(ii) $\dfrac{\log\Delta_W(\xi)}{1+\xi^2} \in L^1(\mathbf{R})$.

By Proposition 5.5 we can define a function h_W by

$$h_W(\zeta) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+\lambda\zeta}{\lambda-\zeta}\frac{\log\Delta_W(\lambda)}{1+\lambda^2}d\lambda\right] \qquad (\mathrm{Im}\ z > 0). \tag{5.7}$$

Theorem 5.6. *It holds that*

$$h_W(\zeta) = \frac{\alpha}{\sqrt{2\pi}}\frac{F_\nu(\zeta)}{F_\sigma(\zeta)} \qquad (\mathrm{Im}\ z > 0). \tag{5.8}$$

Proof of Proposition 5.5 *and Theorem* 5.6. Since Δ_W coincides with the spectral density of αI, where I is the Kubo noise of a stationary Gaussian process with correlation function R_σ, the results follow from Proposition 4.1 and Theorem 4.2. □

By Theorem 5.6, we see that the boundary value $h_W(\xi) := \lim_{\eta\downarrow 0} h_W(\xi + i\eta)$ satisfies

$$h_W(\xi) = \frac{\alpha}{\sqrt{2\pi}}\frac{F_\nu(\xi)}{F_\sigma(\xi)} \qquad (\xi \in \mathbf{R} \setminus \{0\}). \tag{5.9}$$

Let W be one of the stationary Gaussian random distributions with spectral density Δ_W. Then Proposition 5.5 (ii) implies that W is purely nondeterministic. Hence it follows from general theory that there exists a unique one-dimensional Brownian motion $(B(t) : t \in \mathbf{R})$, called *the canonical Brownian motion* of W, such that $B(0) = 0$ and

$$W(\phi) = \sqrt{2\pi} \int_{-\infty}^{\infty} (h_W \hat{\phi})\tilde{}(t) dB(t) \qquad (\phi \in \mathcal{D}(\mathbf{R})), \tag{5.10}$$

$$M_t(W) = M_t(\dot{B}) \qquad \text{for any } t \in \mathbf{R}. \tag{5.11}$$

Now we consider the following equation:

$$\dot{X} = -\gamma * \dot{X} + W. \tag{5.12}$$

We are concerned with a stationary random distribution X which satisfies (5.12) as well as the causality condition:

$$M_t(X) = M_t(W) \qquad \text{for any } t \in \mathbf{R}. \tag{5.13}$$

Theorem 5.7. *There exists a unique stationary random distribution X which satisfies* (5.12)–(5.13)*. The solution X is a purely nondeterministic, stationary Gaussian process with reflection positivity. The solution X is given by*

$$X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} K_\nu(t-s) dB(s) \qquad (t \in \mathbf{R}), \tag{5.14}$$

where $\nu = S^{-1}(L^{-1}(\alpha, \rho))$ and $(B(t))$ is the canonical Brownian motion of W.

Proof. Let X be a stationary random distribution which satisfies (5.13). Then by (5.11) it holds that

$$M_t(X) = M_t(\dot{B}) \qquad \text{for any } t \in \mathbf{R}. \tag{5.15}$$

Hence, by general theory of purely nondeterministic stationary processes, X has a representation of the form (4.18) with some g which satisfies (4.19) for some $k \in \mathbf{N} \cup \{0\}$. Then by Proposition 5.1, (5.4), (5.9) and (5.10), we have for any $\phi \in \mathcal{D}(\mathbf{R})$,

$$\dot{X}(\phi) + (\gamma * \dot{X})(\phi) - W(\phi) = \alpha \int_{-\infty}^{\infty} \left\{ (g - F_\nu) \frac{\hat{\phi}}{F_\sigma} \right\}\tilde{}(t) dB(t).$$

Therefore X satisfies (5.12) if and only if $g = F_\nu$, so that the stationary Gaussian process given by (5.14) is the unique solution of (5.12)–(5.13). □

Remark 5.8. If we omit the condition (5.13), then the uniqueness does not hold. In fact, if $(X(t) : t \in \mathbf{R})$ is a solution of (5.12) and if a is an arbitrary time-independent random variable in $L^2(\Omega)$ with expectation 0 which is independent of X, then $(X(t) + a)$ also satisfies (5.12).

Remark 5.9. The equation (5.12) can be formally rewritten as

$$\dot{X}(t) = -\int_{-\infty}^{t} \gamma(t-s) \dot{X}(s) ds + W(t). \tag{5.16}$$

Theorem 5.10. *Let X be the stationary Gaussian process which satisfies* (5.12)–(5.13)*, and let R be the correlation function of X. Then*

(i) *$R = R_\sigma$ with $\sigma = L^{-1}(\alpha, \rho)$;*
(ii) *$R(0) = \sqrt{2\pi}\alpha$;*

(iii) $\displaystyle \frac{1}{R(0)}\int_0^\infty e^{i\zeta t}R(t)dt = \left[-i\zeta - i\zeta\int_0^\infty e^{i\zeta t}\gamma(t)dt\right]^{-1} \qquad (\mathrm{Im}\ z > 0)$;

(iv) $\displaystyle \int_0^\infty R(t)dt = \infty$;

(v) $\displaystyle X = \lim_{M\to\infty}\frac{1}{R(0)}(\chi_{(0,M)}R) * W$ *as random distributions.*

Proof. The assertion (i) follows from Theorem 5.7, and (ii)–(iv) from (i) and Theorem 3.2. Since $(1/\alpha)W$ coincides with the Kubo noise of X, (v) follows from (ii) and Theorem 4.7. ☐

Example 5.11. Let $0 < p < 1$. We define a Borel measure ρ on $(0,\infty)$ by $\rho(d\lambda) = \lambda^{p-1}d\lambda/\Gamma(p)$. Then for any $\alpha > 0$, the pair (α,ρ) is in $\mathcal{L}_\infty$, and we see that

$$\gamma(t) = \chi_{(0,\infty)}(t)\int_0^\infty e^{-\lambda t}\rho(d\lambda) = \chi_{(0,\infty)}(t)\frac{1}{t^p}, \tag{5.17}$$

$$\Delta_W(\xi) = \frac{\sqrt{2\pi}\alpha\xi}{\pi}\int_0^{\infty-}\gamma(t)\sin\xi t dt = \frac{\sqrt{2\pi}\alpha\Gamma(1-p)\cos(\frac{p\pi}{2})}{\pi}|\xi|^p. \tag{5.18}$$

6. Long-time behaviors

In this section, we investigate the relation between the long-time behavior of γ in the equation (5.12) and that of the correlation function R of the solution X. We follow the notations of Bingham–Goldie–Teugels [2] on regular variation.

Theorem 6.1. *Let R be the correlation function of the stationary process X which satisfies* (5.12)–(5.13)*. Let $0 < p < 1$, and let l be a slowly varying function. Then the following are equivalent:*

$$\gamma(t) \sim t^{-p}l(t) \quad (t\to\infty), \tag{6.1}$$

$$R(t) \sim \frac{R(0)\sin(p\pi)}{\pi}\cdot\frac{t^{-(1-p)}}{l(t)} \quad (t\to\infty). \tag{6.2}$$

Proof. We only prove the assertion (6.1) $\Rightarrow$ (6.2); the part (6.2) $\Rightarrow$ (6.1) can be proved similarly. By taking $\zeta = i\eta$ in Theorem 5.10 (iii), we have

$$\int_0^\infty e^{-\eta t}R(t)dt = R(0)(\eta + \eta\int_0^\infty e^{-\eta t}\gamma(t)dt)^{-1} \qquad (\eta > 0). \tag{6.3}$$

By Karamata's Tauberian Theorem (cf. Bingham–Goldie–Teugels [2], Theorem 1.7.6), (6.1) implies

$$\eta\int_0^\infty e^{-\eta t}\gamma(t)dt \sim \Gamma(1-p)\eta^p l(1/\eta) \quad (\eta\to 0+),$$

and hence by (6.3),

$$\int_0^\infty e^{-\eta t}R(t)dt \sim \frac{R(0)}{\Gamma(1-p)}\frac{\eta^{-p}}{l(1/\eta)} \quad (\eta\to 0+).$$

Again by Karamata's Tauberian Theorem,

$$R(t) \sim \frac{1}{\Gamma(1-p)\Gamma(p)}\frac{t^{-(1-p)}}{l(t)} \quad (t\to\infty).$$

Since $\Gamma(1-p)\Gamma(p) = \pi/\sin(p\pi)$, we obtain (6.2). $\square$

Next we consider the cases which correspond to $p = 0$ or 1 in (6.1).

Lemma 6.2. *Let f be a positive, non-increasing function in $L^1_{loc}[0,\infty)$. Then*

$$\int_0^\infty e^{-\eta t} f(t)dt = \eta \int_0^\infty e^{-\eta t} \left(\int_0^t f(s)ds \right) dt \qquad (\eta > 0). \tag{6.4}$$

Proof. Since

$$\int_0^t f(s)d \le \int_0^1 f(s)dt + f(1+)t \qquad (t > 0),$$

we obtain the result by integration by parts. $\square$

Let l be a slowly varying function such that $\int^\infty l(s)ds/s = \infty$, and choose M so that $l \in L^1_{loc}[M,\infty)$. We put

$$\widetilde{l}(t) = \int_M^t l(s)ds/s \qquad (t \ge M). \tag{6.5}$$

We note that the asymptotic behavior of $\widetilde{l}$ does not depend on the choice of M. By Proposition 1.5.9a in Bingham–Goldie–Teugels [2], we see that $\widetilde{l}$ is also slowly varying.

Theorem 6.3. *Let R be the correlation function of the stationary process X which satisfies* (5.12)–(5.13)*. Let l be a slowly varying function. Then*

(i) $\gamma(t) \sim l(t) \quad (t \to \infty)$ *implies* $\displaystyle\int_0^t R(s)ds \sim \frac{R(0)}{l(t)} \quad (t \to \infty)$;

(ii) $R(t) \sim R(0)\dfrac{l(t)}{t} \quad (t \to \infty)$ *implies* $\gamma(t) \sim \dfrac{1}{\widetilde{l}(t)} \quad (t \to \infty)$;

(iii) $\gamma(t) \sim \dfrac{l(t)}{t} \quad (t \to \infty)$ *implies* $R(t) \sim \dfrac{R(0)}{\widetilde{l}(t)} \quad (t \to \infty)$;

(iv) $R(t) \sim R(0)l(t) \quad (t \to \infty)$ *implies* $\displaystyle\int_0^t \gamma(s)ds \sim \frac{1}{l(t)} \quad (t \to \infty)$.

Proof. (i) By Karamata's Tauberian Theorem, $\gamma(t) \sim l(t)$ $(t \to \infty)$ implies

$$\eta \int_0^\infty e^{-\eta t}\gamma(t)dt \sim l(1/\eta) \qquad (\eta \to 0+),$$

so by (6.3) and Lemma 6.2,

$$\eta \int_0^\infty e^{-\eta t} \left(\int_0^t R(s)ds \right) dt \sim R(0)/l(1/\eta) \qquad (\eta \to 0+).$$

Then Karamata's Tauberian Theorem yields $\int_0^t R(s)ds \sim R(0)/l(t) \quad (t \to \infty)$.

(ii) Since $\int_0^\infty R(t)dt = \infty$, $R(t) \sim R(0)l(t)/t$ $(t \to \infty)$ implies $\int_0^t R(s)ds \sim R(0)\widetilde{l}(t)$ $(t \to \infty)$, so Karamata's Tauberian Theorem yields

$$\eta \int_0^\infty e^{-\eta t} \left(\int_0^t R(s)ds \right) dt \sim R(0)\widetilde{l}(1/\eta) \qquad (\eta \to 0+).$$

Then by Lemma 6.2 and (6.3) we see that

$$\eta \int_0^\infty e^{-\eta t}\gamma(t)dt \sim 1/\widetilde{l}(1/\eta) \qquad (\eta \to 0+).$$

Hence again by Karamata's Tauberian Theorem we obtain $\gamma(t) \sim 1/\widetilde{l}(t)$ $(t \to \infty)$.

The assertions (iii) and (iv) can be proved in the same way. □

Example 6.4. We define a Borel measure ρ on $(0,\infty)$ by $\rho(d\lambda) = e^{-\lambda}d\lambda$. Then for any $\alpha > 0$, the pair (α, ρ) is in $\mathcal{L}_\infty$, and we have $\gamma(t) = \chi_{(0,\infty)}(t)/(1+t)$. Since $\gamma(t) \sim 1/t$ as $t \to \infty$, Theorem 6.3 (iii) shows that $R(t) \sim R(0)/\log t$ as $t \to \infty$.

Now we recall the results of Okabe [11], [15] and Inoue [5] to compare with ours. We state their results in accordance with our formulation. Let $\alpha > 0$, $\beta > 0$, and ρ be a Borel measure in M_0. We put $\gamma(t) = K_\rho(t)$. Let W be a stationary Gaussian random distribution with spectral density Δ_W of the form

$$\Delta_W(\xi) = \frac{\sqrt{2\pi}\alpha}{\pi}\left[\beta + \int_0^\infty \frac{\xi^2}{\lambda^2+\xi^2}\rho(d\lambda)\right] \qquad (\xi \in \mathbf{R}\setminus\{0\}).$$

We consider the following equation:

$$\dot{X} = -\beta X - \gamma * \dot{X} + W \tag{6.6}$$

with

$$M_t(X) = M_t(W) \qquad \text{for any } t \in \mathbf{R}. \tag{6.7}$$

Let X be the unique stationary process which satisfies (6.6)–(6.7); the existence of X is due to Okabe [11], and the uniqueness of X can be proved similarly as Theorem 5.7. Let R be the correlation function of X. Let $0 < p < \infty$, and l be a slowly varying function. Then (cf. Okabe [15], Theorem 3.1 and Inoue [5], Theorem 2.2)

$$\gamma(t) \sim t^{-p}l(t) \qquad (t \to \infty) \tag{6.8}$$

if and only if

$$R(t) \sim \frac{R(0)p}{\beta^2}t^{-(1+p)}l(t) \qquad (t \to \infty). \tag{6.9}$$

From this we see that, as far as $\beta > 0$, the exponent of R is equal to $p+1$ if the exponent of γ is p. On the other hand in view of Theorem 6.1 if $0 < p < 1$ and $\beta = 0$, then the exponent of R is equal to $1-p$. Therefore in the case $\beta = 0$ the decay of γ influences the decay of R in a different way from the case $\beta > 0$.

Now according to Theorem 6.3 we supplement the results of Okabe [15] and Inoue [5]. If l is slowly varying and $\int_M^\infty l(s)ds/s < \infty$, then we put

$$\bar{l}(t) = \int_t^\infty l(s)ds/s \qquad (t \geq M). \tag{6.10}$$

By Proposition 1.5.9b in Bingham–Goldie–Teugels [2], we see that $\bar{l}$ is also slowly varying.

Theorem 6.5. *Let R be the correlation function of the stationary process X which satisfies* (6.6)–(6.7). *Let l be a slowly varying function. Then*

(i) $\gamma(t) \sim l(t) \quad (t \to \infty)$ *implies* $\displaystyle\int_t^\infty R(s)ds \sim \frac{R(0)}{\beta^2} l(t) \quad (t \to \infty)$;

(ii) $R(t) \sim \dfrac{R(0)}{\beta^2}\dfrac{l(t)}{t} \quad (t \to \infty)$ *implies* $\gamma(t) \sim \bar{l}(t)$ $(t \to \infty)$.

Proof. The following proof is similar to the one of Theorem 4.1 in Okabe [15]. First by Theorem 8.5 in Okabe [11] we have

$$\int_0^\infty e^{-\eta t} R(t)dt = R(0)\left[\beta + \eta + \eta \int_0^\infty e^{-\eta t}\gamma(t)dt\right]^{-1} \qquad (\eta > 0). \tag{6.11}$$

(i) By integration by parts and (6.11), we have

$$\eta \int_0^\infty e^{-\eta t}\left(\int_t^\infty R(s)ds\right)dt = \frac{R(0)(\eta + \eta \int_0^\infty e^{-\eta t}\gamma(t)dt)}{\beta(\beta + \eta + \eta \int_0^\infty e^{-\eta t}\gamma(t)dt)} \qquad (\eta > 0),$$

where we used the fact $\int_0^\infty R(t)dt = R(0)/\beta$. Hence by Karamata's Tauberian Theorem, $\gamma(t) \sim l(t)$ $(t \to \infty)$ implies

$$\int_t^\infty R(t)dt \sim \frac{R(0)}{\beta^2} l(t) \qquad (t \to \infty).$$

(ii) Since $\int_0^\infty R(t)dt < \infty$,

$$R(t) \sim \frac{R(0)}{\beta^2}\frac{l(t)}{t} \ (t \to \infty)$$

implies

$$\int_t^\infty R(s)ds \sim \frac{R(0)}{\beta^2}\bar{l}(t) \qquad (t \to \infty).$$

From (6.11) we have

$$\eta \int_0^\infty e^{-\eta t}\gamma(t)dt = \{\beta\eta \int_0^\infty e^{-\eta t}\left(\int_t^\infty R(s)ds\right)dt\}\{\int_0^\infty e^{-\eta t}R(t)dt\}^{-1} - \eta,$$

so by Karamata's Tauberian Theorem we obtain $\gamma(t) \sim \bar{l}(t)$ $(t \to \infty)$. □

Example 6.6. Let $q > 0$. We define a Borel measure σ on $(0, \infty)$ by

$$\sigma(d\lambda) = \frac{\chi_{(0,1/2)}(\lambda)}{(-\log \lambda)^{(1+q)}} d\lambda. \tag{6.12}$$

Then since $\sigma((0,\infty)) < \infty$ and $\int_0^\infty \lambda^{-1}\sigma(d\lambda) < \infty$, it follows from Okabe [11] that there exists a triplet (α, β, ρ) such that the correlation function R of the solution of (6.6)-(6.7) coincides with R_σ. By Karamata's Tauberian Theorem (a version of Theorem 1.7.6 in Bingham–Goldie–Teugels [2] with $x \to 0+$, $s \to \infty$) we see that

$$R(t) \sim \frac{1}{t(\log t)^{1+q}} \qquad (t \to \infty). \tag{6.13}$$

Since

$$\int_t^\infty \frac{1}{s(\log s)^{1+q}} ds = \frac{1}{q(\log t)^q} \qquad (t > 1),$$

Theorem 6.5 (ii) yields

$$\gamma(t) \sim \frac{\beta^2}{qR(0)} \cdot \frac{1}{(\log t)^q} \qquad (t \to \infty). \tag{6.14}$$

Finally we return to the equation (5.12), and show the relation between the long-time behavior of $\gamma(t)$ and the behavior of $\Delta_W(\xi)$ as $\xi \to 0+$.

Theorem 6.7. *Let* $0 < p < 1$*, and* l *be a slowly varying function. Let* R *be the correlation function of the solution of* (5.12)–(5.13)*. Then the following are equivalent:*

$$\gamma(t) \sim t^{-p} l(t) \qquad (t \to \infty), \tag{6.15}$$

$$\Delta_W(\xi) \sim \frac{R(0)\Gamma(1-p)\cos(\frac{p\pi}{2})}{\pi} \xi^p l(1/\xi) \qquad (\xi \to 0+). \tag{6.16}$$

Proof. Since $\gamma \in L^1_{loc}[0, \infty)$, the theorem follows immediately from (5.6), Theorem 5.10 (ii) and the theorem of Pitman (cf. Bingham–Goldie–Teugels [2], Theorem 4.10.3). □

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