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The Distribution of Stress, Velocity and Temperature in Glaciers

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Abstract

Glacier ice is, in the first approximation, an incompressible isotropic material having non-linear viscosity, and the problem of its motion may be solved on the basis of the flow theory, using relations among strain rate, stress and temperature in ice which have been established experimentally.

In order to determine the distribution of stress, velocity and temperature in ice of a glacier, it is necessary to solve the system of the following equations of the theory of continuous media: the equation of continuity, the three equations of equilibrium, the six compatibility conditions and the equation of energy.

Until recently, exact analytical solutions of the system of the above equations were known only for one and two-dimensional cases under unrealistically simplified assumptions.

On the basis of new data concerning the rheological properties of ice, the existing solutions of one-dimensional problems were improved, and a solution of the problem of the distribution of stress, velocity and temperature in a non-linearly viscous ice slab was obtained for a simple laminar motion, where the dissipiation of energy and the dependency of creep rate on temperature are taken into account.

For isothermal viscous ice, a solution of the two-dimensional problem was obtained, giving for the first time the possibility of an approximation of stress and velocity distribution in the longitudinal cross section of a glacier.

I. Introduction

Glaciers are streams of ice of atmospheric origin. An important problem of the glacier science is to know the laws governing relations among their sizes, forms, properties, the processes characterising them, and outer conditions.

Mass-exchange condition at the outer boundary of an ice body may be written as

$$\frac{a}{\rho}+v_n-\frac{\partial h}{\partial t}=0\,, \tag{1}$$

where $a=\partial^2 m/\partial s\,\partial t$ is the specific rate of mass-exchange, that is, the income of mass m from the outer medium on a unit area of the surface s of the ice body in a unit interval of time t, and is negative in case of ablation; ρ is the density of the ice; v_n is the component of the velocity vector $\boldsymbol{v}$ along the outward normal $\boldsymbol{n}$ of the surface; h is the coordinate of the surface on the axis $\boldsymbol{n}$. In case of a non-moving ice body, eq. (1) is reduced to $a/\rho=\partial h/\partial t$, while in case of no mass-exchange $v_n=\partial h/\partial t$. In actual glaciers, things are more complicated. In order to explain, reconstruct or predict their sizes and forms, it is necessary to know the distribution of the rate of mass-exchange a

and the field of velocity of motion $\boldsymbol{v}$. The determinations of a and v may be considered as independent problems.

The field of velocity vector can be determined if the distribution of stress and the dependency of strain-rate on it are known. The distribution of stress is, in turn, dependent on the rheological properties of ice and on forces acting on ice, and hence, in the gravity field, on the size and form of the glacier.

II. Rheological Properties of Ice

Glacier ice (including snow, firn and moraine-containing ice) is an inhomogeneous, anisotropic and non-linear material. If we were to attempt a complete description of its complex rheological properties, it would be necessary to use integral operators, extended over a part of the system's history, or even over all of it, together with anisotropic tensor functions. However, this is impossible because of the lack of our present knowledge and, moreover, it would probably be unadvisable because the accuracy of the obtained results may not deserve the complicated procedures of calculation. Judging from the accuracy of the existing data, it may be justified to regard glacier ice, in solving the problem of its motion, as a material such as:

a) isotropic, that is, the principal axes of stress tensor being parallel to those of strain-rate tensor;

b) incompressible, that is, the density being assumed constant, $\rho=0.9$ g/cm^3 (Shumskiy, 1963), and the settling and slide of snow-firn cover being treated by the principle of superposition;

c) fluid, that is, elasticity, relaxation phenomena and non-stationary creep being neglected because of the short relaxation time of ice which is in the order of $10\sim10^2$ min.

Under such assumptions, the problem of the motion of ice with a gradually changing velocity can be solved, if we neglect the rapid pulsations, on the basis of the theory of flow (stationary creep) of an isotropic and incompressible material, whose non-homogenity is only caused by variations in temperature. The stationary flow of glacier ice must be considered as the tertiary creep, which occurs after a strain of several percents and is due to a special structure of ice resulting from recrystalization in the course of such a strain (Steineman, 1958).

It has been established experimentally that the strain rate of ice is dependent on the second (quadratic) invariant of stress tensor and on temperature (Glen, 1956, and others) and practically not on the first invariant (Rigsby, 1958; Steineman, 1958). The role of the third (cubic) invariant is not clear. Thus, by virtue of the above mentioned assumptions and experimental facts, the rheological equation of ice can be written as

$$\varepsilon_{ij}=f(\theta,\sigma)\,\sigma_{ij}\,, \quad (i,j=1,2,3)\,, \tag{2}$$

where ε_{ij} and σ_{ij} are components of the deviators of strain-rate and stress, respectively, θ is the temperature and σ is the generalized stress:

$$\sigma=\frac{1}{\sqrt{6}}\sqrt{\sum_{i\neq j}\{(\sigma_{ii}-\sigma_{jj})^2+6\sigma_{ij}^2\}}\geqslant 0\,.$$

From the experimental results obtained by Steineman (1958), the present author found that the rheological function $f(\theta, \sigma)$ for the tertiary creep of ice is most well represented by

$$f(\theta, \sigma) = B \exp(\kappa\theta) \cdot (1+K\sigma^3), \tag{3}$$

where $B=0.13\ \mathrm{cm^2 \cdot kg\text{-}wt^{-1} \cdot year^{-1}}$, $\kappa=0.282\ \mathrm{degree^{-1}}$ and $K=5.2\times 10^{-2}\ \mathrm{cm^6 \cdot kg\text{-}wt^{-3}}$.* Because of the small value of K, when the flow velocity is small, ice approaches a viscous fluid with the viscosity of $\eta=(2B\exp(\kappa\theta))^{-1}$, while it approaches to an ideal plastic material, when the flow velocity is high.

The relationship between the flow velocity and the water content of ice at its melting point remains unknown.

III. Equations of the Theory of Continuous Media

In order to determine eleven unknowns, that is, the mean normal stress s, the six components of the stress deviator σ_{ij} (or three invariants and three angles of Euler), the three components of the velocity vector v_i (or the length and the two angles) and the temperature θ, it is necessary to solve the system of the following eleven equations (i, $j, k=1, 2, 3$):

The equation of continuity (reduced to the condition of incompressibility),

$$\frac{\partial v_i}{\partial x_i} = 0 . \tag{4}$$

The three equations of motion (reduced to the equations of quasi-static equilibrium because of the negligibly small inertial forces),

$$\frac{\partial s_{ij}}{\partial x_j} + \rho F_i = 0 , \tag{5}$$

where s_{ij} are the components of the stress tensor:

$$s_{ij} = s\delta_{ij} + \sigma_{ij} , \quad \delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) , \end{cases}$$

and F_i are the components of the body force.

The six conditions of compatibility among the components of the strain-rate tensor (deviator),

$$\frac{\partial^2 \varepsilon_{kk}}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_k}\left(\frac{\partial \varepsilon_{ij}}{\partial x_k} - \frac{\partial \varepsilon_{jk}}{\partial x_i} - \frac{\partial \varepsilon_{ki}}{\partial x_j}\right) = 0 , \quad \begin{cases} i=j\neq k \\ i\neq j\neq k\neq i . \end{cases} \tag{6}$$

The equation of energy (the kinetic energy and the energy of volume change are neglected whereas the inner drainage of energy due to melting is included),

$$\rho c\left(\frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_i}\right) - \lambda \frac{\partial^2 \theta}{\partial x_i^2} - A\,\sigma_{ij}\varepsilon_{ij} - w_{\mathrm{r}} - \rho L\nu = 0 , \tag{7}$$

where c, λ, and L are respectively the specific heat, the thermal conductivity and the latent heat of fusion of ice, while A represents the thermal equivalent of work, w_{r} the

* The formerly calculated values, B_1 and K_1, of the constants (Shumskiy, 1964; Shumskiy and Krenke, 1964) should be corrected to $B=(3/2)B_1$ and $K=3\sqrt{3}\,K_1$, since the values given by Steineman, which were formerly mistaken for those of components of the deviators, actually represent the values of components of the full tensors.

total specific power of sources and drains of radiant energy, and ν the relative rate of freezing of water contained in ice.

In an isothermal glacier of melting temperature, the temperature is not an unknown variable and the energy equation drops out (The first three terms and the sum of the remaining terms are all equal to zero).

Strictly speaking, all glaciers are non-stationary and the ice composing them is generally in a complicated (three-dimensional) stress-strain state. But if we are interested in the mean velocity over a time interval not less than one year, some glaciers can be regarded as stationary (quasi-stationary). Equations (4) to (7) show that the difference between the solution for non-stationary cases and that for stationary cases arises only from the difference in the distribution of temperature, and that in an isothermal glacier no difference exists between the two solutions, and the change with time in the distribution of stress and velocity is caused by the change in the thickness and form of the glacier ($\partial h/\partial t \neq 0$), or also by the change of shear stress at the bottom.

In several cases, stress-strain states in ice of glaciers are simplified and the number of the simultaneous equations to be solved decreases in accordance with the decreases in the number of unknown variables. Examples are: Plane (two-dimensional) stress-strain states (five equations); states of complex and simple laminar flow (three and two equations, respectively); states of uni-axial tension and of symmetric biaxial tension (each with three equations). Plane stress-strain states are observed near the long axis of valley glaciers; states of complex laminar flow are observed on the boundary between tensioned and compressed regions in a transversal cross section with the maximal velocity and thickness of ice, provided that the cross section coincides with the accumulation limit and the width of the glacier does not change along the longitudinal axis; states of simple laminar flow are observed in the same cross section near the longitudinal axis of the glacier, where the thickness and the physical characteristics of the bottom do not change in the transversal direction. Strictly speaking, only the surface of a glacier can be in states of uni-axial or bi-axial tensions. But ice of floating ice shelf is nearly in such a state, though there are small shear stresses on horizontal planes.

On a free (not contacting with rocks) surface, $z=F(x,y)$, the solution must satisfy the boundary condition eq. (1) and the following equations:

$$s_{nn}=-p_{a}\,,\quad \sigma_{nl}=\varepsilon_{nl}=0\,,\quad \theta=\theta_0\,,\quad \frac{\partial\theta}{\partial n}=\gamma_0 \text{ or } w_{r}+q+La_{p}-\lambda\frac{\partial\theta}{\partial n}=0\,, \tag{8}$$

where p_a is the atmospheric (in water, hydrostatic) pressure and $l(l=1,2)$ are tangential axes to the surface, w_r is the total specific power of sources and drains of radiant energy, q is the heat flow from the outer medium to the glacier surface, and a_p is the specific rate of mass-exchange due to phase transformations.

As for the boundary condition at the bottom $z=G(x,y)$ of the glacier, eq. (1) can be regarded as $v_n=-a/\rho$, because practically $\partial h/\partial t=0$ (the rate of the change of bottom relief is generally negligibly small compared with the rate of the change in glacier thickness), and the condition of energy conservation can be written as

$$A\,\sigma_{nl}v_l+\rho L v_n-q-\lambda\frac{\partial\theta}{\partial n}=0\,, \tag{9}$$

where q is the heat flow from the Earth's interior ($q<0$ because the outward normal to the bottom is in a downward direction). The condition of non-slip, $v_l=0$, adopted in hydrodynamics is in general not applicable to glaciers, because even at temperatures lower than the melting point, ice can move over thrust planes at the boundary between ice and bed rock or moraine-containing ice (leaving pieces of pure ice in depressions). For glaciers melting at the bottom, the existing hypotheses (Weertman, 1957, 1962, 1964; Lliboutry, 1959; Kamb and LaChapelle, 1964) determine the velocity of sliding over the bottom as a function of shear stress on the bottom and the roughness of the bottom, which is an unknown variable.

Therefore, a suitable theory of the distribution of stress and velocity on the bottom, may it be a confirmation of one of the existing hypotheses or an elaboration of a new hypothesis, can be obtained only by determining the values of parameters by applying the obtained solutions we are seeking for to actual glaciers.

IV. Known Analytical Solutions

Because of the complexity of the system of the equations, which determines the fields of stress, velocity and temperature in glaciers, analytical solutions are possible only for certain simplified cases. Simplifications are attained by:

a) Simplifying the properties of ice,

b) Taking simple stress-strain states and simplifying boundary conditions, and

c) Separately solving the mechanical equations and the equation of energy for an arbitrarily given distribution of temperature or velocity.

Below are listed the exact analytical solutions of which the present author is aware:

1. The distribution of velocity in case of complex laminar flows of incompressible isothermal viscous ice in a canal of an arbitrary cross section, including as a particular case, a simple laminar flow in an inclined parallel-sided plate (Somigliana, 1921).

2. The distribution of velocity in cases of a simple laminar flow and of a concentric laminar flow in a semi-cylindrical canal of incompressible isothermal ice which satisfies the power flow law (Nye, 1952).

3. The distribution of velocity in cases of uniaxial tension and of symmetric biaxial tension in a floating plate of incompressible ice which satisfies the power flow law with the fluidity depending on temperature, whose vertical distribution is arbitrarily given (Weertman, 1957).

4. The distribution of temperature in case of a simple laminar flow of viscous ice with a constant surface temperature and without any slip and melting on the bottom; internal heat generation is taken into account but the dependence of flow rate on temperature is neglected (Lagally, 1932).

5. The distribution of temperature in semi-infinite ice moving downward with a constant velocity; for a linear vertical distribution of temperature at the beginning and for the surface temperature changing in time linearly or sinusoidally (Benfield, 1949, 1951). If coordinates moving with ice in x-direction are used, the term $\rho c(\partial\theta/\partial t)$ of eq. (7) defines the heat transfer along x-axis $\rho c v_x(\partial\theta/\partial x)$, and this solution can be used to consider the influence of heat transfer in two perpendicular directions

(Robin, 1955).

6. The distribution of temperature in ice moving downward with the velocity linearly decreasing to zero at the bottom; the rate of bottom melting is defined, but the difference of the bottom velocity from zero due to melting is not taken into account, and for this reason in the case of bottom melting the solution is not exact; inner heat generation is neglected (Robin, 1955).

7. The distribution of temperature in ice moving vertically with velocities linearly changing from a given constant value at the surface to a value depending on the rate of bottom melting or freezing defined by the solution; inner heat production is neglected (Zotikov, 1962, 1964).

Application of these exact solutions to real glaciers with conditions different from the foreseen ones lead to errors difficult to be estimated. Besides, there are approximate solutions which contain assumptions not conformable to initial preconditions. For example, the widely adopted solution for stresses in a glacier with thickness h changing along the longitudinal axis x,

$$\sigma_{zx}(z) = -\rho g z \tan\alpha \text{ or } \sigma_{zx}(z) = -\rho g z \frac{\mathrm{d}h}{\mathrm{d}x},$$

(α is the angle of the inclination of the surface and z is the depth from the surface), unduly assumes $\sigma_{xx}=\sigma_{zz}=(\sigma_{zx})_{z=0}=0$ and disregards the plane force $\sigma_{zx}(0)-\int_0^z \left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}\right)\mathrm{d}z$, though it may be of the same order as the body force which are taken into consideration. On the other side, Nye (1951, 1957) gave solutions of the problem of plane flow in an infinite, homogeneously inclined plate of isothermal, ideally plastic ice and of ice satisfying the power flow law. But for such conditions the plane stress-strain state with $\sigma_{xx}=-\sigma_{zz}\neq 0$ cannot arise. Nye made use of the solution by Prandtle of the problem of compressing a plate, adopting the upper boundary conditions equivalent to taking off the compressing force: $s_{zz}=0$. In actual cases, longitudinal stress arises from the variation in the inclination of the bottom and the surface in different parts of a glacier, with the result that the stress changes in a longitudinal direction in contradiction to the results obtained by Nye.

Thus, except for the solution by Somigliana on the velocity distribution in complex laminar flow and the solution by Benfield on the temperature distribution in downward moving ice with changing surface temperature, all others are the solutions of one-dimensional problems.

In connection with the incompleteness of the existing theories, it is of great interest to elaborate solutions for conditions more similar to actual ones, first of all, by analytical methods.

V. Improvements of Solutions of One-Dimensional Problems

Solutions of one-dimensional problems, containing rheological relations, can be improved by taking account of eqs. (2) and (3).

In the state of a simple laminar flow in a direction of x in planes perpendicular to the axis z, we have

$$\sigma = |\sigma_{zx}|\,, \quad \varepsilon_{zx} = B\exp(\kappa\theta)\cdot(1+K|\sigma_{zx}|^3)\,\sigma_{zx}\,,$$

$$v_y = v_z = 0\,, \quad \frac{\mathrm{d}v_x}{\mathrm{d}z} = \varepsilon_{zx}+\omega_y = 2\varepsilon_{zx}\,,$$

where ω_y is the rate of angular rotation: $\omega_y = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)$. Substituting, in these equations, the value of shear stress $\sigma_{zx} = -\rho g z \sin\alpha$ (z is the depth from the surface), which is obtained from eq. (5) under the boundary condition on the surface, eq. (8), and integrating, we have for $\theta=0$

$$v_x(z) = v_0 - B\rho g \sin\alpha\, z^2\left(1+\frac{2}{5}K|\rho g z\sin\alpha|^3\right), \tag{10}$$

where $v_0 = v_x(0)$ is the velocity on the surface. For a negative constant temperature θ of ice, $B\exp(\kappa\theta)$ is substituted for B in eq. (10).

If the change in temperature with depth is given as $\theta=\theta(z)$, we have

$$v_x(z) = v_0 - 2B\rho g\sin\alpha\left[\int_0^z z\,e^{\kappa\theta(z)}\,\mathrm{d}z + K|\rho g\sin\alpha|^3\int_0^z z^4\,e^{\kappa\theta(z)}\,\mathrm{d}z\right]. \tag{11}$$

For a linear change in temperature, $\theta(z)=\theta_0+\gamma z$ (γ is constant), the integrals are expressed in terms of elementary functions:

$$\begin{aligned} v_x(z) = v_0 - 2Bd\rho g\sin\alpha\exp(\kappa\theta_0)\,\{(z-d)\exp(\kappa\gamma z)\\ +d+K|\rho g\sin\alpha|^3[\exp(\kappa\gamma z)(z^4-4d\,\{z^3-3d[z^2-2dz+2d^2]\})-24d^4]\}\,, \end{aligned} \tag{12}$$

where

$$d = 1/\kappa\gamma\,.$$

For a concentric laminar flow in a semi-cylindrical canal, a completely analogous solution is possible, because, by transforming to the cylindrical coordinate system, x, φ, r ($r=\sqrt{y^2+z^2}$ is the distance from the axis of the cylinder), the only non-zero component of the stress deviator is σ_{rx}, and hence, $\sigma=|\sigma_{rx}|$, and eq. (5) along x axis,

$$\frac{\mathrm{d}\sigma_{rx}}{\mathrm{d}r} + \frac{\sigma_{rx}}{r} + \rho g\sin\alpha = 0\,,$$

is integrated in the following form:

$$\sigma_{rx}(r) = -\frac{1}{2}\rho g r\sin\alpha\,.$$

In particular, for zero temperature,

$$v_x(r) = v_0 - \frac{1}{2}B\rho g\sin\alpha\, r^2\left(1+\frac{1}{20}K|\rho g r\sin\alpha|^3\right). \tag{13}$$

If the temperature distribution is not given, it is determined by eq. (7), which, in a stationary non-isothermal one-dimensional case without radiant energy exchange, is reduced to

$$\lambda\frac{\mathrm{d}\theta}{\mathrm{d}z^2} + 2A\,\sigma_{zx}\varepsilon_{zx} = 0\,,$$

where σ_{zx} and ε_{zx} are determined by eqs. (5), (2) and (3). Hence,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} + 2\frac{A}{\lambda}B(\rho g\sin\alpha)^2(1+K|\rho g z\sin\alpha|^3)\exp(\kappa\theta) = 0\,. \tag{14}$$

If the value of function $\theta(z)$ obtained by integrating eq. (14) (by numerical or other approximate methods) is substituted in eq. (11), we have the possibility of determining $v_x(z)$.

It should be noted, however, that, as mentioned above, the state of simple shear is observed only in narrow limited parts of the cross sections of glaciers.

In the state of laminar flow, the velocity of motion and the quantity of flowing ice do not change along the glacier, namely, the rate of income of ice to a cross section is equal to that of the outflow. Hence, in a laminar flow (that is, in the state of shear) a glacier cannot play any role in the natural circulation of water: that is, to expell the surplus of ice from one region and to release it into the surrounding media in another region. Moreover, at the starting point of a glacier, ice does not slide along the bottom. Therefore, in a laminar flow, sliding could not occur over the entire length of the glacier and it would not affect the bed rock at all. Hence, however closely the stress-strain state of ice might be simulated by the state of laminar flow, we would arrive at principally incorrect results, if we disregard the difference between the two states all along the glacier. This compels us to pay special attention to the solutions of the two-dimensional problem.

For uniaxial tension of a parallel-sided floating plate in the x direction, we have

$$\sigma_{xx} = -\sigma_{zz} = \sigma \text{ and hence } s_{xx} - s_{zz} = 2\sigma_{xx}\,,$$

from eqs. (2) and (4), and from eq. (5),

$$s_{zz} = -\rho g z\,.$$

Hence,

$$s_{xx} = 2\sigma_{xx} - \rho g z\,,$$

and

$$\int_0^h s_{xx}(z)\,\mathrm{d}z = 2\int_0^h \sigma_{xx}(z)\,\mathrm{d}z - \frac{1}{2}\rho g h^2\,, \tag{15}$$

(z is the depth from the surface). On the other hand, as shown by Weertman (1957 a), from the hydrostatic equilibrium condition of a plate in water of density ρ_{w}, we have

$$\int_0^h s_{xx}(z)\,\mathrm{d}z = -\int_{(1-\rho/\rho_{\mathrm{w}})h}^h \rho_{\mathrm{w}} g\left[z - \left(1 - \frac{\rho}{\rho_{\mathrm{w}}}\right)h\right]\mathrm{d}z = -\frac{1}{2}\rho g \frac{\rho}{\rho_{\mathrm{w}}} h^2\,. \tag{16}$$

From eqs. (15) and (16),

$$\int_0^h \sigma_{xx}(z)\,\mathrm{d}z = \frac{1}{4}\rho g\left(1 - \frac{\rho}{\rho_{\mathrm{w}}}\right)h^2\,,$$

$$\bar{\sigma}_{xx} = \frac{1}{4}\rho g\left(1 - \frac{\rho}{\rho_{\mathrm{w}}}\right)h\,.$$

By substituting these values in eqs. (2) and (3), the extension rate of the plate, which is constant along x and z axes, is obtained as

$$\varepsilon_{xx} = \frac{1}{4} B \exp\left[\kappa\theta(\bar{z})\right]\rho g\left(1 - \frac{\rho}{\rho_{\mathrm{w}}}\right)h\left\{1 + \frac{1}{64}K\left[\rho g\left(1 - \frac{\rho}{\rho_{\mathrm{w}}}\right)h\right]^3\right\}, \tag{17}$$

where

$$\theta(\bar{z}) = \ln\left\{\frac{1}{[1 + K\bar{\sigma}_{xx}^3]\,\bar{\sigma}_{xx}h}\int_0^h \exp\left[\theta(z)\right]\left[1 + K\sigma_{xx}^3(z)\right]\sigma_{xx}(z)\,\mathrm{d}z\right\} \simeq \frac{1}{h}\int_0^h \theta(z)\,\mathrm{d}z\,.$$

In case of symmetric biaxial extensions, we have

$$\sigma_{ii} + \sigma_{zz} = 0 \quad (i = 1, 2)\,, \quad s_{xx} = 3\sigma_{xx} - \rho g z\,, \quad \sigma = \sqrt{3}\,\sigma_{xx}\,,$$

$$\bar{\sigma}_{xx} = \bar{\sigma}_{yy} = \frac{1}{6}\rho g\left(1-\frac{\rho}{\rho_w}\right)h ,$$

and

$$\varepsilon_{xx} = \varepsilon_{yy} = \frac{1}{6} B \exp\left[\kappa\theta(\bar{z})\right]\rho g\left(1-\frac{\rho}{\rho_w}\right)h\left\{1+\frac{1}{24\sqrt{3}}K\left[\rho g\left(1-\frac{\rho}{\rho_w}\right)h\right]^3\right\}. \quad (18)$$

VI. Solution of Two-Dimensional Problem

Since analytical solution of the complete plane problem may probably be impossible, it seems to be advisable to make the necessary and sufficient simplifications to solve the problem. Such simplifications were found to be the homogenization and the linearization of the media: the solution becomes possible, if ice is regarded as homogeneous (isothermal) and linear (viscous) material. The former condition may be realized at the temperature of melting point and the latter may approximately be realized for small stresses and strain rates (For instance, the errors may be less than 10% for $\sigma \leqslant 1.25$ kg-wt/cm^2 and $\varepsilon \leqslant 0.18$ year^{-1}).

For incompressible ice in a plane stress-strain state in the xz-plane,

$$\sigma_{xy} = \sigma_{yy} = \sigma_{yz} = 0\,, \quad \varepsilon_{xy} = \varepsilon_{yy} = \varepsilon_{yz} = 0\,,$$

and hence from eqs. (2) to (4)

$$\varepsilon_{xx} = -\varepsilon_{zz}\,, \quad \sigma_{xx} = -\sigma_{zz}\,.$$

Differentiating the first equation of (5) by z and the second by x, and subtracting one of the obtained equations from the other, and further expressing σ_{zz} by $-\sigma_{xx}$, we obtain

$$2\frac{\partial^2 \sigma_{xx}}{\partial x \partial z} = \frac{\partial^2 \sigma_{zx}}{\partial x^2} - \frac{\partial^2 \sigma_{zx}}{\partial z^2}\,. \quad (19)$$

Since for a homogeneous viscous material $\sigma_{ij} = 2\eta\varepsilon_{ij}$,

$$2\frac{\partial^2 \varepsilon_{xx}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_{zx}}{\partial x^2} - \frac{\partial^2 \varepsilon_{zx}}{\partial z^2}\,. \quad (20)$$

In the state in question, the only equation in eq. (6) which has terms different from zero is

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} - 2\frac{\partial^2 \varepsilon_{zx}}{\partial x \partial z} = 0\,,$$

or expressing ε_{zz} by $-\varepsilon_{xx}$,

$$2\frac{\partial^2 \varepsilon_{zx}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} - \frac{\partial^2 \varepsilon_{xx}}{\partial x^2}\,. \quad (21)$$

Differentiating eqs. (20) and (21) by x and z, we have

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \varepsilon_{zx}}{\partial x \partial z}\right) - 2\frac{\partial^4 \varepsilon_{xx}}{\partial x^2 \partial z^2} - \frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \varepsilon_{zx}}{\partial x \partial z}\right) = 0\,,$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \varepsilon_{xx}}{\partial x \partial z}\right) + 2\frac{\partial^4 \varepsilon_{zx}}{\partial x^2 \partial z^2} - \frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \varepsilon_{xx}}{\partial x \partial z}\right) = 0\,.$$

Substituting the terms in parentheses by the results of eqs. (20) and (21), we have

$$\left.\begin{aligned}
&\frac{\partial^4 \varepsilon_{xx}}{\partial x^4}+2\frac{\partial^4 \varepsilon_{xx}}{\partial x^2 \partial z^2}+\frac{\partial^4 \varepsilon_{xx}}{\partial z^4}=0\,,\\
&\frac{\partial^4 \varepsilon_{zx}}{\partial x^4}+2\frac{\partial^4 \varepsilon_{zx}}{\partial x^2 \partial z^2}+\frac{\partial^4 \varepsilon_{zx}}{\partial z^4}=0\,,\\
\text{or}\quad&\\
&\Delta\Delta\,\varepsilon_{xx}(x,z)=0\,,\\
&\Delta\Delta\,\varepsilon_{zx}(x,z)=0\,.
\end{aligned}\right\}\tag{22}$$

The bi-harmonic equations (22) have the following solution (Goursat, 1898):

$$\varepsilon(x,z)=\mathrm{Re}\,[\bar{\zeta}\chi(\zeta)+\psi(\zeta)]\,,\tag{23}$$

where $\zeta=x+iz$, $\bar{\zeta}=x-iz$, χ and ψ are arbitrary analytic functions (ε denotes either of ε_{xx} and ε_{xz}, and $i=\sqrt{-1}$).

On the outer boundary of the glacier, the unknown functions ε_{xx}, ε_{xz} and their normal gradients must satisfy the boundary conditions, eqs. (1) and (8). In order to determine the form of the functions χ and ψ, the values of ε_{xx} and ε_{xz} and their derivatives must be substituted in eq. (23) and in the following equations, which are obtained by the differentiation and addition or subtraction of eq. (23):

$$\left.\begin{aligned}
&\frac{\partial\varepsilon}{\partial x}+\frac{\partial\varepsilon}{\partial z}i=\chi(\zeta)+\overline{\zeta\chi'(\zeta)}+\overline{\psi'(\zeta)}\,,\\
&\frac{\partial\varepsilon}{\partial x}-\frac{\partial\varepsilon}{\partial z}i=\overline{\chi(\zeta)}+\bar{\zeta}\chi'(\zeta)+\psi'(\zeta)\,,
\end{aligned}\right\}\tag{24}$$

where $\overline{\chi(\zeta)}=\chi(x-zi)$ and so on.

By definitions, $\varepsilon_{ll}=\dfrac{\partial v_l}{\partial l}$ and $\varepsilon_{nl}=\dfrac{1}{2}\left(\dfrac{\partial v_l}{\partial n}+\dfrac{\partial v_n}{\partial l}\right)$. Therefore, $\dfrac{\partial \varepsilon_{ll}}{\partial l}=\dfrac{\partial^2 v_l}{\partial l^2}$, and considering eq. (8), we have $\dfrac{\partial \varepsilon_{nl}}{\partial l}=0$ and $\dfrac{\partial v_l}{\partial n}=-\dfrac{\partial v_n}{\partial l}$. Hence,

$$\frac{\partial \varepsilon_{ll}}{\partial n}=\frac{\partial^2 v_l}{\partial l\,\partial n}=-\frac{\partial^2 v_n}{\partial l^2}\,.$$

On the other hand, from eq. (5) along the $\boldsymbol{l}$ axis and eq. (8) and from $\sigma_{ij}=2\eta\varepsilon_{ij}$, we have

$$\frac{\partial \varepsilon_{nl}}{\partial n}=-\frac{\rho g}{2\eta}\sin\alpha-2\frac{\partial \varepsilon_{ll}}{\partial l}\,,$$

where $\alpha=\angle\boldsymbol{xl}=\angle\boldsymbol{zn}$ (The axis $\boldsymbol{x}$ is taken horizontally and $\boldsymbol{z}$ upwardly). Transforming the components of vectors $\boldsymbol{p}$ and tensors T by the rule

$$p_i'=a_{ij}p_j\,,\quad T_{ij}'=a_{ik}a_{jl}T_{kl}\,,$$

(a_{ij} is the cosine of angle between the new and old axes designated by the first and the second indicators, respectively), we have

$$\begin{aligned}
&(\varepsilon_{xx})_{z=F(x)}=\frac{\partial v_l}{\partial l}\cos 2\alpha\,;\\
&\left(\frac{\partial \varepsilon_{xx}}{\partial x}\right)_{z=F(x)}=-\frac{\rho g}{2\eta}\sin^2\alpha\sin 2\alpha+\frac{\partial^2 v_l}{\partial l^2}(\cos\alpha\cos 2\alpha-2\sin\alpha\sin 2\alpha)\\
&\qquad+\frac{\partial^2 v_n}{\partial l^2}\sin\alpha\cdot\cos 2\alpha-2\frac{\partial v_l}{\partial l}F''(x)\cos^4\alpha\sin 2\alpha\,;
\end{aligned}$$

$$\left.\begin{aligned}
\left(\frac{\partial \varepsilon_{xx}}{\partial z}\right)_{z=F(x)} &= \frac{\rho g}{4\eta}\sin^2 2\alpha+\frac{\partial^2 v_l}{\partial l^2}(\sin\alpha\cdot\cos 2\alpha+2\cos\alpha\cdot\sin 2\alpha)\\
&\quad-\frac{\partial^2 v_n}{\partial l^2}\cos\alpha\cdot\cos 2\alpha-\frac{\partial v_l}{\partial l}F''(x)\cos^2\alpha\cdot\sin^2 2\alpha\,;\\
(\varepsilon_{zx})_{z=F(x)} &= \frac{\partial v_l}{\partial l}\sin 2\alpha\,;\\
\left(\frac{\partial \varepsilon_{zx}}{\partial x}\right)_{z=F(x)} &= \frac{\rho g}{2\eta}\sin^2\alpha\cdot\cos 2\alpha+\frac{\partial^2 v_l}{\partial l^2}(\cos\alpha\cdot\sin 2\alpha+2\sin\alpha\cdot\cos 2\alpha)\\
&\quad+\frac{\partial^2 v_n}{\partial l^2}\cdot\sin\alpha\sin 2\alpha+2\frac{\partial v_l}{\partial l}F''(x)\cos^4\alpha\cos 2\alpha\,;\\
\left(\frac{\partial \varepsilon_{zx}}{\partial z}\right)_{z=F(x)} &= -\frac{\rho g}{8\eta}\sin 4\alpha+\frac{\partial^2 v_l}{\partial l^2}(\sin\alpha\cdot\sin 2\alpha-2\cos\alpha\cdot\cos 2\alpha)\\
&\quad-\frac{\partial^2 v_n}{\partial l^2}\cos\alpha\cdot\sin 2\alpha+\frac{1}{2}\frac{\partial v_l}{\partial l}F''(x)\cos^2\alpha\sin 4\alpha\,.
\end{aligned}\right\}\quad(25)$$

(For the free bottom surface of the floating glacier, $z_1=F_1(x)$, eq. (25) take slightly different forms because $\angle \boldsymbol{zn}=\pi+\alpha$ for $\angle \boldsymbol{xn}=\alpha$).

Thus, if the velocity vector $\boldsymbol{v}$, or the rate of accumulation a and the rate of moving of the surface, $\partial h/\partial t$ (see eq. (1)), on the known free surface of a glacier $z=F(x)$ and $z_1=F_1(x)$ are given, the plane field of strain rate is completely determined in the neighbourhood of the surface by eqs. (23) to (25), and hence the stress field is also determined by these and rheological equations. The velocity field $\boldsymbol{v}(x,z)$ is most easily obtained from the field of strain rate by means of the following equations:

$$\begin{aligned}
v_x(x,z) &= v_x(F^{-1}(z),z)+\int_{F^{-1}(z)}^{x}\varepsilon_{xx}(x,z)\,\mathrm{d}x\,,\\
v_z(x,z) &= v_z(x,F(x))+\int_{F(x)}^{z}\varepsilon_{zz}(x,z)\,\mathrm{d}z\,,
\end{aligned}\qquad(26)$$

($x=F^{-1}(z)$ is the equation of the surface. In order to find $v_x(x,z)$ by the integration procedure from the surface to the depth of z, it is necessary to know, besides $\varepsilon_{xz}(x,z)$, the equation of the angular rate of rotation, $\omega_y(x,z)$, which is not equal to ε_{xz} in this case but is determined through a differential equation). The length of the velocity vector and the inclination of the stream lines are now obtained by the formulae

$$\begin{aligned}
v(x,z) &= \sqrt{v_x^2+v_z^2}\,,\\
\tan\varphi(x,z) &= v_z/v_x\,.
\end{aligned}\qquad(27)$$

Equation (1) with $\partial h/\partial t=0$ and eq. (9) with $\partial\theta/\partial n=0$ determine the contour of the glacier bottom $z=G(x)$, where the unevenness of the contour with the radius of curvature which is at least in the order of the glacier thickness $h=F-G$ is taken into account: Smaller unevennesses of the bottom do not affect the distribution of velocity on the upper surface and should be treated as the "roughness" of the bottom. If there were no bottom-melting or freezing, the angle of the bottom inclination β would coincide with that of the lower stream-line φ_0, but in general cases

$$\begin{aligned}
\beta(x) &= \varphi_0(x)+\arcsin\left(\frac{v_n}{v}\right)_{z=G(x)}\,,\\
G(x) &= G(x_0)+\int_{x_0}^{x}\tan\beta(x)\,\mathrm{d}x\,.
\end{aligned}\qquad(28)$$

The integrated equation of continuity (4)

$$\int_G^F v_x(z)\,\mathrm{d}z = -\int_{(l)} v_n(l)\,\mathrm{d}l - \int_{(l^*)} v_n(l^*)\,\mathrm{d}l^*\,, \tag{29}$$

can be used to control the results. (l and l^* are the upper and the lower contours of the outer surface of the glacier.) In eq. (29), v_n may be expressed also by a and $\partial h/\partial t$ by means of eq. (1).

Because of the absence of natural data, we consider as an example a linear decrease of $\partial v_l/\partial l$ along the surface of a mountain glacier of a typical form as shown in Fig. 1 in which the strain rate decreases from the maximum at the starting point via zero at the middle to the minimum at the end, so that the upper reaches of the glacier are tensioned while the lower are compressed. For simplifying the calculations, the curved contour is substituted by jointed segments so that on each of them the functions χ and ψ of eqs. (23) and (24) become the simplest functions of a complex variable ζ, of which the conjugated harmonic functions of real variables x and z become constants. For $\partial^2 v_n/\partial l^2$ such values were selected as to obtain the typical form of the bottom.

The results of calculations are shown in Figs. 1 and 2 where the junctions of the

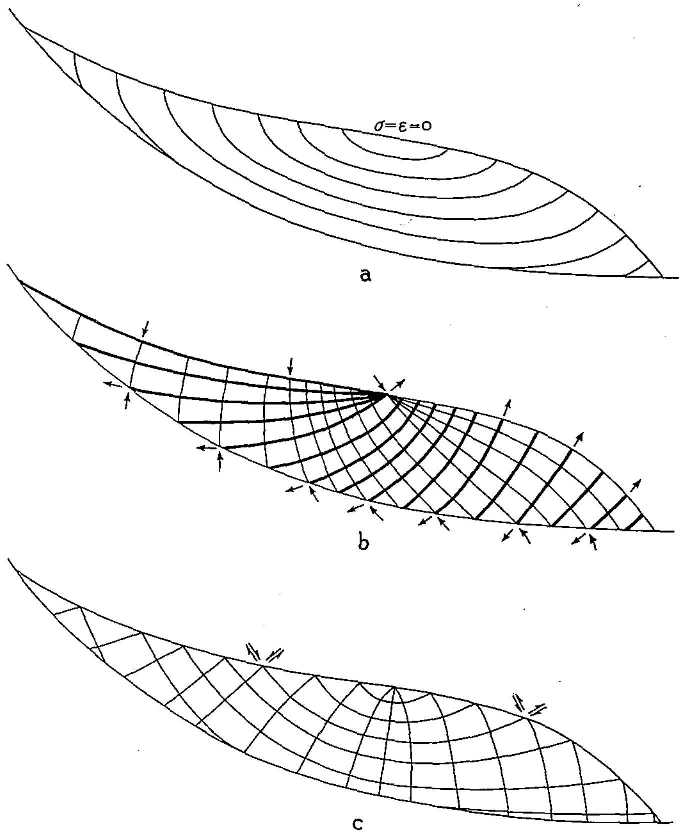

Fig. 1. The distribution of stress and strain rate in a viscous isothermal glacier. a) Lines $\sigma(x,z)=\text{const.}$ $\varepsilon(x,z)=\text{const.}$ b) Lines of maximal tension (thick lines) and those of maximal compression (thin lines). c) Lines of maximal shear

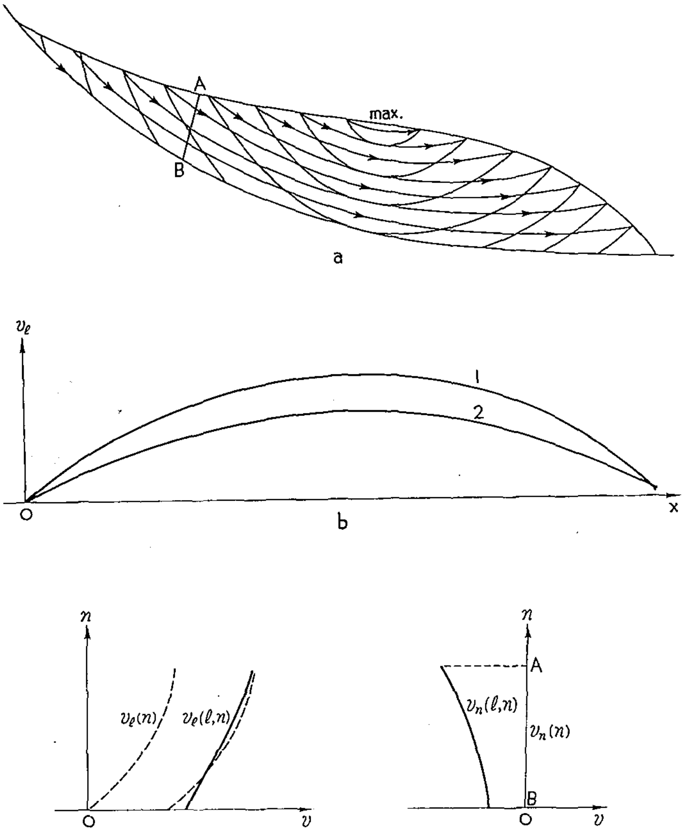

Fig. 2. The distribution of velocity in a viscous isothermal glacier.
a) Iso-speed lines and stream lines (with arrows); the line AB is for Fig. 2 c.
b) The tangential velocity distributions on the upper (1) and on the bottom (2) surface. c) Distributions of v_x and v_n along the cross section AB corresponding to one- and two- dimensional solutions

straight line segments have been rounded.

If the profile of the bottom is given together with the profile of the surface and the velocity vector on it, the system of equations (23) to (28) becomes overdetermined and incompatible. Only a definite relation between $v_l(l)$ and $v_n(l)$ will be consistent with the given contours of the surface and the bottom, and therefore to a given rate of accumulation a, a definite rate of change in the size of the glacier, $\partial h/\partial t$, corresponds. The possibility of differences in the rate of change in the size is connected with differences in the velocity of sliding on the bottom under different shear stresses.

Further analysis of the obtained equations is necessary, as well as solutions of the problem under various conditions, such as when the countour of the bottom, shear stresses on it and the countour of the surface or the rate of accumulation, and so on are given.

In case of heterogeneous nonlinear ice the field of stress and velocity in the glacier which is in a state of two-dimensional stress are determined by the system of equations

(19), (21), (2) and (3). They are reduced to two equations (in stresses or in strain rates), one of which is nonlinear. Therefore, the analytical solutions are impossible, at least in non-isothermal case (Shumskiy, 1966). The solution of this system is being elaborated by numerical methods.

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