Econ. J. of Hokkaido Univ.,
Vol. 36(2007), pp. 1-22

Gale-Nikaido's Lemma in Infinite Dimensional Spaces: Early Attempts by Prof. Nikaido

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As shown in Debreu(1959, Ch.5), Gale-Nikaido's lemma of Gale(1955) and Nikaido(1956a) is the key in proving the existence of a competitive equilibrium in classical economies with a finite number of commodities. Nikaido(1956b, 57b, 59) extend the Gale-Nikaido's lemma in economies with a finite number of commodities to the one in some infinite dimensional spaces such as normed spaces and locally convex topological vector spaces. This is surprising since Gale-Nikaido's lemma is generalized to the infinite dimensional spaces just after Gale-Nikaido's lemma in economies with a finite number of commodities is established. Since the literature on the existence of competitive equilibrium in economies with infinite number of commodities stars after Peleg-Yaari(1970) and Bewley(1972) and it was one of the main topics for 80's in general equilibrium thoery, Nikaido(1956b, 57b ,59) precedes to the literature as Debreu(1954) does. The purpose of this paper is to reconsider the Nikaido(1956b, 57b, 59)'s generalization of Gale-Nikaido's lemma in infinite dimensional spaces from the present state of general equilibrium theory of infinite dimensional spaces.

JEL Classification:C.62,D51
Keywords: Gale-Nikaido's Lemma, Infinite Dimensional Commodity Space, Existence of Competitive Equilibrium, Interiority Problem and Approximated Cone

1. Introduction

In economies with a finite number of commodities, Gale(1955) and Nikaido (1956a) reduce the existence of competitive equilibrium in such economies to the result, called Gale-Nikaido's lemma in the literature, where the non-positive orthant and an image of the correspondence satisfying Walras' law and continuity intersect at some point in the domain. This Gale-Nikaido's lemma is a crucial key to the proof of the existence of competitive equilibrium in economies with a finite number of commodities. Gale's proof for this lemma uses KKM's lemma and Nikaido's proof uses Kakutani's fixed point theorem, which is also employed in Theory of Value(1959, Ch.5), a famous textbook on general equilibrium theory.[1)] As is explained in this Debreu's book, applying this lemma to the market excess demand correspondence yields the existence of competitive equilibrium. Since the set of competitive equilibrium of the original economies are equal to the one of the economies truncated by a large cube, it is enough to establish the existence of competitive equilibrium in the economies within a bounded region. Under the standard conditions, the market excess demand correspondence in the bounded economies satisfied the conditions for Gale-Nikaido's lemma, and hence the bounded economies have competitive equilibrium. This implies that the original unbounded economies also have competitive equilibrium as well.

The above argument is made in economies with a finite number of commodities, and hence the commodity space is $n-$dimensional Euclidean space. Thus, even when the goods are distinguished according to the differences in quality in goods and in the time and the state to be delivered of goods, the number of these differences must be finite. On the other hand, when the number of one of these differences is infinite, there exist an infinite number of commodities in such economies and the commodity spaces of these economies are infinite dimensional. Then the above argument used in $n-$dimensional Euclidean spaces can not apply to this situation since it is applicable only in the cases with a finite number of commodities.

Thus it is an interesting question how to extend and prove Gale-Nikaido's lemma in economies with an infinite number of commodities. This is carried out in Nikaido(1956b, 57b) in an economy with a infinite dimensional norm space as its commodity space and Nikaido(1959) in an economy with an infinite dimensional general locally convex topological vector space as its commodity space. The existence of competitive equilibrium in economies with an infinite number of commodities became an central issue in general equilibrium in 1980's after two path breaking papers by Peleg-Yaari(1970) and Bewley(1972) came out.[2)] The surprising fact is that these Nikaido(56b,57b,59) in infinite dimensional commodity space cases were made just after corresponding Nikaido(56a) in a finite dimensional commodity space case were came out. Thus, Nikaido(56b,57b,59) proceeded at least one decade to the literature on the existence of competitive equilibrium in economies with an infinite number of commodities. There is also Debreu(1954) which considers fundamental theorems of welfare economics in general linear spaces.[3)] The

[1)] Debreu(1956) extends this lemma in the case with the nonnegative orthant to the one with a general convex closed cone. Thus, sometimes the term Gale-Nikaido-Debreu's lemma is used.

[2)] Peleg-Yaari(70) uses Edgeworth equilibrium approach that is based on the result on the non-emptyness of the core and on Debreu-Scarf(63)' limit theorem on the core and the set of competitive equilibrium. Bewley(72) employes the limiting approach that uses the argument of limiting the dimension of the finite dimensional subeconomies and Debreu(63)'s theorem on the existence of competitive equilibrium in economies with a finite number of commodities. The Bewley(72)'s original version(69) uses Negishi's approach that is based on the 2nd fundamental theorem of welfare economics. None of these approach employes excess demand correspondence as its primitive concept as in the case of economies with finite dimensional commodity spaces. Thus, Nikaido(56b,57b,59) is considered as the approach using the excess demand correspondence of the economy as the primitive concept in the existence argument.

[3)] None of Nikaido(56b,57b,59) mentions Debreu(54).

commodity space in Debreu(54) may be infinite dimensional, and it is the first rigorous attempt to consider economies with infinite dimensional commodities. It, however, does not contain any argument on the existence of competitive equilibrium in such economies since even the existence of competitive equilibrium in economies with a finite number of commodities was just proved by Arrow-Debreu(1954) and McKenzie(1954) after Walras(1874,77) and Wald(1945)'s attempt. It is quite interesting that these mathematical oriented scholars, Nikaido and Debreu, treated the similar argument on economies with an infinite number of commodities around the same period.

Although Nikaido(59) establishes Gale-Nikaido's lemma in a locally convex topological vector space case with a acyclic-valued excess supply correspondence, the proof of this case is quite complicated. But the result with convex-valued excess supply correspondence turns out to be quite simple when we follow the suggestion of Nikaido(57b, p.4, remark 2) mentioning that the proof of Gale-Nikaido's lemma in the normed space case with a convex-valued excess supply correspondence is easily extended to the one in the topological vector space case.[4] The purpose of this paper is thus to shows Gale-Nikaido's lemma in the locally convex topological vector space case with a convex-valued excess supply correspondence based on this Nikaido(57b, p.4, remark 2)'s suggestion. Although this proof is very straight forward and much like Bewley(72)' proof, it is unfortunate that any of later papers on the theory of general equilibrium with infinite dimensional commodity spaces have not referred to Nikaido(56b, 57b, 59) as one of early attempts besides Debreu(54).

2. Model

The proof of Gale-Nikaido's lemma in infinite dimension in Nikaido(59) is made with using the general result in topological vector spaces which is applicable to the existence of Nash-equilibrium in abstract economies and minimax theorem in topological vector spaces. It is not direct extension of the one in the normed space case of Nikaido(56b,57b). The proofs of Nikaido(56b,57b) in infinite dimensional normed space are done with the convex valued correspondence, and these proof are very straightforward. Although the proof of this general result applies to the case with acyclic-valued correspondence, which is more general than the convex-valued correspondence, the proof is, however, very complicated. It turns out that, the similar simple proof as in

[4]Bewley(72) uses the method of approximating the original economy with infinite dimensional space by a family of finite dimensional subeconomies to which Arrow-Debreu-McKenzie theorem applied in the proof of the existence of competitive equilibrium in an infinite dimensional commodity space. This approach is same at the bottom in the method of the proof used in Nikaido(57b). Also, Fan-Glicksberg's generalization of Kakutani's fixed point theorem to locally convex topological vector space uses as its proof approximating the original infinite dimensional space by a family of finite dimensional subspaces to which Kakutani's fixed point theorem is applied. In these arguments, there is a common feature in approximating the original space that Kakutani's fixed point theorem, Arrow-Dereu-McKenzie theorem, or Gale-Nikaido's lemma is applied to a family of finite dimensional subeconomies. From this viewpoint, late Professor Nikaido had an opinion that finite dimensional fixed point theorems are much more fundamental than infinite dimensional ones. Note that these three results are known to be equivalent in the finite dimensional setting.

Nikaido(56b,57b) is possible even in the general locally convex topological vector space case, and this is the proof that this paper tried to establish as the main purpose.

The model employed in this paper follows the one in Nikaido(59), although some results obtained in Nikaido(57b) are also mentioned as far as they are necessary for the present purpose. Let E be the commodity space of the economy considered here and a (real Hausdorff) locally convex topological vector space.[5] Let τ be the original locally convex linear topology defined on E. Then there is a fundamental family of convex and balanced $\tau-$neighborhoods of the origin for this topology τ.[6] Let E^* be the dual space of E with respect to τ, i.e., the set of $\tau-$continuous linear functional on E. Let $(\cdot,\cdot) : E \times E^* \to R, (x,p) \in E \times E^* \to x \cdot p$ be the bilinear form on $E \times E^*$. For given $p \in E^*$, the weakest linear topology on E which makes $(\cdot,p) : E \to R$ continuous on E is called the weak topology $\sigma(E,E^*)$ on E. Similarly, for given $x \in E$, the weakest topology on E^* which makes $(x,\cdot) : E^* \to R$ continuous on E^* is called the weak $*$ topology $\sigma(E^*,E)$ on E^*.[7]

For a convex cone $P(\subset E)$, its dual cone P^* is defined as the set $\{p \in E^* : x \cdot p \geqq 0$ holds for any $x \in P\}$. Then P^* is a convex cone and weak $* \ \sigma(E^*,E)$ -closed. Moreover, its second dual cone P^{**} is defined as the set $\{x \in E : x \cdot p \geqq 0$ holds for any $p \in P^*\}(\supset P)$. Then P^{**} is a convex cone and weak $\sigma(E,E^*)$ -closed. For a convex cone, let $< P^* >= P^* \backslash \{0\}$, the set of non-zero $p \in P^*$.

For a given τ-closed convex cone$P(\neq \emptyset)$, let $< P^* >$ be a price set of the economy considered here. Let the excess supply correspondence of the economy be $\phi :< P^* > \to E \backslash \{\emptyset\}$. Here $\phi(p) \neq \emptyset$ is assumed for any $p \in < P^* >$.[8] Then the issue is that under what conditions there is some $\overline{p} \in < P^* >$ satisfying $(\phi(\overline{p}) \cap P) \neq \emptyset$ as in the finite dimensional case. This is indeed considered in Nikaido(56b,57b,59) and is called Gale-Nikaido's lemma in infinite dimension. The purpose of this paper is, thus, to reconsider this result from the viewpoint of present general equilibrium theory of infinite dimensional spaces.

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5) Nikaido(56b,57b) use an infinite dimensional normed space as E, and Nikaido(59) uses a general infinite dimensional locally convex topological vector space as E. It is Debreu(54) that employes a general topological vector space as the commodity space in the general equilibrium literature. Debreu(54) uses a separation theorem of convex sets in general topological vector space, but does not use the weak topology $\sigma(E,E^*)$, the weak $*$ topology $\sigma(E^*,E)$, and the duality theorems on these linear topologies such as Banach-Alaogru theorem. On the other hand, Nikaido(56b,57b) employs the weak topology $\sigma(E,E^*)$, the weak $*$ topology $\sigma(E^*,E)$, and the duality theorem on these linear topologies such as Banach-Alaogru theorem. This theorem says that the norm unit ball is compact in the weak $*$ topology $\sigma(E^*,E)$ and turns out to be crucial, as in Bewley(72), in establishing the price set and the feasible set are compact in the weak $*$ topology.

6) Since this topology is a linear topology, vector operations are continuous with respect to this topology. Moreover, since this topology is local convex, it is generated by a family of semi-norms.

7) $\sigma(E,E^*)$ is the linear topology on E that is generated by $\{|x \cdot p| : p \in E^*\}$ as the family of semi-norms on E, similarly, $\sigma(E^*,E)$ is the linear topology on E^* that is generated by $\{|x \cdot p| : x \in E\}$ as the family of semi-norms on E^*.

8) In infinite dimensional commodity space cases, there are several examples where the excess supply(demand) correspondences may not be defined over the entire price set, so that the value of the excess supply(demand) correspondences is empty for some prices.

3. Several Lemmas

In this section, some lemmas are established which are necessary to show Gale-Nikaido's lemma in infinite dimensional general locally convex topological vector spaces. The following results hold for a locally convex space E.

Lemma 1 For a ($\tau-$)closed convex cone P which is not equal to E, P^* contains points other than 0.[9)]

Lemma 2 For a convex cone P in E, $x \in cl_\tau(P)$ is equivalent to $x \cdot p \geqq 0$ for $p \in P^*$.[10)]

Lemma 3 For a ($\tau-$)interior point u of a convex cone P in E, $u \cdot p > 0$ holds for $p \in P^* \backslash \{0\}$.[11)]

When a convex cone P in E (or E^*) does not contain x and $-x$ except 0 at the same time, it is called *Pointed*. That is, the convex cone does not contain any straight line passing the origin 0, and hence $P \cap (-P) = \{0\}$ holds. Then the following results hold.

Lemma 4 When the ($\tau-$)interior of a convex cone P in E is non-empty, then P^* is pointed.[12)]

Proof. Let $p \in P^* \backslash \{0\}$ and u be an ($\tau-$)interior point of P. Then, lemma 3 implies $-u \cdot p < 0$ and hence $-u \notin P^*$. Thus, P^* is pointed. ∎

Lemma 5 When the ($\tau-$)interior of a convex cone P in E is non-empty and $cl_\tau(P)$ is not E, then $< P^* >$ is non-empty and convex. [13)]

Proof. Since $cl_\tau(P)$ is not E, lemma 1 implies $< (cl_\tau(P))^* > \neq \varnothing$. Since $P \subset cl_\tau(P)$ implies $(cl_\tau(P))^* \subset P^*$, $p \in < (cl_\tau(P))^* >$ gives rise to $p \neq 0$ and $p \in P^*$, and hence $p \in < P^* > \neq \varnothing$ holds. Next, let $p, q \in < P^* >, \lambda \in (0,1)$. The convexity of P^* implies $\lambda p + (1-\lambda)q \in P^*$. Let $u \in int_\tau(P)$, then by lemma 3 $u \cdot p > 0$ and $u \cdot q > 0$ hold and hence $u \cdot (\lambda p + (1-\lambda)q) = \lambda(u \cdot p) + (1-\lambda)(u \cdot q) > 0$ holds. Thus, $(\lambda p + (1-\lambda)q) \neq 0$ and $\lambda p + (1-\lambda)q) \in < P^* >$ hold, and hence $< P^* >$ is convex. ∎

Suppose that P is a ($\tau-$)closed convex cone satisfying $P \cap (-P) \neq P$. Since $P \cap (-P)$ is the maximal linear subspace contained in P, this condition implies that P is not a linear subspace. Moreover, if P contains ($\tau-$)interior point in E, this is equivalent to $P \neq E$. But it is shown in the following that a general

9)This follows from a form of separation theorem of convex sets. Nikaido(57b, lemma 2), Nikaido(59, lemma 4).

10)Duality theorem on convex cone and second dual cone implies $cl_\tau(P) = P^{**}$. Nikaido(57b, lemma 3), Nikaido(59, lemma 5).

11)Suppose $u \cdot p = 0$ holds for $p(\neq 0) \in P^*$. Then the ($\tau-$)contuity of $(\cdot, p)$ and $u \in int_\tau(P)$ implies that $u' \cdot p > 0$ occurs for some $u' \in P$, which is a contradition to $p \in P^*$. Nikaido(57b, lemma 1), Nikaido(59, lemma 6).

12)Nikaido(57b, lemma 4), Nikaido(59, lemma 7).

13)Nikaido(59, lemma 8) supposes that $int_\tau(P)$ is not E, but here it is changed to that $cl_\tau(P)$ is not E. This change does not interfare the argument below as of Nikaido(57b,59).

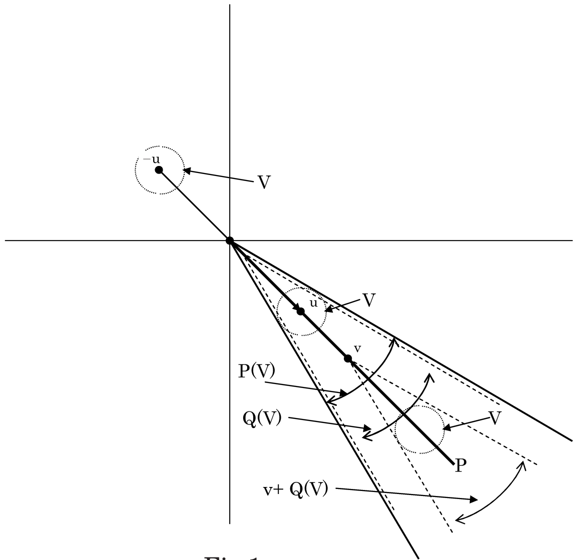

Fig.1

P with the property $P \cap (-P) \neq P$ is approximated from the outside by a family of convex cones which have ($\tau-$)interior points and are not equal to E. Take an arbitrary point $u \in P \backslash (P \cap (-P))$. Of course, $u \neq 0$ follows. Then $-u \notin P$ and the ($\tau-$)closedness of P imply that $(-u + U) \cap P = \varnothing$ holds for some convex balanced neighborhood U of 0. Now for any convex balanced neighborhood $V (\subset U)$ of 0, let $Q(V)$ be the convex cone generated by $u + V$, i.e., $\cup_{\lambda > 0} \lambda (u + V)$, and $P(V) = cl_{\tau}(P + Q(V)) (\supset P)$.[14] Then $P(V)$ is $\tau-$closed, and $(u + V) \subset Q(V) \subset P(V)$ and $u \in P \subset P(V)$ imply that ($\tau-$)interior of $P(V)$ is non-empty. Then the following result follows. See Fig.1.

Lemma 6 (i) $-u \notin P(V)$ holds for any convex balanced neighborhood $V (\subset U)$ of 0. (ii) $P = \cap \{P(V)$: all convex balanced neighborhood $V (\subset U)$ of $0\}$.[15]

Proof. (i) : Suppose $-u \in P(V)$. Then $-u = x + \lambda(u + a) + b$ holds for some $x \in P, \lambda \geqq 0, a, b \in V$. Therefore, $-u = x/(1+\lambda) + \lambda a/(1+\lambda)) + b/(1+\lambda)$ holds. Note that since P is a cone $x/(1 + \lambda) \in P$ holds. Moreover, the convexity of V implies $[\lambda a/(1+\lambda)) + b/(1+\lambda)] \in V$, and hence, the balancedness of V implies $-[\lambda a/(1+\lambda)) + b/(1+\lambda)] \in V$. Thus $(-u - [\lambda a/(1+\lambda) + b/(1+\lambda)] = x/(1+\lambda) \in$ $)(-u + V) \cap P \neq \varnothing$ follows. $V \subset U$, however, implies $(-u + U) \cap P \neq \varnothing$, but this is a contradiction to the choice of U.

(ii) : Since $P(V) \supset P$ holds for any convex balanced neighborhood $V (\subset U)$ of 0 from the definition of $P(V)$, $P \subset \cap \{P(V)$: any convex balanced neighborhood $V (\subset U)$ of $0\}$. Suppose that $y \in P(V)$ holds for any convex balanced neighborhood $V (\subset U)$ of 0 but $y \notin P$ occurs. $y \in P(V)$ implies that there is

[14] A similar closed convex cone constructed as $P + Q(V)$ is employed by MasColell(1986) in the properness condition in the infinite dimensional commodity cases where the interior condition does not hold.

[15] Nikaido(57b, lemma 7), Nikaido(59, lemma 8).

$x_V \in P, \lambda_V \geqq 0, a_V, b_V \in V$ such that $y = x_V + \lambda_V(u + a_V) + b_V$ holds. Then $y - (x_V + \lambda_V u) = \lambda_V a_V + b_V = c_V$ holds. Since P is a convex cone, $x_V + \lambda_V u \in P$ holds. Moreover, $y \notin P$ implies that by the $(\tau-)$closedness of P there is a convex balanced neighborhood $W(\subset U)$ of 0 such that $(y + W) \cap P = \varnothing$ holds. Then, $y - c_V = (x_V + \lambda_V u) \in P$ and the balancedness of W imply $c_V \notin W$. $\lambda_V \to \infty (V \downarrow 0)$ is shown first. For this purpose, it is enough to show for given n that $\lambda_V > n - 1$ holds for any $V \subset W/n$. The convexity of V implies $c_V/(1 + \lambda_V) = [\lambda_V a_V/(1 + \lambda_V)) + b_V/(1 + \lambda_V)] \in V \subset W/n$, and hence $c_V \in W(1 + \lambda_V)/n$ follows. But $c_V \notin W$ implies $(1 + \lambda_V)/n > 1$ and $\lambda_V > n - 1$ follows. $c_V/(1 + \lambda_V) \in V$ implies $c_V/\lambda_V = y/\lambda_V - (x_V/\lambda_V + u) \in (1 + 1/\lambda_V)V$ and hence the balancedness of V implies $x_V/\lambda_V + u = y/\lambda_V - c_V/\lambda_V \in y/\lambda_V + (1 + 1/\lambda_V)V$. Then as $V \downarrow 0$ implies $\lambda_V \to \infty$, $y/\lambda_V + (1 + 1/\lambda_V)V \to 0$ and hence $x_V/\lambda_V + u \to 0$, i.e., $\lim_{V\downarrow 0} x_V/\lambda_V = -u$ follows. Since P is a cone $x_V \in P$ implies $x_V/\lambda_V \in P$, and hence $\lim_{V\downarrow 0} x_V/\lambda_V = -u \in cl_\tau(P) = P$. This is, however, a contradiction to $-u \notin P$. Thus $P \supset \cap\{P(V)$: any convex balanced neighborhood $V(\subset U)$ of $0\}$, and hence $P = \cap\{P(V)$: any convex balanced neighborhood $V(\subset U)$ of $0\}$ holds. ■

Since $P(V)$ is $\tau-$closed in lemma 6 and $-u \notin P(V)$ implies $P(V) \neq E$, lemma 5 is applicable to $P(V)$.

Next consider briefly the relation between upper semi-continuity and closedness of correspondences (or multi-valued mappings). Let S and X be two Hausdorff topological spaces and $\varphi : S \to X \backslash \{\emptyset\}$ be a correspondence (or multi-valued mapping). When for any $p \in S$ and open neighborhood $U(\varphi(p))$ of $\varphi(p)$ there is an open neighborhood $V(p)$ of p such that $q \in V(p)$ implies $\varphi(q) \subset U(\varphi(p))$, then φ is called $upper\ hemi-continuous(u.h.c)$.[16] When the graph of φ, $G_\varphi = \{(p, x) \in S \times X : x \in \varphi(p)\}$ is closed with respect to the product topology on $S \times X$, i.e., $cl(G_\varphi) = G_\varphi$, φ is called *closed*. From the definition continuous functions are $u.h.c.$ when they are viewed as correspondences. Also the composite mapping of two $u.h.c.$ correspondences is $u.h.c.$. Although these two definitions are not equivalent, they become so under some conditions.

Lemma 7 Let S be compact. Then the following two conditions are equivalent. (a) : φ is compact-valued and $u.h.c.$. (b) : φ is *closed* and there is a compact subset T of X such that any $\varphi(p)$ is contained in T.[17]

Proof. $(a) \to (b)$: Let $T = \cup_{p\in S}\varphi(p)$ and $\{V_\lambda\}_{\lambda\in\Lambda}$ be a open cover of T. For any $p \in S$, there are some $V_{\lambda_i}, i = l_1, \cdots, l_p$, such that $\varphi(p) \subset \cup_{i=l_1}^{l_p} V_{\lambda_i}$. Since $\cup_{i=l_1}^{l_p} V_{\lambda_i}$ is an open neighborhood of $\varphi(p)$, $u.h.c.$ of φ implies that there an open neighborhood U_p of p such that $q \in U_p \to \varphi(q) \subset \cup_{i=l_1}^{l_p} V_{\lambda_i}$ holds.

[16] When the similar condition holds for open half space $H(\varphi(p))$ containing $\varphi(p)$ instead of open neighborhood $U(\varphi(p))$ of $\varphi(p)$, φ is called $upper\ demi-continuous(u.d.c)$, and used in Yannelis(1985) which considers a generalization of Gale-Nikaido's lemma in the $u.d.c.$ case. Since open half spaces containing $\varphi(p)$ are open neighborhoods of $\varphi(p)$, $u.h.c.$ implies $u.d.c.$. Although Yannelis(85) employs more general condition $u.d.c.$, it , however, uses the interiority condition to the convex cone P, which is not used here. Thus, it is an open question whether the interiority condition is also dropped in the $u.d.c.$ case as in Nikaido(57b,59).

[17] Nikaido(57b, lemma 8) and Nikaido(59, lemma 3).

Since $\{U_p\}_{p\in S}$ is an open cover of S, the compactness of S implies that there some $p_1,\cdots,p_m \in S$ such that $S=\cup_{i=1}^m U_{p_i}$ holds. Then $q\in U_{p_j}\to\varphi(q)\subset \cup_{i=l_1^{p_j}}^{l_{p_j}} V_{\lambda_i}, j=1,\cdots,m$ implies that $q\in S=\cup_{i=1}^m U_p \to \varphi(q)\subset\cup_{i=1}^m V_{\lambda_i}$, and hence $T=\cup_{p\in S}\varphi(p)\subset\cup_{i=1}^m V_{\lambda_{p_i}}$ holds, where $m=\sum_{i=1}^m l_i$. Thus T has a finite open subcover $\{V_{\lambda_{p_i}}\}_{i=1}^m$ of $\{V_\lambda\}_{\lambda\in\Lambda}$, and $T=\cup_{p\in S}\varphi(p)$ is compact.[18] Next let $(p,x)\in cl(G_\varphi)\backslash(G_\varphi)$. By the compactness of $\varphi(p)$ and $x\notin\varphi(p)$, there are two open neighborhoods W_x of x and $V(\varphi(p))$ of $\varphi(p)$ such that $(V(\varphi(p))\cap W_x)=\emptyset$ holds. $u.h.c.$ of φ implies that there is an open neighborhood U_p of p such that $q\in U_p\to\varphi(q)\subset V(\varphi(p))$ holds. $(p,x)\in cl_\tau(G_\varphi)$ and $(p,x)\in(U_p\times W_x)$ imply that there is some $(q,y)\in[((U_p\times W_x)\cap G_\varphi]$ so that $q\in U_p$ and $y\in(W_x\cap\varphi(q))$ hold. This, however, implies $\emptyset\neq(W_x\cap\varphi(q)))\subset(V(\varphi(p))\cap W_x)=\emptyset$, and a contradiction occurs. Thus, $cl(G_\varphi)\backslash(G_\varphi)=\emptyset$, and hence $(G_\varphi)=cl(G_\varphi)$ follows. Therefore φ is *closed*.

$(b)\to(a)$: Let $x\in cl(\varphi(p))(\subset T)$. Then $(z_V\in)(V\cap\varphi(p))\neq\emptyset$ holds for any open neighborhood V of x. Let U be any open neighborhood of p. Then $(p,x)\in U\times V$ implies $((p,z_V)\in)[(U\times V)\cap G_\varphi]\neq\emptyset$ and hence $(p,x)\in cl(G_\varphi)=G_\varphi$. Thus, $x\in\varphi(p)$ and hence $cl(\varphi(p))=\varphi(p)$ holds. Thus $\varphi(p)$ is closed. Since T is compact $\varphi(p)\subset T$ implies that $\varphi(p)$ is compact. Suppose next that φ is not $u.h.c.$ at p. Then there is an open neighborhood V of $\varphi(p)$ such that there are $p_U\in U$ and $x_U\in V$ satisfying $x_U\notin V$ any open neighborhood U of p. Define by $U'\leqq U\leftrightarrow U\subset U'$ an order over open neighborhoods of p, and consider a net (or generalized sequence) $(p_U),(x_U)$ defined by this order. Then $p_U\to p$ holds and the compactness of $T(\supset(x_U))$implies that (x_U) has a converging subnet (x_{U_λ}) satisfying $x_{U_\lambda}\to y\in T$. $p_{U_\lambda}\to p$ holds as well. $x_U\notin V$ and the closedness of V^c implies $y\notin V$. Since $(p_{U_\lambda},x_{U_\lambda})\in G_\varphi,(p_{U_\lambda},x_{U_\lambda})\to(p,y)$, the *closedness* of φ implies $(p,y)\in G_\varphi$ and $y\in\varphi(p)\subset V$ holds. This is, however, a contradiction to $y\notin V$ and hence φ is $u.h.c.$ at p. ■

The proof of the main theorem of this paper follows the one in Nikaido(57b), which uses original Gale-Nikaido's lemma in $n-$dimensional Euclidean space case established in Nikaido(56a). Here first establishes this basic result in $n-$dimensional Euclidean space case. Let S^n be the unit simplex in R^n, i.e., $\{p\in R^n : p\geqq 0,\sum_{i=1}^n p_i=1\}$ and R_+^n be the nonnegative orthant of R^n, i.e., $\{x\in R^n : x\geqq 0\}$. Moreover let an excess supply conespondence in this case be $\phi: S^n\to R^n\backslash\{\emptyset\}$. Then the following result holds. See Fig.2.[19]

Lemma 8(Gale-Nikaido's lemma:R^n-case) Suppose that the excess supply correspondence $\phi : S^n\to R^n\backslash\{\emptyset\}$ is non-empty compact convex-valued and $u.h.c.$, and that $x\in\phi(p)\to p\cdot x\geqq 0$ holds for $p\in S^n$(weak Walras law). Then there is $\overline{p}\in S^n$ such that $\phi(\overline{p})\cap R_+^n\neq\emptyset$ holds. Thus $\overline{x}\geqq 0$ holds for some $\overline{x}\in\phi(\overline{p})$.[20]

Proof. Since S^n is compact and ϕ is $u.h.c.$ and compact-valued, the first part of the proof of lemma 7 implies that $T=\cup_{p\in S^n}\phi(p)$ is compact and hence $\phi(p)\subset\Delta$ holds for any $p\in S^n$ if Δ is chosen as a sufficiently large cube

[18] This part implies that if S is compact and $\varphi: S\to X$ is $u.h.c.$ then $T=\cup_{p\in S}\varphi(p)$ is compact.

[19] In Fig.2, excess demand is used instead of excess supply in the lemma.

[20] Nikaido(57b, auxiliary theorem). The following proof is adapted in Debreu(59, ch.5).

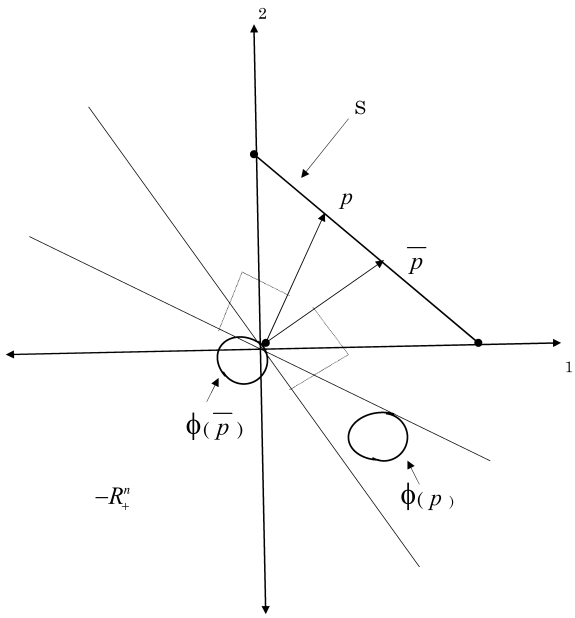

Fig.2

containing T. From lemma 7 ϕ is *closed*. Define the correspondence $\theta : \Delta \to S^n \backslash \{\emptyset\}$ by $\theta(x) = \{p \in S^n : p \cdot x \leqq q \cdot x$ for $q \in S^n\}$. The continuity of inner product and the compactness of S^n imply $\theta(x) \neq \emptyset$. Also the linearity of the inner product implies the convexity of $\theta(x)$. Note that from lemma 7 the *closedness* of θ is equivalent to the *u.h.c.* of θ. Suppose $(x^m, p^m) \in G_\theta, x^m \to x \in \Delta, p^m \to p \in S^n$. Since the definition of $\theta(\cdot)$ implies that $p^m \cdot x^m \leqq q \cdot x^m$ for any fixed $q \in S^n$, $m \to \infty$ implies $p \cdot x \leqq q \cdot x$ by the continuity of inner product. Thus $p \in \theta(x)$ and $(x,p) \in G_\theta$ follows. Then θ is *closed* and hence *u.h.c.*. Next consider the mapping $\chi : S^n \times \Delta \to (S^n \times \Delta)\backslash\{\emptyset\}$ defined by $\chi(p,x) = \theta(x) \times \phi(p)$. Then $\chi(p,x)$ is convex and compact. Moreover, the *closedness* of θ and ϕ imply the *closedness* of χ, and hence χ is *u.h.c.* from the compactness of $S^n \times \Delta$ and lemma 7. Thus, χ is a non-empty convex and compact-valued *u.h.c.* correspondence which maps from $S^n \times \Delta$ into itself. Then by Kakutani's fixed point theorem there is $(\overline{p}, \overline{x}) \in S^n \times \Delta$ satisfying $(\overline{p}, \overline{x}) \in \chi(\overline{p}, \overline{x}) = \theta(\overline{x}) \times \phi(\overline{p})$. Since $\overline{x} \in \phi(\overline{p}) \subset \Delta$ and $\overline{p} \in \theta(\overline{x})$ hold, weak Walras' law implies that $0 \leqq \overline{p} \cdot \overline{x} \leqq q \cdot \overline{x}$ holds for any $q \in S^n$. Then letting $q = e^i = (0, \cdots, 0, 1(i-th), 0, \cdots, 0)$ gives rise to $0 \leqq e^i \cdot \overline{x} = \overline{x_i}, i = 1, \cdots, n$, and hence $(\overline{x} \in)(\phi(\overline{p}) \cap R^n_+) \neq \emptyset$ follows. ■

4. Proof of the Theorem

This section proves and discusses Gale-Nikaido's lemma in an infinite dimensional case established in Nikaido(57b,59) when the excess supply correspondence is convex-valued. As before, let E be a locally convex topological vector space and P be a (non-empty) $\tau-$closed convex cone satisfying

$(-P \cap P) \neq P$. The theorem to be shown is following.

Theorem 1(convex and weak($\sigma(E, E_*)-$)compact-valued case) Suppose that the excess supply correspondence $\varphi :< P^* > \rightarrow E \backslash \{\emptyset\}$ satisfies the following four conditions: (i) $\varphi(p)$ is non-empty, convex, and weak($\sigma(E, E^*)$)$-$compact for any $p \in < P^* >$, (ii) let L be a finite dimensional subcone in P^*, $< P^* > \cap L$ and E be endowed with weak $*(\sigma(E^*, E))$ topology and weak($\sigma(E, E^*)$) topology, and φ be restricted over $< P^* > \cap L$, then the correspondence $\varphi :< P^* > \cap L \rightarrow E \backslash \{\emptyset\}$ is $u.h.c.$, (iii) $\cup_{p \in < P^* >} \varphi(p)$, the image of $< P^* >$ by φ, is weak($\sigma(E, E^*)$)$-$compact, $(iv) x \in \phi(p) \rightarrow p \cdot x \geqq 0$ holds for $p \in < P^* >$(weak Walras' law). Then there is some $\overline{p} \in < P^* >$ satisfying $\phi(\overline{p}) \cap P \neq \emptyset$.

When this theorem is compared with lemma 8 in the previous section, there is an important difference. It is that φ is not $u.h.c.$ over the entire $< P^* >$ as in R^n case of the previous section but is $u.h.c.$ only over any finite dimensional subcone $L \cap < P^* >$. A finite dimensional subcone L of P^* is the intersection between P^* and the finite dimensional linear subspace $[p_1, \cdots, p_m]$ spanned by a finite set of linear independent vectors $\{p_1, \cdots, p_m\}$. Since the locally convex topologies and the Euclidean topology are equivalent on finite dimensional spaces, and the dual space $(R^m)^*$ of a $m-$dimensional Euclidean space R^m is equivalent to R^m, in a sense, condition (ii) says that the similar continuity requirement as in the finite dimensional case also holds over a finite dimensional subcone $L \cap < P^* >$. When $u.h.c.$ of φ holds over $< P^* >$, trivially $u.h.c.$ of φ holds over $L \cap < P^* >$ as well, but the converse is not true.

Proof of theorem. Since $P \subset P(V)$ holds for $P(V)$ to which P is expanded, $P(V)^* \subset P^*$ holds. Also any $P(V)$ contains u as a common $(\tau-)$interior point but does not contain $-u$ from lemma 6. Thus $P(V)$ satisfies the condition of lemma 5 and hence $< P(V)^* >$ is non-empty and convex. Take an arbitrary finite subset $F = \{p_1, \cdots, p_m\}$ from $< P(V)^* >$. The convexity of $< P(V)^* >$ implies $co(F) \subset < P(V)^* >$, where $co(F)$ is the convex hull of F. Define two continuous functions. First, define $\alpha : E \rightarrow R^m$ by $\alpha_i(x) = p_i(x), i = 1, \cdots, m$. Since $p_i \in E^*$ implies that it is weak($\sigma(E, E^*)$)$-$continuous linear functional on E, $\alpha_i : E \rightarrow R$ is also weak($\sigma(E, E^*)$)$-$continuous and hence $\alpha : E \rightarrow R^m$ is weak($\sigma(E, E^*)$)$-$continuous as well. Next define $\beta : S^m \rightarrow < P(V)^* >$ by $\beta(w) = \sum_{i=1}^m w_i p_i$. Since β is a convex combination of $p_i, i = 1, \cdots, m$, the property of linear topology implies that β is continuous when weak $*$ $(\sigma(E^*, E))$ topology endowed on E^* is used for $< P(V)^* >$. Note that $co(F) \subset < P(V)^* >$ implies $\beta(w) \in < P(V)^* >$. Let the composite mapping $\alpha \circ \varphi \circ \beta$ of α,β, and φ be $\phi : S^m \rightarrow R^m$. Then $\phi(w) = \alpha(\varphi(\beta(w)))$ implies that when $\beta(w)$ is a convex combination of $p_1, \cdots, p_m$ with w as its coefficient, $\phi(w)$ is a vector $(p_1(\varphi(\beta(w)), \cdots, p_m(\varphi(\beta(w))))$ of the values evaluated at $p_1, \cdots, p_m$ of the excess supply $\varphi(\beta(w))$ at the price vector $\beta(w)$. Since $\beta(S^m) \subset (L \cap P(V)^*) = ([p_1, \cdots, p_m] \cap P(V)^*)$ holds (iii) implies that φ is $u.h.c.$ on $\beta(S^m)$.

This $\phi : S^m \rightarrow R^m$ satisfies the conditions of lemma 8. Let $\beta(w) = q$. Then $\varphi(q) \neq \emptyset$ implies that the value of $\varphi(q)$ evaluated at each p_i exists and hence $\phi(w) \neq \emptyset$ follows. Let $r'' = \gamma r + (1 - \gamma) r''$ for $r, r' \in \phi(w), \gamma \in (0, 1)$. Then $r, r' \in$

$\phi(w)$ implies that there are $y, y' \in \varphi(\beta(w))$ satisfying $r = \alpha(y), r' = \alpha(y')$. Since $\varphi(\beta(w))$ is convex $\gamma y + (1-\gamma)y' \in \varphi(\beta(w))$ and hence $\alpha(\gamma y + (1-\gamma)y') \in \phi(w)$ holds. Since α is linear $\alpha(\gamma y + (1-\gamma)y') = \gamma\alpha(y) + (1-\gamma)\alpha(y') = \gamma r + (1-\gamma)r' = r'' \in \phi(w)$ holds, and $\phi(w)$ is convex. Since $\varphi(q)$ is weak($\sigma(E, E^*)$)$-$compact and each p_i is weak($\sigma(E, E^*)$)$-$continuous, by Weierstraus theorem $p_i(\varphi(q))(\subset R)$ is compact and $\phi(w)(\subset R^m)$ is compact as well. Since α and β are continuous functions they are $u.h.c.$ when they are viewed as correspondences. Since ϕ is a composition mapping of $u.h.c.$ correspondences and the composition mapping of $u.h.c.$ correspondences is $u.h.c.$, ϕ is $u.h.c.$. Let $r \in \phi(w)$. Then there is $z \in \varphi(\beta(w))$ satisfying $r = \alpha(z) = (p_1(z), \cdots, p_m(z))$. (iv) implies $[\beta(w)](z) \geqq 0$ and hence $w \cdot r = \sum_{i=1}^m p_i(z)w_i = (\sum_{i=1}^m w_i p_i)(z) = [\beta(w)](z) \geqq 0$ holds. Thus weak Walras law holds for ϕ as well.

Since $\phi : S^m \to R^m$ satisfies the conditions of lemma 8, there is $\overline{w} \in S^m$ such that $\phi(\overline{w}) \cap R^m_+ \neq \emptyset$ holds, and hence $(p_1(\overline{x}), \cdots, p_m(\overline{x})) \geqq 0$ holds for some $\overline{x} \in \varphi(\beta(\overline{w}))$. Let $P(V, F) = \{x \in E : p_i(x) \geqq 0, p_i \in F\} = \{x \in E : p_i(x) \geqq 0, i = 1 \cdots, m\}$. Since $\beta(w) \in < P(V)^* >$ implies $\overline{x} \in \varphi(\beta(\overline{w})) \subset \Delta = \cup_{p \in <P(V)^*>} \varphi(p)$, $\overline{x} \in P(V, F)$ and $(P(V, F) \cap \Delta) \neq \emptyset$ follow.[21] Since p_i is weak($\sigma(E, E^*)$)$-$continuous, $P(V, F)$ is weak($\sigma(E, E^*)$)$-$closed. This result holds any finite subset F of $< P(V)^* >$. Consider the family $\{P(V, F) : F \subset < P(V)^* >, \#F < \infty\}$ for such finite subset F. Then this family satisfies the finite intersection property on Δ. Since Δ is weak($\sigma(E, E^*)$)$-$compact, $[(\cap_{F \subset <P(V)^*>} P(V, F)) \cap \Delta] \neq \emptyset$ holds. Since $P(V)$ is a $\tau-$closed convex cone and $\tau-$closedness implies weak($\sigma(E, E^*)$)$-$closedness, lemma 2 implies $P(V) = \{x \in E : p(x) \geqq 0, \forall p \in P(V)^*\} = \{x \in E : p(x) \geqq 0, \forall p \in < P(V)^* > \} \subset P(V, F)(= \{x \in E : p(x) \geqq 0, p \in F, \#F < \infty\})$ and hence $P(V) \subset (\cap_{F \subset <P(V)^*>} P(V, F))$ holds. On the other hand, letting $F = \{p\}$ for $p \in P(V)^*$ implies $(\cap_{F \subset <P(V)^*>} P(V, F)) \subset (\cap_{\{p\} \subset <P(V)^*>} P(V, \{p\})) = \{x \in E : p(x) \geqq 0, \forall p \in < P(V)^* >\} = P(V)$. Thus, $P(V) = (\cap_{F \subset <P(V)^*>} P(V, F))$ holds. Therefore, $[P(V) \cap \Delta] \neq \emptyset$ follows from $[(\cap_{F \subset <P(V)^*>} P(V, F)) \cap \Delta] \neq \emptyset$. This holds for any convex balanced neighborhood V of 0. Since the family $\{P(V)$: any convex balanced neighborhood V of $0\}$ of $P(V)$ also satisfies the finite intersection property on Δ as well, $[(\cap_{V \in N(0)} P(V)) \cap \Delta] \neq \emptyset$ follows. Since (ii) of lemma 6 implies $(\cap_{V \in N(0)} P(V)) = P$, $(P \cap \Delta) \neq \emptyset$ holds. Thus, there is some $\overline{p} \in < P(V)^* >$ satisfying $(\varphi(\overline{p}) \cap P) \neq \emptyset$. This is the result to be shown. ■

Now discuss several aspects of this theorem.[22] First consider the condition where P is chosen so as to be a $\tau-$closed convex cone and to satisfy $((-P) \cap P) \neq P$. When this does not hold, the conclusion that $(\varphi(\overline{p}) \cap P) \neq \emptyset$ holds for some $\overline{p} \in < P(V)^* >$ in the theorem may not hold. Let $E = R^3$ and P be a hyperplane M passing through the origin 0. Then $(P \cap (-P)) = P$ holds and hence $(P \cap (-P)) \neq P$ does not hold. Moreover P^* is a straight line l which passes the origin and is perpendicular to M. Thus, $< P(V)^* > = P(V)^* \backslash \{0\}$ is $l \backslash \{0\}$, a straight line l besides the origin 0. $< P(V)^* >$ is composed of two half straight lines l_1 and l_2 which do not contain the origin 0. Take and fix

[21] This result is shown in the general case where $\varphi(x)$ is acyclic in Nikaido(59) with using the general result which is applicable to the existence of Nash equilibrium in abstract economies and minimax theorem. Here, however, in the case where $\varphi(x)$ is convex, the same result follows much easily with the method of proof used in Nikaido(57b).

[22] The following disccusions are taken from Nikaido(1957b, 59).

$x_1 \in l_1$ and $x_2 \in l_2$, and let $\varphi(p) = x_1(x_2)$ for $p \in l_1(l_2)$. Then φ satisfies the condition $(i) - (iv)$ of the above theorem. In particular, *u.h.c.* of φ follows since disconnectedness of l_1 and l_2 at the origin 0 makes the value of φ able to jump between x_1 and x_2 at the origin 0. But the definition of φ implies $(\varphi(< P(V)^* >) \cap M) = (\varphi(l \backslash \{0\}) \cap M)(\subset (l \backslash \{0\}) \cap M)) = \emptyset$, and hence the conclusion of the theorem does not hold.

Next consider the conditions (ii) and (iii). In the finite dimensional case of $E = R^m$, $(R^m)^* = R^m$ implies $P^* \subset R^m$ and (ii) becomes that φ is *u.h.c.* over entire P^*. Define $\psi :< P^* > \rightarrow R^m$ by $\psi(p) = \varphi(p/||p||)$. Then φ is restricted to $B^m \cap P^*$ where $B^m = \{p \in R^m : ||p|| = 1\}$ is the unit ball in R^m. Since B^m and hence $B^m \cap P^*$ is compact, lemma 7 implies that $\psi(< P^* >) = \cup_{p \in < P^* >} \psi(p) = \cup_{q \in (B^m \cap P^*)} \varphi(q) = \varphi(B^m \cap P^*)$ is compact. Since $p/||p||$ is continuous on $< P^* >$ and the composition mapping of *u.h.c.* correspondences is also *u.h.c.*, ψ is *u.h.c.* on $< P^* >$. Moreover (i) and (iv) follow automatically from the assumption, $\psi :< P^* > \rightarrow R^m$ satisfies the conditions $(i) - (iv)$ of the theorem. Thus $(\psi(\overline{p}) \cap P) = (\varphi(\overline{p}/||\overline{p}||) \cap P) \neq \emptyset$ holds for some $\overline{p} \in < P(V)^* >$. That is, $(\varphi(\overline{p}/||\overline{p}|| \cap P) \neq \emptyset$ holds for some $\overline{p}/||\overline{p}|| \in < P^* >$ when $\varphi :< P^* > \rightarrow R^m$ satisfies only (i),(ii), and (iv). In the finite dimensional case of $E = R^m$, the condition (iii) of the theorem is unnecessary to φ. Note that P^{**} has interior points when P^* is pointed. Thus when a closed convex cone which is pointed is used for P^* from the beginning and the corresponding P^{**} is used for P, then Debreu(56)'s generalization of Gale-Nikaido's lemma in the S^m case of lemma 7 to the one in the case of the pointed closed convex cone P^* case follows.[23)]

Consider the case where E is a general infinite dimensional locally convex topological space. Suppose that $int_\tau(P) \neq \emptyset$ holds and (ii) is strengthened to $(ii)'$ $\varphi :< P^* > \rightarrow E$ is *u.h.c.* on $< P^* >$ where $< P^* >$ and E are endowed with the weak $*(\sigma(E^*, E))$ topology and the weak $(\sigma(E, E^*))$ topology, respectively.[24)] $(ii)' \rightarrow (ii)$ holds automatically. Take $u \in int_\tau(P)$ and let $C_u = \{p \in P^* : p(u) = 1\}$. Since u is a $\tau-$interior point of P, $(u + W) \subset P$ holds for a convex balanced neighborhood W of the origin 0. Then the balancedness of W implies $p(u + w) \geqq 0, p(u - w) \geqq 0$ for any $w \in W$ and $p \in P^*$. By the definition $p \in P^*$ implies $p(u) \geqq 0$. Since $p \neq 0$ implies $p(w) \neq 0$ for some $w \in W \backslash \{0\}$, $p(u) = 0$ does not hold. Thus, $p(u) > 0$ holds for $p \in < P^* >$. Therefore $p/p(u) \in C_u$ holds for $p \in < P^* >$. Since $q(u+w) = q(u)+q(w) = 1+q(w) \geqq 0$ holds for any $q \in C_u$ and $w \in W$, $q(w) \geqq -1$ holds for $w \in W$ and $q \in C_u$. Since the balancedness of W implies $-w \in W$, $q(-w) \geqq -1$ holds and hence $q(w) \leqq 1$ holds. Thus, $|q(w)| \leqq 1$ follows. This implies that C_u is contained in the polar W° of W. Since W° is weak $*(\sigma(E^*, E))-$compact by Alaogru-Bourbaki theorem and C_u is weak $*(\sigma(E^*, E))-$closed by the weak $*(\sigma(E^*, E))-$continuity of u on E^*, C_u is weak $*(\sigma(E^*, E))-$compact as well.[25)] Define $\psi :< P^* > \rightarrow E$ by $\psi(p) = \varphi(p/p(u))$. Then $\psi(< P^* >) = \varphi(C_u)$ holds. Since C_u is weak $*(\sigma(E^*, E))-$compact and φ is *u.h.c.* on C_u with

[23)]Nikaido(68, theorem 3.12(*ii*)).

[24)]Although Nikaido(59) considers this fact only in the normed space case, in fact, it holds even in this general infinite dimensional locally convex space case.

[25)]By Alaogru-Banach theorem $U^o = \{p \in E^* : |p \cdot x| \leqq 1\}$, the polar of U, is weak $*(\sigma(E^*, E))-$compact for a neighborhood U of the origin 0 in E.

respect to weak $*(\sigma(E^*, E))-$topology, lemma 7 implies that $\varphi(C_u)$ is weak $*(\sigma(E^*, E))-$compact. Since $p/p(u)$ is weak $*(\sigma(E^*, E))-$continuous on $< P^* >$, ψ is $u.h.c.$ on $< P^* >$ with respect to weak $*(\sigma(E^*, E))-$topology. Moreover, (i) and (iv) hold from the assumption. Thus, $\psi :< P^* >\rightarrow E$ satisfies the conditions $(i)-(iv)$ of the theorem. Therefore $(\psi(\overline{p})\cap P) = (\varphi(\overline{p}/\overline{p}(u))\cap P) \neq \emptyset$ holds for some $\overline{p} \in< P(V)^* >$. That is, $(\varphi(\overline{p}/\overline{p}(u)) \cap P) \neq \emptyset$ holds for $\overline{p}/\overline{p}(u) \in< P^* >$ when $\varphi :< P^* >\rightarrow E$ satisfies only (i), $(ii)'$, and (iv). In this situation, the condition (iii) is unnecessary to φ.

Florenzano(83) shows that $(\varphi(\overline{p}/\overline{p}(u)) \cap P) \neq \emptyset$ holds for $\overline{p}/\overline{p}(u) \in C_u$ in the case where E is a locally convex space, $int_\tau(P) \neq \emptyset$ holds, and $\varphi : C_u \rightarrow E$ satisfies (i),$(ii)'$, and (iv).[26] Note that this result of Florenzano(83) holds under $int_\tau(P) \neq \emptyset$, but the above theorem holds as long as $(P \cap (-P)) \neq P$ holds even without $int_\tau(P) \neq \emptyset$.

Although Nikaido(56b) in the normed space case is unavailable nowadays unfortunately, the result obtained in Nikaido(56b), however, may be stated as following from the part of Nikaido(57b) which mentioned the former. That is, Nikaido(56b) proves Gale-Nikaido's lemma in the case where E is an infinite dimensional normed space, P satisfies the interiority condition $int_\tau(P) \neq \emptyset$, and $(ii)'$ $\varphi :< P^* >\rightarrow E$ is $u.h.c.$ when $< P^* >$ is endowed with weak $*(\sigma(E^*, E))-$topology and E is with weak $(\sigma(E, E^*))-$topology.[27] And the method of the proof employed is similar to the one used in this paper as of Nikaido(57b).[28] It uses finite dimensional Gale-Nikaido's lemma and an approximation of the entire space by a family of finite dimensional subspaces. Note that the method of the proof for $\varphi :< P^* >\rightarrow E$ in this section uses the weak $(\sigma(E, E^*))-$compactness of $\Delta = \cup_{p\in<P(V)^*>}\varphi(p)$ stated as the condition (iii), and it does not use the weak $*(\sigma(E^*, E))-$compactness of C_u as in Nikaido(56b) and Florenzano(83). Note, however, that (iii) may not hold for some economies with infinite dimensional commodity spaces and only a weaker condition $(iii)'$ $\cup_{p\in<P^*>} \varphi(p)$, the image of $< P^* >$ by φ, is weak$(\sigma(E, E^*))-$relatively compact holds although condition (i), (ii), and (iv) hold. Thus, it is quite interesting to prove Gale-Nikaido's lemma in infinite dimensions when condition (iii) is weakened to $(iii)'$. The purpose of Flo-

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[26]Florenzano(83, corollary 1, p. 213-4). The proof of its original result, Florenzano(83, lemma 1, p. 212), is almost same as of the proof here since it uses approximation of the original space by a family of finite dimensional subspaces to which finite dimensional Gale-Nikaido's lemma applies. The choice of the original linear topology, however, are made a bit differently so that the result obtained applies to the Banach predual space case such as l_∞. Infinite dimensional Gale-Nikaido's lemma is further generalized by Yannelis(85) and Mehta-Tarafdar(87) as to the continuity requirement for the correspondence. Also Aliprantis-Brown(83) considers excess demand functions in economies with topological vector lattices as the commodity spaces and establishes the existence of competitive equilibrium in such economies. These results are further generalized in Urai(2000).

[27]Nikaido(57b, p.4, Remark 2, p.11-2).

[28]From the part of Nikaido(57b) which mentioned the content of Nikaido(56b), what Nikaido(56b) tried to do is to establish infinite dimensional Gale-Nikaido' lemma in an economy where $int_\tau(P) \neq \emptyset$, the interioiry condition, is used and weak$(\sigma(E, E^*)-)$ topology is used on E. When Prof. Nikado constructed an economy with l_∞ as its conommodity space and tried to prove the existence of competitive equilibrium in such an economy, he went further to sharpen his version of infinite dimensional Gale-Nikaido's lemma in a different direction from those of Nikaido(57b,59).

renzano(83) is indeed to consider the existence of competitive equilibrium in such situations, and establishes the one in economies with Banach predual spaces such as L_∞ for the infinite dimensional commodity spaces when (iii) is replaced by $(iii)'$. From this viewpoint, in the next section, Gale-Nikaido's lemma in an infinite dimension is extended to the case where $(iii)'$ holds.

5. A Generalization:Weak($\sigma(E,E^*)$)—Relatively Compact-Valued Case

This section considers the case with $(iii)'$: weak $(\sigma(E,E^*))-$relative compactness of $\Gamma = \cup_{p\in<(-P)^*>}\varphi(p)$.[29] This is a generalization of theorem 1 with (iii) : weak $(\sigma(E,E^*))-$compactness of $\Gamma = \cup_{p\in<(-P)^*>}\varphi(p)$. The proof of this result also uses the methods similar to the one used in the proof of Theorem 1 in the previous section where the finite dimensional approximation of the original space based on (ii) is used. There is, however, a difference in the proof. Although the existence of $\overline{p} \in< (P)^* >$ satisfying $(\varphi(\overline{p}) \cap P) \neq \emptyset$ is established directly from (iii) in the proof of Theorem 1, when $(iii)'$ is used instead of (iii) in the proof of the theorem in this section, the existence of $\overline{p} \in< (P)^* >$ satisfying $(\varphi(\overline{p}) \cap P) \neq \emptyset$ is established indirectly as the limit of a net of prices. For this purpose, it is necessary to make the price set weak $*(\sigma(E^*,E))-$compact, and hence, to use as the commodity space a locally convex topological space that has $\tau-$closed convex cone P with $int_\tau(P) \neq \emptyset$. The argument using lemma 6 goes through in the case of finite dimensional spaces, but the similar argument does not in the case of infinity dimensional spaces treated here, and hence, $int_\tau(P) \neq \emptyset$ is required as shown below. [30]

Let's pick an arbitrary element $u \in int_\tau(P) \neq \emptyset$, and make $\Delta^* = \{p \in P^* : p \cdot u = 1\}$ as the price space. Then the argument in the last part of the previous section implies that Δ^* is weak $*(\sigma(E^*,E))-$compact. As already mentioned above, the existence of an equilibrium price in Δ^* is shown not directly but indirectly in a sense that it is a limit of a net in Δ^*. This requires to consider the limit of a net in the graph of the excess supply correspondence, and hence it is necessary to consider the extension of the graph of the excess supply correspondence so that this extended graph indeed contains the limit of a net in the graph of the original excess supply correspondence.

Let (S,τ_S) and (X,τ_X) be two Hausdorff topological spaces, and $\varphi : S \to X\backslash\{\emptyset\}$ be a correspondence from S into X. Consider the closure $cl_{\tau_S\times\tau_X}(G_\varphi)(\subset S\times X)$ of the graph $G_\varphi = \{(p,x) \in S\times X : x \in \varphi(p)\}$ of φ in $S\times X$ with respect to the product topology $\tau_S\times\tau_X$. Define the correspondence $\widetilde{\varphi} : S \to X\backslash\{\emptyset\}$ by $x \in S \to \widetilde{\varphi}(p) = \{x \in X : (p,x) \in cl_{\tau_S\times\tau_X}(G_\varphi)\}$. Since $(p,x) \in G_\varphi \subset cl_{\tau_S\times\tau_X}(G_\varphi)$ holds from the definition, $x \in \varphi(p) \subset \widetilde{\varphi}(p)$ follows $\forall p \in \Delta^*$, and hence $\widetilde{\varphi} : S \to X\backslash\{\emptyset\}$ is considered as an extension of $\varphi : S \to X\backslash\{\emptyset\}$. Of course, $\widetilde{\varphi}(p) = \varphi(p)$ does not necessarily hold and only $\widetilde{\varphi}(p) \supset \varphi(p)$ holds even when $\varphi(p)$ is τ_X-closed. When, however, $\varphi : \Delta^* \to E$ is $u.h.c.$ on Δ^*

[29] Florenzano(83) also considers a similar case to here and the commodity space E is a dual space of another topological vector space (G,τ^G) so that $E = (G,\tau^G)^*$ holds. This is corresponding to the case where $G = l_1$ and $E = l_\infty$ holds.

[30] Of courese, when the excess supply correspondence is defined on a weak $*(\sigma(E^*,E)-)$compact price set as in the finite dimensional case, the approximating argument based on lamma 6 still works and $int_\tau(P) \neq \emptyset$ is unnecessary.

as in $(ii)'$ in the last part of the previous section, lemma 7 and the weak $*(\sigma(E^*, E))-$compactness of Δ^* imply the weak $*(\sigma(E, E^*))-$compactness of $\varphi(\Delta^*)$, and hence $cl_{\sigma(E^*,E)\times\sigma(E,E^*)}(G_\varphi) = G_\varphi$ holds. This implies $\widetilde{\varphi}(p) = \varphi(p)$ $\forall p \in \Delta^*$ so that it is enough to use the original $\varphi : \Delta^* \to E\backslash\{\emptyset\}$, and it is not necessary to use the extended $\widetilde{\varphi} : \Delta^* \to E\backslash\{\emptyset\}$.

Now let's consider the generalization of theorem 1 in the case of weak $(\sigma(E, E^*))-$ relative compact-valued case. Let E be a locally convex topological vector space and $P(\neq E)$ be a $\tau-$, and hence, weak $(\sigma(E, E^*))-$closed convex cone with $int_\tau(P) \neq \emptyset$. Let $\varphi : \Delta^* \to E\backslash\{\emptyset\}$ be an excess supply correspondence where the price set Δ^* is defined to be $\{p \in P^* : p \cdot u = 1\}$ for some $u \in int_\tau(P)$. Δ^* is weak $*(\sigma(E^*, E))-$compact as already mentioned. The modified version of the Gale-Nikaido's lemma in an infinite dimensional space is following.

Theorem 2(convex and weak($\sigma(E, E_*)$)$-$relatively compact-valued case) Suppose that the excess supply correspondence $\varphi : \Delta^* \to E\backslash\{\emptyset\}$ satisfies the following four conditions: $(i)'$ $\varphi(p)$ is non-empty and convex for any $p \in \Delta^*$, $(ii)''$ let L be an arbitrary finite dimensional subcone in E^*, $\Delta^* \cap L$ and E be endowed with weak $*(\sigma(E^*, E))$ topology and weak($\sigma(E, E^*)$) topology, and φ be restricted over $\Delta^* \cap L$, then the correspondence $\varphi : \Delta^* \cap L \to E\backslash\{\emptyset\}$ is $u.h.c.$, $(iii)'$ $\cup_{p\in\Delta^*} \varphi(p)$, the image of Δ^* by φ, is contained in a weak($\sigma(E, E^*)$)$-$compact subset Ψ, (iv) $x \in \phi(p) \to p \cdot x \geqq 0$ holds for $p \in \Delta^*$ (weak Walras' law). Let define the extended correspondence $\widetilde{\varphi} : \Delta^* \to \Psi$ by $p \in \Delta^* \to \widetilde{\varphi}(p) = \{x \in \Psi : \exists x \in E, (p, x) \in cl_{\sigma(E^*,E)\times\sigma(E,E^*)}(G_\varphi)\}$. Then there is some $\widetilde{p} \in \Delta^*$ satisfying $\widetilde{\varphi}(\overline{p}) \cap P \neq \emptyset$.[31)]

First of all, φ is not $u.h.c.$ over the entire Δ^* as in theorem 1 of the previous section but is $u.h.c.$ only over any finite dimensional subcone $L \cap \Delta^*$. Since a finite dimensional subcone L of Δ^* is the intersection between Δ^* and the finite dimensional linear subspace $[p_1, \cdots, p_m]$ spanned by a finite set of linear independent vectors $\{p_1, \cdots, p_m\}$ in $< P^* >$, $\{p_1, \cdots, p_m\}$ are adjusted proportionally so that each p_i is included as p'_i in Δ^*. The crucial difference from theorem 1 is that $\Gamma = \cup_{p\in\Delta^*}\varphi(p)$ is assumed not to be weak($\sigma(E, E^*)$)$-$compact but to be included in a weak($\sigma(E, E^*)$)$-$compact set Ψ. The excess supply correspondence φ is defined over Δ^* and, as in theorem 1, the result of lemma 7 in the case of the finite dimensional spaces also holds here since $(ii)''$ implies that the conditions of lemma 7 holds on a finite dimensional convex subcone L. This does not, however, necessarily imply that the excess supply correspondence φ has the closed graph in $\Delta^* \times \Psi$. Although, as seen below, an equilibrium price is found as the limit of a net in Δ^* so that $(p, x) \to (\widetilde{p}, \widetilde{x})$ holds for $(p, x) \in G_\varphi$, $(\widetilde{p}, \widetilde{x}) \in G_\varphi$ does not necessarily hold and only $(\widetilde{p}, \widetilde{x}) \in cl_{\sigma(E^*,E)\times\sigma(E,E^*)}(G_\varphi)$ holds. This is the reason why the excess supply correspondence φ is extended to $\widetilde{\varphi}$. Although $\varphi(\overline{p}) \cap P \neq \emptyset$ is unable to be established, $\widetilde{\varphi}(\overline{p}) \cap P \neq \emptyset$ is definitely able to be established once G_φ is extended to $cl_{\sigma(E^*,E)\times\sigma(E,E^*)}(G_\varphi)$ on $\Delta^* \times \Psi$.

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31) The result is corresponding to Florenzano(83, lemma 1, pp.212-3).

Proof of Theorem 2. Let $F = \{p_1, \cdots, p_m\}$ be a finite set of arbitrarily selected points in Δ^*. Since Δ^* is convex, the convex full $co(F)$ of F is included in Δ^*. Define two continuous functions. First, define $\alpha : E \to R^m$ by $\alpha_i(x) = p_i(x), i = 1, \cdots, m$. Since $p_i \in E^*$ is a weak($\sigma(E, E^*)$)$-$continuous linear functional on E, $\alpha_i : E \to R$ is also weak($\sigma(E, E^*)$)$-$continuous, and hence so is $\alpha : E \to R^m$. Next define $\beta : S^m \to \Delta^*$ by $\beta(w) = \sum_{i=1}^m w_i p_i$. Since β is a convex combination of $p_i, i = 1, \cdots, m$, the basic property of linear topology implies that β is continuous with respect to the weak$*$($\sigma(E^*, E)$)$-$topology on E^*. Also $co(F) \subset \Delta^*$ implies $\beta(w) \in \Delta^*$. Let $\phi : S^m \to R^m$ be the composite mapping of α, β, and φ. Then $\phi(w) = \alpha(\varphi(\beta(w)))$ means that $\phi(w)$ is $(p_1(\varphi(\beta(w))), \cdots, p_m(\varphi(\beta(w))))$. This is the vector of evaluation with $p_1, \cdots, p_m$ of the set of excess supply $\varphi(\beta(w))$ at the price vector $\beta(w)$ which is a convex hull of $p_1, \cdots, p_m$ with its weight equal to w. Since $\beta(S^m) = co(F) \subset \Delta^*$ holds, (iii) implies that ϕ is u.h.c. on $\beta(S^m)$. As in the proof of theorem 1, it is easily show that ϕ satisfies the conditions of lemma 8, and hence, there is $\overline{w_F} \in S^m$ so that $\phi(\overline{w_F}) \cap (R^m_+) \neq \emptyset$ holds. Thus, $(p_1(\overline{x_F}), \cdots, p_m(\overline{x_F})) \geqq 0$ holds for some $\overline{x_F} \in \varphi(\beta(\overline{w_F}))$. Since $\beta(\overline{w_F}) \in \Delta^*$ implies $\overline{x_F} \in \varphi(\beta(\overline{w_F})) \subset \Gamma = \cup_{p \in \Delta^*}\varphi(p) \subset \Psi$ and Ψ is weak($\sigma(E, E^*)$)$-$compact, $\overline{x_F} \in \Gamma \subset \overline{\Gamma} = cl_{\sigma(E,E^*)}(\Gamma) \subset \Psi$ holds. Also $\overline{x_F} \in P(F)$ holds for $P(F) = \{x \in E : p_i(x) \geqq 0, p_i \in F\} = \{x \in E : p_i(x) \geqq 0, i = 1 \cdots, m\}$. Thus, $\overline{x_F} \in (\Gamma \cap P(F)) \subset (\overline{\Gamma} \cap P(F)) \neq \emptyset$ holds.[32] Since each p_i is weak($\sigma(E, E^*)$)$-$continuous $i = 1 \cdots, m$, $P(F)$ is weak($\sigma(E, E^*)$)$-$closed. This result holds for any finite subset F in Δ^*. Consider the family of sets $\{P(F) : F \subset \Delta^*, \#F < \infty\}$. Then it satisfies finite intersection property on $\overline{\Gamma}$ since any finite union of finite subsets is also a finite subset. Since Ψ is weak($\sigma(E, E^*)$)$-$compact and hence $\overline{\Gamma}$ is weak($\sigma(E, E^*)$)$-$compact, $\cap\{\overline{\Gamma} \cap P(F) : F \subset \Delta^*, \#F < \infty)\} = [(\cap_{F \subset \Delta^*, \#F<\infty} P(F)) \cap \overline{\Gamma}] \neq \emptyset$ holds from the property of compact sets. Since P is weak($\sigma(E, E^*)$)$-$closed convex cone, lemma 2 implies that $P = \{x \in E : p(x) \geqq 0, \forall p \in P^*\} = \{x \in E : p(x) \geqq 0, \forall p \in < P^* > \} \subset \{x \in E : p(x) \geqq 0, p \in F, \#F < \infty, F \subset \Delta^*\}$ holds for $P^* = \{p \in E^* : p(x) \geqq 0, \forall p \in P\}$ when E is equipped with the weak($\sigma(E, E^*)$)$-$topology. Thus, $P \subset (\cap_{F \subset \Delta^*, \#F<\infty} P(F))$ holds. Since, however, letting $F = \{p\}$ for $p \in \Delta^*$ implies $(\cap_{F \subset \Delta^*, \#F<\infty} P(F)) \subset (\cap_{\{p\} \subset \Delta^*} P(\{p\})) = \{x \in E : p(x) \geqq 0, \forall p \in < P^* >\} = P$, and hence, $(\cap_{F \subset \Delta^*, \#F<\infty} P(F)) \subset P$ holds. Thus, $P = (\cap_{F \subset \Delta^*, \#F<\infty} P(F))$ and hence, $[(\cap_{F \subset \Delta^*, \#F<\infty} P(F)) \cap \overline{\Gamma}] \neq \emptyset$ implies $(P \cap \overline{\Gamma}) \neq \emptyset$. Let $N^*(0)$ be the weak($\sigma(E, E^*)$)$-$neighborhood system of 0. Choose $\overline{x} \in (P \cap \overline{\Gamma})$. Then $\overline{x} \in \overline{\Gamma}$ implies that $[(\overline{x} + V) \cap \Gamma] \neq \emptyset$ holds for any weak($\sigma(E, E^*)$)$-$neighborhood V of 0, and hence, there is $x_V \in \Gamma$ for any $V \in N^*(0)$ satisfying $x_V \in (\overline{x} + V)$. $x_V \in \Gamma = \cup_{p \in \Delta^*}\varphi(p)$ implies that there is $p_V \in \Delta^*$ satisfying $x_V \in \varphi(p_V)$. Define a directed set $(N^*(0), \prec)$ by $V \prec V' \iff V' \subset V$. Then $(p_V, x_V)_\prec$ forms a net according to this order $\prec$. Since there is $x_V \in \Gamma$ for any $V \in N^*(0)$ satisfying $x_V \in (\overline{x} + V)$, $x_V \to \overline{x}(\prec\uparrow)$ follows. Moreover, since Δ^* and $\overline{\Gamma}$ are weak($\sigma(E, E^*)$)$-$compact and weak $*$($\sigma(E^*, E^*)$)$-$compact, respectively, the net $(p_V, x_V)_\prec$ has a converging subnet, which is again expressed as $(p_V, x_V)_\prec$ for simplicity, $(p_V, x_V) \to (\overline{\pi}, \overline{x}) \in \Delta^* \times \overline{\Gamma}$ holds as $\prec\uparrow$. Since $x_V \in \varphi(p_V)$ and $p_V \in \Delta^*$ imply $(p_V, x_V) \in G_r(\varphi)$, $(\overline{\pi}, \overline{x}) \in (cl_{\sigma(E^*,E)\times\sigma(E,E^*)}(G_r(\varphi)))(\subset \Delta^* \times \overline{\Gamma})$

[32] Up to here, the proof based on Nikaido(56b,57b,59) and the proof of Florenzano(83) are same. They are different in the follwoing argement. The argment of Florenzano(83) in the following argment is also shown later to make clear the difference of these two.

holds and hence $\overline{x} \in \widetilde{\varphi}(\overline{\pi})$ holds. Then, $\overline{x} \in (P \cap \overline{\Gamma})$ implies $\overline{x} \in (\widetilde{\varphi}(\overline{\pi}) \cap P)$. Therefore, $(\overline{x} \in)(\widetilde{\varphi}(\overline{\pi}) \cap P) \neq \emptyset$ holds for some $\overline{\pi} \in \Delta^*$, which is to be shown. ■

Although the above proof follows the one used in Nikaido(57b, 59) which is also used in the proof of theorem 1, it uses the weak $*(\sigma(E^*, E))$ −compactness of Δ^*. Note that theorem 1 uses the weak$(\sigma(E, E^*))$−compactness of Γ. As its consequence, $\overline{\pi} \in \Delta^*$ still holds but $\overline{x} \in (\varphi(\overline{\pi}) \cap P)$ does not necessarily hold. When $(P \cap (-P)) \neq P$ is used instead of $int_\tau(P) \neq \emptyset$ so that Δ^* is not necessarily weak $*(\sigma(E^*, E^*))$−compact, it is possible to show only $(P \cap \overline{\Gamma}) \neq \emptyset$, i.e., there is $x \in P$ such that for any $V \in N^*(0)$ $x_V \in (\varphi(p_V) \cap (x + V)$ holds for some $p_V \in < P^* >$. Although $x_V \to x$ holds, (p_V) does not necessarily have the weak $*(\sigma(E^*, E))$−limit since Δ^* is not necessarily weak $*(\sigma(E^*, E))$−compact. Also when $\Delta_V = \{p \in (P(V))^* : p \cdot u = 1\}(\subset \Delta^*)$ is used, it is weak $*(\sigma(E^*, E))$−compact and $V \supset V' \iff \Delta_V \subset \Delta_{V'}$ holds. Then Δ^* is approximated from inside by $\{\Delta_V : V \in N^*(0)\}$. But although $p_V \in \Delta_V$ gives rise to a net $(p_V : V \in N^*(0))$ in $(\cup_{V \in N^*(0)} \Delta_V \subset)\Delta^*$, since $\cup_{V \in N^*(0)} \Delta_V$ and Δ^* are not necessarily weak $*(\sigma(E^*, E))$−compact, the net $(p_V : V \in N^*(0))$ does not necessarily have the weak $*(\sigma(E^*, E))$−limit in $\cup_{V \in N^*(0)} \Delta_V$ or in Δ^*. Thus, in any case, the assumption $u \in int_\tau(P)$ makes Δ^* weak $*(\sigma(E^*, E))$−compact, and hence, (p_V) has a converging subnet whose limit is in Δ^*. In the case of finite dimensional space, on the other hand, the compactness of unit ball $B = \{p \subset R^n : ||p|| = 1\}$ implies that, even when P is approximated by P_V from outside, $||p_V|| = 1$ gives rise to a converging subsequence $p_V \to \overline{p}, ||\overline{p}|| = 1$. Thus, existence of such a $\overline{p}$ holds not like the case of infinite dimensional spaces.

The method of proof used in Florenzano(83, p.213) may be used to the argument below $(P(F) \cap \overline{\Gamma}) \neq \emptyset$ in the above proof. Consider the family $\boldsymbol{F} = \{F : F \subset \Delta^-, \#F < \infty, (P(F) \cap \overline{\Gamma}) \neq \emptyset\}$ and define $F \prec F' \iff F' \subset F$. Then $(\boldsymbol{F}, \prec)$ is a directed set and hence $(\overline{x_F}, \beta(\overline{w_F}))_\prec$ satisfying $\overline{x_F} \in (P(F) \cap \Psi), \beta(\overline{w_F}) \in \Delta^*$ becomes a net. Since Δ^* and Ψ are weak $*(\sigma(E^*, E))$−compact and weak$(\sigma(E, E^*))$−compact, respectively, $(\overline{x_F}, \beta(\overline{w_F}))_\prec$ has a converging subnet so that $(\overline{x_F}, \beta(\overline{w_F})) \to (\overline{x}, \overline{\pi}) \in \Psi \times \Delta^*$ holds. Since $\overline{x_F} \in \varphi(\beta(\overline{w_F}))$ and $\beta(\overline{w_F}) \in \Delta^*$ imply $(\beta(\overline{w_F}), \overline{x_F}) \in G_r(\varphi)$, $(\overline{\pi}, \overline{x}) \in cl_{\sigma(E^*,E) \times \sigma(E,E^*)}(G_r(\varphi))$ and hence $\overline{x} \in \widetilde{\varphi}(\overline{\pi})$ holds. The object is to show $\overline{x} \in P$. Suppose $\overline{x} \notin P$. Then the weak$(\sigma(E, E^*))$−closedness of P and lemma 1 give rise to the existence $q \in (P)^* \backslash \{0\}$ with satisfying $q \cdot \overline{x} < 0 \leqq q \cdot y \ \forall y \in P$. $q \in P^* \backslash \{0\}$ implies $q \cdot u > 0$ and hence $q/q \cdot u = q' \in \Delta^*$ and $q' \cdot \overline{x} < 0$ hold. From the definition of weak $*(\sigma(E^*, E))$−topology and $\overline{x_F} \to \overline{x}$, $q' \cdot \overline{x_F} \to q' \cdot \overline{x} > 0 (\prec\uparrow)$ holds. Thus, $q' \cdot \overline{x_{F'}} > 0$ holds for any sufficiently further, in the sense of $\prec$, F' with $q' \in F'$, and hence $x_{F'} \notin P(F')$ holds for such F'. This is, however, a contradiction since $\overline{x_F} \in P(F)$ holds for any finite set F in Δ^*. Thus, $\overline{x} \in P$ holds and hence $(\overline{x} \in)(\widetilde{\varphi}(\overline{\pi}) \cap P) \neq \emptyset$ holds for some $\overline{\pi} \in \Delta^*$. This Florenzano(1983)'s argument uses the family $\boldsymbol{F} = \{F : F \subset \Delta^-, \#F < \infty, (P(F) \cap \overline{\Gamma}) \neq \emptyset\}$, on the other hand, the proof of theorem 2 uses the weak$(\sigma(E, E^*))$−neighborhood system of 0, $N^*(0)$.

6. Conclusion

So far, this paper shows in a quite simple way Gale-Nikaido's lemma in the locally convex topological vector space case with a convex-valued excess supply correspondence. Here follows a suggestion of Nikaido(57b, p.4, remark 2) which mentioned that the proof of Gale-Nikaido's lemma in normed space case with a convex-valued excess supply correspondence can be easily extended to the one in a topological vector space case. Although this proof is very straight forward, it is, however, unfortunate that any of later papers on the theory of general equilibrium with infinite dimensional commodity spaces have not referred to Nikaido(56b, 57b, 59) as one of early attempts besides Debreu(54).

Let relate this proof to the other similar proofs. Bewley(72) uses the method of approximating the original economy with infinite dimensional space by a family of finite dimensional subeconomies to which Arrow-Debreu-McKenzie theorem applied in the proof of the existence of competitive equilibrium in an infinite dimensional commodity space. This approach is same at the bottom in the method of the proof used in Nikaido(56b, 57b). Also, Fan-Glicksberg's generalization of Kakutani's fixed point theorem to locally convex topological vector space uses as its proof approximating the original infinite dimensional space by a family of finite dimensional subspaces where Kakutani's fixed point theorem in the finite dimensional case is applied. In these arguments, there is a common feature in approximating the original space that Kakutani's fixed point theorem, Arrow-Dereu-McKenzie theorem, or Gale-Nikaido's lemma is applied to a family of its finite dimensional subeconomies. [33)]

There seems several reasons why Nikaido (56b, 57b, 59) is neglected by the literature on the theory of general equilibrium with infinite dimensional commodity spaces. The first reason is that Nikaido(56b, 57b) are in the form of discussion papers and were not published in any journals, and Nikaido(59) is published in a mathematical journal, not economics journals, issued by the mathematical society of Japan.[34)] The second reason is although Nikaido(59) is published in a journal, its tittle did not have any words suggesting economic theory on infinite dimensional commodity spaces not like Nikaido(56b), and the proof of Gale-Nikaido's lemma in topological vector space case of Nikaido(59) is too general to follow since the excess supply correspondence is assumed to be acyclic-valued instead of convex valued. As shown in this paper, the proof in the case with convex-valued excess supply correspondence is very straight forward and easy to follow even in the general topological vector space case. The third reason is that any of Nikaido(56b, 57b, 59) did not contain any interesting economic examples which uses infinite dimensional commodity spaces. It was desirable to construct an economic example

[33)] From this viewpoint, late Professor Nikaido had an opinion that finite dimensional fixed point theorem is much more fundamental than infinite dimensional ones. Note also that these three results are known to be equivalent in the finite dimensional setting.

[34)] Nikaido(59) is referred in Nikaido(1968) which is a famous text book on general equilibrium theory and referred by many general equilibrium theorist including authors on infinite dimensional commodity spaces. Thus, when the tittle of Nikaido(59) contains words suggesting infinite dimensional commodity space such as Nikaido(56b), Nikaido(59) may be referred as one of the early attempts in infinite dimensional commodity space besides Debreu(54) by some general equilibrium theorist on infinite dimensional commodity spaces.

which satisfy the conditions of Gale-Nikaido's lemma in the infinite dimensional spaces case. As in Debreu(54), l_∞ may be employed as the underlying commodity space since l_∞^+ has $||\cdot||_\infty-$interior points so that lemma 6 is unnecessary. Thus when even the simplest version of Gale-Nikaido's lemma in infinite dimension such as Nikaido(56b) also includes an economically interesting example which satisfies the condition of this Gale-Nikaido's lemma and is published, it would give a strong impact to the literature definitely and treated as one of the early literature on infinite dimensional commodity space such as Debreu(54). Indeed, Florenzano(83) constructs an economic example where a pure exchange economy in l_∞ gives rise to an excess demand function which satisfies the conditions necessary to apply an infinite dimension version of Gale-Nikaido's lemma. Thus, it is quite interesting to construct an example of an economy where an excess supply function satisfies the condition of Gale-Nikaido's lemma in this paper. Note that the original Gale-Nikaido's lemma in finite dimension is used as a means to establish the existence of competitive equilibrium and it is not the object itself as shown in Gale(55), Nikaido(56a), and Debreu(59).

In any case, although Nikaido(56b, 57b, 59) were neglected in the literature on the theory of general equilibrium with infinite dimensional commodity spaces, once they are reviewed from the present viewpoint, they made to the literature quite an important contribution in a sense that Gale-Nikaido's lemma is proved in an infinite dimensional commodity space even just after Gale-Nikaido' lemma in finite dimensional spaces is proved. Although they have been neglected so far and did not construct any interesting economic examples which use infinite dimensional commodity spaces, the contributions made by Nikaido(56b, 57b, 59) do not loose any significances to the literature on the theory of general equilibrium with infinite dimensional commodity spaces.

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