# On the Applications of the Theory of Semi-Group Operators on the Treatment of One-Dimensional Physical Systems*

by

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## Introduction

The mathematical theory of semi-group operators assures us that the general solutions of the typical partial differential equations which appear in mathematical physics can be expressed in the form of semi-group transformations ; that is, the quantity which describes the state of the physical system subject to such a partial differential equation can be written, as a function of the space coordinate $P$ and the time $t$, in the form

$$F(P,t)=\boldsymbol{T}(t)\,F(P), \tag{1}$$

where $F(P)$ is the quantity which describes the state of the system at the initial time $t=0$, and $\boldsymbol{T}(t)$ is a semi-group operator such that

$$\boldsymbol{T}(t_1+t_2)=\boldsymbol{T}(t_2)\,\boldsymbol{T}(t_1), \tag{2}$$

in accordance with the principle of the scientific determinism, or the so-called Huygens' principle. Skipping all the mathematical complexity, such an operator should at least formally be written in a transcendental exponential form :

$$\boldsymbol{T}(t)=e^{t\boldsymbol{A}}, \tag{3}$$

where $\boldsymbol{A}$ is the so-called infinitesimal generator of the operator $\boldsymbol{T}(t)$. Putting (3) into (1) and carrying out the formal differentiation with respect to $t$, we have

$$\frac{\partial F(P,t)}{\partial t}=\boldsymbol{A}\,F(P,t). \tag{4}$$

For example, the general solution of the diffusion (heat conduction) equation

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$$\frac{\partial u}{\partial t}=\frac{1}{4}\frac{\partial^2 u}{\partial x^2},\ u(x,0)=f(x), \tag{5}$$

which is given by

$$u(x,t)=(\pi t)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} e^{-(x-\xi)^2/t}f(\xi)\,d\xi,\quad t>0, \tag{6}$$

can be put into the form :

$$u(x,t)=\boldsymbol{T}(t)[f], \tag{7}$$

where $\boldsymbol{T}(t)$ is the semi-group operator with its infinitesimal generator

$$\boldsymbol{A}=\frac{1}{4}\frac{\partial^2}{\partial x^2}, \tag{8}$$

in harmony with the expression (4).

The same result is obtained in the case of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},\quad u(x,0)=f_1(x),\quad u_t(x,0)=f_2(x), \tag{9}$$

whose solution is

$$u(x,t)=\frac{1}{2}[f_1(x+t)+f_1(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}f_2(\xi)\,d\xi. \tag{10}$$

If we consider the quantities $u(x,t)$ and $u_t(x,t)$ as the components of the vector $F(x,t)$ :

$$F(x,t)=\begin{pmatrix}u(x,t)\\ u_t(x,t)\end{pmatrix}, \tag{11}$$

and the quantities $f_1(x)$ and $f_2(x)$ as the components of the vector :

$$F(x)=\begin{pmatrix}f_1(x)\\ f_2(x)\end{pmatrix}=\begin{pmatrix}u(x,0)\\ u_t(x,0)\end{pmatrix}, \tag{12}$$

the formula (10) can be put into the symbolic form

$$F(x,t)=\boldsymbol{T}(t)\,F(x), \tag{13}$$

where $\boldsymbol{T}(t)$ is the operator matrix

$$\boldsymbol{T}(t)=\begin{pmatrix}\mu'(t) & \mu(t)\\ \mu''(t) & \mu'(t)\end{pmatrix}, \tag{14}$$

with

$$\mu(t)=\frac{1}{2}\int_{x-t}^{x+t} f(\xi)\,d\xi, \tag{15}$$

which has the semi-group property. Its infinitesimal generator is calculated to be

$$A=\begin{pmatrix} 0 & I \\ \dfrac{\partial^2}{\partial x^2} & 0 \end{pmatrix}. \tag{16}$$

The solutions of the three-dimensional diffusion and wave equations can also be written in the form of the semi-group transformation of the initial values.[1]

Here we note that, at least in the one-dimensional case, there is a complete symmetry between space and time, and we may interchange the space and time variables in the above argument. In the case of the wave equation, we may rewrite it in the form

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}, \tag{9'}$$

and construct the solution in the following form

$$F(x,t)=T(x)F(t), \tag{17}$$

where

$$F(x,t)=\begin{pmatrix} u(t,x) \\ u_t(t,x) \end{pmatrix}, \qquad F(t)=\begin{pmatrix} u(t,0) \\ u_x(t,0) \end{pmatrix}, \tag{18}$$

with the infinitesimal generator

$$A=\begin{pmatrix} 0 & I \\ \dfrac{\partial^2}{\partial t^2} & 0 \end{pmatrix}. \tag{19}$$

Also in the case of the diffusion equation, we obtain the solution in the above form, since it involves the second derivative with respect to $x$, as in the wave equation. The infinitesimal generator will now become

$$A=4\begin{pmatrix} 0 & I \\ \dfrac{\partial}{\partial t} & 0 \end{pmatrix}. \tag{20}$$

In some problems of physics, such as that of light transmission through an optical system, or of the heat conduction in an extended medium, we are concerned

1) E. Hille: Functional Analysis and Semi-Groups, Chap XX.

with the spacial development of the system rather than the temporal one. In the case of light transmission we are interested in the question : In what wave form, or in what intensity, or in what state of polarisation, does the light wave emerge from one end of an optical system such as a crystal plate, when light of known wave character falls upon the other end ? In the case of heat conduction, the question that arouses concern is, for example, what sort of the time variation of temperature will be expected in the upper atmosphere when we know it on the earth. In any case the solutions of the form (17) and (18) are to be required.

The mathematical formulation for the case of one-dimensional optical systems has already been carried out to a considerable extent by R. C. Jones[3] and others. They constructed a matrix calculation which consists in deducing the emerging light vector from the incident one through the operation of so-called $\boldsymbol{M}$-matrix. It is a remarkable fact that this $\boldsymbol{M}$-matrix has the semi-group property, and has an infinitesimal generator which is called the $\boldsymbol{N}$-matrix. Being suggested by this fact, we attempted to re-investigate their calculus from the point of view of the theory of semi-group transformations, and arrived at the conclusion that Jones' calculus can be deduced from the wave equation, at least in case the optical activity is absent.

The recently developed theory of optical anti-reflection films, which is based upon the analogy to the well-kown four-terminal circuit theory, and leads to a fundamental transformation relation, similar to Jones' calculus, between the inicident and the out-going light, can also be deduced from the above-mentioned general basis.

S. A. Schelkunoff[2] suggested that the concept of impedance, which has been extensively used in the electrical circuit theory, could be carried over into various physical systems, and pointed out the possibility of the more or less unified formulation of mechanical, electrical, thermal, and optical problems. But the operator formulation such as that in optical case has not yet been constructed in other cases. In the present paper the formulation will be extended to the thermal case (see §3).

Our deduction of Jones' $\boldsymbol{M}$-matrix or the transformation relation for the case of optical films would seem much more involved than the respective original deduction, and of little use in practice, since it apparently introduces only a superfluous complexity. We suppose, however, that it not only gives a unified methodological basis for existing several separate treatments of the same or different physical objects, for example, the afore-mentioned Jones' calculus and the theory of anti-reflection films, but also provides a new method of treatment of other one-dimensional extended systems. We surmise that this method could be applied extensively to the treatment of the so-called distributed systems, i. e. the systems

2) S. A. Schelkunoff : Bell System Tech. Journal, Vol 17, p. 17 (1938).

which have both space and time dependence and are described in terms of partial differential equations.

In §1 is given a simple derivation of Jones' matrix for an optically inactive medium. This is then derived in §2 from the point of view of the theory of semi-group transformations. The derivation of the transfer relation in the theory of anti-reflection films is also contained in the same section. §3 deals with the thermal case. Finally in §4 the surmise respecting the general treatment of the distributed systems is given briefly.

## 1. Jones' $M$-Matrix in a Simple Case.

R. C. Jones[3] has shown that the effect of a crystal plate, or any other light-transmitting homogeneous medium, on the collimated beam of monochromatic polarized light, can always be described by a two-by-two matrix operating on the electric vector of the incident light. We take up here the simple case of the crystal plate which is optically inactive and whose principal axes of absorption and those of refraction coincide with each other (a partial polariser), and suppose that a polarized monochromatic plane wave of light falls normally upon the plane surface of such a crystal extending semi-infinitely. Let us take the positive $z$-axis in the direction of light propagation, and $x$- and $y$-axes in the plane of the surface of the crystal.

Then the propagation of the light wave in the crystal must be described by the one-dimensional Maxwell's equation :

$$\frac{\partial^2 \boldsymbol{E}}{\partial z^2} = \frac{\varepsilon}{c^2} \ddot{\boldsymbol{E}}, \tag{21}$$

where $\boldsymbol{E}$ is the two-component vector $(E_x, E_y)$, and $\varepsilon$ should be considered as the two-by-two matrix :

$$\varepsilon = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{pmatrix}. \tag{22}$$

If, according to the assumption of monochromatic wave, we put

$$\boldsymbol{E}(z, t) = \boldsymbol{E}(z)\, e^{i\omega' t}, \tag{23}$$

the equation reduces to

$$\frac{d^2 \boldsymbol{E}}{\partial z^2} = -\frac{\omega'^2}{c^2} \varepsilon \boldsymbol{E}. \tag{24}$$

3) J. O. S. A. **31,** 488-503 (1941) ; **32,** 486-493 (1942), **37,** 107-112 (1947) ; **38,** 671 (1948).

Rewriting this for the separate components of $\boldsymbol{E}$, we have

$$\left.\begin{aligned} \frac{d^2 E_x}{d z^2} &= -\frac{\omega'^2}{c^2}(\varepsilon_{xx} E_x + \varepsilon_{xy} E_y), \\ \frac{d^2 E_y}{d z^2} &= -\frac{\omega'^2}{c^2}(\varepsilon_{yx} E_x + \varepsilon_{yy} E_y). \end{aligned}\right\} \tag{25}$$

Now we rotate the $y$- and $x$-axes by an angle $\delta$, so that the matrix (22) becomes

$$\varepsilon = \begin{pmatrix} n_x^2 & 0 \\ 0 & n_y^2 \end{pmatrix}, \tag{26}$$

the transformation matrix and the inverse of it being

$$\boldsymbol{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix}, \tag{27}$$

and

$$\boldsymbol{C}^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}, \tag{28}$$

respectively. If we denote the components of the electric vector with respect to the rotated axes by the upper indices 0, we have

$$\boldsymbol{E}^0 = \boldsymbol{C}\,\boldsymbol{E},$$

or

$$\left.\begin{aligned} E_x{}^0 &= \cos\delta\, E_x + \sin\delta\, E_y, \\ E_y{}^0 &= -\sin\delta\, E_x + \cos\delta\, E_y, \end{aligned}\right\} \tag{29}$$

and (25) becomes

$$\left.\begin{aligned} \frac{d^2 E_x{}^0}{d z^2} &= -\frac{\omega'^2}{c^2} n_x^2 E_x{}^0, \\ \frac{d^2 E_y{}^0}{d z^2} &= -\frac{\omega'^2}{c^2} n_y^2 E_y{}^0. \end{aligned}\right\} \tag{30}$$

These equations are evidently satisfied by

$$\left.\begin{aligned} E_x{}^0 &= E_{x0}{}^0\, e^{-i\frac{\omega'}{c} n_x z}, \\ E_y{}^0 &= E_{y0}{}^0\, e^{-i\frac{\omega'}{c} n_y z}, \end{aligned}\right\} \tag{31}$$

where the lower indices 0 indicate the initial values of $E^0$'s at $z=0$. Transforming back to the original coordinate system, using (27) and (28), we get

$$\left.\begin{aligned} E_x &= E_{x0}\left(e^{-i\frac{\omega'}{c}n_x z}\cos^2\delta + e^{-i\frac{\omega'}{c}n_y z}\sin^2\delta\right) \\ &\quad + E_{y0}\left(e^{-i\frac{\omega'}{c}n_x z} - e^{-i\frac{\omega'}{c}n_y z}\right)\sin\delta\cos\delta\,, \\ E_y &= E_{x0}\left(e^{-i\frac{\omega'}{c}n_x z} - e^{-i\frac{\omega'}{c}n_y z}\right)\sin\delta\cos\delta \\ &\quad + E_{y0}\left(e^{-i\frac{\omega'}{c}n_x z}\sin^2\delta + e^{-i\frac{\omega'}{c}n_y z}\cos^2\delta\right). \end{aligned}\right\} \tag{32}$$

Thus we obtained the transformation :

$$\boldsymbol{E}(z) = \boldsymbol{M}(z)\,\boldsymbol{E}_0\,, \tag{33}$$

$\boldsymbol{M}$ being the matrix

$$\boldsymbol{M} = \begin{pmatrix} N_x\cos^2\delta + N_y\sin^2\delta & (N_x - N_y)\sin\delta\cos\delta \\ (N_x - N_y)\sin\delta\cos\delta & N_x\sin^2\delta + N_y\cos^2\delta \end{pmatrix}, \tag{34}$$

where for simplicity we replaced the exponential expressions by abbreviations $N$. This is the Jones' transformation matrix in the present case.

By a simple calculation it is shown that

$$\boldsymbol{M}(z_1)\,\boldsymbol{M}(z_2) = \boldsymbol{M}(z_1 + z_2)\,, \tag{35}$$

that is, the operator $\boldsymbol{M}$ has the semi-group property. Moreover, we see that

$$\lim_{z\to 0}\boldsymbol{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \boldsymbol{I}. \tag{36}$$

Thus it is expected that there exists a corresponding infinitesimal generator

$$\boldsymbol{N} = \lim_{h\to 0}\frac{1}{h}\,[\boldsymbol{M}(h) - \boldsymbol{I}], \tag{37}$$

which generates $\boldsymbol{M}$ in the manner

$$\boldsymbol{M} = e^{z\boldsymbol{N}}\,. \tag{38}$$

In fact we get, by direct calculation,

$$\boldsymbol{N} = -i\,\frac{\omega'}{c}\begin{pmatrix} n_x\cos^2\delta + n_y\sin^2\delta & (n_x - n_y)\sin\delta\cos\delta \\ (n_x - n_y)\sin\delta\cos\delta & n_x\sin^2\delta + n_y\cos^2\delta \end{pmatrix}. \tag{39}$$

Differentiating (32) with respect to $z$, we obtain

$$\left.\begin{aligned}\frac{dE_x}{dz} &= E_{x0}\left(-i\frac{\omega'}{c}n_x N_x \cos^2\delta - i\frac{\omega'}{c}n_y N_y \sin^2\delta\right)\\&\quad + E_{y0}\left(-i\frac{\omega'}{c}n_x N_x + i\frac{\omega'}{c}n_y N_y\right)\sin\delta\cos\delta\,,\\\frac{dE_y}{dz} &= E_{x0}\left(-i\frac{\omega'}{c}n_x N_x + i\frac{\omega'}{c}n_y N_y\right)\sin\delta\cos\delta\\&\quad + E_{y0}\left(-i\frac{\omega'}{c}n_x N_x \sin^2\delta - i\frac{\omega'}{c}n_y N_y \cos^2\delta\right),\end{aligned}\right\} \tag{40}$$

or briefly,

$$\frac{d\boldsymbol{E}}{dz} = \boldsymbol{N}\boldsymbol{E}. \tag{41}$$

This is just in harmony with the expression (38).

## 2. Deduction of the *M*-Matrix and the Transfer Relation in the Theory of Optical Anti-Reflection Films on the Unified Basis of the Theory of Semi-Groups.

Now we abandon the assumption made in §1 that the incident light is strictly monocromatic and consider a more general case, i. e. the case in which the incident light has an arbitrary initial wave-form, but we neglect at this moment the dispersion phenomenon, or the frequency dependence of the dielectric constant, the consideration of which will later prove to be more than is necessary.

Under these circumstances the wave equations with reference to the principal axes are

$$\begin{aligned}\frac{\partial^2 E_x{}^0}{\partial z^2} &= \frac{n_x{}^2}{c^2}\ddot{E}_x{}^0\,,\\\frac{\partial^2 E_y{}^0}{\partial z^2} &= \frac{n_y{}^2}{c^2}\ddot{E}_y{}^0\,,\end{aligned} \tag{42}$$

with the conditions

$$E_x{}^0(t,z)|_{z=0} = E_{x1}{}^0(t)\,,\quad \frac{\partial}{\partial z}E_x{}^0(t,z)|_{z=0} = E_{x2}{}^0(t)\,. \tag{43}$$

The general solution is given by

$$\begin{aligned}E_x{}^0(t,z) &= \frac{1}{2}\left[E_{x1}{}^0\left(t+\frac{n_x}{c}z\right)+E_{x1}{}^0\left(t-\frac{n_x}{c}z\right)\right]+\frac{1}{2}\frac{c}{n_x}\int_{t-\frac{n_x}{c}z}^{t+\frac{n_x}{c}z}E_{x2}{}^0(\xi)\,d\xi\,,\\E_y{}^0(t,z) &= \frac{1}{2}\left[E_{y1}{}^0\left(t+\frac{n_y}{c}z\right)+E_{y1}{}^0\left(t-\frac{n_y}{c}z\right)\right]+\frac{1}{2}\frac{c}{n_y}\int_{t-\frac{n_y}{c}z}^{t+\frac{n_y}{c}z}E_{y2}{}^0(\xi)\,d\xi\,.\end{aligned} \tag{44}$$

If we define the two-component vectors as

$$\left.\begin{aligned} F_x^0(t,z) &= \begin{pmatrix} E_x^0(t,z) \\ \frac{\partial}{\partial z} E_x^0(t,z) \end{pmatrix}, \quad F_x^0(t) = \begin{pmatrix} E_{x1}^0(t) \\ E_{x2}^0(t) \end{pmatrix}, \\ F_y^0(t,z) &= \begin{pmatrix} E_y^0(t,z) \\ \frac{\partial}{\partial z} E_y^0(t,z) \end{pmatrix}, \quad F_y^0(t) = \begin{pmatrix} E_{y1}^0(t) \\ E_{y2}^0(t) \end{pmatrix}, \end{aligned}\right\} \tag{45}$$

and the corresponding matrix operators as

$$\begin{aligned} \boldsymbol{T}_x(z) &= \frac{c}{n_x}\begin{pmatrix} \mu_x'(z) & \mu_x(z) \\ \mu_x''(z) & \mu_x'(z) \end{pmatrix}, \\ \boldsymbol{T}_y(z) &= \frac{c}{n_y}\begin{pmatrix} \mu_y'(z) & \mu_y(z) \\ \mu_y''(z) & \mu_y'(z) \end{pmatrix}, \end{aligned} \tag{46}$$

in which

$$\left.\begin{aligned} \mu_x(z)[f] &= \frac{1}{2}\int_{t-\frac{n_x}{c}z}^{t+\frac{n_x}{c}z} f(\xi)\,d\xi, \\ \mu_y(z)[f] &= \frac{1}{2}\int_{t-\frac{n_y}{c}z}^{t+\frac{n_y}{c}z} f(\xi)\,d\xi, \end{aligned}\right\} \tag{47}$$

the equation (44) can be written in the form :

$$\left.\begin{aligned} F_x^0(t,z) &= \boldsymbol{T}_x(z)[F_x^0(t)], \\ F_y^0(t,z) &= \boldsymbol{T}_y(z)[F_y^0(t)]. \end{aligned}\right\} \tag{48}$$

Transforming back to the original coordinate system, we obtain

$$\left.\begin{aligned} F_x(t,z) &= [\boldsymbol{T}_x(z)\cos^2\delta + \boldsymbol{T}_y(z)\sin^2\delta]\,F_x(t) \\ &\qquad + \sin\delta\cos\delta\,[\boldsymbol{T}_x(z) - \boldsymbol{T}_y(z)]\,F_y(t), \\ F_y(t,z) &= \sin\delta\cos\delta\,[\boldsymbol{T}_x(z) - \boldsymbol{T}_y(z)]\,F_x(t) \\ &\qquad + [\sin^2\delta\,\boldsymbol{T}_x(z) + \cos^2\delta\,\boldsymbol{T}_y(z)]\,F_y(t). \end{aligned}\right\} \tag{49}$$

If we again introduce the new vectors

$$F(t,z) = \begin{pmatrix} F_x(t,z) \\ F_y(t,z) \end{pmatrix}, \qquad F(t) = \begin{pmatrix} F_x(t) \\ F_y(t) \end{pmatrix}, \tag{50}$$

and the operator

$$
\boldsymbol{T}(z) \equiv c \left( \begin{array}{cc}
\frac{\mu_x'}{n_x} \cos^2 \delta + \frac{\mu_y'}{n_y} \sin^2 \delta & \frac{\mu_x}{n_x} \cos^2 \delta + \frac{\mu_y}{n_y} \sin^2 \delta \\
\frac{\mu_x''}{n_x} \cos^2 \delta + \frac{\mu_y''}{n_y} \sin^2 \delta & \frac{\mu_x'}{n_x} \cos^2 \delta + \frac{\mu_y'}{n_y} \sin^2 \delta \\
\sin \delta \cos \delta \left( \frac{\mu_x'}{n_x} - \frac{\mu_y'}{n_y} \right) & \sin \delta \cos \delta \left( \frac{\mu_x}{n_x} - \frac{\mu_y}{n_y} \right) \\
\sin \delta \cos \delta \left( \frac{\mu_x''}{n_x} - \frac{\mu_y''}{n_y} \right) & \sin \delta \cos \delta \left( \frac{\mu_x'}{n_x} - \frac{\mu_y'}{n_y} \right)
\end{array} \right.
$$

$$
\left. \begin{array}{cc}
\sin \delta \cos \delta \left( \frac{\mu_x'}{n_x} - \frac{\mu_y'}{n_y} \right) & \sin \delta \cos \delta \left( \frac{\mu_x}{n_x} - \frac{\mu_y}{n_y} \right) \\
\sin \delta \cos \delta \left( \frac{\mu_x''}{n_x} - \frac{\mu_y''}{n_y} \right) & \sin \delta \cos \delta \left( \frac{\mu_x'}{n_x} - \frac{\mu_y'}{n_y} \right) \\
\frac{\mu_x'}{n_x} \sin^2 \delta + \frac{\mu_y'}{n_y} \cos^2 \delta & \frac{\mu_x}{n_x} \sin^2 \delta + \frac{\mu_y}{n_y} \cos^2 \delta \\
\frac{\mu_x''}{n_x} \sin^2 \delta + \frac{\mu_y''}{n_y} \cos^2 \delta & \frac{\mu_x'}{n_x} \sin^2 \delta + \frac{\mu_y'}{n_y} \cos^2 \delta
\end{array} \right), \tag{51}
$$

the equations (49) become equivalent to the simple operational equation

$$
F(t, z) = \boldsymbol{T}(z) F(t). \tag{52}
$$

Obviously $\boldsymbol{T}(0)$ is the unit matrix. Laborious direct calculations show that $\boldsymbol{T}(z)$ has the semi-group property (2). Hence it must have the infinitesimal generator, which in fact is calculated to be

$$
\boldsymbol{A} = \frac{1}{c^2} \begin{pmatrix}
0 & c^2 & 0 & 0 \\
(\cos^2 \delta \, n_x^2 + \sin^2 \delta \, n_y^2) \frac{\partial^2}{\partial t^2} & 0 & \cos \delta \sin \delta \, (n_x^2 - n_y^2) \frac{\partial^2}{\partial t^2} & 0 \\
0 & 0 & 0 & c^2 \\
\sin \delta \cos \delta \, (n_x^2 - n_y^2) \frac{\partial^2}{\partial t^2} & 0 & (\sin^2 \delta \, n_x^2 + \cos^2 \delta \, n_y^2) \frac{\partial^2}{\partial t^2} & 0
\end{pmatrix} \tag{53}
$$

Now we apply the so-called theorem of factor-function transformation established in the theory of semi-group transformations : Let $f(\omega, z)$ and $f(\omega)$ be the Fourier-transforms of $F(t, z)$ and $F(t)$, respectively, with regard to the variable $t$, and $B(\omega) f(\omega)$ the Fourier-transform of $\boldsymbol{A} F(t)$. Then, if the operator $\boldsymbol{T}(z)$ commutes with the real translation of the independent variable of the operand function,

$$
f(\omega, z) = e^{zB(\omega)} f(\omega). \tag{54}
$$

That is, the Fourier spectrum of the time variation of the quantity $F$, which describes the system at $z$, can be obtained by the semi-group transformation from the one at $z=0$. Moreover, the operator reduces to a mere multiplier.

In our case the commutability condition is evidently satisfied. The Fourier transform of the operator $\boldsymbol{A}$ is given by

$$B(\omega)=\frac{1}{c^2}\begin{pmatrix} 0 & c^2 & 0 & 0 \\ (\cos^2\delta\, n_x^2+\sin^2\delta\, n_y^2)(i\omega)^2 & 0 & \cos\delta\sin\delta\,(n_x^2-n_y^2)(i\omega)^2 & 0 \\ 0 & 0 & 0 & c^2 \\ \sin\delta\cos\delta\,(n_x^2-n_y^2)(i\omega)^2 & 0 & (\sin^2\delta\, n_x^2+\cos^2\delta\, n_y^2)(i\omega)^2 & 0 \end{pmatrix} \tag{55}$$

$e^{zB(\omega)}$ can be calculated by carrying out the lengthy calculation using the development

$$e^{zB(\omega)}=\boldsymbol{I}+Bz+\frac{B^2z^2}{2\,!}+\frac{B^3z^3}{3\,!}+\cdots\cdots\cdots, \tag{56}$$

or from the direct Fourier transformation of the operator $\boldsymbol{T}(z)$. The result is:

$$e^{zB(\omega)}=$$

$$\begin{pmatrix}
\cos^2\delta\cos\frac{n_x\omega z}{c}+\sin^2\delta\cos\frac{n_y\omega z}{c} & \frac{c}{\omega}\left[\frac{\cos^2\delta}{n_x}\sin\frac{n_x\omega z}{c}+\frac{\sin^2\delta}{n_y}\sin\frac{n_y\omega z}{c}\right] & \sin\delta\cos\delta\left[\cos\frac{n_x\omega z}{c}-\cos\frac{n_y\omega z}{c}\right] & \frac{c\sin\delta\cos\delta}{\omega}\left[\frac{1}{n_x}\sin\frac{n_x\omega z}{c}-\frac{1}{n_y}\sin\frac{n_y\omega z}{c}\right] \\
-\frac{\omega}{c}\left[n_x\cos^2\delta\sin\frac{n_x\omega z}{c}+n_y\sin^2\delta\sin\frac{n_y\omega z}{c}\right] & \cos^2\delta\cos\frac{n_x\omega z}{c}+\sin^2\delta\cos\frac{n_y\omega z}{c} & -\frac{\omega}{c}\sin\delta\cos\delta\left[n_x\sin\frac{n_x\omega z}{c}+n_y\sin\frac{n_y\omega z}{c}\right] & \cos\delta\sin\delta\left[\cos\frac{n_x\omega z}{c}-\mathrm{oos}\frac{n_y\omega z}{c}\right] \\
\cos\delta\sin\delta\left[\cos\frac{n_x\omega z}{c}-\cos\frac{n_y\omega z}{c}\right] & \frac{c\cos\delta\sin\delta}{\omega}\left[\frac{1}{n_x}\sin\frac{n_x\omega z}{c}-\frac{1}{n_y}\sin\frac{n_y\omega z}{c}\right] & \sin^2\delta\cos\frac{n_x\omega z}{c}+\cos^2\delta\cos\frac{n_y\omega z}{c} & \frac{c}{\omega}\left[\frac{\sin^2\delta}{n_x}\sin\frac{n_x\omega z}{c}+\frac{\cos^2\delta}{n_y}\sin\frac{n_y\omega z}{c}\right] \\
-\frac{\omega}{c}\sin\delta\cos\delta\left[n_x\sin\frac{n_x\omega z}{c}+n_y\sin\frac{n_y\omega z}{c}\right] & \sin\delta\cos\delta\left[\cos\frac{n_x\omega z}{c}-\cos\frac{n_y\omega z}{c}\right] & -\frac{\omega}{c}\left[n_x\sin^2\delta\sin\frac{n_x\omega z}{c}+n_y\cos^2\delta\sin\frac{n_y\omega z}{c}\right] & \sin^2\delta\cos\frac{n_x\omega z}{c}+\cos^2\delta\cos\frac{n_y\omega z}{c}
\end{pmatrix} \tag{57}$$

The amplitude spectrum of the light at $z=z$ can be calculated from the one at $z=0$ by the transformation operator (57). Here we see that, since this operator represents a mere multiplier, the transformation formula (52) will still be valid if the principal indices are not constant but functions of frequency $\omega$, so that the neglect of the dispersion assumed at the outset is completely justified and involves no deficiency. We may expect that also in such a case the transform of the operator (57), $n$'s being considered as functions of $\omega$, gives the correct transformation operator in the time language, although it brings out an intractable complexity.

Now we can show that the transformation matrix (57) in fact reduces to Jones' $\boldsymbol{M}$-matrix when we assume the strictly monochromatic incident light. We use the relation which is reciprocal to (29):

$$\left.\begin{aligned} E_x &= \cos\delta\, E_x{}^0 - \sin\delta\, E_y{}^0\,, \\ E_y &= \sin\delta\, E_x{}^0 + \cos\delta\, E_y{}^0\,. \end{aligned}\right\} \tag{58}$$

Substituting for $E_x{}^0$ and $E_y{}^0$ the values given by (31) and transforming both sides of these equations to their respective Fourier mates and thereafter letting $z\to 0$, we get

$$\left.\begin{aligned} E_{x1}(\omega) &= \delta(\omega-\omega')\cos\delta\, E_{x0}{}^0 - \delta(\omega-\omega')\sin\delta\, E_{y0}{}^0\,, \\ E_{y1}(\omega) &= \delta(\omega-\omega')\sin\delta\, E_{x0}{}^0 - \delta(\omega-\omega')\cos\delta\, E_{x0}{}^0\,, \end{aligned}\right\} \tag{59}$$

and, performing the same procedure after differentiation of (58) with respect to $z$,

$$\left.\begin{aligned} E_{x2}(\omega) &= -i\frac{\omega'}{c}n_x\,\delta(\omega-\omega')\cos\delta\, E_{x0}{}^0 + i\frac{\omega'}{c}n_y\delta(\omega-\omega')\sin\delta\, E_{y0}{}^0\,, \\ E_{y2}(\omega) &= -i\frac{\omega'}{c}n_x\,\delta(\omega-\omega')\sin\delta\, E_{x0}{}^0 - i\frac{\omega'}{c}n_y\delta(\omega-\omega')\cos\delta\, E_{y0}{}^0\,. \end{aligned}\right\} \tag{60}$$

Operating (57) on the vector $(E_{x1}, E_{x2}, E_{y1}, E_{y2})$, we find, after some calculations, that the result can be written in the form of (33) and (34). Thus we have shown that Jones' $\boldsymbol{M}$-matrix here treated is nothing other than a special case of our general scheme. It might be expected that the general Jones' matrix can also be deduced in this way, although the derivation would become much involved.

Next we turn to the theory of anti-reflection film.

If the medium is isotropic, $n_x=n_y=n$, and we can put $\delta=0$. Then (55) reduces to

$$B(\omega) = \begin{pmatrix} 0 & 1 \\ \dfrac{(i\omega)^2 n^2}{c^2} & 0 \end{pmatrix}, \tag{61}$$

and we get the transformation matrix

$$e^{zB(\omega)} = \begin{pmatrix} \cos \frac{n\omega}{c} z & \frac{1}{\omega n} \sin \frac{n\omega}{c} z \\ -\omega n \sin \frac{n\omega z}{c} & \cos \frac{n\omega}{c} z \end{pmatrix} \tag{62}$$

between the vectors

$$f(\omega, z) = \begin{pmatrix} E(\omega, z) \\ \frac{\partial}{\partial z} E(\omega, z) \end{pmatrix} \quad \text{and} \quad f(\omega) = \begin{pmatrix} E_1(\omega) \\ E_2(\omega) \end{pmatrix}. \tag{63}$$

If we again assume the monochromatic light, and put

$$E(z, t) = E(z)\, e^{i\omega' t}, \quad H(z, t) = H(z)\, e^{i\left(\omega' t + \frac{\pi}{2}\right)}, \tag{64}$$

we have, according to the Maxwell's equation,

$$\frac{\partial E}{\partial z} = \frac{\omega'}{c} H, \tag{65}$$

$H$ being the magnetic vector. Hence (63) becomes

$$f(\omega, z) = \begin{pmatrix} E(\omega, z) \\ \frac{\omega}{c} H(\omega, z) \end{pmatrix}, \qquad f(\omega) = \begin{pmatrix} E(0) \\ \frac{\omega}{c} H(0) \end{pmatrix} \delta(\omega - \omega'). \tag{66}$$

Thus we get the transformation relation

$$\begin{pmatrix} E(\omega, z) \\ H(\omega, z) \end{pmatrix} = \begin{pmatrix} \cos \frac{n\omega' z}{c} & \frac{1}{nc} \sin \frac{n\omega' z}{c} \\ -nc \sin \frac{n\omega' z}{c} & \cos \frac{n\omega' z}{c} \end{pmatrix} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}. \tag{67}$$

If we put $\sigma = \frac{i\omega' n}{c}$, this becomes

$$\begin{pmatrix} E(z) \\ H(z) \end{pmatrix} = \begin{pmatrix} \cos h\sigma z & i \frac{1}{nc} \sin h\sigma z \\ -inc \sin h\sigma z & \cos h\sigma z \end{pmatrix} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}, \tag{68}$$

which is the transformation relation that appears in the theory of anti-reflection films.

We have thus completed the unified derivation of Jones' matrix and the transformation matrix in the theory of anti-reflection films.

## 3. The Case of Heat Conduction.

The fundamental differential equation for the one-dimensional heat flow is

$$\frac{\partial^2 \theta}{\partial z^2} = \tau \frac{\partial \theta}{\partial t}, \tag{69}$$

where $\tau$ is the reciprocal of thermal conductivity $\kappa$. According to the general circumstance stated in the introduction, the solution of this equation, when the temporal variations of temperature and its gradient at $z=0$ are given, must be written in the form

$$\Theta(z,t) = \boldsymbol{T}(z)\,\Phi(t), \tag{70}$$

where

$$\Theta(z,t) = \begin{pmatrix} \theta(z,t) \\ \dfrac{\partial}{\partial z}\theta(z,t) \end{pmatrix}, \qquad \Phi(t) = \begin{pmatrix} \theta(0,t) \\ \dfrac{\partial}{\partial z}\theta(0,t) \end{pmatrix} = \begin{pmatrix} \varphi(t) \\ \varphi_1(t) \end{pmatrix}, \tag{71}$$

and

$$\boldsymbol{T}(z) = e^{zA}, \qquad \boldsymbol{A} = \begin{pmatrix} 0 & I \\ \tau\dfrac{\partial}{\partial t} & 0 \end{pmatrix}. \tag{72}$$

Applying the theorem of factor-function transformation, we can also write

$$\Theta(z,\omega) = e^{zB(\omega)}\,\Phi(\omega), \tag{73}$$

where

$$B(\omega) = \begin{pmatrix} 0 & I \\ \tau(i\omega) & 0 \end{pmatrix}. \tag{74}$$

Calculating $e^{zB(\omega)}$ in the same way as in §2, we find

$$e^{zB(\omega)} = \begin{pmatrix} \cos h\, z\sqrt{\tau i\omega} & \dfrac{1}{\sqrt{\tau i \omega}} \sin h\, z\sqrt{\tau i\omega} \\ \sqrt{\tau i\omega}\, \sin h\, z\sqrt{\tau i\omega} & \cos h\, z\sqrt{\tau i\omega} \end{pmatrix}, \tag{75}$$

which is closely analogous to the formula (62) or (68).

If we assume the strictly harmonic temporal variation of temperature at any point in space, and put

$$\theta = e^{i\omega' t}\,\psi(z), \tag{76}$$

the equation (69) reduces to

$$\frac{\partial^2 \psi}{\partial z^2} = i\omega' \tau \psi(z), \tag{77}$$

the physically acceptable solution of which is given by

$$\psi(z) = A\, e^{-\sqrt{i\omega'\tau}\, z}. \tag{78}$$

Thus the matrix $e^{zB(\omega)}$ reduces as before to a simple multiplier

$$e^{-\sqrt{\tau i\omega'}\, z} \tag{79}$$

on $A$, the amplitude of the sinusoidal variation at $z=0$, and the formula (78) itself represents the semi-group transformation relation.

Multiplying both sides of (78) by $e^{i\omega' t}$, we get

$$\begin{aligned}
\theta(z,t) &= A\, e^{i\omega' t} e^{-\sqrt{\tau i\omega'}\, z} \\
&= A\, e^{i\omega' t} e^{-z\sqrt{\frac{\tau\omega'}{2}}} e^{iz\sqrt{\frac{\tau\omega'}{2}}} \\
&= A\, e^{-z\sqrt{\frac{\tau\omega'}{2}}} (\cos\omega' t + i \sin\omega' t)\left(\cos z\sqrt{\frac{\tau\omega}{2}} - i \sin z\sqrt{\frac{\tau\omega}{2}}\right),
\end{aligned} \tag{80}$$

the real part of which is

$$= A\, e^{-z\sqrt{\frac{\tau\omega}{2}}} \cos\left(\omega t - z\sqrt{\frac{\tau\omega}{2}}\right). \tag{81}$$

This is the stationary solution when the sinusoidal temperature variation $A\cos\omega t$ is applied at the surface of a solid extending semi-infinitely.

The general form (75) contains the components which are physically insignificant. If we leave out these components, we have

$$S(\omega) = \begin{pmatrix} \dfrac{e^{-z\sqrt{i\omega\tau}}}{2} & \dfrac{-e^{-z\sqrt{i\omega\tau}}}{2\sqrt{\tau i\omega}} \\[2ex] -\dfrac{\sqrt{\tau i\omega}}{2} e^{-z\sqrt{i\tau\omega}} & \dfrac{e^{-z\sqrt{i\omega\tau}}}{2} \end{pmatrix}. \tag{82}$$

As can easily be verified, this expression also has the semi-group property. Thus the physically acceptable solution will be given by

$$\Theta(\omega, z) = S(\omega)\Phi(\omega). \tag{83}$$

Since the Fourier-transforms of the elements of $S(\omega)$ are, respectively, given by

$$\left.\begin{aligned}
&\frac{e^{-z\sqrt{i\omega\tau}}}{2} \to \frac{1}{2}\,\frac{z\sqrt{\tau}}{2\sqrt{\pi}\,t^{3/2}}\,e^{-\frac{\tau z^2}{4t}}, \quad t>0,\\
&\frac{-e^{-z\sqrt{i\omega\tau}}}{2\sqrt{\tau i\omega}} \to -\frac{1}{2\sqrt{\tau}}\,\frac{1}{\sqrt{\pi}\,t^{1/2}}\,e^{-\frac{\tau z^2}{4t}}, \quad t>0,\\
&-\frac{\sqrt{\tau i\omega}}{2}\,e^{-z\sqrt{i\omega\tau}} \to -\frac{\sqrt{\tau}}{2}\left(\frac{z^2\tau}{2t}-1\right)\frac{1}{2\sqrt{\pi}t^{3/2}}\,e^{-\frac{\tau z^2}{4t}}, \quad t>0,
\end{aligned}\right\} \tag{84}$$

the general solution of the problem comes out to be

$$\theta(t,z)=\frac{\sqrt{\tau}\,z}{4\sqrt{\pi}}\int_{-\infty}^{t}\varphi(t')\frac{e^{-\frac{\tau z^2}{4(t-t')}}}{(t-t')^{3/2}}\,dt' - \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{t}\varphi_1(t')\frac{e^{-\frac{\tau z^2}{4(t-t')}}}{(t-t')^{1/2}}\,dt', \tag{85}$$

whose verification is an easy matter.

It is to be expected that the matrix $S(\omega)$ would serve for the determination of thermal condutivity or diffusion constant of the atmosphere, for example, when the spectrum of the temporal variation of the temperature and its gradient are known on the earth and in the upper atmosphere.

## 4 General Problems.

Physical systems can be classified into two categories, namely lumped constant systems and distributed ones. Lumped constant system is the one in which the "response" of the system to a certain forcing can be described strictly or approximately in terms of ordinary differential equation with constant coefficients, and their properties are completely determined by these coefficients. On the contrary, in distributed systems the response propagates with a finite velocity in that system, and their *overall* properties cannot be described by ordinary differential equations but by partial differential equations such as the wave equations and diffusion equations treated above.

It is an important problem to determine or surmise the properties of physical systems, which are at first unknown to us, from the responses of such systems to certain appropriate forcings. In cases of lumped constant systems, the methods of solving the problem has been well formulated. The most interesting and generally applicable method is due to Prof. K. Imahori[4], which consists in finding the impendace function $Z(i\omega)$ in the form of the polynomial of $i\omega$, such that

$$Z(i\omega)=a_0+a_1(i\omega)+a_2(i\omega)^2+\cdots\cdots\quad +a_n(i\omega)^n. \tag{86}$$

The ordinary differential equation which describes the system is then found to be

4) K. Imahori and K. Suhara : On the Statistical Method in the Brain-Wave Study. Part I : Folia Psch. et Neul. Jap. Vol. 3, No. 2 137 (1949).

$$L\{x(t)\} = 0\,,$$

or (87)

$$= p(t)\,,$$

according as the problem is causal or statistical ($p(t)$ is the so-called random function with white spectrum), where

$$L = a_0 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} + \cdots\cdots\cdot \quad + a_n \frac{d^n}{dt^n}\,. \tag{88}$$

This method has been successfully applied to the analysis of the brain waves and the properties of dielectrics.

In the case of distributed systems such a systematic method is not known as yet. I suppose that the method described in the preceding sections would provide the first step to approach it, at least in one-dimensional case.

If we assume that the partial differential equation which describes the system in question contains spacial derivative only in the second order, its general form becomes

$$\frac{\partial^2 u}{\partial z^2} = \sum_{i=0}^{n} a_i \left(\frac{\partial}{\partial t}\right)^i u = L'\{u\}\,. \tag{89}$$

There is no a priori reason that the assumption made above is legitimate. It is only a matter of convenience for considering the problem in relation to the treatment of the preceding sections. We might choose as well the first or third or any higher order spacial derivatives, or any linear combinations of them. The judgment would depend upon the physical intuition, and its justification should be found in the practical usefulness of the result. Moreover, it seems most convenient to restrict ourselves in only *one* spacial term, in order to carry out the proposed method effectively. That is, we must in any case *assume* the only one order of spacial derivative.* We presume that it does not offer any serious difficulty anyhow, since in most cases the thing required is not the strict determination of the properties of the system but only an approximate surmise of them.

Thus if we assume the equation (89), with the condition

$$\left.\begin{aligned} &u(0,t) = u_1(t)\,, \\ &\frac{\partial}{\partial z} u(z,t)\,|_{z=0} = u_2(t)\,, \end{aligned}\right\} \tag{90}$$

its solution may be, as before, written in the form :

* This point should, however, be examined more carefully. We leave it to a later paper.

$$U(z,t)=\boldsymbol{T}(z)\,U(t), \tag{91}$$

where

$$U(z,t)=\begin{pmatrix} u(z,t) \\ \frac{\partial}{\partial z}u(z,t)\end{pmatrix}, \qquad U(t)=\begin{pmatrix} u_1(t) \\ u_2(t)\end{pmatrix}, \tag{92}$$

and

$$\boldsymbol{T}(z)=e^{z\boldsymbol{A}}, \qquad \boldsymbol{A}=\begin{pmatrix} 0 & I \\ L' & 0\end{pmatrix}. \tag{93}$$

Again applying the theorem of factor-function transformation, and designating the Fourier-transforms of $U$'s by $V$'s, we have

$$V(z,\omega)=e^{zB(\omega)}\,V(\omega), \tag{94}$$

where

$$B(\omega)=\begin{pmatrix} 0 & I \\ P(i\omega) & 0\end{pmatrix}, \tag{95}$$

$$e^{zB(i\omega)}=\begin{pmatrix} \cos h\, z\sqrt{P(i\omega)} & \frac{1}{\sqrt{P(i\omega)}}\sin h\, z\sqrt{P(i\omega)} \\ \sqrt{P(i\omega)}\sin h\, z\sqrt{P(i\omega)} & \cos h z\sqrt{P(i\omega)}\end{pmatrix}, \tag{96}$$

$$P(i\omega)=\sum_{i=0}^{n} a_\nu (i\omega)^\nu . \tag{97}$$

We see that if we know the spectra of temporal variation of the quantity which describes our system at $z=0$ and other points in the system, we can calculate, at least in principle, the polynomial $P(i\omega)$, and hence the coefficients in the differential equation (89), strictly or approximately, and thus the problem will be completely solved, apart from the above-stated ambiguity, which would practically be tolerated.

It is to be noted that there is a marked parallelism between Prof. Imahori's method of treatment of the lumped constant systems and our method probably usable in treating the distributed systems. Our polynomial $P(i\omega)$ exactly corresponds to the impedance function $Z(i\omega)$.

If the assumption be made that the temporal variation of $U$'s is strictly sinusoidal, we obtain, as before, the semi-group transformation relation between the amplitude $A$ at $z=0$, and that at $z=z$:

$$A(z) = e^{-\sqrt{P(i\omega')}z} A \,. \tag{98}$$

Apart from the afore-mentioned problem concerning the spacial terms, there remain a lot of problems which must further be investigated. It is, for instance, necessary to introduce the statistical elements into the theory in order to be able to treat the statistical problems. The stability problem may also be of interest. Further it is naturally required to put the method to the test by applying it to practical problems. All these details are left to future investigations.

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Note added in Proof: The generalization of the operational method to the case in which the number of the spacial derivatives is arbitrary, and the introduction of the statistical elements into the scheme have already been achieved. These led to the most general theory of Brownian motion, which can treat statistically perturbed distributed systems as well as lumped-constant systems, and be applied to the prediction problem of such systems: The paper dealing with these generalisations is now in press.