<table>
<thead>
<tr>
<th>Title</th>
<th>NORMAL VIBRATIONS OF SODIUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>TOYA, Tomiyuki</td>
</tr>
<tr>
<td>Citation</td>
<td>JOURNAL OF THE RESEARCH INSTITUTE FOR CATALYSIS HOKKAIDO UNIVERSITY = 北海道大学触媒研究所紀要</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1958-10</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/24680">http://hdl.handle.net/2115/24680</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin</td>
</tr>
<tr>
<td>File Information</td>
<td>6(2)_P183-195.pdf</td>
</tr>
</tbody>
</table>
NORMAL VIBRATIONS OF SODIUM*)

By

Tomiyuki TOYA**)

(Received September 15, 1958)

§ 1. Introduction

Frequencies of the normal vibrations of a metal are usually calculated by assuming short range forces alone, taking interactions of neighbouring ions only into account. Force constants are introduced as parameters, which are adjusted to the observed elastic constants or the Debye temperature.

This treatment is not, however, very reliable as exemplified by its application to the computation of the frequencies of Cu. JACOBSEN has evaluated nine force constants according to the theory of BORN and BECHIE between nearest, second nearest and third nearest neighbouring ions of Cu from the frequency versus wave vector relations determined by the temperature-diffuse X-ray scattering. The force constants between nearest and third nearest neighbours have thus been found of the same order of magnitude, indicating the longe range character of the forces between ions in metals.

The potential between ions in a metal is the sum of (i) the coulomb repulsion $v_c$, (ii) the interaction induced by valence electrons $v_E$, inclusive of the electron-electron interactions, and (iii) the sum $v_R$ of the exchange repulsion and van der Waals attraction. The $v_c$ and $v_E$ are those of long range force so that all interactions of ions in the distance must be reckoned except in the case of the complete shielding. However, the compensation by valence electrons for the perturbing ionic field accompanying lattice vibrations is not complete, so that the resultant perturbing field is of primary importance, as discussed in the foregoing paper.

The frequencies of lattice vibrations of a metal may be evaluated without introducing any arbitrary parameters, if the contributions $v_c$, $v_E$ and $v_R$ to the adiabatic potential are computed directly. The con-

*) Supported in part by the Grant in Aid of the Fundamental Research of the Ministry of Education,

**) Research Institute for Catalysis, Hokkaido University, Sapporo.
tribution $v_C$ may be computed by the Ewald's method\footnote{Ewald's method} and that $v_n$, which decreases very rapidly with distance, by taking only the neighbouring ions into account. The contribution $v_x$ has been derived in the foregoing paper\footnote{WIGNER, SEITZ\textsuperscript{6} and FUCHS\textsuperscript{3}}, by the extention of the Hartree-Fock method to the electron-lattice vibration interaction. The frequencies of the normal vibrations of Na which propagate along the three principal directions are thus calculated in the present paper and compared with those from usual calculations mentioned above. The appropriate elastic constants at the long wave-length limit are found coincident with those derived by WIGNER, SEITZ\textsuperscript{6} and FUCHS\textsuperscript{3}, while the formulation is verified for short wave-length.

§ 2. The Equations for the Normal Modes and Frequencies

The displacement $u_i(u_x, u_y, u_z)$ of an ion from the equilibrium position $R_i$, accompanying the lattice vibration of wave number $q$, is given as

$$u_i = N^{-1/2} e_q \left\{ a_q \exp (i q R_i) + a_q^* \exp (-i q R_i) \right\}, \quad (2.1)$$

where $a_q$ is the amplitude of the wave, $e_q$ its polarization vector and $N$ is the number of ions in unit volume. The adiabatic potential $\delta \mathcal{W}$ for the vibration is

$$\delta \mathcal{W} = \frac{1}{4} \sum_{i \neq i'} \sum_{x y} \left\{ v(R_i - R_{i'}) \right\}_{x y} (u_{ix} - u_{i'x})(u_{iy} - u_{i'y}). \quad (2.2)$$

The $v(R_i - R_{i'})$ in Eq. (2.2) is the interaction potential between ions at positions $R_i$ and $R_{i'}$, which consist of three parts, i.e.,

$$v = v_x + v_c + v_n,$$

where $v_x, v_c$ and $v_n$ are defined in §1, and the subscripts $x y, etc.$ of $\{v(R=(X, Y, Z))\}_{x y}$ signify partial differential coefficient $\partial v / \partial X \partial Y, etc.$ The dipole-dipole or dipole-quadrupole interactions between ion-cores are neglected in comparison with the exchange repulsions.

Defining $v_x(R_i=0) + v_c(R_i=0)$ and $v_n(R_i=0)$ as

$$\sum_i \left\{ v_x(R_i) + v_c(R_i) \right\}_{x y} = 0 , \quad (2.3a)$$
$$\sum_i \left\{ v_n(R_i) \right\}_{x y} = 0 , \quad (2.3b)$$

we have from Eq. (2.2)
Normal Vibrations of Sodium

\[ \delta \wedge \nu = \frac{1}{2} \sum \sum \{v (R_{i} - R_{j}) \}_{xy} u_{ix} u_{iy}, \quad (2.4) \]

or, substituting Eq. (2.1) for \( u_{i} \) into Eq. (2.4),

\[ \delta \wedge \nu = \delta \wedge \nu^{E} + \delta \wedge \nu^{C} + \delta \wedge \nu^{R} \]

\[ \delta \wedge \nu^{E} = - \frac{1}{2} |a_{i}|^{2} \sum \sum \{[xy]^{E} + [xy]^{C}\} e_{xi} e_{yi}, \quad (2.5E) \]

\[ \delta \wedge \nu^{C} = - \frac{1}{2} |a_{i}|^{2} \sum \sum \{[xy]^{E} + [xy]^{C}\} e_{xi} e_{yi}, \quad (2.5C) \]

\[ \delta \wedge \nu^{R} = - \frac{1}{2} |a_{i}|^{2} \sum \sum \{[xy]^{E} + [xy]^{C}\} e_{xi} e_{yi}, \quad (2.5R) \]

where coupling coefficients \([xy]^{E}, etc.,\) are given as,

\[ [xy]^{E} = \sum \{v_{E}(R_{i})\}_{xy} \exp (iqR_{i}), \quad (2.6E) \]

\[ [xy]^{C} = \sum \{v_{C}(R_{i})\}_{xy} \exp (iqR_{i}), \quad (2.6C) \]

\[ [xy]^{R} = \sum \{v_{R}(R_{i})\}_{xy} \exp (iqR_{i}). \quad (2.6R) \]

For a crystal of cubic symmetry, \([xy]^{E} = [xy]^{C}, etc.,\) and the equations for the normal modes of vibration of wave number vector \( q \) are given from Eq. (2.5) as (\( M: \) the atomic mass)

\[ [xx] e_{xx} + [xy] e_{xy} + [xz] e_{xz} = - Mo\omega^{2} e_{xx}, \]

\[ [yx] e_{yx} + [yy] e_{yy} + [yz] e_{yz} = - Mo\omega^{2} e_{yy}, \]

\[ [zx] e_{zx} + [zy] e_{zy} + [zz] e_{zz} = - Mo\omega^{2} e_{zz}, \quad (2.7) \]

where \([xy]^{E}+[xy]^{C}+[xy]^{R}\) and the circular frequencies \( \omega \) are given by the secular equation


Eqs. (2.7) and (2.8) are simplified by the geometrical symmetry for the waves propagating along the principal directions, i.e., \( q = (q, 0, 0), q = (q, q, 0) \) and \( q = (q, q, q). \)

(i) \( (q, 0, 0) \); in this case \([yy] = [zz]\) and \([xy] = [yz] = [xz] = 0\). The modes and their frequencies are given as,

\[ (L) \quad e_{q} = (1, 0, 0), \quad Mo^{2} = -[xx]. \quad (2.9L) \]
\( (T_1) \ e_\omega = (0, 1, 0), \ M_\omega^s = -[yy], \quad (2.9T_1) \\
(T_2) \ e_\omega = (0, 0, 1), \ M_\omega^s = -[yy]. \quad (2.9T_2) \\
(ii) \ (q, q, 0); \ [xx] = [yy], \ [xz] = [yz] = 0, \text{ and hence,} \\
\( L \) \ e_\omega = (1/\sqrt{2}, 1/\sqrt{2}, 0), \ M_\omega^s = -\{[xx]+[xy]\}, \quad (2.10L) \\
(T_1) \ e_\omega = (1/\sqrt{2}, -1/\sqrt{2}, 0), \ M_\omega^s = -\{[xx]-[xy]\}, \quad (2.10T_1) \\
(T_2) \ e_\omega = (0, 0, 1), \ M_\omega^s = -[zz]. \quad (2.10T_2) \\
(iii) \ (q, q, q); \text{ this case is symmetrical in } x, y, z. \text{ Therefore it follows} \\
\text{that } [xx]=[yy]=[zz] \text{ and } [xy]=[yz]=[xz], \text{ and hence} \\
\( L \) \ e_\omega = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}), \ M_\omega^s = -\{[xx]+2[xy]\}, \quad (2.11L) \\
(T_1) \ e_\omega = (1/\sqrt{2}, -1/\sqrt{2}, 0), \ M_\omega^s = -\{[xx]-[xy]\}, \quad (2.11T_1) \\
(T_2) \ e_\omega = (1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}), \ M_\omega^s = -\{[xz]-[yz]\}. \quad (2.11T_2) \\

The transversal modes \( T_1 \) and \( T_2 \) for \( q=(q, 0, 0) \) and \( q=(q, q, q) \) degenerate \( \text{as readily seen from the symmetry properties.} \)

§ 3. The Coupling Coefficients \([xy]\)

The coefficients \([xy]\) are given explicitly in this section. The \([xy]^\sigma\) is given immediately from Eqs. (4.11), (5.1), (7.3), (7.9) and (7.10) in the preceding paper\(^{eb}\) for the contribution from the valence electrons to the adiabatic potential, as

\[
[xy]^\sigma = [xy]^{\sigma t} + [xy]^{\sigma t} , \quad (3.1)
\]

\[
[xy]^{\sigma t} = Ne\left\{ + \frac{4\pi}{3} \delta_{xy} \right\} , \quad (3.1\text{E1})
\]

\[
[xy]^{\sigma t} = Ne\left\{ -4\pi \sum_a \frac{(q_a+K_a)(q_x+K_{xy})}{|q+K_a|} G(t)^2 F'(t)^{-1} f(t) \right\} , \quad (3.1\text{E2})
\]

where \( K_a(K_{x_n}, K_{y_n}, K_{z_n}) \) is the reciprocal lattice vector and \( t=|q+K_a|/2k_F, \) \( k_F \) being the wave number vector of an electron at the Fermi surface.

With the wave function

\[ \psi(k) = U_\sigma(r) \exp(ikr) \]

of an electron of wave number vector \( k \) and of its Hartree energy \( E_\sigma(k) = E_{\sigma} + (\hbar^2/2m)k^2 \), the functions \( G(t), F(t) \) and \( f(t) \) are given as discussed in the preceding paper\(^{eb}.\)
Normal Vibrations of Sodium

\[ G(t) = \left\{1 + \gamma (V(r) - E_0) \right\} \frac{3}{\beta \zeta} e^{\frac{3}{\beta \zeta} t} g(2k_F r, t), \]  
\[ F(t) = \left\{ (D_e/D)(2/\beta) t^2 + (1 - B e) f(t) \right\}, \]

and

\[ f(t) = \frac{1}{2} + \frac{1 - e^t}{4t} \log \left| \frac{1+t}{1-t} \right|, \]

where \( \gamma = U^2(r) \), \( \beta = e^2 k_F \pi \zeta \), \( \zeta \) the Fermi energy in the Hartree approximation, \( g(x) = 3 \frac{\sin x - x \cos x}{x^3} \), \( V(r) \) the Hartree potential energy at the surface of atomic sphere of radius \( r \), \( D \) or \( D_0 \) the density of states with or without the exchange and correlation energies taken into account, and \( B \), which allows for the effect of the exchange and correlation interactions on the screening field, is assumed to be independent of \( q + K_n \). The function \( G(t) \) should vanish at \( K_n (h \approx 0) \), but that given above is not vanish, though its value at \( t = |K_n|/2k_F \) is so small (\( \approx 0.0005 \)) that we may disregard the deviations.

The \([xy]^c\) is readily formulated by the Ewald's method with the result,

\[ [xy]^c = N e^l \left\{ -G_{xy} + H_{xy} + \left\{ \frac{16}{3\sqrt{\pi}} e^2 \delta_{xy} \right\} \right\}, \]

for (b. c. c.),

\[ [xy]^c = N e^l \left\{ -G_{xy} + H_{xy} + \left\{ \frac{8}{3\sqrt{\pi}} e^2 \delta_{xy} \right\} \right\}, \]

for (f. c. c.),

where

\[ G_{xy} = 4\pi \sum_{\lambda} \left( h_{2x} + \tilde{h}_{2x} \right) \left( h_{2y} + \tilde{h}_{2y} \right) \exp \left\{ -\frac{\pi^2}{4\beta^2} (h + \tilde{q})^2 \right\} \]

and

\[ H_{xy} = \frac{4}{2j} \sum_{l \nu} \left\{ -f(l) \delta_{xy} + g(l) \frac{L_x}{l^2} \cos \pi (\tilde{q}, l) \right\}, \]

for (b. c. c.),

\[ (f. c. c.), \]

with

\[ f(l) = \frac{2}{\sqrt{\pi}} e^{\frac{\epsilon l}{\beta}} \left( \frac{\varphi(\epsilon l)}{\beta} \right), \]

\[ g(l) = \frac{4}{\sqrt{\pi}} e^{\epsilon l} + \frac{6}{\sqrt{\pi}} e^{\epsilon l} \left( \frac{\varphi(\epsilon l)}{\beta} \right), \]

and

\[ \varphi(\epsilon, l) = 1 - \frac{2}{\sqrt{\pi} \xi} \int_{0}^{\xi} e^{-\epsilon t} dt. \]
In the above equations, the lattice point vector $R_i$, the reciprocal lattice point vector $K_h$, and the wave number vector $q$ are expressed as ($2\pi$: the lattice constant)

$$R_i = r_i(l_x, l_y, l_z), \quad l = \sqrt{l_x^2 + l_y^2 + l_z^2}, \quad (l_x, l_y, l_z) \text{ are all odd or all even for b.c.c.}$$

$$K_h = \pi/r_i(h_x, h_y, h_z), \quad (h_x + h_y + h_z) \text{ is even for b.c.c.}$$

and

$$q = \pi/r_i(q_x, q_y, q_z),$$

respectively. The constant $\varepsilon$ should be chosen so that the two series $\sum H$ and $\sum' \varepsilon$ rapidly converge.

The $v_R$ is given, according to Born and Mayer\(^a\),

$$v_R(R_i) = C_n b \cdot \exp \left\{ (2r_R - R_i)/\rho \right\}, \quad (3.6)$$

where $b=10^{-18}$ erg., $\rho=0.345 \cdot 10^{-6}$ cm, $C_n=1.25$ and $r_R=0.875 \cdot 10^{-6}$ cm.

The $[xy]^R$ is expressed from Eqs. (3.6) and (2.6R), as

$$[xy]^R = \frac{C_n b}{\rho} \sum_{l=0} \exp \left\{ (2r_R - R_i)/\rho \right\} \left\{ \frac{1}{R_i} \right\} \cdot \exp (iqR_i - 1). \quad (3.7)$$

By the definitions (2.3a) and (2.3b), the $[xy]$'s have to satisfy the equations

$$\lim_{q \to 0} \left\{ [xy]^R + [xy]^F \right\} = 0 \quad (3.8)$$

and

$$\lim_{q \to 0} [xy]^R = 0, \quad (3.9)$$

as actually the case according to Eqs. (3.1), (3.5) and (3.7), since

$$\lim [xy]^R = Ne^0 \left\{ \frac{4\pi}{3} \delta_{xy} - \pi \sum_{h} \frac{K_h K_{hp}}{|K_h|^2} \right\},$$

neglecting the contributions from $G(t=K_h/2k_R)$, $h \neq 0$, which do not vanish because of the approximate formulation, and

$$\lim [xy]^F = Ne^0 \left\{ \frac{-4\pi}{3} \delta_{xy} + 4\pi \sum_{h} \frac{K_h K_{hp}}{|K_h|^2} \right\}.$$

Eq. (3.9) is apparent from Eq. (3.7).
§ 4. Numerical Results

Frequencies are calculated as functions of wave number vector by Eqs. (2.8), (2.9), (2.10), (2.11), (3.1), (3.5) and (3.7) along the (100), (110), and (111) directions. The results are given in Table I, II, and III, in which \( C \), \( R \), \( EI \), \( EII \) or \( T \) means the contribution from \([xy]^C \), \([xy]^R \), \([xy]^E \), \([xy]^EII \), or their total \([xy]\) in units of \( Ne^2/M \) respectively, and the circular frequency \( \omega \) is given in the last column in units of \((Ne^2/M)^{1/2}\). The column \( \hbar \approx 0 \) of \( EII \) gives the contributions from Umklapp processes satisfying the condition \(|q+\mathbf{K}| < |\mathbf{K}_{\mathit{Umklapp}}|\). For Na, \( \tau = 1 \), \( V(r_s) - E_c = 0.08 \text{eV}, \, \zeta_e = 3.17 \text{eV}, \, D/D_o = 0.90, \, B = 1.25 \) and \( r_s = 3.96 \text{Bohr radius}^6 \).

The contributions from exchange repulsions are about 10% of the adiabatic potentials as shown in the table, and negative for the modes \( T \), of \((q, q, 0)\), from which we may conclude that the b.c.c. structure.

### Table I. Frequencies \( \omega \) versus wave number vector \( \mathbf{q} = (\pi/r_o)(\hat{q}, 0, 0) \)

The \( C \) or \( R \) denotes the contributions from the coulomb repulsion between ions (Eq. (3.5)) or those from the exchange repulsions between ion-cores (Eq. (3.7)). The \( EI \) or \( EII \) is the values of Eq. (3.1 \( EI \)) or (Eq. 3.1 \( EII \)), representing the electron-ion and electron-electron interactions, and \( T = C + R + EI + EII \). The units are \((Ne^2/M) = 1.54 \times 10^{36} \text{sec}^4\). The circular frequencies are given in the last column in units of \((Ne^2/M)^{1/2} = 1.24 \times 10^{14} \text{sec}^2\).

<table>
<thead>
<tr>
<th>( \hat{q} )</th>
<th>( C )</th>
<th>( R )</th>
<th>( EI )</th>
<th>( EII )</th>
<th>( T )</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>7.791</td>
<td>0.044</td>
<td>4.189</td>
<td>-11.523</td>
<td>-0.050</td>
<td>0.451</td>
</tr>
<tr>
<td>0.4</td>
<td>6.062</td>
<td>0.142</td>
<td>4.189</td>
<td>-8.800</td>
<td>-0.190</td>
<td>1.403</td>
</tr>
<tr>
<td>0.6</td>
<td>3.501</td>
<td>0.233</td>
<td>4.189</td>
<td>-5.430</td>
<td>-0.239</td>
<td>2.252</td>
</tr>
<tr>
<td>0.8</td>
<td>1.041</td>
<td>0.282</td>
<td>4.189</td>
<td>-2.522</td>
<td>-0.223</td>
<td>2.738</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>0.295</td>
<td>4.189</td>
<td>-0.798</td>
<td>-0.798</td>
<td>2.888</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( q )</th>
<th>( C )</th>
<th>( R )</th>
<th>( EI )</th>
<th>( EII )</th>
<th>( T )</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>-3.895</td>
<td>0.027</td>
<td>4.189</td>
<td>0</td>
<td>-0.039</td>
<td>0.282</td>
</tr>
<tr>
<td>0.4</td>
<td>-3.031</td>
<td>0.098</td>
<td>4.189</td>
<td>0</td>
<td>-0.263</td>
<td>0.993</td>
</tr>
<tr>
<td>0.6</td>
<td>-1.750</td>
<td>0.189</td>
<td>4.189</td>
<td>0</td>
<td>-0.748</td>
<td>1.880</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.521</td>
<td>0.265</td>
<td>4.189</td>
<td>0</td>
<td>-1.330</td>
<td>2.603</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>0.295</td>
<td>4.189</td>
<td>0</td>
<td>-1.596</td>
<td>2.888</td>
</tr>
</tbody>
</table>
is unstable for metals with large exchange repulsion. It is interesting to note that $[xy]^\text{eff}$ contributes appreciably to the transversal modes of shorter wave-length through Umklapp processes.

The elastic constants may be determined from the ratio of the frequency and the wave number in the long wave-length limit; i.e., $c_{11}$ or $c_{44}$ from the ratio of the $L$ or $T_1$ branch of $(q,0,0)$, and $c_{11}-c_{44}$ from the ratio of the $T_1$ branch of $(q,q,0)$. The $c_{11}$ or the compressibility $\varepsilon = \{c_{11}-(2/3)(c_{11}-c_{44})\}^{-1}$ thus determined coincides with that given by the theory of Wigner and Seitz$^6$, as we have seen in §§6 and 7 of the foregoing paper.

**Table II.** Frequencies $\omega$ versus wave number vector $(q=\pi/r_e)(\bar{q},\bar{q},0)$

| $\bar{q}$ | $C$ | $R$ | $EI$ | $EII$ | $T$ | $\omega$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h=0$</td>
<td>$h\equiv 0$ (Umklapp)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>8.216</td>
<td>0.034</td>
<td>4.189</td>
<td>-12.035</td>
<td>-0.043</td>
<td>0.351</td>
</tr>
<tr>
<td>0.2</td>
<td>7.791</td>
<td>0.121</td>
<td>4.189</td>
<td>-10.548</td>
<td>-0.270</td>
<td>1.283</td>
</tr>
<tr>
<td>0.3</td>
<td>7.256</td>
<td>0.230</td>
<td>4.189</td>
<td>-8.401</td>
<td>-0.861</td>
<td>2.413</td>
</tr>
<tr>
<td>0.4</td>
<td>6.818</td>
<td>0.318</td>
<td>4.189</td>
<td>-6.003</td>
<td>-2.006</td>
<td>3.312</td>
</tr>
<tr>
<td>0.5</td>
<td>6.649</td>
<td>0.351</td>
<td>4.189</td>
<td>-3.766</td>
<td>-3.766</td>
<td>3.657</td>
</tr>
</tbody>
</table>

**Transversal branch** $(T_1)$, $e_q = (1/\sqrt{2}, 1/\sqrt{2}, 0)$.

| $\bar{q}$ | $C$ | $R$ | $EI$ | $EII$ | $T$ | $\omega$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=0$</td>
<td>$h\equiv 0$ (Umklapp)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>-4.170</td>
<td>-0.004</td>
<td>4.189</td>
<td>0</td>
<td>-0.009</td>
<td>0.006</td>
</tr>
<tr>
<td>0.2</td>
<td>-4.118</td>
<td>-0.016</td>
<td>4.189</td>
<td>0</td>
<td>-0.029</td>
<td>0.026</td>
</tr>
<tr>
<td>0.3</td>
<td>-4.051</td>
<td>-0.030</td>
<td>4.189</td>
<td>0</td>
<td>-0.062</td>
<td>0.046</td>
</tr>
<tr>
<td>0.4</td>
<td>-3.994</td>
<td>-0.042</td>
<td>4.189</td>
<td>0</td>
<td>-0.096</td>
<td>0.057</td>
</tr>
<tr>
<td>0.5</td>
<td>-3.972</td>
<td>-0.046</td>
<td>4.189</td>
<td>0</td>
<td>-0.111</td>
<td>0.060</td>
</tr>
</tbody>
</table>

**Transversal branch** $(T_2)$, $e_q = (0,0,1)$.

| $\bar{q}$ | $C$ | $R$ | $EI$ | $EII$ | $T$ | $\omega$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=0$</td>
<td>$h\equiv 0$ (Umklapp)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>-4.047</td>
<td>0.013</td>
<td>4.189</td>
<td>0</td>
<td>-0.018</td>
<td>0.137</td>
</tr>
<tr>
<td>0.2</td>
<td>-3.673</td>
<td>0.048</td>
<td>4.189</td>
<td>0</td>
<td>-0.058</td>
<td>0.506</td>
</tr>
<tr>
<td>0.3</td>
<td>-3.305</td>
<td>0.091</td>
<td>4.189</td>
<td>0</td>
<td>-0.124</td>
<td>0.951</td>
</tr>
<tr>
<td>0.4</td>
<td>-2.824</td>
<td>0.125</td>
<td>4.189</td>
<td>0</td>
<td>-0.192</td>
<td>1.298</td>
</tr>
<tr>
<td>0.5</td>
<td>-2.677</td>
<td>0.139</td>
<td>4.189</td>
<td>0</td>
<td>-0.222</td>
<td>1.429</td>
</tr>
</tbody>
</table>
Normal Vibrations of Sodium

TABLE III. Frequencies $\omega$ versus wave number vector $\mathbf{q}=(\pi/r_o)(\mathbf{q}, \mathbf{q}, \mathbf{q})$

Longitudinal branch ($L$), $c_l = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

<table>
<thead>
<tr>
<th>$\mathbf{q}$</th>
<th>$C$</th>
<th>$R$</th>
<th>$EI$</th>
<th>$EII$ $h=0$</th>
<th>$T$</th>
<th>$\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=0$ (Umklapp)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>8.178</td>
<td>0.061</td>
<td>4.189</td>
<td>-11.776</td>
<td>-0.062</td>
<td>0.590</td>
</tr>
<tr>
<td>0.2</td>
<td>7.433</td>
<td>0.196</td>
<td>4.189</td>
<td>-9.544</td>
<td>-0.322</td>
<td>1.952</td>
</tr>
<tr>
<td>0.3</td>
<td>5.826</td>
<td>0.300</td>
<td>4.189</td>
<td>-6.786</td>
<td>-0.697</td>
<td>2.832</td>
</tr>
<tr>
<td>0.4</td>
<td>3.306</td>
<td>0.296</td>
<td>4.189</td>
<td>-3.975</td>
<td>-0.879</td>
<td>2.837</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>0.190</td>
<td>4.189</td>
<td>-1.832</td>
<td>-0.611</td>
<td>1.942</td>
</tr>
</tbody>
</table>

Transversal branch ($T$, or $T_2$), $c_0 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ or $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

<table>
<thead>
<tr>
<th>$\mathbf{q}$</th>
<th>$C$</th>
<th>$R$</th>
<th>$EI$</th>
<th>$EII$ $h=0$</th>
<th>$T$</th>
<th>$\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=0$ (Umklapp)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>-4.089</td>
<td>0.007</td>
<td>4.189</td>
<td>0</td>
<td>-0.0215</td>
<td>0.0855</td>
</tr>
<tr>
<td>0.2</td>
<td>-3.716</td>
<td>0.029</td>
<td>4.189</td>
<td>0</td>
<td>-0.164</td>
<td>0.338</td>
</tr>
<tr>
<td>0.3</td>
<td>-2.913</td>
<td>0.071</td>
<td>4.189</td>
<td>0</td>
<td>-0.576</td>
<td>0.771</td>
</tr>
<tr>
<td>0.4</td>
<td>-1.603</td>
<td>0.129</td>
<td>4.189</td>
<td>0</td>
<td>-1.373</td>
<td>1.345</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>0.190</td>
<td>4.189</td>
<td>0</td>
<td>-2.443</td>
<td>1.942</td>
</tr>
</tbody>
</table>

The $c_u$ is derived by expanding the coefficient $[yy]$ of $(q,0,0)$ with respect to $q^2$. The constant term (signified by subscript 0) in the expansion is

$$[yy]_0 = [yy]_0^c + [yy]_0^e + [yy]_0^p$$

$$= Ne^c \left[ -\frac{4\pi}{3} \sum_{n} \frac{h_n^2}{h^2} \exp \left( -\frac{\pi^2}{4\epsilon^2} h^2 \right) \
+ 4 \sum_{l=0}^\infty \left( -f(l) + g(l) \frac{l_p^2}{l^2} \right) + \frac{16}{3\sqrt{\pi}} \epsilon^2 \right] + 0$$

$$= Ne^c \left[ -\frac{4\pi}{3} + \frac{4\pi}{3} \right] = 0, \quad (4.1)$$

and the coefficient of $q^2$ (signified with subscript 2)

$$[yy]_2 = [yy]_2^c + [yy]_2^e + [yy]_2^p \quad (4.2)$$

with

$$[yy]_2^p \approx 0, \quad (4.2E)$$

and
\[
[yy]^{e}_e = Ne^2 \left\{ 4\pi \sum_{l=0}^{\infty} \frac{h^2_y}{h^2} \left( 4 \frac{h^2_x}{h^4} - \frac{1}{h^2} + \frac{\pi^4}{8\varepsilon^4} h^2_x - \frac{\pi^2}{4\varepsilon^2} \right) h^2 + \frac{\pi^2}{\varepsilon^2} \right\} \exp\left( -\frac{\pi^2}{4\varepsilon^2} h^2 \right) - 2\pi^2 \sum_{l=0}^{\infty} \left\{ f(l) + g(l) \frac{L^2}{F} \right\} \right\}
\]

(4.2C)

The \(c_4\) is now given as,

\[
c_{44} = -\frac{Ne^2}{\pi^2} \left\{ [yy]^e_e + [yy]^e_e \right\},
\]

which is exactly the same expression as given by Fuchs\textsuperscript{7}. In the case of alkali metals, \(V(r_s) - E_s\) is very small in general, so that \([yy]^e_e\) is nearly zero, which justifies the assumption made by Fuchs in his

---

Fig. 1  Circular frequencies \(\omega\) versus wave number vector \(q = \pi/\rho \langle q, 0, 0 \rangle\).
derivation of the shear moduli. In the case of noble metals (Cu, Ag and Au), \( V(r_s) - E_0 \) is of the order of \( 3 \sim 4 \) eV and the coulomb term \([yy]_0^E\) is compensated a good deal by \([yy]_0^F\), as assumed by Zener and Huntington.\(^{15}\)

The \( c_{11} - c_{12} \) is given similarly as

\[
c_{11} - c_{12} = - \frac{N r_s^2}{\pi^2} \left\{ [xx]_0 - [xy]_0 \right\},
\]

where \([xx]_0\) or \([xy]_0\) is the coefficient of \(q^2\) in the expansion of \([xx]\) or \([xy]\) of \((q, q, 0)\). The result is the same as that obtained by Fuchs, as readily verified similarly as in the case of \( c_{44} \).

Figs. I, II and III show the results in comparison with the calculations by Bauer\(^{15}\) and by Bhatia\(^{12}\). Bauer has assumed central forces and introduced force constants \( a \) and \( \tau \) corresponding to the interactions

---

Fig. 2 Circular frequencies \( \omega \) versus wave number vector \( q = \pi r_s \) \((q, q, 0)\).

Fig. 3 Circular frequencies \( \omega \) versus wave number vector \( q = \pi r_s \) \((q, q, q)\).
between the nearest and second nearest neighbours, which are determined as \(a=r_c a_n\) and \(r=a/4\). The dispersion relations by Bauer are given by dotted line. Bhatia, on the other hand, has taken into account in a simple way the effects of valence electrons on the ionic motion, but only the interactions between nearest neighbours. Three parameters introduced was adjusted by the three independent elastic constants \(c_{11}, c_{12}\) and \(c_{44}\). Broken lines in Figs. show the results by Bhatia.

§ 5. Verification of the Formulation for Short Wave-Length

We will now examine the validity of our formulation for shorter wave-length. For the mode \(q=(\pi/r, 0, 0), e_q=(1, 0, 0)\) or \(q=(0, \pi/r, 0), e_q=(1, 0, 0)\) in the metals of b.c.c. structure, each ion is displaced alternately by \(+u\) or \(-u\) in the direction of \(x\)-axis from the equilibrium position, where \(u=N^{-\frac{1}{3}}(a_q + a_q^*)\) by Eq. (1). The contribution from the coulomb repulsions of ions vanishes as seen in Table I, and those of valence electrons to the potential for the vibration may be calculated by cellular method\(^\text{13}\), admitting that the coulomb energy of valence electrons is approximately balanced by the exchange and correlation energies just as in perfect crystals. The circular frequency is given as

\[
M_0 \omega^2 = 2 \varepsilon(u)/u^2 + \text{(increase of the exchange repulsions), (5.1)}
\]

where \(\varepsilon(u)\) is the perturbing energy when an ion is displaced from the centre of the atomic sphere, given, by taking the boundary condition as \(\left(\frac{d^2 v}{dr^2}\right)_{r=r_s} = 0\), as

\[
2 \varepsilon(u)/u^2 = \left(\frac{1}{r_s}\right)\left(\frac{dv_1}{dr}\right)_{r=r_s} - \left(2m/k^2\right)\{v_1(r_s) - \varepsilon_s\}^2
\]

\[
= Ne^2 \left\{\frac{4\pi/3 - (2\pi/3)(r_s/a_H)^3(e^2/2a_H)^{-2}}{(v_1(r_s) - \varepsilon_s)^2} \right\}
\]

(5.2)

where \(v_1(r)\) (\(-e^2/r\) for \(r \geq r_s\)) is the potential of an ion, and \(\varepsilon_s\) is the energy of an electron with an ion at the centre of the atomic sphere and the same boundary condition as above. For Na, \(r_s = 3.96a_H\) and \(v_1(r_s) - \varepsilon = 1.48\) eV, which give by Eqs. (5.1) and (5.2),

\[
\omega = 1.73 \left(\frac{Ne^2}{M}\right)^{\frac{1}{2}}
\]

in good agreement with \(\omega = 1.70 \left(\frac{Ne^2}{M}\right)^{\frac{1}{2}}\) in Table I.
Normal Vibrations of Sodium

Acknowledgement

The author would like to express his sincere thanks to Professor J. Horii for his continual encouragement and interests to the present work. He would also like to acknowledge the contributions of Mrs. F. Enomoto and Mr. M. Fujimoto, who carried out many of the computations in the present works.

References

4a) T. Toya, Busseiron Kenkyu 60, 129 (1953) [in Japanese].
8) See Ref. 6, Chapter II.
9) Table I and II of Ref. 4b).