Theoretical Studies on Motion of Snow Kicked up by a Snowplow*

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Abstract

Theoretical studies on motion of snow kicked up by the front edge of a snowplow reveal that the motion is subject to great changes when advancing speed V of the snowplow approaches to or coincides with velocity c of a weak plastic wave propagating within snow lying before the snowplow and that the velocity of the kicked up snow and the force exerted by the snow on the front edge of the snowplow are obtained as functions of $\gamma=(V/c)^2$.

Contents

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I. Introduction

A snowplow consists of a steel blade or blades attached to the front of a motor-driven vehicle for the purpose of clearing snow away from highways or railroads. The Japanese National Railways is planning to run a high-speed electric railcar train, SHINKANSEN, in snowy districts of Japan in the near future. A snowplow must be attached to the front of the train to remove snow deposited on tracks, but many problems will arise from the speed of the train which exceeds 200km/hr. The present paper was prepared to predict motions of snow kicked up by the snowplow running at the high speeds.

Any kind of snowplow first kicks up snow with its front edge. This 'kick up' has characters common to all kinds of snowplows, whereas subsequent movements of the kicked up snow vary with the shape of a snowplow. Theoretical studies of phenomena of the kick up described in this paper will give force $\boldsymbol{F}'$ exerted by snow on the front edge of a snowplow and velocity $\boldsymbol{v}_1$ which the snow gets immediately after the kick up. Velocity $\boldsymbol{v}_1$ provides an initial condition for the movements of the snow ascending from the front edge of the snowplow along its surface varying in shape with the snowplow. After having finished the movement on the surface of the snowplow, the snow flies off into the air striking the body of the vehicle or not doing so ; velocity $\boldsymbol{v}_1'$ of the flying off snow changes diversely depending upon the shape and configuration of both the snowplow and vehicle. Force $\boldsymbol{F}_V'$ of the snow on the vehicle can be determined when velocity $\boldsymbol{v}_1'$ is known;force $\boldsymbol{F}'$ is no more than a part of force $\boldsymbol{F}_V'$, but force $\boldsymbol{F}'$ plays an important role in straining the joints which connect the snowplow to the body of the vehicle.

The snowplow compresses plastically the snow lying before it and the compressed state propagates forward in the snow with velocity c of the weak plastic wave. The phenomenon of the kick up appears very differently according as advancing speed V of the snowplow is smaller or greater than c, with the result that both force $\boldsymbol{F}'$ and velocity $\boldsymbol{v}_1$ undergo great changes in magnitude and direction when V increases or decreases past the value of c. Role of velocity c corresponds to that of sound velocity in the aerodynamics:it is well known that motion of the air around an airplane and force of the air on it are subjected to abrupt and great changes when speed of the airplane goes over the sound velocity. Speed V shall be called 'low speed' and 'high speed' respectively according as it is smaller or greater than c, while it is called 'critical speed' when it coincides with c. The present theory will be developed with stress laid on the phenomena occurring in the kick up when V lies near the critical speed. Although the value of c is not exactly known at present, it may be thought to be a few tens or several tens of meters per second. Therefore, when V lies near c, the inertial force of the snow will be comparable with or greater than internal stresses induced within the snow by the snowplow. No regard is paid in this paper to cases of the 'very low speed' of the snowplow where V is so small that the inertial force of the snow practically disappears.

II. Assumptions and two types of kick up

The following assumptions (I)~(IX) are made in this section for simplification and easiness in development of the theory. One more assumption (X) will be added in section VII. It is noted that the snow kicked up by the snowplow is in almost all cases newly fallen soft snow of small densities about $0.1 g/cm^3$. But very few experimental results have yet been obtained on mechanical properties of the soft snow , while a good many results are available concerning those of compact snow of densities greater than $0.2 g/cm^3$, as seen from the reviews prepared by Mellor.[5)6)] The difference between densities 0.1 and $0.2 g/cm^3$ is accompanied with a remarkable difference in the mechanical properties of snow. Such being the case, assumptions will below be made on mechanical properties of the soft snow by using the experimental results concerning compact snow as basises and by introducing suppositions which seem reasonable from the physical point of view.

II. 1. *Assumptions on motion and shape of the snowplow and an assumption on motion of the snow*

(I) A vehicle with a snowplow in its front runs straight at constant speed V on a flat ground covered evenly with a layer of uniform snow. The snowplow has a rectangular shape with the longer sides lying horizontally and its plane slanting to the horizontal. The lower horizontal side constitutes the front edges of the snowplow, lying several centimeters above the ground. Let h_0 denote the height of the surface of the snow layer above level of the front edge and ρ_0 denote the uniform density of the snow.

(II) The same front edge lies perpendicularlly to the advancing direction of the vehicle, while the surface of the snowplow makes scooping angle α with the horizontal, α being smaller than 90°.

(III) The snow undergoes uniaxial compression with the axis always in the direction of its instant motion. When the snow is nearing the snowplow, it is compressed uniaxially in the advancing direction of the vehicle with no vertical elongation. When the snow is kicked up by the front edge of the snowplow, the course of its motion is sharply curved. It is assumed that, even in this very short interval of the kick up, the compression of snow is uniaxial and its axis changes in direction so as to follow the course of motion of the snow. This assumption of uniaxial compression is based upon the fact that the inertial force of snow balances a greater part of internal stresses raised in the snow by the snowplow. When the vehicle moves at a 'very low speed', the internal stresses must keep balance among themselves with the result that the above assumption becomes absurd.

II. 2. *Two types of kick up : flow type and spray type*

The Railway Technical Research Institute of the Japanese National Railways designed many snowplows of various shapes and tested their working conditions by attaching them to the front of a diesel-engine locomotive. Figures 1 and 2 give some of the photographs taken in the testing experiments, which show moving states of the snow kicked up by a V-shaped snowplow with a narrow horizontal penthouse, a narrow horizontal steel plate, all along the upper margin of its blades. It is known from these photographs that there are two types of kick up, which will be named 'flow type' and 'spray type'. The former is characterized by the flowing up of the kicked up snow over the surface of the snowplow, maintaining its own consistence and keeping in contact with the surface, while the latter is characterized by pulverization of snow by the kick up and flying of the pulverized snow in a direction deviating from the horizontal by more than scooping angle α of the snowplow. Schematic pictures of the two types of kick up are shown in Figs. 4 (a) and 4 (b).

Kick up of the flow type is seen in photographs 1, 2 and 3 of Fig. 1 which are three

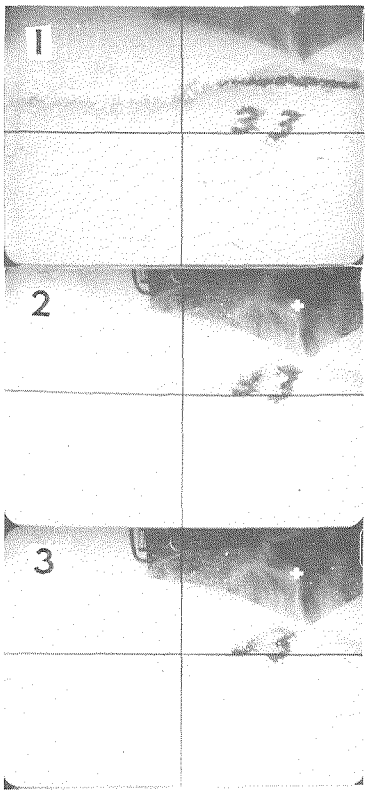

Fig. 2 Kick up of the spray type. Locomotive's speed is 82km/hr. (Photo. by K. Shinojima, Japanese National Railways)

Fig. 1 Kick up of the flow type. Locomotive's speed is 77km/hr. (Photo. by O. Takahashi, Japanese National Railways)

frames taken out from a kinematographic film. The black figure in the upper right quarter of each photograph shows the lowest portion of the locomotive's front and the snowplow is attached to it under an automatic coupler of which the front is marked by the white cross. The numerical figure 33 written on the surface of snow rises on the surface of the approaching snowplow without being crushed, which shows that the kicked up snow flows up the surface of the snowplow in the form of a layer. It is seen in the photographs that the snow is flying horizontally off the snowplow ; this is because the penthouse of the snowplow compels it to do so by bending sharply the course of its flow. Figure 2 gives a photograph of the kick up of the spray type. When the snow is kicked up by the front edge of the snowplow, it is pulverized and sprayed in directions so inclined forward that it flies high without being hindered by the penthouse.

In the above testing experiments the railroad was not covered with naturally fallen snow, but with snow raked together upon it from its both sides. There might be such cases in which the kick up of snow was of the mixed type; namely, the kick up was partly of the flow type and partly of the spray type. But it is assumed that

(IV) for all kinds of snow the kick up is purely either of the flow type or of the spray type.

II. 3. *Kinosita's experiments on the uniaxial compression of snow*[1,2]

Kinosita stood a circular pillar of compact snow about 10 cm high on an iron table and compressed the pillar by lowering at constant speeds an iron plate placed upon its top. The speeds of compression were of the order of mm/min. When the speeds of compression were less than a specific speed which varied from 3 to 8 mm/min among the samples, the pillar was compressed uniformly from top to bottom . But, when the speed of compression was increased beyond the specific speed, the uniform compression stopped, and the snow pillar came instead to be lessened in height by intermittent destructions at its top end or at its bottom end. The intermittent destructions occurred in a thin layer of snow adjacent to the lowering iron plate or to the iron table, the destroyed snow rushing out in the form of powder from the margin of the end plane of the pillar. Kinosita called the former type of compression "plastic" while he called the latter type of compression "destructive" or "brittle".

Figure 3 shows the relationship between uniaxial compressive strees p and uniaxial compressive strain ε produced in the snow pillars of the above experiments. As ε increases in proportion to time t counted from the start of compression, the axis of ε may be thought as that of t. Curve of p versus ε changes its appearance in the order of (1), (2)..., (6) as the speed of compression increases; curve (4) in Fig.(b) is a sideways expanded duplication of curve (4) in Fig.(a). The plastic compression gives curves (1), (2) and (3), while the brittle compression gives curves (4), (5) and (6) : the speed of compression goes across the specific speed while curve (3) turns to curve (4). In Fig.3(b) of the brittle compression, p rises first from 0 in proportion to ε until it attains a certain value p^* and then drops down to a

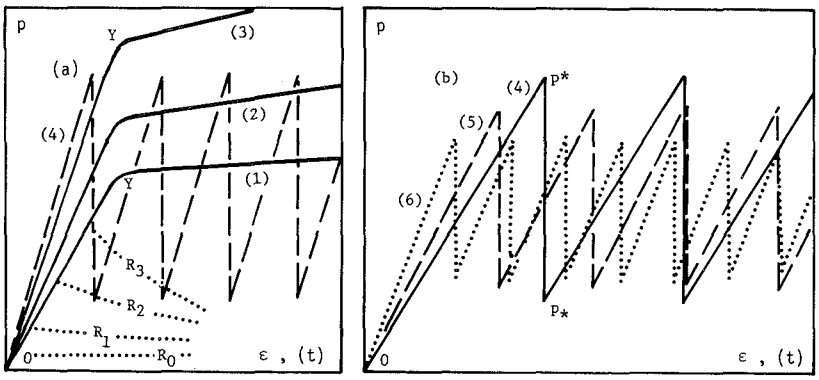

Fig. 3 Stress p and strain ε in uniaxial compression of a pillar of snow. The curve changes in shape in the order of (1), (2), ..., (6) as the time rate $\dot{\varepsilon}$ of change in ε increases.

smaller value p_*. After this, the same rise and drop of p repeat with the only difference that the rises start not from 0 but from p_*. Everytime p reaches p^* the snow pillar is destroyed at one of its ends and removal of a thin layer of snow from there causes p to drop from p^* down to p_*.

It is the compression due to the lowering iron plate that causes the first rise as well as the subsequent repeating rises of p, while the time rate of change $\dot{\varepsilon}$ in ε is proportional to the lowering speed of the iron plate which was above called the speed of compression. Therefore, it can be said that, when $\dot{\varepsilon}$ is greater than a value specific to the snow, the relation

$$p = k\varepsilon \quad \text{with } k = \text{constant and } p < p^*$$

holds for the loading process in uniaxial compression of the snow. Magnitudes of p^*, p_* and k depend upon the sort of snow as well as upon the speed of compression. With increasing speed of compression, p_* and k increase whereas p^* decreases.

II. 4. *Assumptions on the uniaxial stress and strain raised in the snow by the snowplow*

A snowplow pushes snow at far greater speeds than the iron plate used in the Kinosita's experiments for compressing the snow pillar. Accordingly, if there is no other type of compression than the plastic and brittle types defined by Kinosita, compression of the snow caused by the snowplow must be of the brittle type. It is soft snow of about 0.1g/cm^3 in density that the snowplow must remove, whereas the snow used in the Kinosita's experiments was compact and had densities over 0.2g/cm^3. But the results of the Kinosita's experiments will probably be applicable also to the soft snow. Although $\dot{\varepsilon}$ is of the same value throughout the snow pillar in the Kinosita's experiments, $\dot{\varepsilon}$ varies its value from point to point in the snow which is being compressed by the snowplow. Change in advance velocity

V of the snowplow also causes $\dot{\varepsilon}$ to change, and any change in $\dot{\varepsilon}$ brings changes to the values of k and p^*. The former positional variability of these constants will complicate the theory very much. Therefore, use will be made of k and p^* averaged with respect to position which are functions of V.

Thus the following assumptions (V) and (VI) are made on uniaxial stress p and strain ε raised in the snow by the snowplow.

(V) In a loading process, the relation

$$p = k\varepsilon \tag{1}$$

holds, where k is a constant which depends on nature of the snow and speed V of the snowplow.

(VI) When p and ε reach respectively critical values p_P and ε_P, the snow is pulverized, where p_P is the averaged value of p^* and ε_P is connected with p_P by $p_P = k\varepsilon_P$. Critical stress p_P and strain ε_P shall respectively be called 'stress of pulverization' and 'strain of pulverization'. They depend on V and nature of the snow.

II. 5. *Assumption on unloading of the stress*

As shown in Fig.3(a), an increase in the speed of compression brings about actually an increase in k together with an upward shift of point Y, bending point of the curve of the plastic compression. These facts suggest that the loading process referred to in assumption (V) is not purely elastic as may be considered so from the form of Eq.(1). Wakahama compressed a plate of single crystal of ice by the same method as used by Kinosita, and found that compressive stress p in the plate grew with increasing strain ε by following a curve similar in shape to curve (1) in Fig.3(a). He found also that, if the compression was stopped before p reached point Y, p relaxed as curves R_1, R_2 and R_3 show with time t as an abscissa.[7] Curve R_0, however, stretches horizontally, showing that the compression is purely elastic if and only if p is kept below a certain small value. From such experimental results it may well be supposed that elasticity involved in compression is weak in the case of snow which is a collection of small particles of single crystal of ice joined to one another with much free space between; it will, in paticular, be very weak in the case of soft snow of very low density which the snowplow is to kick up. Thus, in disregard of the elasticity in which any decrease in p is accompanied by a decrease in ε proportional to itself, it is assumed that

(VII) in an unloading process, p decreases with no change in ε, the compression of snow is purely plastic in the limited sense that no elasticity is involved in it. (It is not plastic in the ordinary sense, because the snow is pulverized when p reaches p_P.) Equation (1) is still applicable in this case if k is put equal to infinity.

The relation between p and ε assumed in assumptions (V) and (VII) can be graphed by a right-angled triangle of which the oblique side gives Eq.(1). It is well known that stress in snow relaxes with time rather quickly, and consequently the above relation must include time

if it should be more realistic. But time is excluded from the relation in the present theory to keep it from becoming complicated.

II. 6. *Assumptions on generation of the spray type kick up*

In this theory the flow type kick up is considered fundamental while the spray type is a modification of it. The modification is assumed to be brought about by a phenomenon that

(VIII) the snow is pulverized when its uniaxial stress p reaches p_P or its uniaxial strain ε reaches ε_P during the flow type kick up, and by another phenomenon that

(IX) the pulverized snow keeps velocity vector v_P which the flowing snow had just before the pulverization and reflects on the surface of the snowplow to fly upwards as shown in Fig.4(b). In a kick up of spray type, shape of the snowplow has no influence on the motion of the kicked up snow, because it flies off the surface of the snowplow after the reflection.

In the following sections equation of motion of the snow will first be derived for the flow type kick up and its solution will be obtained with no limit imposed on the value of p in Eq. (1). As the actual curve of p versus ε will resemble curve (3) in Fig.3(a) if the snow is not pulverized, the solution will be wrong in its portion where p exceeds p_Y, value of p at point Y indicated on that curve. But this brings no difficulty, because the snow is pulverized at $p=p_P$ with the result that the wrong portion is eliminated by the transformation of kick up

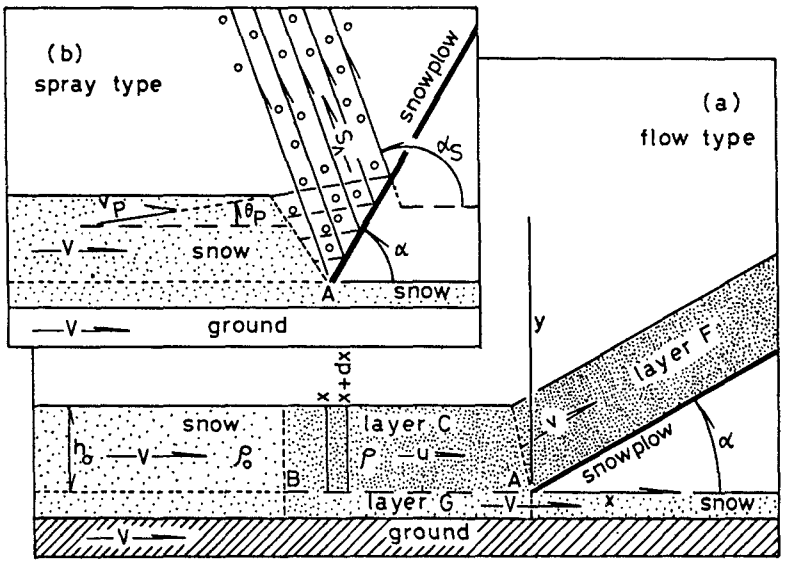

Fig. 4 Schematic diagrams for the two types of kick up : diagram (a) for the flow type and diagram (b) for the spray type. In diagram (b) θ_P may be read α_P while v_P and v_S are the same in magnitude.

from the flow type into the spray type. The eliminated portion will be called 'virtual kick up of the flow type' in the following.

As the snow is compressed fairly rapidly by the snowplow, the air contained in the snow is compelled to rush out of it. Pulverization of snow must greatly be promoted by this rushing out of the air. A number of other influences the air in snow may have on the motion of the snow are disregarded in this paper.

III. Compression of snow in front of the snowplow: Forecompression

III. 1. *Use of coordinates moving with the snowplow*

Studies will be made in this section III of the compression which the snow undergoes before it is kicked up by the front edge of the snowplow. This compression shall be called 'forecompression'.

The motion of snow is described in reference to coordinates x, y and z of which the origin is fixed to and moves together with the front edge of the snowplow shown by point A in Fig. 4(a). When looked at from this moving frame of coordinates, the snowplow stand still, the snow and the ground run towards the snowplow at its advancing speed V and all the phenomena occurring in the snow appear stationary. The axis of coordinate x is laid horizontally in the running direction of the snow and the ground while the axis of coordinate y is stood upright. The edge of the snowplow lies parallel to the axis of coordinate z. As things do not change in the direction of z, the snow as well as the snowplow can be regarded to have the width of a unit length.

III. 2. *Equation of motion for the snow undergoing the forecompression*

The snow layer running towards the snowplow begins to be compressed in its upper part which lies above the level of the front edge of the snowplow, when it reaches a point positioned some distance before the snowplow. The remaining lower part escapes from the compression and continues running together with the ground at speed V to pass through the gap under the front edge of the snowplow. The upper part shall be called 'layer C' because it is being compressed, whereas the name 'layer G' shall be given to the lower part because it continues to be conveyed by the ground. Thus the forecompression is limited to layer C, which keeps its initial thickness h_0 in spite of the forecompression by assumption (III).

If ρ and u denote the density and velocity of the snow in layer C respectively, they are functions of x which is negative and satisfy the following two relations

$$\rho_0 V = \rho u \tag{2}$$

and

$$\rho_0=\rho(1-\varepsilon) \tag{3}$$

where ρ_0 is the density of the original snow lying ahead of the forecompressed snow. Elimination of ρ and $\rho_0$from them gives the formula

$$\varepsilon=1-(u/V). \tag{4}$$

Two forces expressed by

$$df_1=-h_0(dp/dx)dx \quad \text{and} \quad df_2=\mu h_0\rho g dx$$

act on an element of layer C which extends from x to $x+dx$ as shown in Fig.4(a). Force df_1 is due to uniaxial stress p given rise to in layer C by the forecompression, while df_2 is a frictional force exerted on the element through its bottom by layer G. Letter g gives the acceleration of gravity and μ gives the coefficient of friction between layers C and G. Force df_2 is positive, because layer G runs faster than the element in the positive derection of x. The element has a mass equal to $\rho h_0 dx$, while its acceleration is given by $u\,(du/dx)$ because u is independent of time t. Thus the equation of motion

$$\rho u(du/dx)=-dp/dx+\mu\rho g \tag{5}$$

is obtained for the snow in layer C by equating the product of the above mass and acceleration to the sum of forces df_1 and df_2.

III. 3. *Integration of the equation of motion*

The forecompression is a loading process because the snow is more and more compressed as it comes nearer the snowplow. Therefore relation (1) in assumption (V) holds between p and ε and Eq.(4) transforms the relation into

$$p=k\{1-(u/V)\}. \tag{6}$$

By use of Eqs.(2) and (6), the equation of motion (5) can be rewritten as

$$\{1-(1/\gamma)\}\ du/dx=\mu g/u \tag{7}$$

$$\text{with} \quad \gamma=\rho_0 V^2/k \tag{8}$$

If the point at which the forecompression starts is called point B and its distance from point A is denoted by l, the condition

$$u=V \quad \text{for} \quad x=x_B=-l \tag{9}$$

is imposed, becouse point B is the very point at which the velocity of the snow begins to decrease from V and point A makes the origin of coordinate x. Then, if u decreases down

to u_A at point A, solution of Eq.(7) is obtained in the form

$$(1-\gamma)(u^2-u_A{}^2)=-2\mu g\gamma x \tag{10}$$

and application of condition (9) to this solution gives

$$l=(1-\gamma)\ (V^2-u_A^2)/2\mu\gamma g \tag{11}$$

which is the length of the forecompressed snow.

As $V>u>u_A$ and $l>-x>0$, it is necessary that

$$\gamma<1 \tag{12}$$

for both of Eqs.(10) and (11) to be consistent, which means that the forecompression makes its appearance only when condition (12) is fulfilled. Velocity u_A can be expressed as a function of γ by use of the equations which will be obtained later. Replacement of u_A in Eq. (11) with that function shows that l is 0 at $\gamma=0$ and $\gamma=1$ while it attains a maximum between these two values of γ.[4)]

Length l of the forecompressed snow as given by Eq.(11) is 0 for $\gamma=1$ and becomes negative in value when γ exceeds 1. Since negative l has no physical meaning, it follows that, if $\gamma\geqq 1$, the forecompression disappears and the snow layer advances until it reaches the snowplow with its natural state unaltered, the ground doing no other thing than to convey the snow layer. On the contrary, in case of $\gamma<1$, the ground does work on the snow in layer C by exerting the frictional force on it through layer G. As will be seen later, whether the ground does work or not brings about great differences in the motion of the snow.

III. 4. *Weak plastic wave of compression propagating in the snow*

Suppose a semi-infinitely long rod of elastic material of which the density is ρ_0 and in which uniaxial stress p and uniaxial strain ε are connected by $p=k\varepsilon$. If pressure $+\Delta p$ so small as to produce no appreciable increase in density of the material is at time $t=0$ applied to the end of the rod and the application continues thereafter, a wave of stress $+\Delta p$ propagates along the rod with its front advancing at speed $c=\sqrt{k/\rho_0}$, with the result that the rod is at time t stressed and strained with $+\Delta p$ and $+\Delta\varepsilon=+\Delta p/k$ throughout distance $D=ct$ off the end. Now let a continuous negative pressure $-\Delta p$ be applied to the end of the rod from time t_0 on. This means stop of the application of pressure $+\Delta p$ at $t=t_0$ and start of another wave of negative stress $-\Delta p$ at the same moment in superposition upon the wave of $+\Delta p$. Let the waves of $+\Delta p$ and $-\Delta p$ respectively be called 'plus wave' and 'minus wave'. Then the front of the minus wave runs after that of the plus wave at the same speed as c, canselling the stress and the strain which the latter wave has induced. This results in that there is advancing a piece of plus wave $D_0=ct_0$ long, bounded by the fronts of the plus and minus waves.

The situation becomes different in the following point, when the rod is plastic in the

meaning as mentioned in assumptions (V) and (VII).As the application of pressure $-\Delta p$ is an unloading process, k must be infinitely great in this case, which makes speed of the minus wave grow infinitely. Therefore the front of the minus wave runs infinitely fast to cancell instantly at time t_0 all the stress which the plus wave until then induced in region A extending $D_0=ct_0$ from the end of the rod. But the strain which accompanied the stress is left as it is, because the unloading of the stress must occur with no change in strain. After time t_0, the plus and minus waves propagate simultaneously at the same speed cancelling each other, with the result that they produce no effect. Thus the plastic rod stays after time t_0 unstrained everywhere except in region A where it is left strained with $+\Delta\varepsilon=+\Delta p/k$. Although such a difference comes into appearance when the application of pressure $+\Delta p$ is stopped after a finite time interval, speed c of the plus wave is the same whether the rod is elastic or plastic, because the same relation $p=k\varepsilon$ holds true for any loading process occurring in both the rods.

Since the snow to be kicked up by the snowplow is regarded plastic as stated in assumptions (V) and (VII), when a weak compression is created in the snow, a weakly compressed state must propagate in it at a speed given by

$$c=\sqrt{k/\rho_0}, \tag{13}$$

This is the velocity of the weak plastic wave referred to in the introduction and γ defined by formula (8) can be expressed as

$$\gamma=(V/c)^2. \tag{14}$$

If the snow makes a head-on collision with a plate without escaping sideward unlike in the case of striking the snowplow, the snow is pulverized and the pluverized snow is strongly compacted before the plate. This will result in production of an intensely compressed state which propagates as a wave in the snow left intact. As term 'plastic wave' seems to be used for such a wave of intense plastic compression[6], adjective 'weak' is added to the name to be given to the above described wave of plastic compression propagating at speed c. Hereafter, γ will be called as 'velocity variable' and "$\gamma<1$" and "$\gamma>1$" will respectively be used in place of sentences "V is a low speed" and "V is a high speed".

III. 5. *The forecompression of snow and the weak plastic wave*

It is further shown that, if pressure p applied to the end of the rod described in the previous article is not very weak, a wave of stress p propagates still with the same velocity as $c=\sqrt{k/\rho_0}$ as long as stress p and strain ε grow in accordance with the relation of $p=k\varepsilon$ and provided that the rod does not expand laterally in spite of the longitudinal compression due to stress p. Thus the snowplow can compress the snow before it arrives when the snow is coming at a speed smaller than c, that is, when $V<c$ or $\gamma<1$, because the compression which the snowplow produces goes up the stream formed by the coming snow. In the otherwise case of $V>c$ or $\gamma>1$, the compression cannot go up the stream and the snowplow

fails to compress the snow before its arrival. Such a situation of appearance or disappearance of the forecompression of snow is in good accord with the result obtained at the end of the preceding article (3) and provides the reason for dividing the advancing speed V of the snowplow into low and high speeds as defined in the introduction.

If c_E denotes velocity of weak elastic wave of a material, velocity c of weak plastic wave of the same material is generally smaller than c_E. Mellor calculated c_E of snow using its elastic constants and density ρ_0, while he presented a figure (Ref.6,Fig.5) which shows a general tendency of change in c_E in the region of ρ_0 extending from 0.25 to 0.85 g/cm^3. An extension of the tendency into the region of ρ_0 lower than 0.25 g/cm^3 will induce one to believe that c_E is at $\rho_0 = 0.1$ g/cm^3 (density of soft snow) reduced to a value much smaller than sound velocity in the air. Kojima compressed lightly a column of snow whose density lies between 0.16 and 0.19 g/cm^3, and determined its Young's modulus at different temperature below 0℃.[4] Value of c_E calculated from Kojima's data ranges from 37 m/s at 0℃ to 97 m/s at −20℃.[10] Based upon what is stated above, c of the soft snow to be kicked up by the snowplow is estimated in magnitude at a few tens or several tens of m/s.

III. 6. *Total dragging force on the forecompreessed snow*

If F_G is the total frictional force with which layer G drags the forecompressed snow composing layer C, then F_G is given by

$$F_G = \int_{-l}^{0} \mathrm{d}f_2 = \mu g h_0 \int_{-l}^{0} \rho \mathrm{d}x.$$

Density ρ can be expressed as a function of x through Eqs. (2) and (10), which yields

$$F_G = h_0 k (1-\gamma) \{1-(u_A/V)\} \qquad \text{for} \qquad \gamma < 1. \tag{15}$$

This formula holds true only for $\gamma < 1$; F_G vanishes for $\gamma > 1$. As it is the running ground that makes the source of this total dragging force, its symbol is suffixed with G.

IV. Details of the phenomenon of kick up

IV. 1. *Narrow zone of kick up at the front edge of the snowplow*

Section IV devotes some considerations to a violent change to which the snow is subjected while it is being kicked up by the front edge of the snowplow. Although the kick up is a violent phenomenon, its actual process must be continuous as described below. Figure 5 is an enlarged picture of the deformation occurring to the snow during the kick up near the front edge of the snowplow shown by point A. The original snow cover advances from the left to the right. The forecompression starts at point B in upper layer of h_0 in

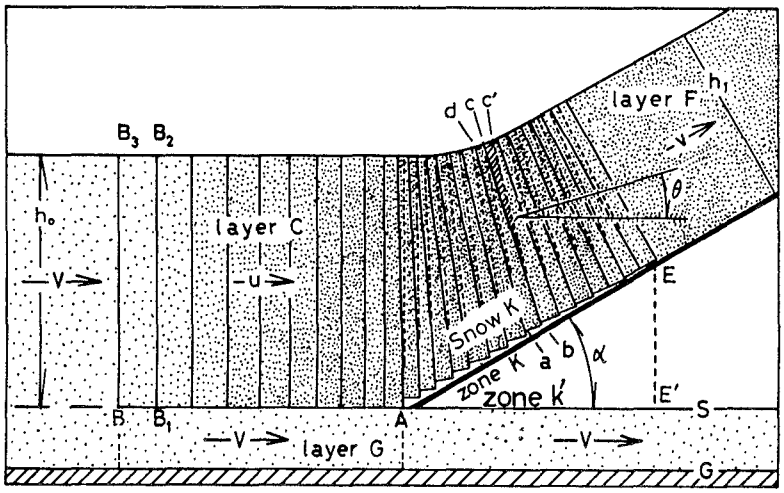

Fig. 5 Schematic diagram showing the process of uniaxial compression of snow caused by snowplow AE.

thickness of the snow and an element of snow enclosed in a thin rectangle $BB_1B_2B_3$ becomes thinner and thinner as it moves to the right, to climb, after arrival at point A, the surface of the snowplow shown by thick line AE.

The surface of the snowplow tilts the element and shortens its top more than its bottom with the result that the contour of the element becomes a trapezoid abcd shown for instance in Fig. 5. But such a deformation of the element is ignored in the present theory. The element is supposed to preserve rectangular contour abc'd in spite of a geometrical contradiction that the space of hatched triangle bcc' is doubly occupied by the element and the next one on the right. Such a neglect of the deformation is necessary for making assumption (III) of uniaxial compression in section II applicable. The thinner the snow cover is, the weaker the above geometrical contradicton becomes and the nearer assumption (III) approaches the truth.

The element of snow changes, during the kick up, its height and thikness as it tilts and, as shown in Fig. 5, these tilt and changes increase continuously as the element climbs higher the surface of the snowplow. This continuous increase is due to the force which each point of the surface exerts on the element. It is assumed that the action of the force, consequently also the above tilt and changes caused by it, are limited within a zone extending a short distance from point A to point E with the result that the process of the kick up is completed at point E. Although point E appears to be located distant from point A in Fig. 5, they are actually positioned so near that the continuous process of the kick up looks sharp and violent. The narrow zone described above shall be called 'zone K', an abbreviation of 'zone of kick up'.

If θ denotes the angle by which the element of snow tilts in zone K, the velocity vector of the element should also make the same angle θ with the horizontal according to assumption (III).

IV. 2. *Replacement of the actual kick up with an imaginary process*

Rapid changes occur in the velocity and compressive stress of the element of snow during the short interval in which the element passes zone K. The actual kick up is so complicated a process that it makes exact determination of the velocity and the stress almost impossible. Therefore an approximate determination will be made on them by replacing the actual kick up by an imaginary process which is much simpler than the actual one.

It is imagined that the snowplow disappeared and left, on horizontal plane S shown in Fig.5, x-components of the forces which zone K had been giving to the snow before the disappearance of the snowplow. Plane S is not the surface of moving layer G, but is an imaginary plane which stands still. As the snowplow is absent, the snow in layer C goes straight over point A to proceed on plane S with the x-components of the forces acting on its bottom. Such is the imaginary process mentioned above. The snow will perform in this imaginary process a motion not far different from the x-component of the actual motion occurring on zone K.

In Fig.5 point E′ is located immediately under point E and a part of plane S lying between points A and E′ shall be called 'zone K″. If p' and $f_x dx$ give respectively the stress of the imaginary snow on zone K′ and the force which an element dx of zone K′ exerts on the bottom of the snow, the following equation of motion is given to the imaginary snow

$$\rho u\,(du/dx) = -dp'/dx + f_x/h_0\,. \tag{16}$$

It is noted that f_x is always negative, because the forces of zone K are inclined in their direction towards the negative side of coordinate x.

IV. 3. *Tendency of changes in the compressive stress of snow in zones K and K′*

If the compressive strain of the imaginary snow is denoted by ε', a relationship between ε' and p' can be expressed by

$$dp' = \varkappa d\varepsilon'. \tag{17}$$

If $dp'/dx > 0$ in Eq. (16),then $\varkappa$ is equal to k shown in Eq. (1), because the snow is more and more compressed as it advances so that the compression belongs to loading processes. When $dp'/dx < 0$, the compression of snow is unloading and $\varkappa$ becomes infinitely great, because $d\varepsilon'$ must vanish for any negative value of dp'. Therefore $dp'/dx > 0$ is combined with $\varkappa = k$, while $dp'/dx < 0$ is combined with $\varkappa = \infty$. Equations (2), (4) and (17) transform Eq. (16) into

$$\{1-(\rho_0 V^2/\varkappa)\}\ dp'/dx = f_x/h_0. \tag{18}$$

Suppose that $dp'/dx>0$ and $\varkappa=k$. Then Eq. (18) is written as

$$(1-\gamma)\, dp'/dx=f_X/h_0, \tag{19}$$

and this equation requires that $\gamma>1$, for otherwise its left side cannot be negative. Thus, when V is a high speed, compressive stress p' of the imaginary snow increases as it advances on zone K'. This situation leaves cases of $\gamma<1$ for the combination of $dp'/dx<0$ and $\varkappa=\infty$. Therefore, when V is a low speed, p'decreases as the imaginary snow goes on zone K' and Eq.(18) takes the form

$$dp'/dx=f_X/h_0. \tag{20}$$

It is assumed that p' gives compressive stress p of the actual snow on zone K for the same value of x;that is, p at any point N on zone K is the same as p' at the point which lies on zone K' immediately under point N. Through this assumption Eqs.(19) and (20) are respectively rewritten into

$$(1-\gamma)\, dp/dx=f_X/h_0 \qquad \text{for} \qquad \gamma>1 \tag{21}$$

and

$$dp/dx=f_X/h_0 \qquad \text{for} \qquad \gamma<1. \tag{22}$$

Thus it is known that, in case of $\gamma>1$, compressive stress p starts increasing at point A of Fig. 5 to continue increasing on zone K, whereas p increases, in case of $\gamma<1$, in the region of forecompression and attains maximum p_A at point A to decrease thereafter on zone K.

IV. 4. *Running resistance due to the kick up*

If vector $\boldsymbol{F}$ is used for representing the force with which the front edge of the snowplow kicks up the snow, $-\boldsymbol{F}$ is equal to $\boldsymbol{F}'$defined in the introduction. Therefore F_X, x-component of F, constitutes, when reversed in sign, running resistance due to the kick up. It is noted that the snowplow is thought to be a unit length in width as mentioned at the end of Article (1) of section III. As the kick up is caused by the forces which every point of zone K exerts on the snow, it is their resultant that produces $\boldsymbol{F}$.
Accordingly F_X is given by

$$F_x=\int_A^{E'} f_x dx<0\,. \tag{23}$$

Equations (21) and (22) give respectively, when integrated over zone K'.

$$(\gamma-1)\, p_E = -F_X/h_0 \qquad \text{for} \qquad \gamma>1 \tag{24}$$

and

$$p_A - p_E = -F_X/h_0 \qquad \text{for} \qquad \gamma<1\,, \tag{25}$$

where p_A and p_E show respectively p at points A and E in Fig. 5. If $\gamma > 1$, then $p_A = 0$, because the original snow layer arrives at point A with no compression. For this reason p_A is absent in Eq. (24). Equations (24) and (25) will importantly be used in later sections.

When the kick up is of the flow type, from point E on, the snow flows in the form of a layer up the surface of the snowplow and disintegrates to fly off the upper margin of the snowplow. This layer of the flowing up snow shall be called 'layer F'. The present theory will need afterwards p_E'', compressive stress which the snow of layer F has at point E, namely on the lower end plane of the layer. But it is very difficult to make a definite formula for p_E'', because it depends upon the shape and size of the surface of the snowplow which vary from one snowplow to another. Such being the case, p_E'' is put equal to zero in the present theory in order to have its results in definite forms.

Since p gradually decreases upwards in layer F and vanishes at or near the upper margin of the snowplow where the snow disintegrates, p_E'' would be small if not null, when the snowplow is a small one as attached to the front of a locomotive or of an electric car. This offers some support to the assumption of putting $p_E'' = 0$.

V. Flow type kick up at low speeds

V. 1. *The law of momentum applied to the kick up*

In the introduction, vector $\boldsymbol{v}_1$ represented velocity with respect to the ground which the snow has at point E of Fig. 5, that is, at the moment when the kick up finishes. If the advancing velocity of the snowplow is denoted by vector $\boldsymbol{V}$, then relation $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{V}$ gives the velocity at point E of the snow observed from the frame of coordinates moving together with the snowplow. Force $\boldsymbol{F}$ remains the same if the coordinates are changed into the moving ones. In this and the following sections, $\boldsymbol{v}$ and $\boldsymbol{F}$ will be obtained as functions of γ by use of the laws of energy and momentum.

Section V limits studies to the flow type kick up at low speeds. So the forecompression of snow occurs and layer C exists. Name 'snow K' shall be given to such a portion of the snow that lies on zone K and is being kicked up.

Three forces $h_0 p_A \boldsymbol{i}$, $-h_1 p_E'' \boldsymbol{s}$ and $\boldsymbol{F}$ act on snow K, where $\boldsymbol{i}$ and $\boldsymbol{s}$ stand respectively for unit vectors directed along the positive axis of x and along the surface of the snowplow upwards, while h_1 gives the thickness of layer F at its lower end. The second force vanishes, because $p_E'' = 0$ as was assumed above. Thus

$$h_0 p_A + F_X \tag{26}$$

makes the x-component of the forces acting upon snow K.

In a unit time, the x-component of momentum of snow K increases by the amount

$$h_1\rho_1 v \cdot v\cos\alpha - h_0\rho_A u_A \cdot u_A,$$

which is the same as

$$h_0\rho_0 V\ (v\cos\alpha - u_A), \tag{27}$$

because $h_1\rho_1 v = h_0\rho_A u_A = h_0\rho_0 V$ by the law of conservation of mass. Here v is the magnitude of vector $\boldsymbol{v}$ while ρ_A and ρ_1 give respectively the density of the snow at points A and E of Fig. 5,values of h_1 and ρ_1 being common to snow K and layer F at point E. The law of momentum requires formulae (26) and (27) to be equal to each other, which results in

$$p_A + (F_X/h_0) = \rho_0 V\ (v\cos\alpha - u_A). \tag{28}$$

V. 2. *Velocities u_A and v*

Stress p diminishes upwards on zone K and coincides at point E with p_E'' which was assumed to be zero. Therefore p_E must be zero and Eq. (25) is reduced to

$$p_A + (F_X/h_0) = 0. \tag{29}$$

As the left side of Eq. (28) vanishes by virtue of Eq.(29), there follows the equation

$$u_A = v\cos\alpha \quad \text{for} \quad \gamma < 1. \tag{30}$$

The law of energy will, in the following articles, give a function which relates u_A with V, and Eq. (30) becomes useful for transforming that function into the one which relates v with V.

V. 3. *Definition of a thermodynamic system and work done on it*

According to the definitions made in article (2) of section III, the forecompressed snow forms layer C while the snow lying between layer C and the ground forms layer G. A combination of snow K, layers C and G is considered as a thermodynamic system, to which the law of energy will be applied.

During infinitesimal time interval dt, the ground performs on the system work in the amount of

$$\mathrm{d}w_G = F_G V\ \mathrm{d}t = (1 - \gamma)\ \{1 - (u_A/V)\}\ h_0 k V\ \mathrm{d}t, \tag{31}$$

where F_G is the force given by formula (15). The system has three end planes: the vertical plane limiting at point B of Fig. 5 layers C and G at their left ends ; the vertical plane which makes the right boundary of layer G at point A; and the oblique plane cutting the snow into snow K and layer F at point E. No work is added to the system through these end planes, because compressive stress p vanishes upon all of them. If height EE' in Fig. 5 is denoted by δh, gravity does work nearly equal to $-\rho_0 h_0 V\mathrm{d}t \cdot \delta h$. This work is disregarded, because

δh is very small. Zone K will perform work by exerting a frictional force upon the system. This work is also disregarded because zone K is very small in area and its surface made of steel creates only a small friction against the snow. Thus there remains dW_G as the only work done on the system from the outside.

V. 4. *Increase in kinetic energy and plastic internal energy of the thermodynamic system*

Kinetic energy of the thermodynamic system increases in dt by the amount

$$dU_K = (1/2)\,\rho_0 h_0 V\,dt\,\{v^2 - V^2\} = -(1/2)\,\{1 - (v/V)^2\}\,\rho_0 h_0 V^3 dt. \quad (32)$$

There are two causes effecting an increase in internal energy of the thermodynamic system : the plastic compression of layer C and the friction between layers C and G.

In layer C compressive stress p starts from zero at point B to rise up to the maximum p_A at point A. Subsequently p drops in snow K down to zero at point E. Such changes in p cause the following changes in the internal energy of the snow. The internal energy per unit volume increases in layer C from zero at point B to the maximum of $(1/2\,k)\,p_A^2$ at point A, and maintains this maximum in snow K up to point E. The maintenace of the maximum is due to assumption (VII) of section II; this assumption requires that the drop of p in snow K should occur with no change in compressive strain ε, namely that p should not do any work while it is dropping. In layer G no plastic internal energy is produced, because layer G is subject to no compression. Thus plastic compression occurring in layer C and in snow K raises during dt the internal energy of the system by the amount

$$dU_P = (1/2\,k)\,p_A^2 h_0 V\,dt = (1/2\gamma)\,\{1-(u_A/V)\}^2\,h_0\rho_0 V^3 dt, \quad (33)$$

the middle term being converted to the last term by Eqs. (1) and (4). If the snow was elastic, p would do negative work on snow K owing to a decrease in ε and the maximum energy $(1/2k)p_A^2$ would be converted to the kinetic energy of the snow, with the result that there remained no elastic internal energy at point E. Such an elastic situation would make the above dU_P vanish.

The production of plastic internal energy dU_P raises the temperature of layer C and snow K. But there is no time for the energy to escape from them in the form of heat, because the snow runs quickly through the space occupied by them. Thus the appearance of p in layer C and the disappearance of p in snow K turn out to be adiabatic processes.

V. 5. *Friction as a cause for increase in internal energy of the thermodynamic system*

Element dx of layer C shown in Fig.4 (a) makes during dt a displacement $(V-u)\,dt$ relative to layer G, being dragged to the right by a frictional force of $\mu\rho h_0 g\,dx$. Therefore the total frictional force between layers C and G does during dt the work equal in amount to

$$\mathrm{d}U_{\mathrm{F}}=\int_{-l}^{0}(V-u)\mathrm{d}t\cdot\mu\rho h_0 g\mathrm{d}x=\frac{1-\gamma}{2\gamma}\left(1-\frac{u_{\mathrm{A}}}{V}\right)^2 h_0\rho_0 V^3\mathrm{d}t. \tag{34}$$

The integration of the middle term can be accomplished by using Eqs. (2) and (10) for expressing ρ and u as functions of x.

The above work results in an increase in internal energy of the snow composing two thin layers of which the one forms the bottom of layer C while the other forms the top of layer G. Although the thin layers rise in temperature, the internal energy created in them has no time to flow away from the system as a frictional heat. Thus $\mathrm{d}U_{\mathrm{F}}$ causes an increase in internal energy of the thermodynamic system due to a friction between layers C and G.

V. 6. *Vectors* $\boldsymbol{v}$ *and* $\boldsymbol{F}$ *expressed as functions of* V *with* α *as a parameter*

As energy changes occurring in the thermodynamic system are adiabatic, it is required by law of energy that the equation

$$\mathrm{d}W_{\mathrm{G}} = \mathrm{d}U_{\mathrm{K}} + \mathrm{d}U_{\mathrm{P}} + \mathrm{d}U_{\mathrm{F}} \tag{35}$$

holds true. Each term in it can be expressed by a function of v and V by use of Eq. (30), namely by the replacement of u_{A} with $v\cos\alpha$ in Eqs.(31), (32), (33) and (34). Then Eq. (35) yields the following formula which relates v/V with $\gamma=(V/c)^2=p_0V^2/k$:

$$v/V=2\cos\alpha/(2\cos^2\alpha+\gamma\sin^2\alpha) \qquad \text{for} \qquad \gamma<1. \tag{36}$$

This relationship between v/V and γ is shown by the dotted curves and the solid segments marked F in the left half of Fig. 6. Advancing speed V of the snowplow is the 'very low speed' in the domain of Fig. 6 extending on the left side of the broken straight line drawn vertically at $\gamma=0.3$. Therefore it would be better to impose little worth on the portions of curves lying in this domain.

As the kick up is of the flow type, vector $\boldsymbol{v}$ has its direction parallel with the surface of the snowplow ; that is, the direction of $\boldsymbol{v}$ makes with the horizontal an angle equal to α, the scooping angle of the snowplow.

The snow exerts on the front edge of the snowplow a force given by $-\boldsymbol{F}$. This force resists the advance of the snowplow with its horizontal component $F_{\mathrm{R}}=-F_{\mathrm{X}}$, while it pushes down the front edge of the snowplow with its vertical component reversed in sign, namely with $F_{\mathrm{D}}=(-\boldsymbol{F})_{-\mathrm{y}}=F_{\mathrm{y}}$. Equation (29) gives

$$F_{\mathrm{R}}=-F_{\mathrm{X}}=h_0p_{\mathrm{A}}=h_0k\varepsilon_{\mathrm{A}}=h_0k\ \{1-(u_{\mathrm{A}}/V)\}$$

$$=h_0k\ \{1-(v/V)\cos\alpha\} \qquad \text{for} \qquad \gamma<1, \tag{37}$$

in which the successive transformations of p_{A} can be performed by use of Eqs. (1), (4) and (30). It follows directly from the law of momentum that

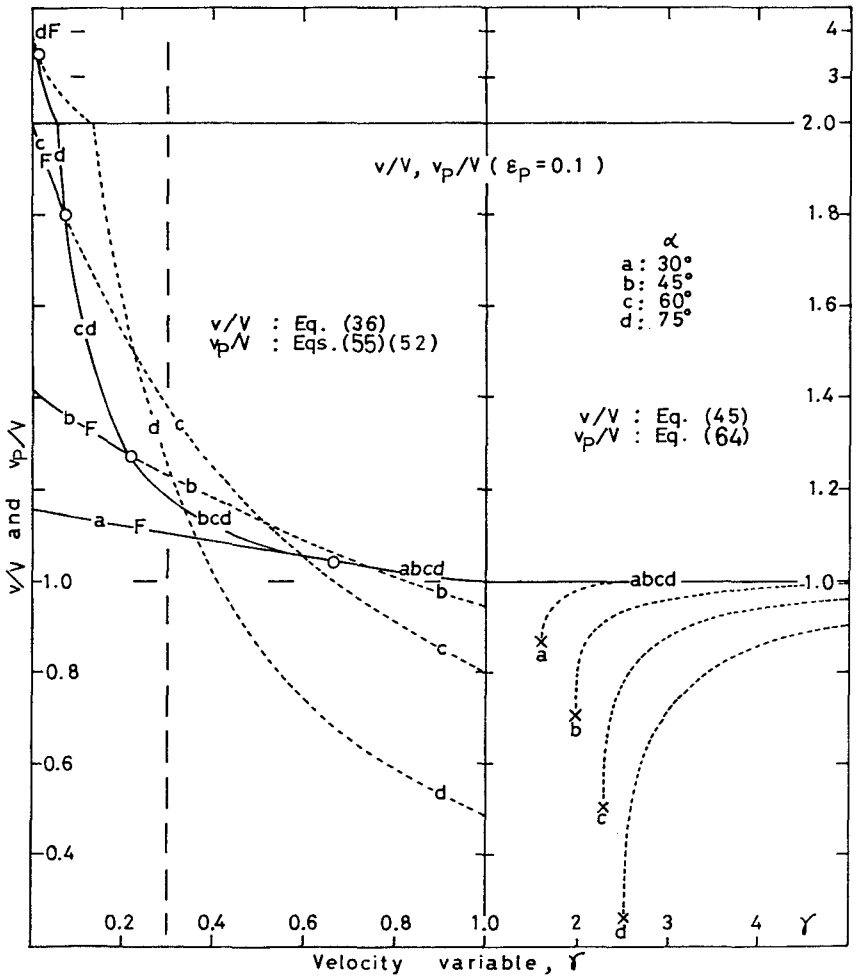

Fig. 6 Relations between v/V, v_P/V and velocity variable $\gamma = V/c$, for the cases of $\varepsilon_P = 0.1$ and of four particular values of α. Letters a, b, c and d stand respectively for 30°, 45°, 60° and 75° of α. Velocity variable γ is scaled differently along the axis of abscissa according as it is smaller or greater than unity. The relations should be roughly estimated to the left of the broken line standing upright at $\gamma = 0.3$. Kick up is of the flow type on the dotted curves and the portions marked F of the solid curves: they give the relations between v/V and γ. The otherwise portions of the solid curves belong to the spray type, giving the relations between v_P/V and γ. Small circles give the transition points between the two types of kick up. Letters, symbols and curves will be used in the same meaning as above also in the following Figures 7 to 11.

$$F_D = F_y = h_0 \rho_0 V \cdot v \sin \alpha \qquad \text{for} \qquad \gamma < 1. \tag{38}$$

If F_R and F_D are written in the form of

$$F_R = C\rho_0 h_0 V^2 \qquad \text{and} \qquad F_D = D\rho_0 h_0 V^2, \tag{39}$$

coefficient C of running resistance and coefficient D of pushing down force are given by

$$C = \{1-(v/V)\cos\alpha\}/\gamma = 1/(\gamma + 2\cot^2\alpha) \qquad \text{for} \qquad \gamma < 1 \tag{40}$$

and

$$D = (v/V)\sin\alpha \qquad \text{for} \qquad \gamma < 1. \tag{41}$$

Equation (36) converts the middle term of Eq. (40) into the right term which is a simple function of γ. Coefficient D can also be expressed as a function of γ by use of Eq. (36). Curves of C and D versus γ will be shown in Figs. 10 and 11 in section VIII.

VI. Flow type kick up at high speeds

VI. 1. *Abrupt drop of p at the rear margin of the zone of kick up*

As shown in articles (3) and (4) of section IV, in case of $\gamma > 1$, compressive strees p of the snow is zero at point A, namely at the front edge of the snowplow, and increases to reach the maximum value p_E at the rear margin of zone K, that is, at point E of Fig.5. After attaining p_E, the value of p must abruptly drop down to zero, because $p''_E = 0$; that is, because a compressive stress in layer F is zero at point E as assumed at the end of section IV. By virtue of such situations, no force is applied to snow K through its front and rear boundary planes, the only force applied to it being force $\boldsymbol{F}$ with which zone K kicks up the bottom of snow K. Therefore the law of momentum gives

$$h_0 \rho_0 V\ (v \cos\alpha - V) = F_X,$$

which is rewritten as

$$-(F_X/h_0) = \rho_0 V^2 \{1-(v/V)\cos\alpha\} \qquad \text{for} \qquad \gamma > 1. \tag{42}$$

If dW_G, dU_K, dU_P and dU_F are used in the same sense as in articles (3) ~(6) of section V, here in the case of $\gamma > 1$

$$dW_G = dU_F = 0,\ dU_P = (1/2k)\ p_E^2 h_0 V\ dt, \tag{43}$$

while dU_K which remains the same is given by formula (32).

The above expression for dU_P can be transformed by Eqs. (24) and (42) into

$$dU_P = k\ \{\gamma^2/2(\gamma-1)^2\}\ \{1-(v/V)\cos\alpha\}^2 h_0 V dt \qquad \text{for} \qquad \gamma > 1. \tag{44}$$

VI. 2. *Vectors* $\boldsymbol{v}$ *and* $\boldsymbol{F}$ *expressed as functions of* V *with* α *as a parameter*

Respective subustitutions of functions (32) and (44) for dU_K and dU_P in Eq. (35) give

$$\frac{v}{V} = \frac{\gamma\cos\alpha + (\gamma-1)\sqrt{(\gamma-1)^2 - \gamma\sin^2\alpha}}{\gamma\cos^2\alpha + (\gamma-1)^2} \qquad \text{for} \qquad \gamma > \gamma_M \tag{45}$$

where $\gamma_M = A + \sqrt{A^2-1}$ with $A = 1 + (\sin^2\alpha/2)$. Ratio v/V is imaginary and the kick up of the flow type is impossible when γ lies in the region from 1 to γ_M. The kick up is realized as that of the spray type in this region as will be shown in section VIII. Dotted curves a, b, c and d in the right half of Fig.6 show the relationship between v/V and γ given by Eq.(45). Letters a, b, c and d indicate that $\alpha$is 30°, 45°, 60° and 75° respectively. The curves start upright from point (γ_M, cos α) and bend gradually to the right as γ increases to approach asymptotically the horizontal line of $v/V = 1$.

Equation (42) gives

$$C = -F_X/h_0\rho_0 V^2 = 1-(v/V)\cdot\cos\alpha \qquad \text{for} \qquad \gamma > \gamma_M > 1 \tag{46}$$

for the coefficent of running resistance. Equation (38) is applicable also to the present case of high speeds, and D, the coefficient of pushing down force, is given by

$$D = (v/V)\sin\alpha \qquad \text{for } \gamma > \gamma_M > 1. \tag{47}$$

Both coefficients C and D can be expressed as functions of γ through Eq. (45).

VII. Spray type kick up at low speeds

VII. 1. *Condition for a kick up to be of the spray type*

As stated in assumption (VIII) of section II, a kick up of the flow type is turned to that of the spray type when uniaxial strain ε of the snow reaches ε_P, the strain of pulverization. Therefore, if letter ε_m is used for the maximum value which ε attains during a flow type kick up,

$$\varepsilon_m > \varepsilon_P \qquad \text{for} \qquad 0 < \gamma < \infty \tag{48}$$

gives a condition in which the kick up turns from the flow type to the spray type.

For a fixed value of γ, the greater scooping angle α of the snowplow is, the greater ε_m becomes. Therefore, if a snowplow of which scooping angle α' is variable is imagined, ε_m will reach ε_p and the snow will begin to be pulverized when α' is enlarged from a small value up to a certain value α_p. Accordingly the above condion (48) can be rewritten as

$$\alpha > \alpha_P \quad \text{for} \quad 0 < \gamma < \infty. \tag{49}$$

Let α_P be called 'angle of pulverization'.

VII. 2. *Angle of pulverization for the kick up at low speeds*

When V is a low speed, ε_m is given by $\varepsilon_A = p_A/k$, because uniaxial compressive stress p of the snow reaches its maximum at point A of Fig. 5 as shown in section III and IV. Equations (4) and (30) give

$$\varepsilon_m = \varepsilon_A = 1-(u_A/V) = 1-(v/V)\cos\alpha \quad \text{for} \quad \gamma < 1 \tag{50}$$

and this is transformed by Eq. (36) into

$$\varepsilon_m = \gamma/(\gamma + 2\cot^2\alpha) \quad \text{for} \quad \gamma < 1. \tag{51}$$

Therefore angle of pulverization α_P fulfills the equation

$$\varepsilon_P = \gamma/(\gamma + 2\cot^2\alpha_P) \quad \text{for} \quad \gamma < 1. \tag{52}$$

Strain ε_P of pulverization which is one of characteristic constants of the snow, cannot be greater than unity by the definition of compressive strain. Plotted in the left half of Fig. 7 are α_P against γ for three values 0.1, 0.5 and 0.75 of ε_P. Curves (Ref. 3, Fig.11(a) ; Ref. 8, Fig. 8) of p versus ε obtained by falling a metal cylinder onto a block of soft snow suggest that ε_P may come up to 0.5 which is very large for a value of strain. In Fig. 7 curves of α_P go down with increasing γ and meet at the points shown by small circles with the horizontal lines which represent particular values 30°, 45°, 60° and 75° of α. On the left of the small circles α_P is greater than α, the kick up being of the flow type, while on the right of them α_P is smaller than α, the kick up being of the spray type. The points shown by small circles shall be called 'first transition points' ; adjective 'first' is added because another transition point will be found in the region of $\gamma > 1$. If γ at a first transition point is denoted by γ_T, it fulfills Eq. (52) in which α_P is replaced with α.

VII. 3. *Angle of spraying*

As shown in Fig. 4 (b), the pulverized snow reflects on the surface of snowplow and flies off into the air in the direction which deviates from the horizontaly by angle α_S. Angle α_S shall be called 'angle of spraying'. If the snow has velocity vector $\boldsymbol{v}_P$ just before the pulverization the same $\boldsymbol{v}_P$ gives the velocity vector of the pulverized snow immediately after the pulverization by assumption (IX) of section II. Let the direction of $\boldsymbol{v}_P$ make angle θ_P with the horizontal. Then a simple geometrical consideration yields the relation

$$\alpha_S = 2\alpha - \theta_P. \tag{53}$$

When α is greater than α_P, namely when ε_A is greater than ε_P, then ε reaches ε_P in region

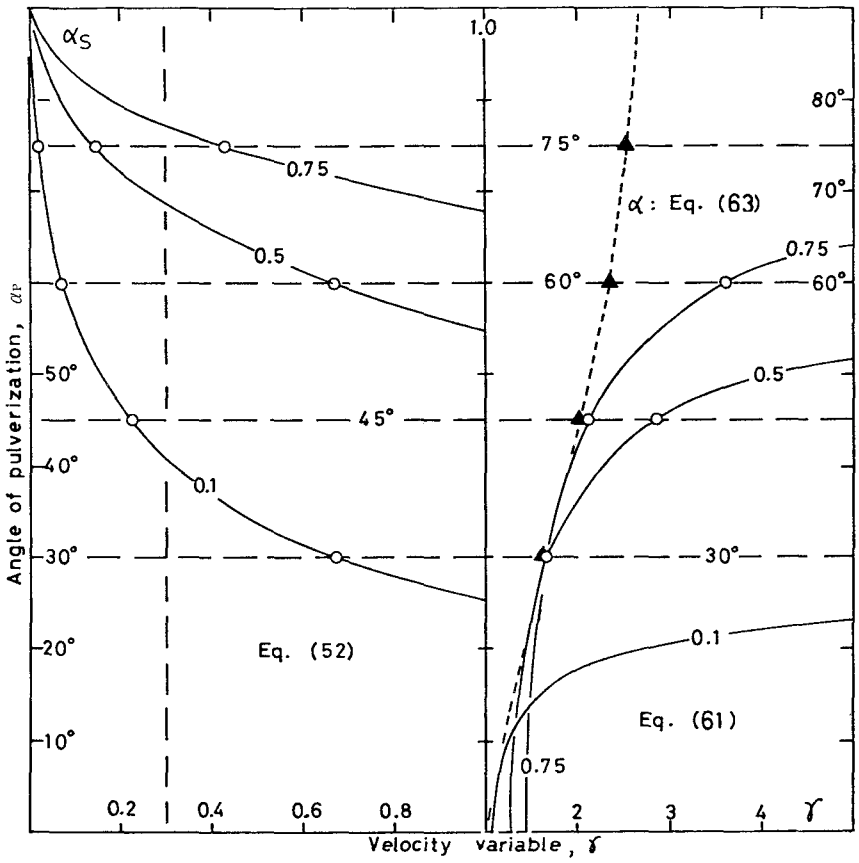

Fig. 7 Relations between angle α_P of pulverization and γ for three particular values 0.1, 0.5 and 0.75 of strain ε_P of pulverization. Velocity variable γ is symboled γ_B at the points where the curves meet the axis of abscissa in the region of $\gamma > 1$. Angle α_P is put equal to zero for $\gamma_B > \gamma > 1$.

BA of forecompression of Fig. 5, that is, while the snow is advancing as layer C, because ε is increasing in the region of compression to attain its maximum ε_A at point A. Therefore the snow is pulverized before it arrives at point A which represents the front edge of the snowplow. Angle θ is zero in the region of forecompression as shown in Fig. 5, so θ_P must be zero. Thus α_S becomes 2α by Eq. (53).

But, in actuality, an upward bend of the snow and an increase in θ will begin in the region of forecompression, because it is nothing but the scooping action of the snowplow that gives rise to the forecompression. Accordingly, although θ was put to zero in the region of forecompression on calculating p and u in the foregoing section, it is assumed, in this case of considering the pulverization of snow, firstly that θ begins increasing to reach θ_P at a point

in the region, secondly that the snow is pulverized at the same point and thirdly that $\theta_P = \alpha_P$. In view of such situations the tenth assumption is made as follows: (X) θ becomes equal to α_p wherever ε grows up to ε_p in the flowing layer of snow, independently of whether γ is greater or smaller than 1.

Then the angle of spraying is given by

$$\alpha_S = 2\alpha - \alpha_P \qquad \text{for} \qquad \alpha_P < \alpha, \tag{54}$$

which can, for $\gamma < 1$, be expressed as a function of γ and ε_P through Eq. (52).

VII. 4. *Velocity of the sprayed snow*

By virtue of the tenth assumption, the flowing motion of the snow preceding the pulverization can be regarded as a flow type kick up caused by an imaginary snowplow of which the scooping angle is equal to α_P. Therefore replacement of α with α_P in Eq.(36) gives a formula for v_P/V, in which v_P is the magnitude of vector $\boldsymbol{v}_P$. Equation (52) transforms the formula thus obtained into

$$v_P/V = (1-\varepsilon_P)/\cos\alpha_P \qquad \text{for} \qquad \gamma < 1. \tag{55}$$

It is noted that v_P/V is independent of α, because α_P is a function only of γ and ε_P as seen from Eq. (52). It is assumed that the pulverized snow loses no energy at its reflection on the surface of the snowplow. Then v_P of Eq. (55) and α_S of Eq. (54) give respectively the magnitude and the direction of velocity vector $\boldsymbol{v}_S$ of the sprayed snow, that is, of the velocity vector which the pulverized snow has after the reflection.

The above $\boldsymbol{v}_S$ and α_S are the velocity vector and the angle which are observed by an observer M moving together with the snowplow. Figure 8 shows schematically the motion of the sprayed snow with respect to an observer M′ who is standing on the ground. Point A is the front edge of the snowplow and observer M′ will see the sprayed snow form a streak going out from point A towards point B′. But the sprayed snow is not flying along the

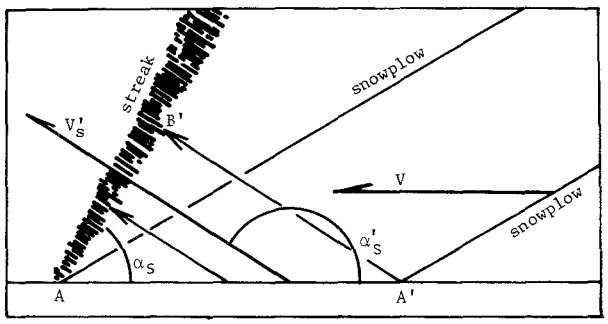

Fig. 8 Motion of the sprayed snow as looked at by an observer standing on the ground.

streak. In the figure, point A′ is the point at which the front edge was positioned a short time interval τ before, while $\boldsymbol{v}_S'$ shows the velocity vector of the sprayed snow with respect to observer M′ which is given by $\boldsymbol{v}_S + V$, if the velocity vector of the snowplow is denoted by $\boldsymbol{V}$. Displacement vector A′B′ is so drawn that it equals $\tau \boldsymbol{v}_S'$. Thus portion AB′ of the streak is formed by the snow particles which are flying in the direction of $\boldsymbol{v}_S'$ after having been sprayed by the front edge while it moved from point A′ to point A. Moving observer M sees also the same streak, but to him the snow particles appear to fly on and along the streak, the angle between the streak and the horizontal being equal to α_S. Angle α_S' in the figure is the angle of spraying for standig observer M′. Although α_S can be smaller than a right angle, α_S' is always greater than a right angle because the snow is set in motion anyway by the snowplow moving leftward in the figure.

VII. 5. *Coefficient C_S of running resistance and coefficient D_S of pushing down force*

Resistance F_R defined in article (6) of section V turns out to be composed of two parts when the kick up is of the spray type. The one part is the resistance on the imaginary snowplow with scooping angle α_P, and its value can be determined by replacing v and α respectively with v_P and α_P in Eq. (37). The other part is equal to the x-component of the momentum which the pulverized snow loses at the reflection on the surface of the snowplow. Thus F_R is given by

$$F_R = h_0 k \ \{1-(v_P/V) \cos \alpha_P\} + h_0 \rho_0 V \cdot v_P \ (\cos \alpha_P - \cos \alpha_S) \qquad \text{for} \quad \gamma < 1. \tag{56}$$

Coefficient C_S of running resistance can be drawn from this equation in the from

$$C_S = F_R / h_0 \rho_0 V^2 = 1 + \ \{(1-\gamma)/\gamma\} \ \varepsilon_P - (v_P/V) \cos \alpha_P \qquad \text{for} \quad \gamma < 1, \tag{57}$$

by use of Eq. (8) and of the relation $\varepsilon_P = 1-(v_P/V) \cos \alpha_P$ which follows from Eq. (4).

Coefficient D_S of pushing down force is given by

$$D_S = h_0 \rho_0 V \cdot v_P \sin \alpha_S / h_0 \rho_0 V^2 = (v_P/V) \sin \alpha_S \qquad \text{for} \quad \gamma < 1. \tag{58}$$

VIII. Spray type kick up at high speeds

VIII. 1. *Angle of pulverization*

Inequalities (48) and (49) give, also in case of high speeds, the condition for the kick up to be of the spray type. As the value of p_E makes the maximum of p in this case, angle α_P of pulverization can be determined for $\gamma > 1$ if a relationship is found among α, γ and $p_E = k\varepsilon_E$.

Equations (24) and (42) give

$$\varepsilon_E = \{\gamma/(\gamma-1)\}\ \{1-(v/V)\cos\alpha\} \qquad \text{for} \qquad \gamma>1. \tag{59}$$

which is a result due to the law of momentum. The law of energy yields

$$1-(v/V)^2=\varepsilon^2{}_E/\gamma \qquad \text{for} \qquad \gamma>1 \tag{60}$$

which can be drawn from the combination of Eqs. (8), (32), (35) and (43). The relationship among α, γ and ε_E referred to above follows from Eqs. (59) and (60) when v/V is eliminated from them, and, the respective replacement in the relationship of α and ε_E with α_P *and* ε_P yields

$$\cos\alpha_P = \{\varepsilon_P+(1-\varepsilon_P)\gamma\}/\sqrt{\gamma(\gamma-\varepsilon_P{}^2)} \quad \text{for} \quad \gamma>\gamma_B>1 \tag{61}$$

which is an equation corresponding to Eq. (52) obtained for $\gamma<1$ in the previous section. Meaning of γ_B will be given below.

Curves of α_P versus γ are drawn through Eq. (61) for three particular values of ε_P in the right half of Fig. 7. The curves start upwards from the horizontal axis of γ ; at the starting points γ has the value

$$\gamma_B = \{1+\sqrt{1+4\varepsilon_P/(2-\varepsilon_P)}\}/2>1. \tag{62}$$

If the right side of Eq. (61) is represented by $f(\gamma;\ \varepsilon_P)$, $f(\infty\ ;\ \varepsilon_P)=1-\varepsilon_P$, $f(\gamma_B;\ \varepsilon_P)=1$ and $f(1;\ \varepsilon_P)=1/\sqrt{1-\varepsilon_P^2}>1$. Therefore α_P is positive for $\gamma>\gamma_B$, vanishes when $\gamma=\gamma_B$ and becomes nonexistent when γ falls below γ_B. Such being the case, it is assumed that α_P is zero in the region of γ extending from 1 to γ_B.

In the right half of Fig. 7, at the points shown by small circles the curves for α_P cross the horizontal broken lines which represent the particular values chosen for α. For the same reason described in articles (1) and (2) of the previous section, the above points shown by the small circles shall be called 'second transition points' for $\gamma>1$. The kick up is of the spray type to the left of them, while it is of the flow type to the right of them. Value γ_T'of γ at a second transtion point is given by the root of Eq. (61) in which α_P is put equal to α.

VIII. 2. *Spray type kick up in the region of γ where the flow type kick up is forbidden*

As noted in article (2) of section VI, the flow type kick up is forbidden for $\gamma_M>\gamma>1$, in which γ_M is given by

$$\gamma_M = A+\sqrt{A^2-1} \qquad \text{with} \qquad A=1+(\sin^2\alpha/2). \tag{63}$$

In the right half of Fig.7, the dotted curve marked α gives, with γ_M as an abscissa, the relationship between α and γ_M shown by Eqs. (63). The crosses marked a, b, c and d in the right half of Fig. 6 coincide respectively in γ with the small black triangles in Fig.7 at which the dotted curve meets the horizontal broken lines representing the four chosen values of α.

Conditions (48) and (49) cannot be applied in the region of γ where it is smaller than γ_M, because no flow type kick up exists there : they are the conditions for a kick up to be transformed from the flow type to the spray type. But the snowplow must anyways kick up the snow and the kick up is realized as a spray type one in this region of γ, because α_P exists there and existence of α_P guarantees the possibility of occurrence for a spray type kick up. Indeed, a spray type kick up is essentially a flow type kick up caused by an imaginary snowplow of scooping angle α_P and its occurrence should not depend upon whether or not the actual snowplow of scooping angle α succeeds in causing a flow type kick up. Thus it results that kick up is of the spray type for γ lying between 1 and γ_T', abscissa of the second transition points.

VIII. 3. *Graphs of* α_S

Angle of spraying is given by $\alpha_S = 2\alpha - \alpha_P$. In Figs. 9 (a) and 9 (b), curves of α_S versus γ are drawn for $0<\gamma<5$ by using the equations obtained for α_P in articles (2) and (3) of the previous section and in the foregoing articles of this section. Values 0.1, 0.5 and 0.75 are chosen for ε_P while values 30°, 45°, 60° and 75° are chosen for α. These values of α are indicated on the curves respectively by letters a, b, c and d; for instance, letter c attached to a curve shows that its α is 60°. In Fig. 9 (b), solid curves are for $\varepsilon_P = 0.5$, while thick broken curves are used for $\varepsilon_P = 0.75$. The equations used for drawing the curves are shown by their numbers in the figures.

Each of the curves in Figs. 9 has two transition points marked by small circles, the first transition point in the region of $\gamma<1$ and the second transition point in the region of $\gamma>1$. The kick up is of the spray type in the region of γ lying between the two transition points ; this region shall be called 'region S'. As α_S coincides with α for the kick up of the flow type, the curves are reduced to the horizontal lines marked with letter F in both the right and the left outsides of region S. Dotted horizontal segments in region S show $\alpha_S(=\alpha)$ of the virtual kicks up of the flow type which fail to turn to those of the spray type.

When increasing γ reaches the critical point of $\gamma=1$, all the curves rise discontinuously up to 2α which is the maximum, and turn sharply downwards at $\gamma=\gamma_B$ after having maintained the maximum value for some interval. The broader is region S, the smaller ε_P is and the greater α is. This is because smaller ε_P and greater α make the snow easier to be pulverized.

VIII. 4. *Velocity* v_S *of the sprayed snow which is equal to* v_P *in magnitude*

Velocity v_P fulfills Eq (60) in which ε_E is replaced with ε_P, namely

$$1-(v_P/V)^2=\varepsilon_P^2/\gamma \qquad \text{for} \qquad \gamma_T'>\gamma>1, \tag{64}$$

while Eqs. (52) and (55) determine v_P/V for $1>\gamma>\gamma_T$. It can be shown that the two (v_P/V)'s thus determined have the same value at $\gamma=1$. Therefore v_P/V is continuous at $\gamma=1$,

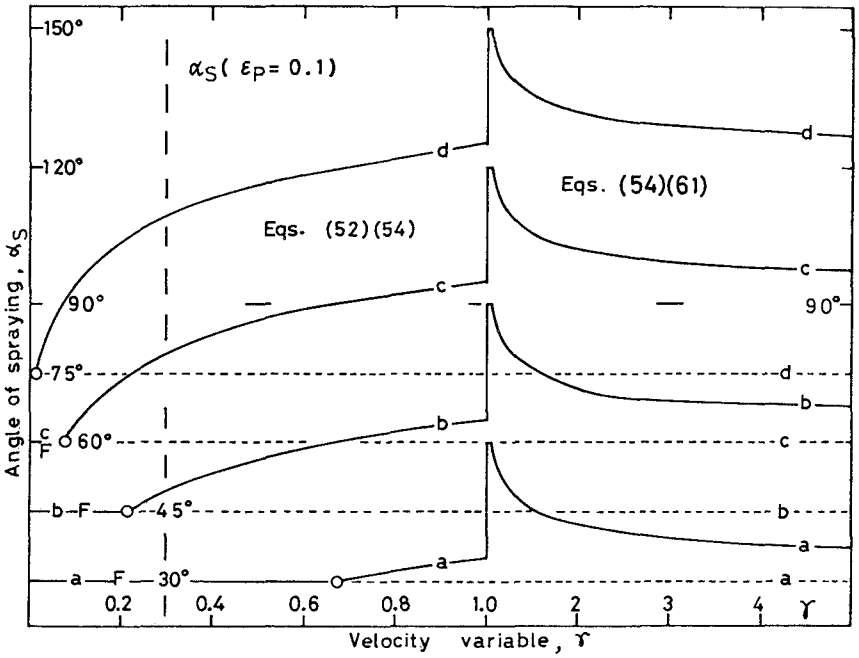

Fig. 9(a) Curves of angle α_S of spraying versus γ for $\varepsilon_P=0.1$.

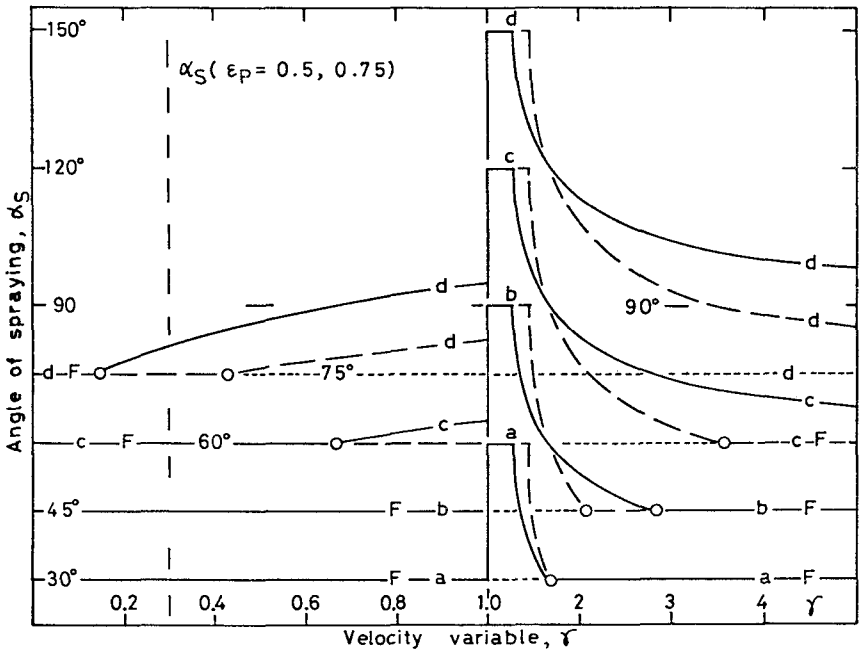

Fig. 9(b) Curves of angle α_S of spraying versus γ. The solid curves are for $\varepsilon_P=0.5$ while the broken curves are for $\varepsilon_P=0.75$.

whereas $d(v_P/V)/d\gamma$ is not.

Curves of v/V and v_P/V versus γ are shown in Fig.6 for $\varepsilon_P=0.1$ and for the four particular values of α in the region of γ extending from 0 to 5. The letters attached to the curves are used in the same meaning as in Fig. 9 ; thus v/V and v_P/V are given respectively by such portions of the solid curves that are furnished and not furnished with letter F. Dotted curves give v/V of the virtual flow type kick up which fails to occur because of the transformation into the spray type kick up.

Independently of whether γ is greater or smaller than unity, a single curve answers the purpose of expressing v_P/V if ε_P is fixed, because v_P/V does not depend upon α. In Fig.6, a curve of v_P/V starts from near the upper left corner of the figure, descends rightwards through the first transition points shown by small circles and then stretches almost horizontally in the region of $\gamma>1$. As snow of $\varepsilon_P=0.1$ is easy to pulverize, the kick up is turned from the flow type to the spray type even for $\alpha=30°$, the smallest value chosen for α, in the region of $5>\gamma>1$ with the result that no second transition point is found there. But, it can be seen from Fig. 7 and Fig. 9 (b) that, if ε_P is increased to 0.5 or 0.75, transition points make their appearance also in this region.

VIII. 5. *Coefficient C_S of running resistance and coefficient D_S of pushing down force*

Coefficients C_S and D_S are used in the same sense as described at the end of the preceding section.

Formulae shown below are obtained for C_S and D_S by the same reasoning as made in deriving those for C and D in articles (1) and (2) of section VI:

$$C_S=1-(v_P/V)\cos\alpha_S \qquad \text{for} \qquad \gamma>1 \tag{65}$$

and

$$D_S=(v_P/V)\sin\alpha_S \qquad \text{for} \qquad \gamma>1. \tag{66}$$

They can be expressed as functions of γ through the relationships hitherto known which connect v_P and α_S with γ.

For $\varepsilon_P=0.1$, coefficients C and C_S are plotted in Fig. 10 against γ ranging from 0 to 5. The portions of curves with and without letter F are respectively for C and C_S, while dotted curves give C of the virtual kick up of the flow type. When γ passes unity in increasing, the value of coefficient C_S make a sudden rise but does not exceed 2, theoretical limit for the coefficient of running resistance in case when no forecompression of the snow takes place.

It can be seen from Fig. 10 that the transformation of the kick up from the virtual flow type to the spray type brings about a great increase in value of the coefficient. But this increase is reduced as ε_P is enlarged above 0.1, because any enlargement of ε_P makes the transformation more unlikely by narrowing the region of γ in which it occurs. On the other hand, the same enlargement of ε_P enhances the sudden rise at $\gamma=1$ of the coefficient.

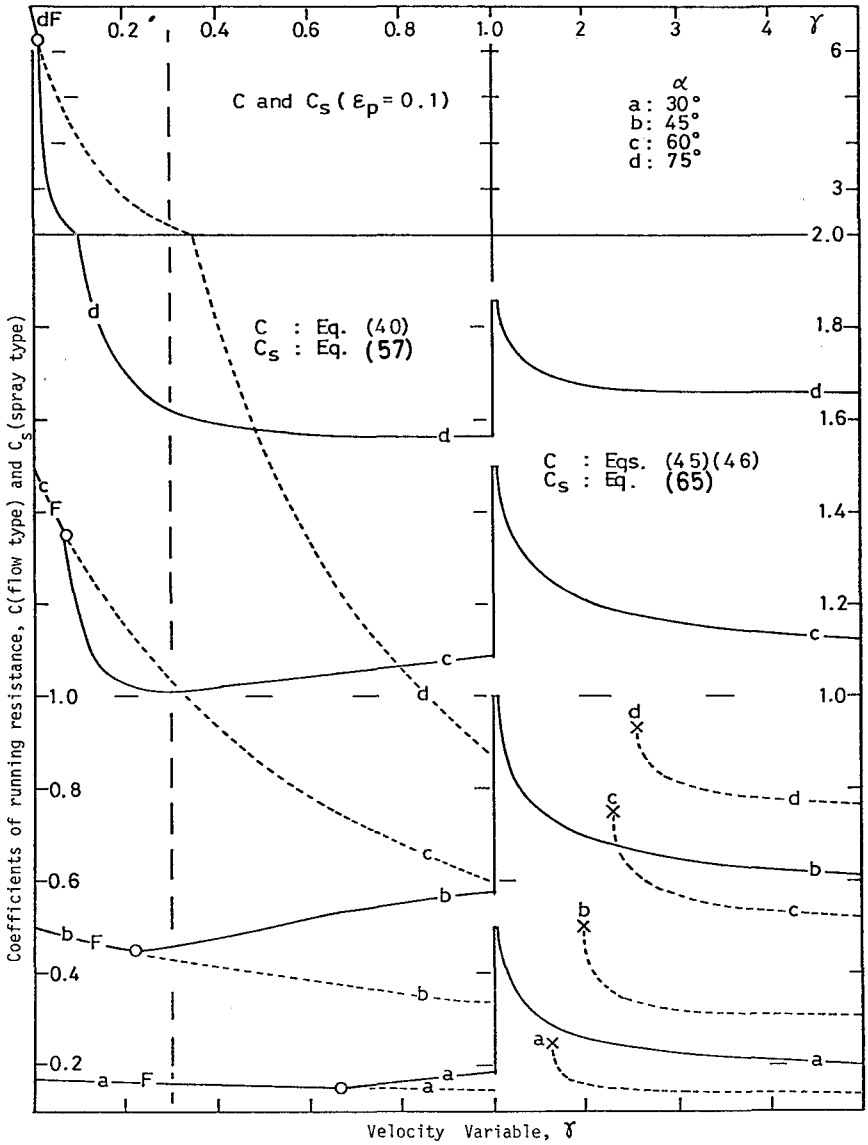

Fig. 10 Coefficient of running resistance versus γ. The dotted curves and the portions marked F of the solid curves give C, the coefficient for the flow type. The otherwise portions of the solid curves give C_s, the coefficent for the spray type.

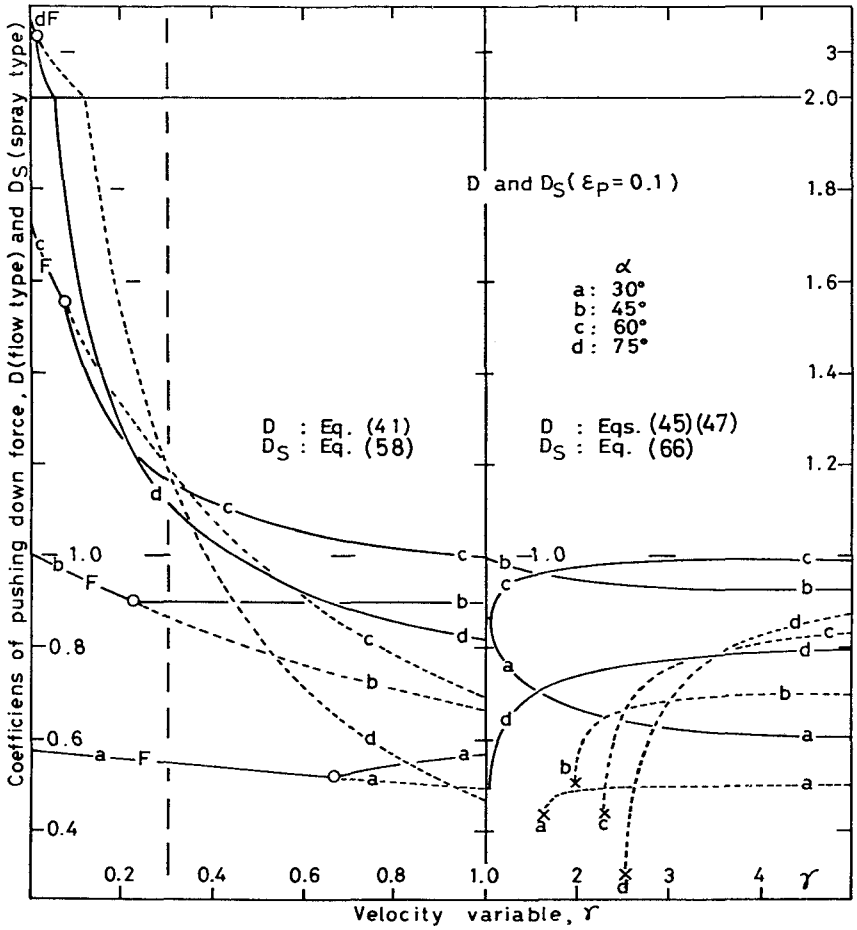

Fig. 11 Coefficient of pushing down force, the force with which the snow pushes down the front edge of the snowplow. Symbols and curves should be read in the same meaning as in Fig. 10, except that D and D_S are respectively used for the coefficients of pushing down force with respect to the flow type and the spray type.

Curves of D and D_S versus γ are shown for $\varepsilon_P=0.1$ in Fig. 11. A change in the value of α alters the shape and configuration of the curves in a rather complicated way.

Curves for $\varepsilon_P=0.5$, 0.75 *of* v/V, v_P/V as well as of the coefficients are found in the papers 9) and 10) cited in the references.

IX. Conclusion

The Japanese National Railways is constructing in snowy districts of Japan new railways called 'Shin-Kansen' on which a train of electric railcars is to run at speeds exceeding 200 km/hr with a snowplow attached to its front. Besides anxiety about enlargement of the resistance raised by the snow to speed, the Railways entertains a fear lest the snow scooped up and shot out by the snowplow should hit and damage requisite equipments installed near the rails.

The following results were obtained through the theoretical studies made in the present paper on motion of the snow kicked up by the front edge of snowplow. The motion is to be observed from the frame of coordinates moving together with the snowplow. (1) The kicked up snow turns its flying direction forward discontinuously by several tens of degrees, when advancing speed V of the snowplow increases past critical speed c, propagation velocity of a weak plastic wave in the original snow lying before the snowplow. Besides the above discontinuity at the critical speed, the direction of the flying snow undergoes rather great changs while V changes in the neighbourhood of c. (2) If the kick up of snow were not subject to transformation from the flow type to the spray type, no change would occur in the direction of motion of the kicked up snow. Therefore the transformation constitutes the cause for the above changes in the flying direction. The smaller scooping angle of the snowplow is, the more unlikely the transformation is to occur. (3) The same transformation, however, erases a discontinuity which may well be in the value of velocity of the flying snow, the transformation makes the value change continuously at the critical speed. (4) The coefficient of running resistance due to the kick up of snow makes an abrupt jump to the maximum value at the critical speed. The maximum value rises as the scooping angle increases, but does not go over 2, the greatest value which the coefficient can reach in the absence of forecompression of the snow. When the snow is forecompressed at low speeds, an additional resistance is produced by friction at the base of the forecompressed snow. This makes it possible for the coefficient to have values greater than 2.

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