Flexural and Other Properties of Sea Ice Sheets

Condensed Version

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Abstract

Physics of sea ice has now advanced to the point that one can predict composite properties of sea ice sheets on a rational basis. This leads to a synthesis on the basis of past analysis. For practical purposes nomograms or computer techniques can be developed, with easily measured quantities as input. The foundation for such methods is laid in this paper.

The composite properties discussed are: zero strength and deformation layer, neutral layer, average crushing strength and Young's modulus, calculation of zero, first and second order moments, modulus of inertia, section modulus, flexural strength, plate rigidity, action radius as well as failure mechanism. The principal assumption underlying the future construction of nomograms from equations is a linear temperature profile and constant salinity. For computer techniques a more flexible polynomial approach is suggested.

The basical physical parameter invariably is the brine volume which can be calculated from temperature and salinity. Only surface temperature could be used under simplifying assumptions, but profile data at least for four points are suggested for computer techniques.

Comparison with test data is now proceeding.

NOTATIONS

A_E	$\dfrac{1-E/E_0}{\eta S}-1=\dfrac{\theta}{\theta_1}$
$\bar{A}_E$	Average of A_E over entire or partial profile
A_σ	$\dfrac{1-\sigma/\sigma_0}{\sqrt{\eta S}}$
$\bar{A}_\sigma$	Average of A_σ over entire or partial profile
D	Flexural rigidity eq. (8.20)
E	Young's modulus of sea ice
E_0	"Basic" Young's modulus of sea ice for zero brine volume comparable to Young's modulus for fresh ice
J	$M_2-M_0z_n^2$ Moment of inertia eq. (8.13)
J_b	An integral, eq. (5.3)
M	Bending moment
M_0	$\int_{z_l}^{z_u}\dfrac{E}{E_0}\,dz$ Zero moment of fictitious cross section
M_1	$\int_{z_l}^{z_u}z\dfrac{E}{E_0}\,dz$ First moment
M_2	$\int_{z_l}^{z_u}z^2\dfrac{E}{E_0}\,dz$ Second moment

M_l	Bending moment with tension at bottom
M_u	Bending moment with tension at top
S	Salinity, here expressed in absolute ratios
S_b	Salinity of brine in equilibrium with given temperature
S_0	Section modulus
S_l	Section modulus for lower fibre
S_u	Section modulus for upper fibre
U	$-\theta_1-\theta$
V	$1-\frac{\nu_1}{\nu_0}$
V_1	$\frac{1}{\nu_0}\left(\frac{a_1}{2}+\frac{a_2}{3}+\frac{a_3}{4}\right)$
V_2	$\frac{2}{\nu_0}\left(\frac{a_1}{3}+\frac{a_2}{4}+\frac{a_3}{5}\right)$
V_3	$\frac{3}{\nu_0}\left(\frac{a_1}{4}+\frac{a_2}{5}+\frac{a_3}{6}\right)$
$V_1(z)$	$\frac{1}{\nu_0}\left(\frac{a_1}{2}z+\frac{a_2}{3}z^2+\frac{a_3}{4}z^3\right)$
$V_2(z)$	$\frac{2}{\nu_0}\left(\frac{a_1}{3}z+\frac{a_2}{4}z^2+\frac{a_3}{5}z^3\right)$
$V_3(z)$	$\frac{3}{\nu_0}\left(\frac{a_1}{4}z+\frac{a_2}{5}z^2+\frac{a_3}{6}z^3\right)$
X	$-\theta$
a_0, a_1, a_2, a_3	Coefficients of third order polynomial
b_0, b_1, b_2, b_3	Coefficients of polynomial, eq. (5.1)
b_r	Brine content by weight
h	Thickness of sea ice, disregarding skeleton layer
h_{sk}	$\approx$2.5 cm Thickness of skeleton layer at the sea ice bottom
l	αl_0 Action radius of sea ice eq. (8.21)
l_0	"Basic" action radius of sea ice eq. (8.23)
l_1	Action width, eq. (8.22)
y_1, y_2, y_3, y_4	Individual values of y, eq. (4.2)
$\bar{y}$	Average over profile or part of it
z	Upwards counted distance from "bridging" layer (where the grooves of the skeleton layer close off), assuming the ice thickness h as unity (disregarding skeleton layer)
z_l	Lower limit of integration for z
z_n	$\frac{M_1}{M_0}$ Distance to neutral layer
z_u	Upper limit of integration for z
α	Reduction of action radius for sea ice, eq. (8.24)
γ_b	Density of brine
γ_i	"Theoretical" density of sea ice, without air bubbles; can be computed from phase relations

$\Delta, \Delta_1, \Delta_2, \Delta^2, \Delta_1^2, \Delta^3$	See eq. (4.2)
η	$\dfrac{\gamma_i}{\gamma_b \nu_0}$
θ	Ice temperature; also temperature of brine under phase equilibrium
θ_0	Ice surface temperature
θ_1	-54.11 for sea ice brine; constant in eq. (2.3)
θ_b	$\theta_w - (\theta_w - \theta_0)\dfrac{h_{sk}}{h + h_{sk}}$ Temperature at bridging layer
θ_l	Temperature at lower limit of integration
θ_u	Temperature at upper limit of integration
θ_w	Water temperature
κ	Foundation modulus (density of water)
μ	Poisson's ratio
ν	$\nu(1)S$ "Volume porosity"; relative volume of brine in sea ice
ν_0	Brine volume for which strength or deformation extrapolates to zero
ν_1	Brine volume at bridging layer
ν_2, ν_3	Volume of brine for $z=1/3$ or $z=2/3$
ν_4	Volume of brine at surface
$\nu(1)$	Relative volume of brine for $S=1‰$; a function of temperature
σ	Strength of sea ice (may be tensile or crushing strength)
$\bar{\sigma}$	Average strength over a profile or a portion of it
σ_0	"Basic strength" of sea ice without brine pockets, comparable to strength of fresh ice
σ_l	Flexural strength with tension at bottom
σ_u	Flexural strength with tension at top

I. Introduction

One of the interesting problems in the physics of sea ice is the response of sea ice sheets to forces. This may be the crushing of a sheet against an obstacle. The flexural response to forces may lead to breakage or just to oscillations such as air-coupled waves.

Sea ice, a mixture of fresh ice, liquid brine and solid salts, is not only strongly anisotropic but changes its basic physical characteristics rapidly with depth. At the bottom it has almost the consistency of mush while at the surface it may approach the characteristics of fresh ice depending upon salinity, age and weather. Although it is obvious that the usual assumptions regarding isotropy and homogeneity, which enter theoretical mechanics, are violated in this case nevertheless the considerable vertical variation of properties can be handled. Papers dealing with the mechanics of sea ice have virtually not considered this complication up to now.

The foundation of a rational system in sea ice physics is its phase diagram or composition. Numerical values have been published by Assur (1958) but these require further experimentation and refinement.

In the following sections composite properties of sea ice sheets are summarized.

II. The Zero Strength Layer in Sea Ice Sheets

Depending upon salinity and temperature one may find that the strength diminishes to negligible values near the bottom or near the top under spring-time conditions. The skeleton layer of about 2.5 cm thickness at the sea ice bottom where no bridging occurs between elementary platelets has been customarily disregarded. However, additional layers may be involved.

We assume a general strength relation

$$\frac{\sigma}{\sigma_0} = 1 - \sqrt{\frac{\nu}{\nu_0}} \,. \tag{2.1}$$

Equation (2.1) can be used not only for tensile but also for crushing strength as recently shown by Peyton (1966).

For no strength $\nu = \nu_0$. This is a nominal condition since eq. (2.1) includes only the first term of a general expansion which is sufficient for practical purposes. Assur (1958) has shown the dependence of ν_0 on the petrographic structure of sea ice. Significant advances have been made in the recent past in determining its dependence on growth velocity and salinity.

Brine content by weight is

$$b_r = \frac{S}{S_b} \,. \tag{2.2}$$

Up to -8.2°C according to Assur (1958)

$$\theta = \theta_1 \frac{S}{1 - S_b} \,, \tag{2.3}$$

or

$$\frac{1}{S_b} = 1 + \frac{\theta_1}{\theta} \,. \tag{2.4}$$

Assume a linear temperature profile throughout the ice thickness

$$\theta = \theta_b - \Delta\theta z \,. \tag{2.5}$$

The relative brine volume is

$$\nu = b_r \frac{\gamma_i}{\gamma_b} = \frac{S}{S_b} \frac{\gamma_i}{\gamma_b} \,. \tag{2.6}$$

From eqs. (2.6) and (2.4)

$$\nu = S\left(1 + \frac{\theta_1}{\theta}\right) \frac{\gamma_i}{\gamma_b} \,. \tag{2.7}$$

Setting this equal to ν_0 one finds

$$\theta_z = \frac{\theta_1}{\dfrac{\nu_0 \gamma_b}{S \gamma_i} - 1} \,, \tag{2.8}$$

for the temperature at the zero strength layer which can be tabulated in terms of salinity. θ_z is virtually proportional to S with an excellent approximation. For the position of the zero strength layer

$$z_0 = \frac{\theta_b - \theta_z}{\theta_b - \theta_0} \,. \tag{2.9}$$

The value of z_0 which may be negative, located between 0 and 1 or even above 1 characterizes the ice sheet from its initial growth until decay in Spring.

III. Average Crushing Strength of a Sea Ice Sheet

Combining eq. (2.1) with eq. (2.7) we have

$$\frac{\sigma}{\sigma_0} = 1 - \sqrt{\left(1+\frac{\theta_1}{\theta}\right)\frac{\gamma_i S}{\gamma_b \nu_0}}, \tag{3.1}$$

which can be also written as

$$\frac{\sigma}{\sigma_0} = 1 - \sqrt{\frac{\theta_1+\theta}{\theta}\eta S}, \tag{3.2}$$

with

$$\eta = \frac{\gamma_i}{\gamma_b \nu_0}.$$

Define a function

$$A_\sigma = \frac{1-\sigma/\sigma_0}{\sqrt{\eta S}} = \sqrt{\frac{\theta_1+\theta}{\theta}} = \sqrt{\frac{U}{X}}, \tag{3.3}$$

with

$$U = -\theta_1 - \theta, \tag{3.4a}$$

and

$$X = -\theta. \tag{3.4b}$$

The average strength $\bar{\sigma}$ between the temperatures θ_l and θ_u can be obtained by integrating eq. (3.3) over θ. This can be done since θ is assumed a linear function of z. Furthermore A is a linear function of σ. Performing the integration one finds

$$(\theta_u-\theta_l)\bar{A}_\sigma = \int_{\theta_l}^{\theta_u}\sqrt{\frac{U}{X}}\,d\theta = \left|-\sqrt{\theta_1\theta(1+\theta/\theta_1)}\right|_{\theta_l}^{\theta_u} - \theta_1 \ln\left(\sqrt{\frac{\theta_u}{\theta_l}}\,\frac{1-\sqrt{1+\theta_1/\theta_u}}{1-\sqrt{1+\theta_1/\theta_l}}\right). \tag{3.5}$$

If the computational zero strength layer from eq. (2.9) is outside the thickness h we have $\theta_u=\theta_0$ and $\theta_l=\theta_b$ and

$$(\theta_0-\theta_b)\bar{A}_\sigma = \sqrt{-\theta_1}\left(\sqrt{-\theta_b(1+\theta_b/\theta_1)} - \sqrt{-\theta_0(1+\theta_0/_1\theta)}\right) - \theta_1 \ln\left(\sqrt{\frac{\theta_0}{\theta_b}}\,\frac{1-\sqrt{1+\theta_1/\theta_0}}{1-\sqrt{1+\theta_1/\theta_b}}\right). \tag{3.6}$$

The average strength $\bar{\sigma}$ itself can be found from

$$\bar{\sigma}/\sigma_0 = 1 - \sqrt{\eta S}\,\bar{A}_\sigma. \tag{3.7}$$

The approach, as outlined, has the advantage that nomograms can be constructed theoretically which show the crushing strength of sea ice sheets in comparison to fresh ice sheets as a function of surface temperature, salinity and water temperature. The disadvantages are the limitations imposed (limited temperature range, linear temperature profile, uniform salinity). A more empirical way using polynomials may be more flexible for general use especially in conjunction with computer techniques.

IV. **Average Profile Property**—Polynomial approach

Let the profile property be presented by a third order polynomial which is sufficient for sea ice.

$$y = a_0 + a_1 z + a_2 z^2 + a_3 z^3 . \tag{4.1}$$

The coefficients can be obtained by least squares from observed or calculated y or one may use selected points. The following is a convenient way. Let y_1 be the profile property for $z=0$, y_2 for $z=1/3$, y_3 for $z=2/3$ and y_4 for $z=1$ which are all equal distant in z.

Let

$$\begin{aligned} \Delta &= y_2 - y_1 \\ & \qquad \Delta^2 = \Delta_1 - \Delta \\ \Delta_1 &= y_3 - y_2 \qquad\qquad \Delta^3 = \Delta_1^2 - \Delta^2 \\ & \qquad \Delta_1^2 = \Delta_2 - \Delta_1 \\ \Delta_2 &= y_4 - y_3 \end{aligned} \tag{4.2}$$

Then by adapting Newton's formula for interpolation

$$a_0 = y_1 , \tag{4.3a}$$

$$a_1 = 3\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3}\right) , \tag{4.3b}$$

$$a_2 = 4.5(\Delta^2 - \Delta^3) , \tag{4.3c}$$

$$a_3 = 4.5\Delta^3 . \tag{4.3d}$$

The average property over the entire profile by integrating eq. (4.1) becomes

$$\bar{y} = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} . \tag{4.4}$$

Similarly the average between limits may be obtained.

V. **Average Crushing Strength**—Polynomial approach

The most convenient approach is to write $\sqrt{\nu}$ as a polynomial

$$\sqrt{\nu} = b_0 + b_1 z + b_2 z^2 + b_3 z^3 . \tag{5.1}$$

Integrating eq. (2.1) with eq. (5.1) over the entire profile

$$\frac{\bar{\sigma}}{\sigma_0} = \int_0^1 \left[1 - \frac{1}{\sqrt{\nu_0}} (b_0 + b_1 z + b_2 z^2 + b_3 z^3)\right] dz = 1 - \sqrt{\frac{\nu_1}{\nu_0}} - \frac{1}{\sqrt{\nu_0}} \left(\frac{b_1}{2} + \frac{b_2}{3} + \frac{b_3}{4}\right), \tag{5.2}$$

where the b coefficients can be obtained from differences similar to eqs. (4.2) and (4.3).

Equation (5.2) assumes, of course, that the full profile has some strength. If this is not the case then one must obtain the difference between two integrals

$$J_b = z\left[1 - \sqrt{\frac{\nu_1}{\nu_0}} - \frac{1}{\sqrt{\nu_0}} \left(\frac{b_1}{2} z + \frac{b_2}{3} z^2 + \frac{b^3}{4} z^3\right)\right], \tag{5.3}$$

evaluated between the limits $z = z_u$ and $z = z_l$. The result divided by $z_u - z_l$ gives the average $\bar{\sigma}/\sigma_0$.

VI. Average Young's Modulus for Sea Ice Sheets

In order to predict the flexural response of a sea ice sheet from basic aspects one has to proceed in a somewhat unconventional way. At first the stress-strain relations must be defined. The viscous behavior as a function of stress and time still causes problems.

Young's modulus for sea ice can be related to brine volume as

$$E = E_0\left(1-\frac{\nu}{\nu_0}\right). \tag{6.1}$$

For evidence one can refer to Langleben and Pounder (1963) who obtained

$$E = (10.00-0.35\nu)\times 10^{10}\ \mathrm{dyn/cm^2}\,, \quad \nu \text{ in } \%\,, \tag{6.2}$$

or

$$\nu_0 = 0.286\,.$$

Abele and Frankenstein (1966) calculated

$$E = (9.01-0.464\nu)\times 10^{10}\ \mathrm{dyn/cm^2}\,, \tag{6.3}$$

or

$$\iota_0 = 0.194\,.$$

The linearity between E and ν was first demonstrated by Tabata (1958) who used volume of air+brine for ν. His Fig. 17 can be presented as

$$E = (9.27-0.282\nu)\times 10^{10}\ \mathrm{dyn/cm^2}\,, \tag{6.4}$$

or

$$\nu_0 = 0.329\,.$$

Equations (6.2)-(6.4) give the *dynamic* Young's modulus. For engineering purposes we may adopt the linearity with ν, assume $\nu_0=0.25$ but must take a different much lower E. Furthermore logic demands to assume the same ν_0 for E in eq. (6.1) as the ν_0 for σ in eq. (2.1). There cannot be any stress without deformation and vice versa.

Combining eq. (6.1) with eq. (2.7) we have

$$\frac{E}{E_0} = 1-\left(1+\frac{\theta_1}{\theta}\right)\frac{\gamma_i S}{\gamma_b \nu_0}\,. \tag{6.5}$$

Assuming a linear temperature profile one can integrate over temperature rather than depth to obtain an average E.

We form a value

$$A_E = \frac{1-E/E_0}{\eta S} - 1 = \frac{\theta}{\theta_1}\,, \tag{6.6}$$

and integrate this over the entire profile to obtain an average $\bar{A}_E$

$$\bar{A}_E = \int_0^1 \frac{\theta_1}{\theta}\,\mathrm{d}z\,. \tag{6.7}$$

From eq. (2.5)

$$\mathrm{d}\theta = -\Delta\theta\,\mathrm{d}z\,, \tag{6.8}$$

which, substituted into eq. (6.7) gives

$$\bar{A}_E = -\frac{1}{\Delta\theta}\int_{\theta_b}^{\theta_0}\frac{\theta_1}{\theta}\,\mathrm{d}\theta = -\frac{\theta_1}{\Delta\theta}\ln\frac{\theta_0}{\theta_b}\,, \tag{6.9}$$

or

$$\frac{\bar{E}}{E_0} = 1-\eta S\left(1-\frac{\theta_1}{\Delta\theta}\ln\frac{\theta_0}{\theta_b}\right). \tag{6.10}$$

VII. Neutral Layer

The neutral layer or axis is defined as the zero deformation layer in pure bending. Its position is given by

$$z_n = \frac{M_1}{M_0}\,. \tag{7.1}$$

The first moment is

$$M_1 = \int_0^1 z\frac{E}{E_0}\,\mathrm{d}z\,. \tag{7.2}$$

Considering eqs. (6.1), (2.7) and (2.5)

$$M_1 = \int_0^1 z\left[1-\left(1+\frac{\theta_1}{\theta_b-\Delta\theta z}\right)\eta S\right]\mathrm{d}z\,, \tag{7.3}$$

which integrated gives

$$M_1 = \frac{1}{2}-\frac{1}{2}\eta S+\eta S\frac{\theta_1}{\Delta\theta}\left[1+\frac{\theta_b}{\Delta\theta}\ln\left(1-\frac{\Delta\theta}{\theta_b}\right)\right]. \tag{7.4}$$

The zero moment can be simply represented as the average $\bar{E}/E_0$ from eq. (6.10) so that

$$z_n = \frac{1}{2}\,\frac{1-\eta S\left\{1-\frac{\theta_1}{\Delta\theta}\left[1+\frac{\theta_b}{\Delta\theta}\ln\left(1-\frac{\Delta\theta}{\theta_b}\right)\right]\right\}}{1-\eta S\left(1-\frac{\theta_1}{\Delta\theta}\ln\frac{\theta_0}{\theta_b}\right)}\,. \tag{7.5}$$

VIII. Flexural Properties of Sea Ice Sheets—Polynomial approach

For computer oriented approaches the following is most convenient.

We express the brine volume computed from observed temperatures and salinities as

$$\nu = a_0+a_1z+a_2z^2+a_3z^3\,, \tag{8.1}$$

with $a_0=\nu_1$, and a_1, a_2, a_3 computed from eq. (4.3) or by least squares.

We base our calculations not on a cross section with unit width as usually done but with a width reduced according to the ratio of Young's moduli.

$$\frac{E}{E_0} = 1-\frac{\nu}{\nu_0}\,. \tag{8.2}$$

For this reduced cross section we calculate the various moments related to the bridging layer $z=0$. The zero moment over the entire profile is

$$M_0 = \int_0^1\frac{E}{E_0}\,\mathrm{d}z = \int_0^1\left[1-\frac{1}{\nu_0}(\nu_1+a_1z+a_2z^2+a_3z^3)\right]\mathrm{d}z\,, \tag{8.3}$$

$$M_0 = 1-\frac{\nu_1}{\nu_0}-\frac{1}{\nu_0}\left(\frac{a_1}{2}+\frac{a_2}{3}+\frac{a_3}{4}\right) = V-V_1\,. \tag{8.4}$$

The first moment is

$$M_1 = \int_0^1 z \frac{E}{E_0} \, dz = \int_0^1 z \left[1 - \frac{1}{\nu_0} (\nu_1 + a_1 z + a_2 z^2 + a_3 z^3) \right] dz \,, \tag{8.5}$$

$$M_1 = \frac{1}{2} \left[1 - \frac{\nu_1}{\nu_0} - \frac{2}{\nu_0} \left(\frac{a_1}{3} + \frac{a_2}{4} + \frac{a_3}{5} \right) \right] = \frac{1}{2} (V - V_2) \,. \tag{8.6}$$

The distance to the neutral layer is

$$z_n = \frac{M_1}{M_0} = \frac{1}{2} \frac{V - V_2}{V - V_1} \,. \tag{8.7}$$

This, of course, postulates the validity of Kirchhoff's law. The position of z_n is quite temperature dependent.

The second moment is

$$M_2 = \int_0^1 z^2 \frac{E}{E_0} \, dz = \int_0^1 z^2 \left[1 - \frac{1}{\nu_0} (\nu_1 + a_1 z + a_2 z^2 + a_3 z^3) \right] dz \,, \tag{8.8}$$

$$M_2 = \frac{1}{3} \left[1 - \frac{\nu_1}{\nu_0} - \frac{3}{\nu_0} \left(\frac{a_1}{4} + \frac{a_2}{5} + \frac{a_3}{6} \right) \right] = \frac{1}{3} (V - V_3) \,. \tag{8.9}$$

We presented the moments for the case when z_n does not lie between 0 and 1 (no zero strength layer). In the general case one must integrate between the limits z_u and z_l which upon integration gives

$$M_0 = \Big| z [V - V_1(z)] \Big|_{z_l}^{z_u} \tag{8.10}$$

$$M_1 = \Big| \frac{z^2}{2} [V - V_2(z)] \Big|_{z_l}^{z_u} \,. \tag{8.11}$$

$$M_2 = \Big| \frac{z^3}{3} [V - V_3(z)] \Big|_{z_l}^{z_u} \,. \tag{8.12}$$

The moment of inertia can be calculated as

$$J = M_2 - M_0 z_n^2 = M_2 - \frac{M_1^2}{M_0} \,. \tag{8.13}$$

The moment of inertia is highly variable for sea ice sheets of the same thickness, depending upon age and weather.

The section modulus for the lower fibre is

$$S_l = \frac{M_2 - M_0 z_n^2}{z_n} = \frac{M_0 M_2}{M_1} - M_1 = \frac{M_0 M_2 - M_1^2}{M_1} \,. \tag{8.14}$$

For the upper fibre

$$S_u = \frac{M_2 - M_0 z_n^2}{1 - z_n} = \frac{1}{\frac{M_0}{M_1} - 1} \left(\frac{M_0 M_2}{M_1} - M_1 \right) = \frac{M_0 M_2 - M_1^2}{M_0 - M_1} \,, \tag{8.15}$$

and

$$\frac{S_u}{S_l} = \frac{M_1}{M_0 - M_1} \,. \tag{8.16}$$

One must consider that the "fictitious" strength of the extreme fibre is equal to the strength of the natural extreme fibre multiplied by E_0/E.

The bending moment with the lower fibre under stress is

$$\frac{\sigma_0\left(1-\sqrt{\frac{\nu_1}{\nu_0}}\right)}{1-\frac{\nu_1}{\nu_0}} S_l = \frac{\sigma_l h^2}{6}. \tag{8.17}$$

Since $h=1$, expressed in z, the lower flexural strength simply becomes

$$\sigma_l = 6\,\frac{\sigma_0\left(1-\sqrt{\frac{\nu_1}{\nu_0}}\right)}{1-\frac{\nu_1}{\nu_0}} S_l. \tag{8.18}$$

Similarly the upper flexural strength is

$$\sigma_u = 6\,\frac{\sigma_0\left(1-\sqrt{\frac{\nu_4}{\nu_0}}\right)}{1-\frac{\nu_4}{\nu_0}} S_u. \tag{8.19}$$

The flexural plate rigidity is

$$D=\frac{Eh^3}{12(1-\mu^2)} \quad \text{or} \quad D=\frac{EJ}{1-\mu^2}, \tag{8.20}$$

for a normal case.

In case of sea ice we must use eq. (8.13) for J and E_0 for E in eq. (8.20). Similar considerations hold for action radius

$$l=\sqrt[4]{\frac{Eh^3}{12\kappa(1-\mu^2)}}, \tag{8.21}$$

and action width

$$l_1=\sqrt{2}\,l. \tag{8.22}$$

The action radius becomes

$$l=\alpha l_0=\alpha\sqrt[4]{\frac{E_0 h^3}{12\kappa(1-\mu^2)}}, \tag{8.23}$$

with

$$\alpha=\sqrt[4]{4(V-V_3)-3\,\frac{(V-V_2)^2}{V-V_1}}, \tag{8.24}$$

if the zero strength layer does not lie within the ice sheet.

Once the action radius is known the behavior of flexural waves can be predicted.

The calculations can be performed in a similar way if the full profile is not available z_0 should lie within 0 and 1. In particular eqs. (8.10)~(8-12) for the moments should be considered.

VIII. Concluding Remarks

The biggest problem is still the failure mechanism. One may assume a finite stress or a function of brine volume which can be absorbed by each layer leading to an elasto-plastic analysis. A more realistic postulate for sea ice, now being tried, is the assumption that partial failure results in the lower layers upon reaching the finite stress without further propagation of a crack through this highly plastic medium. Such an assumption leads to a continuous change in section modulus which can be shown in nomograms. An attempt to derive theoretically time dependent deformation and failure should be made.

The theoretical results can now be compared with numerous tests *in situ* conducted on cantilever and simple beams as well as failure and deflection tests on sheets. Preliminary comparisons have been highly encouraging. The agreement with experience is expected to be better than one would expect for such a highly variable medium as sea ice. Air coupled waves and flexural oscillations can now be derived on a rational basis primarily using equations similar to eqs. (7.19) and (7.20).

Further advances in this direction will depend upon carefully conceived experiments in the laboratory. Some additional thought, are being published in Weeks and Assur (1967).

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