Frequency Dependence of Dynamic Moduli of, and Damping in Snow

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Abstract

The frequency dependence of dynamic properties of the viscoelastic solids is studied based on the theory of linear viscoelasticity. The variation of dynamic moduli, viscosity, and the loss factor with frequency is described under an assumption of linearity in the distribution function of creep curves, and the effects of time limits are studied in detail. Employing a newly developed method of determining the dynamic response of materials, which enables one to obtain the results at any desired frequency, experimental results are obtained from steady state vibration over a large range of frequency. It is shown that the theory developed is approximately obeyed by the test results on snow. In general, the dynamic moduli of snow increase slightly with frequency, the dynamic viscosity shows a decrease, and the loss factor remains more or less constant with frequency. It is shown that the relation of dynamic moduli to frequency can be used to specify the viscoelastic properties of a material.

I. Introduction

A review of literature reveals extensive research conducted in the past on viscoelastic properties of materials in the field of high polymer, but of very little information dealing with snow and ice. Even in the field of polymer research, however, the major efforts seem to have been placed on determination of the viscoelastic properties from the static stress-strain-time relations, and very little has been done in terms of the dynamic properties based on their dependence on excitation frequency.

In the present paper is described a means of determining theoretically the effects of frequency on the dynamic properties of viscoelastic materials, and the experimental results obtained on snow employing a "non-resonance" method are discussed. The first part of the paper is mainly concerned with theoretical methods for determining the frequency-dependent dynamic compliance of the material from the creep function. It is then shown how all other properties of the viscoelastic solids, such as the dynamic modulus of elasticity, can be computed, still using the creep characteristics as the only necessary experimental data. It is quite evident that the relaxation characteristics could have been used just as well, but it was preferred to use the creep function as basic data because determination of the relaxation characteristics is usually more difficult. It is also shown that as far as small deformations are concerned, that is, deformations for which the principle of linear deformation applies, the dynamic modulus or compliance-frequency curve provides a description of all the mechanical properties of viscoelastic materials.

In the second part of the paper is presented the results of preliminary laboratory tests on snow, and the results are discussed in the light of theoretical development described in the first part. A newly developed method of measuring the dynamic response at non-resonant frequencies is also discussed for its experimental validity.

II. Theoretical Considerations

INTRODUCTION

To specify the viscoelastic properties of a material either one of the following three criteria may be used: a creep curve showing the change of strain with time under a constant stress; a relaxation curve showing the change of stress with time under a constant strain; or a dynamic modulus curve showing the variation of the dynamic modulus with frequency. These three criteria are interrelated: if the strain-time relation under alternating stress is known, the results of which may be expressed in terms of the dynamic modulus and damping. In other words, the variation of dynamic modulus and damping can be predicted by knowing or assuming a distribution of relaxation or retardation times, the form of distribution function being given by the stress-strain-time relation. The reverse is true, on the other hand, that the creep or retardation characteristics can be predicted on the basis of a known dynamic modulus curve. In the present study we are primarily concerned with the dynamic modulus and damping characteristics as a function of excitation frequency determined from the stress-strain-time relation, for the relation of the dynamic modulus and damping to frequency is used to specify the viscoelastic properties of snow.

BASIC RELATIONS BETWEEN STRESS AND STRAIN

The behavior of a viscoelastic material is often conveniently represented by a mathematical model, either a Maxwell, a Voigt or any combination of the two. The prediction of a mathematical model to use in advance is a task of extreme difficulty, and the past studies in the field of high polymer (*e.g.* Kolsky, 1960) have shown that most viscoelastic solids do not behave like a single Maxwell or Voigt model, but more complicated models have to be employed to represent the actual behavior of real materials. In the lack of sufficient evidence that this is so with the material like snow and for the sake of simplicity, the present study will be based on the simple Voigt models connected in series.

The stress in a Voigt model may be expressed by

$$\sigma = E\cdot\varepsilon + \eta\dot{\varepsilon}\,, \tag{1}$$

in which ε is strain, $\dot{\varepsilon}$ rate of strain, σ stress, E Young's modulus of the spring, and η is viscosity in the dashpot. The variation of strain with time t under a constant stress may be expressed by

$$\varepsilon = \frac{\sigma}{E}(1-e^{-\frac{t}{\tau}})\,, \tag{2}$$

in which τ is a retardation time and equal to η/E. If a single mechanism with a single τ can not simulate the actual behavior, it must be considered in terms of a series of mechanisms of different τ values. In this case

$$\varepsilon = \sigma \sum^{n} \frac{1}{E_i}(1-e^{-\frac{t}{\tau_i}}), \tag{3}$$

in which n is the number of mechanisms.

The stress-strain relation in the ordinary elastic body is characterized by a modulus of elasticity which is constant at any time of stress application. The stress-strain relation in the viscoelastic body, however, is expressed in complex moduli to account for the dissipation of energy due to the viscosity of the material, which causes the phase difference between the strain and stress. A sinusoidal strain ε produced by the application of alternating stress σ in the steady state may be expressed, neglecting the instantaneous static strain, as

$$\varepsilon = J^*(i\omega)\cdot\sigma, \tag{4}$$

where σ is $\sigma_0 \cdot e^{i\omega t}$,
J^* $J_1(\omega)+iJ_2(\omega)$, called the complex elastic compliance,
ω angular frequency,
σ_0 amplitude of stress.

Application of alternating strain produces in the steady state a sinusoidal stress as expressed by

$$\sigma = E^*(i\omega)\cdot\varepsilon, \tag{5}$$

where $E^*(i\omega)=E_1(\omega)+iE_2(\omega)$, and is called the complex elastic modulus, and $\varepsilon=\varepsilon_0 e^{i\omega t}$ in which ε_0 is the amplitude of strain.

The real part E_1 or J_1 is called the storage modulus or compliance and is related to the elasticity of the material, and is the measure of reversible storage of energy. The imaginary part E_2 or J_2 is called the loss modulus or compliance and is related to the internal friction or damping in the material. The dissipation of energy is expressed in terms of the loss factor, $\tan\delta$, as

$$\tan\delta = \frac{E_2}{E_1} = \frac{J_2}{J_1}, \tag{6}$$

where δ is the phase difference between the stress and strain. The dynamic viscosity is related to the loss factor as follows:

$$\tan\delta = \frac{\eta_1}{\eta_2} = \frac{\omega\eta_1}{E_1} = \frac{E_2}{\omega\eta_2}. \tag{7}$$

Thus,

$$\eta_1 = \frac{E_2}{\omega}; \quad \eta_2 = \frac{E_1}{\omega}. \tag{8}$$

Note all these parameters are frequency dependent and hence the response of a viscoelastic material to forced vibration is also a function of frequency.

All above mentioned equations are particular cases of the general expression which gives the relations between stress, strain and time based on well known Boltzman's principle of superposition (Kolsky, 1960):

$$\varepsilon(t) = \frac{\sigma(t)}{E} + \frac{1}{E}\int_{-\infty}^{t} \frac{d\sigma(t)}{d\tau}\psi(t-\tau)\,d\tau, \tag{9}$$

where $\psi(t)$ is creep function of material,
$\sigma(t)$ $\sigma_0 e^{i\omega t}$,
$\frac{\sigma(t)}{E}$ strain due to elastic property.

The expression for $\sigma(t)$ may also be derived in a similar form using the relaxation function $\bar{\varphi}(t)$ instead of $\psi(t)$. These two equations are mathematically equivalent. Boltzman's principle of superposition gives the most satisfactory solution of a linear viscoelasticity. However, the determination of creep function $\psi(t)$ or relaxation function $\bar{\varphi}(t)$ over a large range of time is difficult and so is the integral when the functions are known. Consequently, alternative forms of stress-strain relation must be used to overcome the difficulty.

The creep function $\psi(t)$ is a continuous uniformly increasing function and can be represented by the following integral for a series of Voigt models:

$$\psi(t)=\int_0^\infty D(\tau)[1-e^{-\frac{t}{\tau}}]d\tau\,, \tag{10}$$

where $D(\tau)$ is termed the distribution function of retardation times. The complex compliance function J^* can be expressed in terms of the creep function as

$$J^*=\int_0^\infty e^{i\omega\tau}\frac{d\psi(\tau)}{d\tau}d\tau\,. \tag{11}$$

Differentiating eq. (10) with respect to time gives

$$\frac{d\psi(t)}{dt}=\int_0^\infty\frac{D(\tau)}{\tau}e^{-\frac{t}{\tau}}d\tau\,.$$

Substituting into eq. (11)

$$J^*=\iint^\infty e^{-i\omega\tau}\frac{D(\tau)}{\tau}e^{-\frac{t}{\tau}}d\tau\,d\tau$$

$$=\int_0^\infty\frac{D(\tau)}{1+i\omega\tau}d\tau\,. \tag{12}$$

In order to evaluate the integrals in eqs. (10) and (12), a definite distribution must be known. One way of analysis would be to write down an analytical expression for the distribution function then compare with the measured curves. The second method consists of deducing the distribution function from the experimentally obtained curves.

In the present study the distribution is assumed to be linear between two limits of time for the general case as prescribed below:

General case:

$$D(\tau)=0 \qquad \tau<\tau_1\,,$$
$$D(\tau)=\beta/\tau \qquad \tau_1<\tau<\tau_2\,,$$
$$D(\tau)=0 \qquad \tau>\tau_2\,,$$

and for a special case:

$$D(\tau)=0 \qquad \tau<\tau_1\,,$$
$$D(\tau)=\beta/\tau \qquad \tau_1<\tau<\infty\,.$$

This assumption of linearity is based on the fact that: (1) it is one of the simplest assumptions leading to an easy solution of the integrals, and (2) such an assumption has been proved by experiment to be valid for ice and snow by Jellinek and Brill (1956). It was shown that most of the creep curves for ice and snow could be expressed in the form of $\varepsilon = A \cdot t^r$, where A and r are constants depending on temperature, density and applied stress, and that theoretical computations of $\varepsilon(t)$ based on the assumed creep function agreed closely with the experimental results.

DYNAMIC MODULUS AND COMPLIANCE-FREQUENCY RELATIONS

In the present study the relation of frequency to the dynamic properties is investigated on the basis of linearity in the material behavior, which is usually expected for small deformations or small rate of deformation. A study of the viscoelastic properties based on the non-linearity is delegated to future research.

i) *General case*

Previously the distribution of creep function was assumed to be linear between the two limits of time. Substituting this distribution function into eq. (10) we obtain for general case

$$\varepsilon(t) = \beta \int_{\tau_1}^{\tau_2} \frac{(1-e^{-\frac{t}{\tau}})}{\tau} d\tau \tag{17}$$

$$= \beta \left[\ln\left(\frac{\tau_2}{\tau_1}\right) + \mathrm{Ei}\left(-\frac{t}{\tau_1}\right) - \mathrm{Ei}\left(-\frac{t}{\tau_2}\right) \right]. \tag{17'}$$

Here β and τ_2 must be chosen to give the best fit to the experimental curve; τ_1 is indeterminate and several values of τ_1 must be considered to obtain the best correlation between the theory and experiment.

It follows from eq. (12) that

$$J^* = \beta \ln \left[\frac{\tau_2}{\tau_1} \cdot \frac{1+i\omega\tau_1}{1+i\omega\tau_2} \right]. \tag{18}$$

Decomposing into real and imaginary component

$$J^* = \frac{\beta}{2} \left[\ln \frac{\tau_2^2}{\tau_1^2} \cdot \frac{1+\omega^2\tau_1^2}{1+\omega^2\tau_2^2} \right] - i\beta \left[\tan^{-1}(\omega\tau_2) - \tan^{-1}(\omega\tau_1) \right], \tag{19}$$

which gives

$$J_1 = \frac{\beta}{2} \left[\ln\left(\frac{\tau_2^2}{\tau_1^2}\right) + \ln\left(\frac{1+\omega^2\tau_1^2}{1+\omega^2\tau_2^2}\right) \right], \tag{20}$$

$$J_2 = -\beta \left[\tan^{-1}(\omega\tau_2) - \tan^{-1}(\omega\tau_1) \right]. \tag{21}$$

The complex modulus E^* can be obtained from the complex compliance J^* as E^* is the reciprocal of J^*. Thus, from eq. (18)

$$E^* = \frac{1}{\beta} \frac{1}{\ln\left[\frac{\tau_2}{\tau_1} \cdot \frac{1+i\omega\tau_1}{1+i\omega\tau_2} \right]}. \tag{22}$$

Separating into real and imaginary part

$$E_1 = \frac{1}{\beta} \frac{\frac{1}{2}\left[\ln \frac{\tau_2^2}{\tau_1^2} + \ln \frac{1+\omega^2\tau_1^2}{1+\omega^2\tau_2^2} \right]}{\left\{ \frac{1}{2} \left[\ln \frac{\tau_2^2}{\tau_1^2} + \ln \frac{1+\omega^2\tau_1^2}{1+\omega^2\tau_2^2} \right] \right\}^2 + \left[\tan^{-1}(\omega\tau_2) - \tan^{-1}(\omega\tau_1) \right]^2}, \tag{23}$$

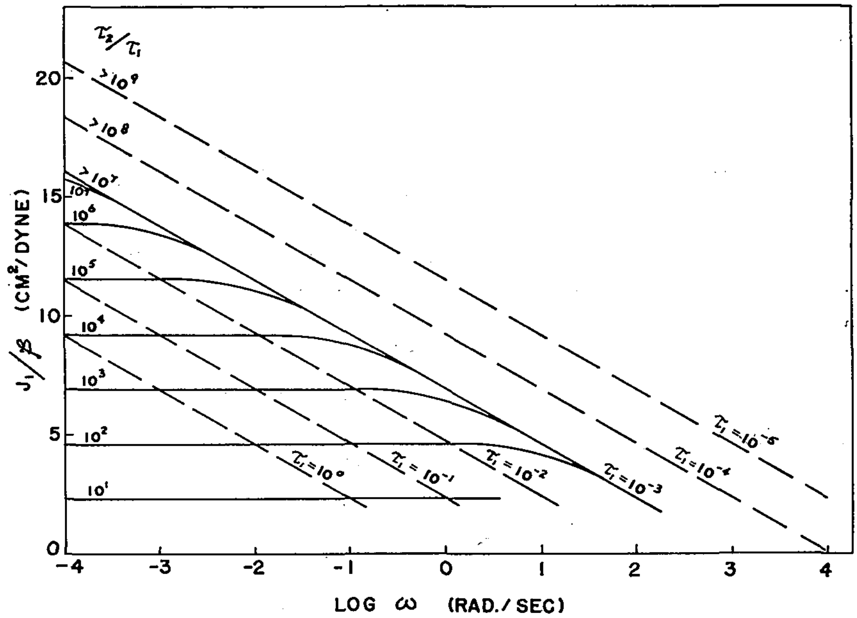

Fig. 1. Storage compliance vs. frequency at various time limits

$$E_2=\frac{1}{\beta}\,\frac{\tan^{-1}(\omega\tau_2)-\tan^{-1}(\omega\tau_1)}{\left\{\frac{1}{2}\left[\ln\frac{\tau_2^2}{\tau_1^2}+\ln\frac{1+\omega^2\tau_1^2}{1+\omega^2\tau_2^2}\right]\right\}^2+\left[\tan^{-1}(\omega\tau_2)-\tan^{-1}(\omega\tau_1)\right]^2}\,. \tag{24}$$

This shows that the complex modulus can be computed not only from the relaxation function (Maxwell model) but also from the creep function (Voigt model) as well.

The digital computer was used to compute J_1 and J_2 as a function of frequency for $10^{-5}\leq\tau_1\leq 10^0$ and for various ratios of τ_2/τ_1. The results are presented in Figs. 1 and 2. It is observed in Fig. 1 that if $\omega\tau_1\ll1$ and $\tau_2>10^4$, then J_1 decreases linearly with log of frequency. In the frequency range of $\omega\tau_1\gg1$ the compliance gradually approaches zero. If τ_2/τ_1 is smaller than 10^4, then J_1 is constant over the range of frequency $\omega\tau_2\ll1$. Beyond this frequency it decreases slowly with increasing frequency merging into the $\tau_2\rightarrow\infty$ envelope. Figure 2 shows that the variation of J_2 with frequency has a form of wave, that is, there is a peak (maximum value of J_2) at a certain frequency or at a frequency range, and J_2 decreases sharply before and after the peak. The sharpness of peak is depended upon the τ_2/τ_1 ratio; as the ratio increases J_2 becomes constant at $\pi/2$ over the range of frequency shown.

The results of computation for the storage modulus E_1 and the dynamic viscosity η as a function of frequency for the values of $\tau_1=10^{-3}$ and $\tau_1=10^{-5}$ have been also obtained from the creep function rather than the relaxation function as described previously. It is noted that E_1 is constant over the range of frequency $\omega\tau_2\ll1$. Beyond this frequency, E_1 increases slowly and then levels off again to resume a constant value. It has been shown also that for a high τ_2/τ_1 ratio the dynamic viscosity decreases with frequency, and when plotted on log-log scale the variation is characterized by a straight line over most of the frequency range considered.

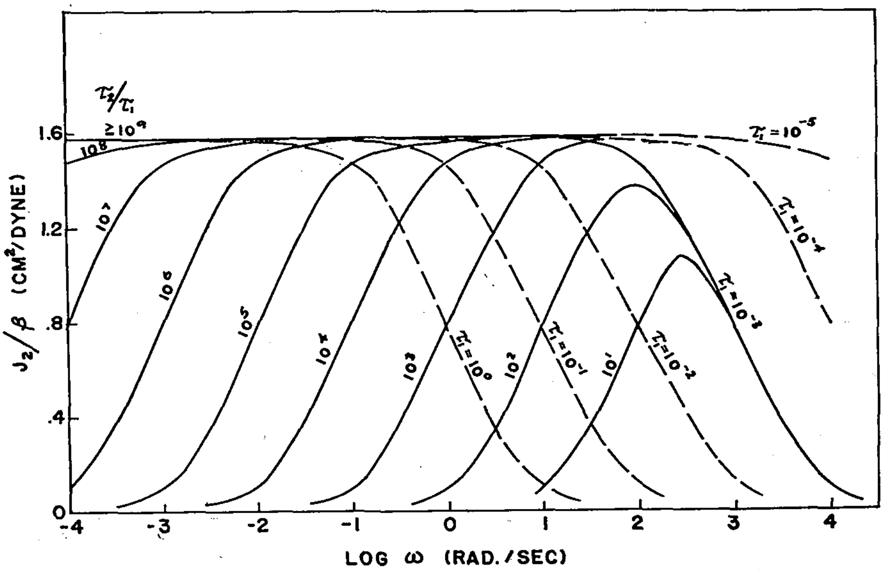

Fig. 2. Loss compliance vs. frequency at various time limits

ii) *Special case*

The special case is when τ_2 is assumed to approach infinity in the prescribed boundary condition. In this case the variation of strain with time can be expressed by

$$\varepsilon(t)=\beta\left[0.5772+\ln\left(\frac{t}{\tau_1}\right)-Ei\left(-\frac{t}{\tau_1}\right)\right], \tag{25}$$

and the complex compliance J^* is

$$J^*=\beta\ln\left[1-i\left(\frac{1}{\omega\tau_1}\right)\right]. \tag{26}$$

Decomposing into real and imaginary component

$$J^*=\frac{\beta}{2}\ln\left(1+1/\omega^2\tau_1^2\right)-i\beta\tan^{-1}(1/\omega\tau_1), \tag{27}$$

Thus

$$J_1=\frac{\beta}{2}\ln\left(1+1/\omega^2\tau_1^2\right),$$

$$J_2=-\beta\tan^{-1}(1/\omega\tau_1). \tag{28}$$

From eq. (26) the complex modulus becomes

$$E^*=\frac{1}{\beta\ln\left[1-i\left(\dfrac{1}{\omega\tau_1}\right)\right]}, \tag{29}$$

and then

$$E_1=\frac{1}{\beta}\,\frac{\dfrac{1}{2}\ln\left(1+1/\omega^2\tau_1^2\right)}{\left[\dfrac{1}{2}\ln\left(1+1/\omega^2\tau_1^2\right)\right]^2+\left[\tan^{-1}(1/\omega\tau_1)\right]^2}, \tag{30}$$

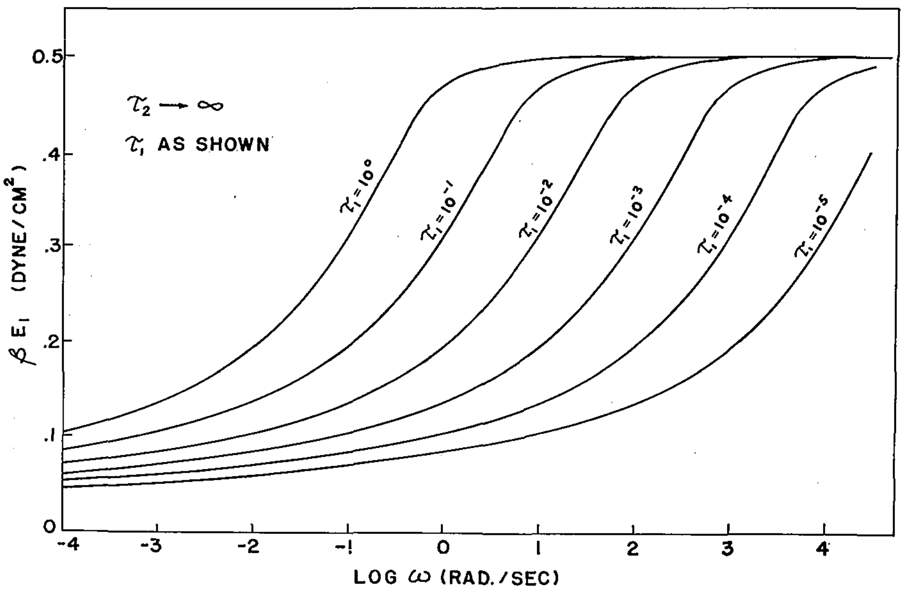

Fig. 3. Storage modulus vs. frequency at various time limits

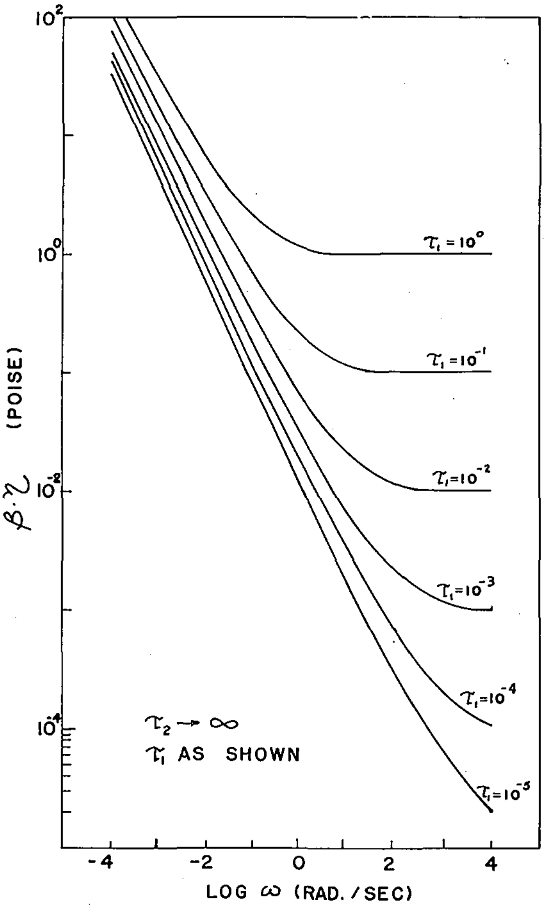

Fig. 4. Dynamic viscosity vs. frequency at various time limits

$$E_2 = \frac{1}{\beta} \frac{\tan^{-1}(1/\omega\tau_1)}{\left[\frac{1}{2}\ln(1+/\omega^2\tau_1^2)\right]^2 + \left[\tan^{-1}(1/\omega\tau_1)\right]^2} . \tag{31}$$

The results of computation for E_1 and η are shown in Figs. 3 and 4, in which the variation of E_1 and η is plotted against the log of frequency for various values of τ_1. It is observed in Fig. 3 that E_1 varies very slightly with frequency in the very low frequency range, then changing into a slowly increasing curve in the region $\omega\tau_1 \ll 1$. In the region of $\omega\tau_1 \gg 1$ it assumes a constant value. It is of interest to note that the increase of E_1 ranges from 5 to 10 times over the range of frequency considered, depending on the τ_1 value. In other words, the dynamic modulus E^* (when neglecting small E_2) would never exceed the static elasticity modulus E by more than approximately 10 times. Figure 4 shows the effect of frequency on the dynamic viscosity η. In the frequency range of $\omega\tau_1 \ll 1$ the decrease of η with frequency is shown to be linear when plotted on the log-log scale, and may be expressed approximately by $1/10^2 \cdot \omega$. In the high frequency range, $\omega\tau_1 \gg 1$, η is constant and is equal to $1/\omega$. This implies that the loss factor $\tan\delta$ is more or less constant or increases slightly with frequency in the range of $\omega\tau_1 \ll 1$ since $\tan\delta = \omega\eta_1/E_1$. In the region of $\omega\tau_1 \gg 1$ the loss factor should increase rapidly with frequency because E_1 increases very slightly, if not at all, with rapidly increasing frequency ω while η remains constant.

The range of frequency usually encountered by the engineer in practice of snow mechanics and foundation engineering is about up to 5 000 cycles per second. In order to see the effect of frequency on the viscoelastic properties of the materials in this frequency range, Figs. 5, 6 and 7 have been prepared. In these figures E^*, η, and $\tan\delta$ are plotted against frequency for the τ_1 values of 10^{-6}, 10^{-8} and 10^{-30} for $\tau_2 = \infty$. Note

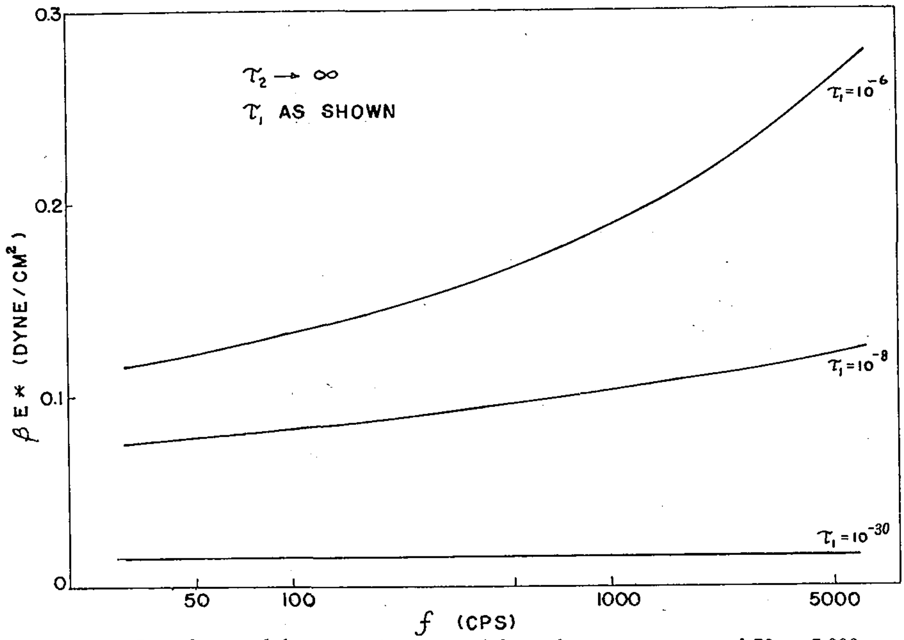

Fig. 5. Complex modulus at various τ_1 and for a frequency range of 50 to 5 000 cps

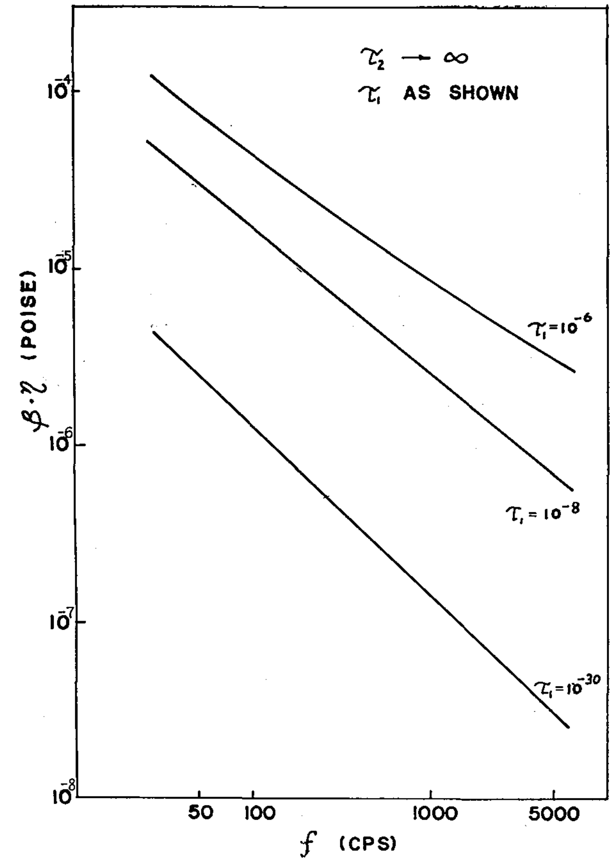

Fig. 6. Dynamic viscosity at various τ_1 and for a frequency range of 50 to 5 000 cps

that $\tau_1=10^{-30}$ shows the condition of $\tau_1 \to 0$. It is seen in Fig. 5 that E^* is a slowly increasing log function of frequency, and over a small range of frequency, say one log cycle, the variation is practically linear with log of frequency. For the value of $\tau_1=10^{-30}$ the variation is almost negligible. The decrease of dynamic viscocity η with log of frequency is practically linear for the frequency range considered. The variation of loss factor $\tan\delta$ is similar to that of the complex modulus E^*: it is a slowly increasing logarithmic curve with frequency, except for very small τ_1, for which the change is insignificant.

Above illustrations show that if there is no upper limit (special case) in $\psi(t)$-log t curve, the relation can be expressed as linear and J^* decreases linearly with log of frequency and the rate of decrease is dependent upon $D(\tau)$ or the slope of $\psi(t)$-log t curve. On the other hand, if there is an upper limit (general case), J^* remains constant for the frequency range $\omega\tau_1 \ll 1$, and decreases at frequencies larger than $\omega\tau_1=1$.

III. Experimental Results

THEORY AND TECHNIQUE EMPLOYED FOR EXPERIMENT

The method that has been used most extensively in the past to measure the dynamic

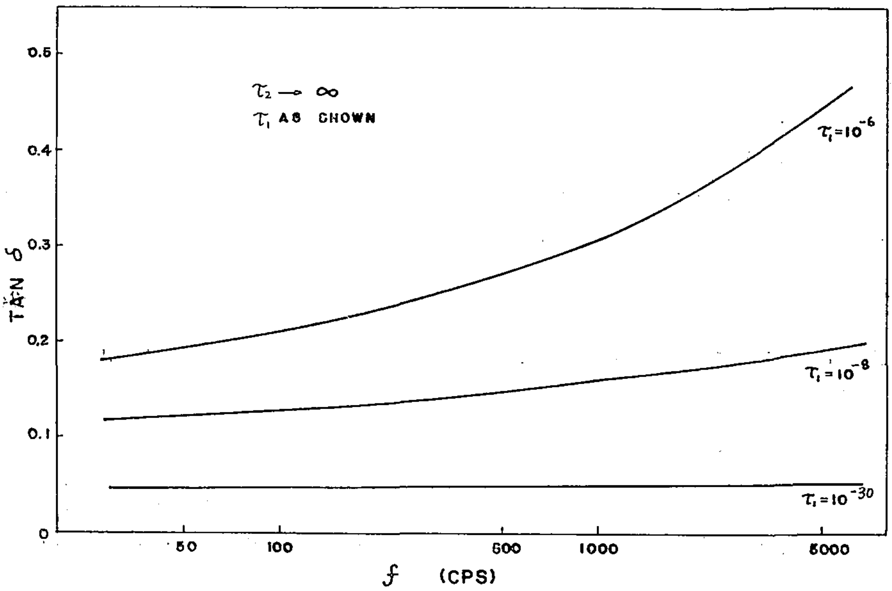

Fig. 7. Loss factor at various τ_1 and for a frequency range of 50 to 5 000 cps

response of materials under forced vibration is that of "resonance", in which a cylindrical column of the material is set into resonance usually by steady-state vibration, and the wave velocities and elastic properties are computed from the dynamic response of the sample. In this method, however, the analysis of test results is rather complicated because the response obtained is not that of the specimen alone but that of a composite body consisting of the specimen and the attached apparatus. The "resonance" method, however, is totally inadequate if the relation of frequency to the dynamic properties is to be determined. The only means of changing frequency in this method would be by varying the length of specimen, which, in cases of some materials, can not be carried out.

The method used in this experiment is based on a concept of the "amplitude ratio" developed originally by Lee (1963). In this method a criterion of amplitude ratio, which is the ratio between the displacement amplitude at the top and the bottom of the specimen, is used rather than the maximum amplitude as in the case of resonance method. There are two distinct adventages of the "amplitude ratio" method over the resonance method: first, it eliminates the difficulty resulting from the coupling of test apparatus to the specimen; secondly, measurements can be made at any desired frequency, including the resonant frequency, in contrast to the resonance method whereby the measurement can only be made at the resonant frequency.

In the "amplitude ratio" method measurements for the amplitude ratio R and the phase shift θ between displacements at the top and the bottom of the specimen are required. These two quantities, R and θ were expressed originally by Lee as follows:

$$R=\left[\sinh^2(\xi k)+\cos^2\xi\right]^{-\frac{1}{2}}, \tag{32}$$

$$\theta = \tan^{-1}\Big[\tan\xi\cdot\tanh(\xi k)\Big], \tag{33}$$

in which
- ξ is dimensionless frequency factor $= \omega l/v$,
- k coefficient of damping $= \tan\delta/2$,
- l length of specimen,
- v wave velocity, shear or longitudinal.

Rearranging eq. (33) and substituting into eq. (32) gives

$$R = \left[\sinh^2(\xi k) + \frac{\tanh^2(\xi k)}{\tanh^2(\xi k)+\tan^2\theta}\right]^{-\frac{1}{2}}. \tag{34}$$

From this equation ξk can be computed by substituting the measured values of R and θ. Then knowing ξk, ξ can be computed from eq. (33) as

$$\xi = \frac{\omega l}{v} = \tan^{-1}\left[\frac{\tan\theta}{\tanh(\xi k)}\right]. \tag{35}$$

The above equation for ξ is valid only for the range of $0\leq\theta\leq\pi/2$. To accomodate the phase angle greater than $\pi/2$ eq. (35) has been amended by this author to read:

$$\xi = n\pi + \tan^{-1}\left[\frac{\tan\theta}{\tanh(\xi k)}\right], \tag{36}$$

for the θ values in the first and third quadrants, and

$$\xi = (n+1)\pi - \tan^{-1}\left[\frac{\tan\theta}{\tanh(\xi k)}\right], \tag{37}$$

for the θ values in the second and fourth quadrants. In these equations n is a factor depending on the measured θ value as shown below:

θ	$0\to\pi$	$\pi\to 2\pi$	$2\pi\to 3\pi$	$\cdots\ \cdots$
n	0	1	2	$\cdots\ \cdots$

Once ξk and ξ are determined, the complex modulus E^* and the coefficient of damping, $\tan\delta/2$, can be computed from

$$\tan\frac{\delta}{2} = \frac{\xi k}{\xi}, \tag{38}$$

$$E^* = \frac{v_l^2\cdot\rho}{1+\tan^2\delta/2}, \tag{39}$$

where $$v_l = \omega l/\xi = 2\pi f l/\xi.$$

At the frequency of fundamental mode it is found from eq. (32) that

$$v_l = 4f_1 l\left(1+\tan^2\frac{\delta}{2}\right),$$

where f_1 is the frequency of fundamental mode, and at this frequency

$$\tan\frac{\delta}{2} = \frac{2}{\pi R}. \tag{40}$$

From eqs. (39) and (40) the dynamic modulus at the resonant frequency can be expressed as

$$E^* = 16 f_1^2 l^2 \rho \left(1 + \tan^2 \frac{\delta}{2}\right). \qquad (41)$$

Thus it is seen that the case of fundamental mode in this method of measurement is only a special case of the entire frequency range. By measuring the amplitude ratio R and the phase shift θ under torsional oscillation the complex shear modulus G^* may be obtained in the same manner.

TEST ON SNOW

i) *Testing apparatus*

Tests on snow were conducted with an apparatus developed by N. Smith (1964) in the Army Cold Regions Research and Engineering Laboratory (CRREL), Hanover, N.H., U.S.A. The test apparatus, as shown in Fig. 8 consisted of a test stand, an exciter, accelerometers, a voltmeter, a frequency meter, and various amplifiers. The test stand

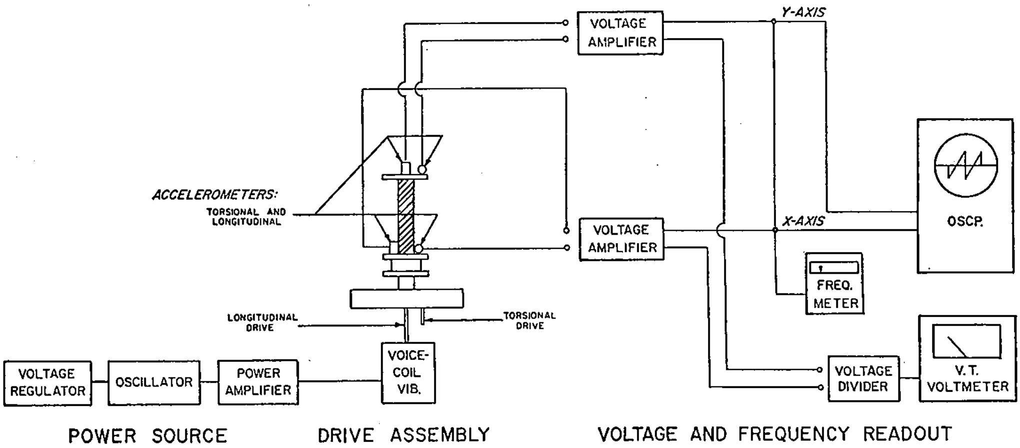

Fig. 8. Dynamic testing apparatus

supports the exciter on the lower platform and the sample vibration table assembly on the upper platform. Vibrations are induced in the sample by the exciter which is driven by an audio-oscillator having a frequency range of 20 to 40 000 cps. Displacements induced by propagated stress waves in the sample were measured with two piezoelectric accelerometers, one at the top and the other at the bottom of the specimen. Voltage readings were made with a sensitive electronic voltmeter and frequency readings by a frequency-meter and discriminator. A decade voltage divider having a voltage ratio range of 0.0001 to 1.0 in steps of 0.0001 was used to measure the ratio of voltage from the bottom accelerometer to that of the top accelerometer. Various amplifiers were used to amplify the output signals from the accelerometers and the oscillator.

ii) *Sample*

The snow used was a laboratory processed snow having an average density of 0.48 g/cm^3. Cylindrical samples were prepared from blocks of snow by a snow sample-cutter. The samples were cut to a nominal 2 inch diameter and had the length varying from 9 to 18 inches. One end of the specimen was slightly melted on a hot-plate and immediately set upright on the metal vibration platform to form a tight bond upon freezing. Two accelerometers were bonded to the specimen in the similar manner. The test room temperature was manitained approximately constant at 25°F. Measurements of the dynamic response were made at various frequencies, which were obtained by either cutting the specimen length with the measurement made at the frequency of fundamental mode, or by making measurements at the frequencies of subsequent harmonics of third, fifth and so on. The measurements at any other frequencies were not attempted for wanting to avoid the use of a phase-meter. The range of frequency tested was from about 800 to 6 000 cps as the limitation of test apparatus restricted the tests to this range.

iii) *Test results and discussion*

In discussing the test results obtained for the dynamic response of a viscoelastic material one must be aware of the fact that many variables are involved in a single test and these variables are interrelated to each other. The major variables to be considered are temperature, density, confining pressure, and dynamic loading conditions such as frequency and input force. Consequently, when the effect of frequency on the dynamic properties of a material alone is considered one must assume that the other variables are kept constant and that the interrelationships among these variables are known. According to previous works (J. Smith, 1964) for instance, that the increase of complex modulus E^* with increasing density of snow can empirically be shown to be $E^*=a\cdot\rho^b$, where a and b are constants.

The results of test on snow are shown in Figs. 9, 10 and 11. Figure 9 shows the variation of complex modulus E^* with frequency. It is observed that the elasticity modulus increases linearly with the logarithm of frequency, and the increase over the

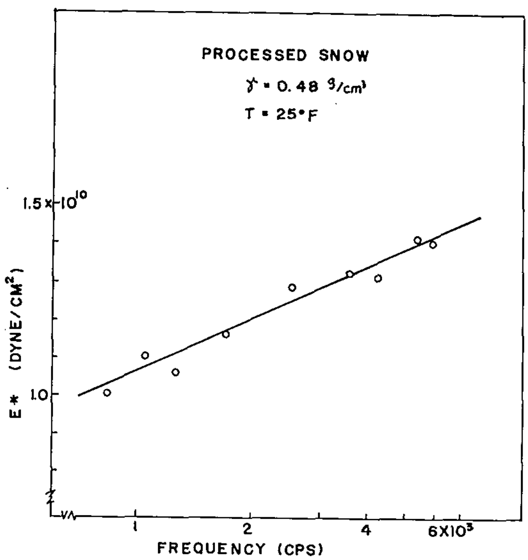

Fig. 9. Loss factor vs. frequency

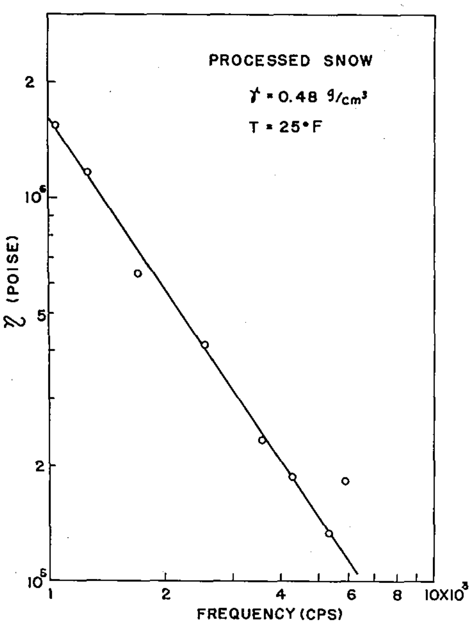

Fig. 10. Dynamic viscosity vs. frequency

entire frequency range considered is about 50%. This linear relation between the modulus and frequency agrees closely with the theoretical results derived on the basis of boundary conditions set forth previously. It was stated in conjunction with Fig. 5 that the modulus E^* is a slowly increasing function of frequency, and over a small range of frequency, say one log cycle, the variation is practically linear with the logarithm of frequency. It may be recalled here that the assumption of a constant β was also shown to be valid experimentally to express the strain-time curve of ice and snow. Consequently, the assumption of linearity in the distribution function is valid for snow, and therefore, the value of β could be used to specify the viscoelastic properties of snow.

The logarithm of dynamic viscosity against the logarithm of frequency is plotted in Fig. 10, which shows a linear relation. This linear relation was theoretically predicted (Fig. 6) and again the correlation betweeen the theory and experiment is reasonably

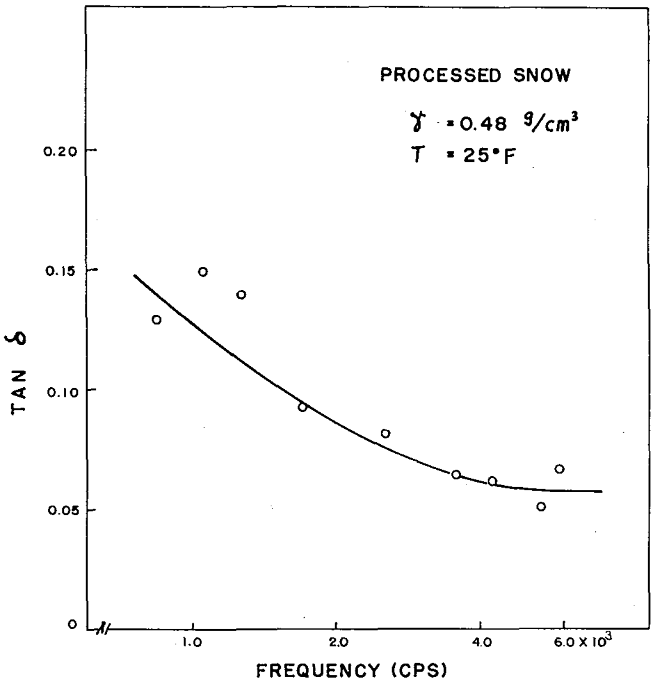

Fig. 11. Complex modulus vs. frequency

good. In Fig. 11 is plotted a curve showing the variation of the loss factor $\tan\delta$ with frequency. The loss factor is shown to be decreasing with increasing frequency, approaching a constant value as the frequency is further increased. Theoretically, however, the loss factor was shown to increase with frequency (Fig. 7) except for very small τ_1, for which case the change was almost negligible. The decrease of loss factor with frequency in the test results may be explained in the light that in the test the decrease of viscosity with frequency $\omega\eta_1$ was not as fast as that of the dynamic modulus E_1 in the equation, $\tan\delta = \omega\eta_1/E_1$, whereas in the theory $\omega\eta_1$ was shown to vary faster with frequency than, if not equal to, E_1.

IV. Conclusions

The variation of dynamic moduli, compliances, viscosity, and the loss factor is shown to be frequency dependent, and the variation of each of these quantities with frequency has been demonstrated on the basis of the theory of linear viscoelasticity. An assumption of linearity between two time limits in the distribution of creep function shows that the variation of dynamic modulus E^* increases with frequency, and is linear over a small range of frequency. The dynamic viscosity decreases with frequency, and varies, in most cases, linearly with frequency when plotted on log-log scale. The variation of loss factor with frequency is controlled by the rate the modulus and viscosity change with frequency.

The method of determining the dynamic properties of viscoelastic materials by measuring the "amplitude ratio" has been proved to be experimentally valid and practical. This method has two distinct advantages over the commonly used "resonance" method in that the dynamic measurements can be made at any desired frequency, and it eliminates the difficulty resulting from the composite body vibration in the conventional method.

The results of preliminary tests show that the dynamic modulus E^* of snow increases linearly with the logarithm of frequency. A linear relation is also obtained between log of the dynamic viscosity and log of frequency. The loss factor shows a decrease with frequency, and this is attributed to a faster change of $\omega\eta_1$ with frequency than that of E_1. These findings closely verify the theory of linear viscoelasticity, in which the distribution of creep function is assumed to be linear between two time limits.

It has been demonstrated both theoretically and experimentally that curves showing the relation between the dynamic moduli and frequency can be used as a criterion to specify the viscoelastic properties of a material. This curve also reveals information on the basic stress-strain-time relation in the material, which is another way of specifying the viscoelastic characteristics of the material.

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