Substructuralized modal logics applied to the two wise girls puzzle

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1 Introduction

In this paper, we will discuss the relation of substructural logics to a logical puzzle called "the two wise girls puzzle" that is presented in Yasugi and Oda [8] . In our ordinary life, we often expand our own beliefs by hearing the utterances expressing some beliefs, and make some inference by use of them. This puzzle focuses on the situation in which the beliefs of several agents influence one another in this way. Using an inference system, Yasugi et al. formulate the inference involving the beliefs held by the agents of the puzzle in question, and discuss the solvability of the puzzle. To begin with, we will review the puzzle and an outline of Yasugi et al's approach to it. Next, we will introduce a system of substructural logics and show that there is a doubtful point to the solvability of the puzzle in the system. Moreover, we will say that there are several candidates for the expression of the modal axiom. We will assume that the readers are familiar with the sequent calculus **LK**.

2 "The two wise girls puzzle"

We will explain the basic setting of the puzzle in question. There are three persons, and we call two of them "agent1" and "agent2." Suppose that agent1 and agent2 are aligned in the same direction and that agent2 is located in front of agent1. Therefore, agent1 can see agent2 and agent2 cannot see agent1, but agent2 can listen to the agent1's voice if agent1 says something. Suppose further that these two agents wear their hats. In this case, neither

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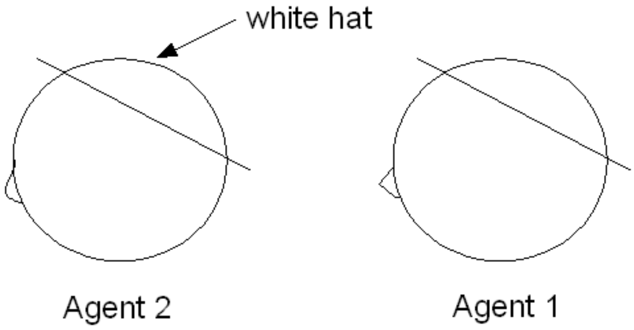
knows the color of the hat of agent2, but the location of these two agents enables agent1 to know the color of her hat. Agent2 wears a white hat but the color of the agent1's hat is not questioned. The observer, who is different from agent1 and agent2, tells the agents that at least one agent wears a white hat. The observer says to agent1, "Do you know if your hat is white?" Then agent1 answers "No! I don't know." After that, the observer asks the agent2 the same question, and then agent2 answers "Yes, I know." If agent2 reaches the correct conclusion, that is, is able to know that she herself wears a white hat, what inference makes her obtain the conclusion in question?



Yasugi and Oda formulated this situation by use of an inference system and explained the question of why agent2 reached the correct conclusion. Their system is the sequent calculus based on a modal propositional logic and is called $\mathbf{KD4^2}$. In addition to the usual logical symbols of propositional logics, this contains the modal operators B_1 and B_2, which are used to express the belief of agent1 and agent2, respectively. Moreover, $\mathbf{KD4^2}$ contains the propositional constant $1W$ which means the proposition that agent1 wears a white hat and the propositional constant $2W$ which means the proposition that agent2 wears a white hat.

The axioms are not different from the ones of the sequent calculus **LK**. As for inference rules, the following two inferences rules $(B_1 \to B_1)$ and $(B_2 \to B_2)$ are added to the ones of the inference system **LK**[1].

$$\frac{\Gamma, B_i\Gamma' \to \Delta}{B_i\Gamma, B_i\Gamma' \to B_i\Delta}(B_i \to B_i) \quad (i = 1, 2)$$

[1]Yasugi and Oda's system does not contain the rules of exchange and contraction in an explicit way. However, since they define the sequent not as the form "the list of formulae $\to$ the list of formulae" but as the form "the set of formulae $\to$ the set of formulae" [8, p. 148] , it would be appropriate to regard the rules as implicitly presupposed. In this paper, when we introduce the system $\mathbf{CFL_eKD4^2}$, we define the sequent as the form "the sequence of formulae $\to$ the sequence of formulae."

Here each of the symbols Γ, Γ' and Δ denotes the set of logical formulae, and Δ has at most one element. In addition, $B_i\Gamma = \{B_i(A)|A \in \Gamma\}(i = 1, 2)$.

These rules are introduced as alternatives of the axioms involving the modal operators **K**, **D** and **4**. When one defines the sequent calculus of modal logics, these three axioms are often introduced as the following form of inference rules:

$$\frac{\Gamma \to A}{B_i\Gamma \to B_i(A)}(B_i\text{-}\mathbf{K}) \quad \frac{\Gamma \to}{B_i\Gamma \to}(B_i\text{-}\mathbf{D}) \quad \frac{B_i\Gamma \to A}{B_i\Gamma \to B_i(A)}(B_i\text{-}\mathbf{4}) \quad (i = 1, 2)$$

These correspond to the axioms **K**, **D** and **4**, respectively[2]. In fact, when $\Gamma' = \emptyset$ and $\Delta \neq \emptyset$, $(B_i \to B_i)$ is $(B_i\text{-}\mathbf{K})$, and when $\Gamma' = \Delta = \emptyset$, $(B_i \to B_i)$ is $(B_i\text{-}\mathbf{D})$, and when $\Gamma = \emptyset$ and $\Delta \neq \emptyset$, $(B_i \to B_i)$ is $(B_i\text{-}\mathbf{4})$. Conversely we can show that $(B_i \to B_i)$ holds by use of $(B_i\text{-}\mathbf{D})$ and $(B_i\text{-}\mathbf{4})$ when $\Delta = \emptyset$, and by use of $(B_i\text{-}\mathbf{K})$ and $(B_i\text{-}\mathbf{4})$ when $\Delta \neq \emptyset$.

By the use of the modal operators B_1 and B_2, they described the sets of sentences expressing the beliefs held by agent1 and agent2. They described Γ_1 the set of sentences denoting beliefs held by agent1 as follows.

$$\Gamma_1 = \{B_1(1W \lor 2W), B_1(2W)\}$$

Here $B_1(1W \lor 2W)$ is the belief held by agent1 when she hears the observer's utterance that at least one agent wears a white hat while $B_1(2W)$ is the belief held by agent1 when she looks at the back of agent2. In contrast, since the agent2's belief changes as agent2 hears the agent1's utterance, they described the situations before and after this change. They described Γ_2 the set of sentences denoting beliefs held by agent2 before she hears the agent1's utterance as follows:

$$\Gamma_2 = \left\{ \begin{array}{c} B_2(1W \lor 2W), B_2(B_1(1W \lor 2W)), \\ B_2(2W \supset B_1(2W)), B_2(\neg 2W \supset B_1(\neg 2W)) \end{array} \right\}$$

$B_2(2W \supset B_1(2W))$ and $B_2(\neg 2W \supset B_1(\neg 2W))$ are the beliefs obtained by agent2 when she considers the location of agent1 and herself. Moreover, they described Γ'_2 the set of sentences denoting beliefs held by agent2 after she hears the agent1's utterance as follows:

$$\Gamma'_2 = \Gamma_2 \cup \{B_2(\neg B_1(2W))\}$$

[2]Strictly speaking, $(B_i\text{-}\mathbf{K})$ corresponds to what is brought by combining $B_i(A \supset B) \supset (B_i(A) \supset B_i(B))$ with the necessitation $A \supset B_i(A)$. Using $(B_i\text{-}\mathbf{K})$, we can show that $(B_i\text{-}\mathbf{D})$ corresponds to the axiom **D**, that is $B_i(A) \supset \neg B_i(\neg A)$. In the same way, using $(B_i\text{-}\mathbf{K})$, we can show that $(B_i\text{-}\mathbf{4})$ corresponds to the axiom **4**, that is $B_i(A) \supset B_i(B_i(A))$.

Γ_2' is an extension of Γ_2. $B_2(\neg B_1(2W))$ is the belief obtained by agent2 when she hears the agent1's answer to the observer, "No, I don't know."

In the following we note the results obtained on the basis of what we have prepared.

(a) $\Gamma_1 \nvdash B_1(1W)$

(b) $\Gamma_1 \nvdash B_1(\neg 1W)$

(c) $\Gamma_2' \vdash B_2(2W)$

(d) $\Gamma_2 \nvdash B_2(2W)$

(a) and (b) show that agent2 does not know whether or not she wears a white hat. This justifies the agent1's answer to the observer, "No, I don't know." (c) and (d) show that agent2 knows that she wears a white hat after she hears the agent1's answer, and that agent2 does not know this until she hears the agent1's answer. The original problem is to show that (c) holds, that is, to show the inference process in which agent2 reaches the correct conclusion. In order to show that (c) holds, Yasugi and Oda presented a procedure to describe the proof figure of the sequent $\Gamma_2' \rightarrow B_2(2W)$. In the later section, we will present a more well-organized proof figure than the one obtained on the basis of this procedure, and use the former for further discussion.

3 Our substructural logic-based approach

Yasugi and Oda's approach was to describe the agent2's inference, using the proof system of modal logics on the basis of the sequent calculus **LK**. In contrast, we will take up a somewhat weaker system than Yasugi and Oda's one, and examine whether or not their result carries over to our adopted system.

The general method for weakening the inference system of modal logics is to change or eliminate the axioms or the inference rules concerning the modal operators in question. Since Yasugi and Oda's system is a variant of the modal logic **KD4**, it would be possible to examine whether or not we obtain the same result, by use of weaker systems **KD** and **K4** than this system or of the systems between each of these two and **KD4**. However, such a restriction would make the connection between the behavior of modal operators and the notion of belief loose. The first reason is that the axiom **D** which asserts that no one holds the contradictory belief can be understood as a postulate to construct a logic for belief. The second reason is that it is natural to posit the axiom **4** which asserts that each one has the ability in which when one consider something, one can think of the relevant consideration itself. Rather, we will not control the inference rules governing the behavior of modal operators, but we will obtain a weaker proof system by restricting the inference rules concerning the structure.

3.1 Substructural modal logic CFL_eKD4^2

The system we adopt is $\mathbf{CFL_eKD4^2}$, the Classical Full Lambek culculus with exchange rules and the modal axioms **K**, **D** and **4** about 2 modal operators[3]. We obtain this system by eliminating the weakening and contraction rules from the set of the structural inference rules of the sequent calculus **LK**, and instead by introducing the propositional constants and the logical connectives and adding the axioms and the inference rules governing these. Here we will explain only the difference from **LK** briefly. For the details, see Watari et al [7] . (Our present paper follows the notation adopted in that paper[4].) The comprehensive description of substructural logics is given in the textbook [4] by Restall.

The substantial difference of our system from **LK** is the presence of binary connectives $*$ and $+$. In linear logics, these are called "multiplicative conjunctive" and "multiplicative disjunctive," respectively. In contrast, the conjunctive connective and the disjunctive connective in **LK** are called "additive conjunctive" and "additive disjunctive," respectively. The notational correspondence between our system and linear logics is in the following.

	Conjunctive		Disjunctive	
	Multiplicative	Additive	Multiplicative	Additive
Linear logics	$\otimes$	&	$\wp$	$\oplus$
Our system	$*$	$\wedge$	$+$	$\vee$

In the system where there are weakening rules and contraction rules, $*$ and $+$ coincide with $\wedge$ and $\vee$, respectively with respect to the behavior. In this sense, $*$ and $+$ are one kind of conjunctive connective and disjunctive connective, respectively, which are brought by the subdivision of conjunction and disjunction by the restriction of the inference rules. The function

[3]This name shows that our system is what is brought by combining the sequent calculi $\mathbf{CFL_e}$ with the modal logics $\mathbf{KD4^2}$. Following this way of naming, we may call Yasugi and Oda's system $\mathbf{LKKD4^2}$. Moreover, since **LK** accords with the system $\mathbf{CFL_{ecw}}$ which is brought by adding the rules of weakening and contraction to $\mathbf{CFL_e}$, we may write the system as $\mathbf{CFL_{ecw}KD4^2}$.

[4]However, as for the modal operators, in order to compare our system with Yasugi and Oda's one easily, we change the relevant notation into B.

of the propositional constants that is introduced newly can be understood in relation to these connectives. We will introduce new logical symbols of $\mathbf{CFL_eKD4^2}$which contains these. Then we will state the axioms and the inference rules governing the behavior of these. Moreover, we will refer to the intuitive meaning carried by these new connectives and finally apply these to the puzzle in question.

The logical symbols added to the system **LK**are the propositional constants $\mathbf{t}$, $\mathbf{f}$, $1W$ and $2W$, the modal operators B_1 and B_2 and the binary connectives $*$ and $+$. The propositional constants $1W$ and $2W$ are not different from the ones explained in the previous section. Modal operators B_1 and B_2 are used again, but the inference rules for them is changed in accordance with the description in Watari et al.'s paper. In our system, the rules $(B_i - \mathbf{K})$, $(B_i - \mathbf{D})$ and $(B_i - \mathbf{4})$ are adopted in place of $B_i \to B_i$ and we have seen in section 2 that they are equivalent.

The symbols $\mathbf{t}$ and $\mathbf{f}$ play a role as the unit elements such that we regard the symbols $*$ and $+$ as binary operations on the set of formulae. That is to say, using the axioms and the inference rules described later, we can show the following. For a formula A,

$$\vdash \mathbf{t} * A \to A \qquad \vdash A \to \mathbf{t} * A \qquad \vdash A * \mathbf{t} \to A \qquad \vdash A \to A * \mathbf{t}$$
$$\vdash \mathbf{f} + A \to A \qquad \vdash A \to \mathbf{f} + A \qquad \vdash A + \mathbf{f} \to A \qquad \vdash A \to A + \mathbf{f}$$

The same thing holds between $\wedge$ and $\top$, and between $\vee$ and $\bot$. That is to say, $\top$ and $\bot$ are the unit elements of $\wedge$ and $\vee$. As in the case of $*$ and $+$, the rules of weakening and contraction cannot differentiate the pair $\mathbf{t}$ and $\top$ and the pair $\mathbf{f}$ and $\bot$. Since in **LK**, $\top$ and $\bot$ can be understood as a true proposition and a false one, respectively, we can think that $\mathbf{t}$ and $\mathbf{f}$ that are introduced newly are one kind of a true proposition and a false one, respectively, which are brought by the restriction of the inference rules.

The axioms and the inference rules of $\mathbf{CFL_eKD4^2}$ that we will use in this paper are in the following. However, here we describe only the axioms and the inference rules added to the system **LK**. As we said in the beginning of this section, the rules of weakening and contraction are eliminated from the inference rules concerning the structure. Therefore we can use only exchange rules and cut rules as the structural inference rules.

- Axioms

$$\to \mathbf{t} \qquad\qquad \mathbf{f} \to \qquad\qquad \Gamma \to \Delta, \top \qquad\qquad \Gamma, \bot \to \Delta$$

- Inference rules

$$\frac{\Gamma \to \Delta}{\Gamma, \mathbf{t} \to \Delta}(\mathbf{t}\mathrm{w}) \qquad\qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, \mathbf{f}}(\mathbf{f}\mathrm{w})$$

$$\frac{\Gamma, A, B \to \Delta}{\Gamma, A * B \to \Delta}(* \text{ left}) \qquad \frac{\Gamma \to A, \Delta \quad \Gamma' \to B, \Delta'}{\Gamma, \Gamma' \to A * B, \Delta, \Delta'}(* \text{ right})$$

$$\frac{\Gamma, A \to \Delta \quad \Gamma', B \to \Delta'}{\Gamma, \Gamma', A + B \to \Delta, \Delta'}(+ \text{ left}) \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A + B}(+ \text{ right})$$

$$\frac{\Gamma \to A}{B_i\Gamma \to B_i(A)}(B_i\text{-}\mathbf{K}) \quad \frac{\Gamma \to}{B_i\Gamma \to}(B_i\text{-}\mathbf{D}) \quad \frac{B_i\Gamma \to A}{B_i\Gamma \to B_i(A)}(B_i\text{-}\mathbf{4}) \quad (i = 1, 2)$$

Here the symbols A and B denote formulae, and each of the symbols Γ, Γ', Δ and Δ' denotes the sequence of formulae, and when $\Gamma = A, B, \cdots$, $B_i\Gamma = B_i(A), B_i(B), \cdots$. Basically, this system depends on Watari et al.'s system, but they differ in that the inference rules concerning modal operators are relativized to each agent in order to apply them to the puzzle in question.

3.2 Resource-sensitivity

For further discussion, we describe the feature of the inference rules concerning the symbol $+$. As we have already said, since the symbol $+$ is one kind of disjunctive connective, it is useful to compare with this the inference rules concerning $\vee$ that is also a disjunctive connective. The difference between these becomes clear, in the left rule concerning $+$, especially in the case where $\Gamma = \Gamma'$ and $\Delta = \Delta'$.

$$\frac{\Gamma, A \to \Delta \quad \Gamma, B \to \Delta}{\Gamma, \Gamma, A + B \to \Delta, \Delta}(+ \text{ left}) \qquad \frac{\Gamma, A \to \Delta \quad \Gamma, B \to \Delta}{\Gamma, A \vee B \to \Delta}(\vee \text{ left})$$

In these two inferences, the premise of the inference coincides while the conclusion of the inference does not. In the left rule concerning $+$, two Γs that are in the premise of the inference appear in the conclusion as they are, while in the left rule concerning $\vee$, two Γs and two Δs are each altered into one symbol. That is to say, we can say that both differ in that one preserves the resources used in the inference while the other does not. The sensitivity of the newly introduced disjunctive connectives — of even this entire proof system — to the increase or decrease of the resources used in the inference is called "resource-sensitivity" or "resource-conciousness." That $+$ and $\vee$ are given as separate disjunctive connectives is a realization of the fact that our system has a resource-sensitive feature.

Such a resource-sensitivity can be seen when we compare the right rule concerning $+$ with the one concerning $\vee$. The right rule concerning $\vee$ has the following two forms:

$$\frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, A \vee B}(\vee \text{ right}) \qquad \frac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \vee B}(\vee \text{ right})$$

In the rule of the latter form, we consider the case where $\Delta = \Delta', A$.

$$\frac{\Gamma \to \Delta', A, B}{\Gamma \to \Delta', A \vee B, B}(\vee \text{ right}) \qquad \frac{\Gamma \to \Delta', A, B}{\Gamma \to \Delta', A + B}(+ \text{ right})$$

Again the premises in the two inferences coincide while the conclusions there do not. When we use the right rule concerning $\vee$, the new symbol B is introduced as a disjunct in the disjunction that appears newly in the conclusion. In contrast, when we use the right rule concerning $+$, the symbol B contained in the premise is used as a disjunct in the disjunction that appears newly in the conclusion. In consequence, when we use the right rule concerning $\vee$, the number of Bs which are prepared as the resources of the inference increases, and when we use the right rule concerning $+$, such an increase does not happen. This fact again shows that the symbol $+$ has the resource-sensitivity.

Since the duality that is seen between $\wedge$ and $\vee$ is also seen in between $*$ and $+$, the symbol $*$ has the resource-sensitivity again. Since $*$ and $+$ are a conjunctive connective and a disjunctive one that are separated from $\wedge$ and $\vee$ by restricting the structural inference rules, $*$ and $+$ can be understood as a resource-sensitive aspect potentially existing in $\wedge$ and $\vee$. By making the notions of conjunction and disjunction more fine-grained in this way, our system obtains a certain "weakness" that will be described later.

For further discussion, we would like to give the symbol $+$ a certain meaning in a naive sense. The characterization by the inference rules certainly is one way of giving it some meaning, but we can give it a naive meaning in a different way from this. One way is to utilize the logical formula which is equivalent to $A + B$ but does not contain $+$[5]. Because we develop our discussion within the proof-theoretic framework, it is appropriate to use the mutual deducibility between two logical formulae as the equivalency. For example, we use the following facts:

$$\vdash A + B \to (\neg A) \supset B \qquad \vdash (\neg A) \supset B \to A + B$$

That is to say, $A+B$ and $(\neg A) \supset B$ involve the relation of mutual deducibility. Thus, since the meaning of $(\neg A) \supset B$ is "if A does not hold, B holds," we can adopt this as the meaning of $A + B$. In contrast, the meaning of the proposition $A \vee B$ containing the disjunctive connectives that is not $+$, is "A holds or B holds." However, in this case, we do not know whether A

[5] It is possible to give a semantics by use of an appropriate algebra [7, pp. 432–434] , but since we intend to apply the logical puzzle described in the natural language to a logical system, we cannot depend on such a semantics which is quite different from the natural language.

holds or B holds [6, p. 23] . The inference rule concerning $\vee$ reflects this consideration well. For example, if we want to deduce C from $A \vee B$, we must show that C is deduced from A and that B is deduced from A. That is because we do not know whether A holds or B holds. This is what the left rule concerning $\vee$ requires, as shown in the following.

$$\frac{A \rightarrow C \quad B \rightarrow C}{A \vee B \rightarrow C}(\vee \text{ left})$$

3.3 Strength of our system

Since Yasugi and Oda's system differs from our system with respect to the logical symbol, it is impossible to compare the sizes of the set of theorems easily. However, the claim that our system is a weak one in a sense can be understood from the fact that we cannot prove representative tautologies as shown in the following. For formulae A, B and C,

- $A \supset (B \supset A)$
- $A \supset (B \supset C) \supset ((A \supset B) \supset (A \supset C))$
- $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)$

For example, when we draw the proof figure by going back from the conclusion, under the situation in which we take A, B and C as formulae, we cannot accomplish the proof figure as in the following figure.

$$\cfrac{\cfrac{\cfrac{?}{A, B \rightarrow A}}{A \rightarrow B \supset A}}{\rightarrow A \supset (B \supset A)}$$

In this case, the impossibility of the use of the rule of weakening prevents us from completing the proof figure. In addition, as for the other two logical formulae, the proof figure cannot be completed, since we do not have the rule of contraction. In contrast, our system is not too weak. As in the case of $(A \wedge B) \supset (A \vee B)$, in our system, we can accomplish the proof which does not require the use of the rules of weakening and contraction in **LK**. Moreover, we can prove some logical formulae which we cannot prove in **LK** without the use of the rules of weakening and contraction, by replacing the conjunctive connective and the disjunctive connective occurring here with $*$ and $+$, respectively. For example, in our system we cannot prove the mutual deducibility between $A \supset B$ and $(\neg A) \vee B$ since it requires the use of the rules of contraction and weakening. However, the mutual deducibility holds between $A \supset B$ and the proposition such that we replace the disjunctive connective appearing in $(\neg A) \vee B$ with $+$. In addition, the law of contradiction and the law of excluded middle are its examples.

4 Applying CFL_eKD4^2 to the puzzle

Based on the preparation developed above, we will examine whether or not Yasugi and Oda's result holds in the system $\mathbf{CFL_eKD4^2}$. In particular, our examination consists in the question of whether or not we can prove that $B_2(2W)$ is deduced from Γ'_2 the set of sentences describing the agent2's belief after she hears the agent1's answer to the observer. That is to say, what matters is whether or not we can draw the proof figure leading to the sequent $\Gamma'_2 \to B_2(2W)$. We construct the proof figure by going back from the conclusion. The following figure is an example of the ones constructed in this way.

$$
\dfrac{
\dfrac{\dfrac{2W \to 2W}{\to \neg 2W, 2W}
\qquad
\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{?}{1W \to 2W, 1W}}{1W, \neg 2W \to 1W}
\qquad
\dfrac{\dfrac{?}{2W \to 2W, 1W}}{2W, \neg 2W \to 1W}
}{1W \vee 2W, \neg 2W \to 1W}}{B_1(1W \vee 2W), B_1(\neg 2W) \to B_1(1W)}}{B_1(1W \vee 2W), \neg B_1(1W), B_1(\neg 2W) \to}
}{B_1(1W \vee 2W), \neg B_1(1W), \neg 2W \supset B_1(\neg 2W) \to 2W}
}{B_2 B_1(1W \vee 2W), B_2 \neg B_1(1W), B_2(\neg 2W \supset B_1(\neg 2W)) \to B_2 2W}
$$

Yasugi and Oda [8, p. 155] pay attention to the fact that the restricted elements of Γ'_2 are sufficient in order to obtain the required figure. Accordingly, we set the necessary elements in the bottom of the figure. As seen in the top of the proof, since we do not have the rule of weakening, we cannot complete this proof figure. Of course, because there are infinitely many ways of drawing the proof figure, we must discuss this matter by preparing the appropriate semantics in order to verify that we cannot prove the relevant matter without the use of the rule of weakening, but we will leave this to further study. Rather, we note the following fact. That is to say, if the symbol $\vee$ appearing in the formula $B_2(B_1(1W \vee 2W))$ contained in the bottom of the figure is the symbol $+$, we can accomplish the proof.

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{1W \to 1W \quad \dfrac{2W \to 2W}{2W, \neg 2W \to}}{1W+2W, \neg 2W \to 1W}}{B_1(1W+2W), B_1(\neg 2W) \to B_1(1W)} \\ \vdots \end{array}}{B_2 B_1(1W+2W), B_2 \neg B_1(1W), B_2(\neg 2W \supset B_1(\neg 2W)) \to B_2(2W)}$$

We have so far described the two kinds of disjunctive connectives, but we have said little things about the question of how we make these two disjunctive connectives distinguished with respect to the use in relation to the ordinary language. Since Γ_2' is what we take over from Yasugi and Oda's previous study, with respect to this element there may be a portion which should be re-described by use of the logical symbol of substructural logics. Now is it correct to replace the disjunctive connectives appearing in the formula $B_2(B_1(1W \vee 2W))$ that is an element of Γ_2'? Do we obtain a certain method for the examination of the correctness in question?

What we encounter here is the problem concerning the correspondence between the formal language and the natural language. Ideally, when we are given a sentence containing a disjunction in the ordinary language, it would be good to have a method for determining whether we should write the disjunction as $\vee$ or as $+$. We gave naive meanings to sentences containing $+$ or $\vee$. That is to say, we showed the possibility that we read $A \vee B$ as "A hold or B holds (but we do not know whether either holds.)" and read $A + B$ as "If A does not hold, B holds." However, this translation seems to be rather weak as a method for the classification of the disjunctive sentences. That is because it seems difficult to avoid the criticism that these two meaning characterization eventually says the same thing. Some people cannot recognize the difference between these two meanings of the disjunctions, by inferring "A holds or B holds, and if A does not hold, it is natural that B holds" and "suppose that if A does not hold, then B holds; A holds or A does not hold; if A holds, we can say that A hold or B holds; even if A does not hold, because we can deduce that B holds from the supposition, again we can say that A holds or B holds." Since there is no strange point in the inference of this person, the problem consists in the meaning given in English to the disjunctive connectives. We need further examination on this point.

5 On the axiom D

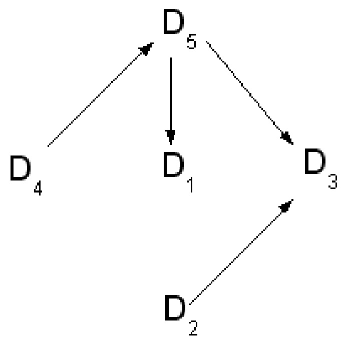
In this section, we will give our own opinion to the modal axiom **D**. We have added the inference rules corresponding to the axioms **K**, **D** and **4** to the sequent calculus $\mathbf{CFL_e}$. The axiom **D** corresponding to the rule B_i-**D** is

$B_i(A) \supset \neg B_i(\neg A)$ where A is a formula. There are other ways for describing the axiom **D**. For example, one of the ways is to state $\neg B_i(A \wedge \neg A)$. The axiom **D** is originally what is introduced in order to express the deontic modality, but because we use B_i as an operator for the expression of the belief, the axiom of the latter form means "It is not believed that A and not A," that is, "No contradiction is believed." We call the former $\mathbf{D_1}$ and the latter $\mathbf{D_2}$. In the system **LK** with the rule B_i-**K**, using the rules of contraction and weakening, we can prove that $\mathbf{D_1}$ and $\mathbf{D_2}$ involve the relation of mutual deducibility. However, in $\mathbf{CFL_e}$ with B_i-**K**, these two deductions fail. Therefore, we cannot identify $\mathbf{D_1}$ with $\mathbf{D_2}$.

In $\mathbf{CFL_e}$, the notion of disjunction is divided into two ones, so that two operations "$\vee$" and "$+$" occur. In addition, "$\perp$" and "$\mathbf{f}$ " are introduced as each unit element of these two operations, respectively. In the system containing the rules of weakening and contraction, as in **LK**, these two can be identified and we can regard these as propositional constants expressing any inconsistent proposition. Moreover, In $\mathbf{CFL_e}$, the notion of conjunction is also divided into two ones, so that two operations "$\wedge$" and "$*$" occur. Therefore if we take $\mathbf{CFL_e}$ as the basis of a proof system, this means that we have much more ways for the expressions of inconsistent propositions. If the proposition "No contradiction is believed" is a meaning of the axiom **D**, the following will work as candidates of the expressions.

- $\mathbf{D_1}$: $B_i(A) \supset \neg B_i(\neg A)$
- $\mathbf{D_2}$: $\neg B_i(A \wedge \neg A)$
- $\mathbf{D_3}$: $\neg B_i(\perp)$
- $\mathbf{D_4}$: $\neg B_i(\mathbf{f})$
- $\mathbf{D_5}$: $\neg B_i(A * \neg A)$

The first two are $\mathbf{D_1}$ and $\mathbf{D_2}$ which has already mentioned. $\mathbf{D_3}$, $\mathbf{D_4}$ and $\mathbf{D_5}$ are what can be obtained by replacing an inconsistent proposition in $\mathbf{D_2}$ with other expressions. We cannot identify any two of these. That is because the deducibility holding between these in the system $\mathbf{CFL_e}$with the rule B_i-**K** is given in the following.



In **LK** with B_i-**K**, all of these involve the relation of mutual deducibility, but in $\mathbf{CFL_e}$ we would be able to choose any one of these expressions. We have adopted $\mathbf{D_1}$, which corresponds to B_i-**D** well. We have no idea of which expression can denote the content of the sentence "No contradiction is believed" in the best way. Since the axiom **D** is introduced as one of requirements satisfied by the belief or the obligation, we may say that the fact that such diverse formulations are possible means that the use of substructurized logics works as help for understanding the notions of belief and obligation. Here what becomes important again is the correspondence between the natural language and the formal language that has been mentioned in the previous section. Do we refer to the multiplicative conjunction or the additive conjunction of a sentence and its denial by the word "contradiction"? The problem of such a correspondence exists in the foundation of our inquiry, and it is this that makes the expressions of the axioms diverse.

6 Summary

We have recapitulated "the two wise girls puzzle" and Yasugi and Oda's approach to it. Next, we have introduced the system of modal substructurized logics involving the resource-sensitive property, and using this system, we have shown that there is a doubtful point as to the solvability of the puzzle established by Yasugi and Oda[6]. Finally, we have pointed out that we can solve the puzzle in the introduced system of substructurized logics, depending on how we establish the correspondence between the connectives inherent to such logics and the counterparts of the ordinary language. In consequence, we have confirmed that the correspondence between substructurized logics and the ordinary language is important. Such an importance as regards the

[6] The epistemic system introduced in Sadrzadeh[5] to analyze the muddy children paradox will provide different solutions to the wise girls puzzle of Yasugi and Oda.

correspondence has also been clear from the fact that this importance even affects the theoretical system characterizing the expressions of the axioms.

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