Chambers of Arrangements of Hyperplanes and Arrow’s Impossibility Theorem

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February 10, 2007

Abstract
Let $A$ be a nonempty real central arrangement of hyperplanes and $\text{Ch}$ be the set of chambers of $A$. Each hyperplane $H$ defines a half-space $H^+$ and the other half-space $H^-$. Let $B = \{+,-\}$. For $H \in A$, define a map $\epsilon_H^+: \text{Ch} \to B$ by $\epsilon_H^+(C) = +$ (if $C \subseteq H^+$) and $\epsilon_H^+(C) = -$ (if $C \subseteq H^-$). Define $\epsilon_H^- = -\epsilon_H^+$. Let $\text{Ch}^m = \text{Ch} \times \text{Ch} \times \cdots \times \text{Ch}$ ($m$ times). Then the maps $\epsilon_H^\pm$ induce the maps $\epsilon_H^\pm: \text{Ch}^m \to B^m$. We will study the admissible maps $\Phi: \text{Ch}^m \to \text{Ch}$ which are compatible with every $\epsilon_H^\pm$. Suppose $|A| \geq 3$ and $m \geq 2$. Then we will show that $A$ is indecomposable if and only if every admissible map is a projection to a component. When $A$ is a braid arrangement, which is indecomposable, this result is equivalent to Arrow’s impossibility theorem in economics. We also determine the set of admissible maps explicitly for every nonempty real central arrangement.

Key words: arrangement of hyperplanes, chambers, braid arrangements, Arrow’s impossibility theorem.

1 Main Results
Let $A = \{H_1, H_2, \ldots, H_n\}$ be a nonempty real central arrangement of hyperplanes in $\mathbb{R}^\ell$. In other words, each hyperplane $H_j$ goes through the origin of $\mathbb{R}^\ell$. In this note, we frequently refer to [OT] for elementary facts about arrangements of hyperplanes, which are usually referred as arrangements for brevity. The connected components of the complement $\mathbb{R}^\ell \setminus \bigcup_{1 \leq j \leq n} H_j$ are called chambers of $A$. Let $\text{Ch} = \text{Ch}(A)$ denote the set of chambers of $A$. For each hyperplane $H_j \in A$, fix a real linear form $\alpha_j$ such that $H_j = \ker(\alpha_j)$. The product $\prod_{j=1}^n \alpha_j$ is called a defining polynomial for $A$. Define

$$H_j^+ = \{x \in \mathbb{R}^\ell \mid \alpha_j(x) > 0\}, \quad H_j^- = \{x \in \mathbb{R}^\ell \mid \alpha_j(x) < 0\} \quad (j = 1, \ldots, n).$$

Throughout this note, let $\sigma$ denote $+$ or $-$. Let $B = \{+,-\}$, which we frequently consider as a multiplicative group of order two in the natural way.

*This research was supported in part by Japan Society for the Promotion of Science.
Let $1 \leq j \leq n$. The maps $\epsilon_j^\sigma : \text{Ch} \to B$ are defined by $\epsilon_j^\sigma(C) = \sigma \tau$ if $C \subseteq H_j^\tau$ ($\sigma, \tau \in B$). Let $m$ be a positive integer. Consider the $m$-time direct products $\text{Ch}^m$ and $B^m$. We let the same symbol $\epsilon_j^\sigma$ also denote the map $\text{Ch}^m \to B^m$ induced from $\epsilon_j^\sigma : \text{Ch} \to B$:

$$\epsilon_j^\sigma(C_1, C_2, \ldots, C_m) = (\epsilon_j^\sigma(C_1), \epsilon_j^\sigma(C_2), \ldots, \epsilon_j^\sigma(C_m))$$

for $(C_1, C_2, \ldots, C_m) \in \text{Ch}^m$.

**Definition 1.1.** A map $\Phi : \text{Ch}^m \to \text{Ch}$ is called an admissible map if there exists a family of maps $\phi_j^\sigma : B^m \to B$ ($1 \leq j \leq n$, $\sigma \in B = \{+, -\}$) which satisfies the following two conditions:

1. $\phi_j^\sigma(\sigma, +, \ldots, +) = +$, and
2. the diagram

\[
\begin{array}{ccc}
\text{Ch}^m & \xrightarrow{\Phi} & \text{Ch} \\
\downarrow{\epsilon_j^\sigma} & & \downarrow{\epsilon_j^\sigma} \\
B^m & \xrightarrow{\phi_j^\sigma} & B
\end{array}
\]

commutes for each $j$, $1 \leq j \leq n$, and $\sigma \in B = \{+, -\}$.

Let $AM(A, m)$ denote the set of all admissible maps determined by $A$ and $m$.

As we will see in Proposition 2.5, when $\Phi$ is an admissible map, a family of maps $\phi_j^\sigma$ ($1 \leq j \leq n$, $\sigma \in B = \{+, -\}$) satisfying the conditions in Definition 1.1 is uniquely determined by $\Phi$, $A$, and $m$.

The main purpose of this note is to study the set $AM(A, m)$ for all $A$ and $m$.

**Definition 1.2.** For $1 \leq h \leq m$, let

$$\Phi = \text{the projection to the } h\text{-th component},$$

$$\phi_j^\sigma = \text{the projection to the } h\text{-th component}.$$

Then it is easy to see that $\Phi$ is an admissible map with a family of maps $\phi_j^\sigma$ ($1 \leq j \leq n$, $\sigma \in B = \{+, -\}$). We call the admissible maps of this type projective admissible maps.

For a central arrangement $A$, define

$$r(A) = \text{codim}_{\mathbb{R}} \bigcap_{1 \leq j \leq n} H_j.$$

**Definition 1.3.** A central arrangement $A$ is said to be decomposable if there exist nonempty arrangements $A_1$ and $A_2$ such that $A = A_1 \cup A_2$ (disjoint) and $r(A) = r(A_1) + r(A_2)$. In this case, write $A = A_1 \uplus A_2$. A central arrangement $A$ is said to be indecomposable if it is not decomposable.
Note that $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ if and only if the defining polynomials for $\mathcal{A}_1$ and $\mathcal{A}_2$ have no common variables after an appropriate linear coordinate change.

**Remark.** It is also known [STV, Theorem 2.4 (2)] that $\mathcal{A}$ is decomposable if and only if its Poincaré polynomial [OT, Definition 2.48] $\pi(\mathcal{A}, t)$ is divisible by $(1 + t)^2$.

We will see in Proposition 2.3 that any nonempty real central arrangement $\mathcal{A}$ can be uniquely (up to order) decomposed into nonempty indecomposable arrangements:

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_r.$$  

The following two theorems completely determine the set $\text{AM}(\mathcal{A}, m)$ of admissible maps.

**Theorem 1.4.** For a nonempty real central arrangement $\mathcal{A}$ with the decomposition (1), there exists a natural bijection

$$\text{AM}(\mathcal{A}, m) \simeq \text{AM}(\mathcal{A}_1, m) \times \text{AM}(\mathcal{A}_2, m) \times \cdots \times \text{AM}(\mathcal{A}_r, m)$$

for each positive integer $m$.

**Theorem 1.5.** Let $\mathcal{A}$ be a nonempty indecomposable real central arrangement and $m$ be a positive integer. Then,

1. if $|\mathcal{A}| = 1$,

   $$\text{AM}(\mathcal{A}, m) = \{ \Phi : \text{Ch}^m \rightarrow \text{Ch} \mid \Phi(C, C, \ldots, C) = C \text{ for each chamber } C \}$$

2. if $|\mathcal{A}| \geq 3$, every admissible map is projective.

(1) if $|\mathcal{A}| = 1$, then $\mathcal{A}$ is decomposable.

**Corollary 1.6.** Decompose a nonempty real central arrangement $\mathcal{A}$ into nonempty indecomposable arrangements as

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_a \uplus \mathcal{B}_1 \uplus \mathcal{B}_2 \uplus \cdots \uplus \mathcal{B}_b$$

with $|\mathcal{A}_p| = 1$ (1 $\leq p \leq a$) and $|\mathcal{B}_q| \geq 3$ (1 $\leq q \leq b$). Then, for each positive integer $m$,

$$|\text{AM}(\mathcal{A}, m)| = (2^a(2^a-2))m^b.$$  

**Remark.** Theorem 1.5 can be regarded as a generalization of Kenneth Arrow’s impossibility theorem ([A, M-CWG]) in economics:

In the impossibility theorem, we assume that a society of $m$ people have $\ell$ policy options and that every individual has his/her own order of preferences on the $\ell$ policy options. A social welfare function can be interpreted as a voting system by which the individual preferences are aggregated into a single societal preference. We require the following two requirements for a reasonable social welfare function:

1. the society prefers the option $i$ to the option $j$ if every individual prefers the option $i$ to the option $j$ (Pareto property), and
2. whether the society
prefers the option $i$ to the option $j$ only depends on which individuals prefer the option $i$ to the option $j$ (pairwise independence).

The conclusion of Arrow’s impossibility theorem is striking: for $\ell \geq 3$, the only social welfare function satisfying the two requirements (A) and (B) is a dictatorship, that is, the societal preference has to be equal to the preference of one particular individual.

In Theorem 1.5, let $\mathcal{A}$ be a braid arrangement in $\mathbb{R}^\ell$ ($\ell \geq 3$), i.e.,

$$\mathcal{A} = \{H_{ij} \mid 1 \leq i < j \leq \ell\}, \text{ where } H_{ij} := \ker(x_i - x_j).$$

The braid arrangements are indecomposable as we will see in Example 2.2. Let $H^{+}_{ij} = \{(x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_i > x_j\}$ and $H^{-}_{ij} = \{(x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_i < x_j\}$. Then each chamber of $\mathcal{A}$ can be uniquely expressed as

$$\{(x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(\ell)}\}$$

for a permutation $\pi$ of $\{1, 2, \ldots, \ell\}$. This gives a one-to-one correspondence between $\text{Ch}(\mathcal{A})$ and the permutation group $S_\ell$ of $\{1, 2, \ldots, \ell\}$. Thus we can interpret an order of preferences on $\ell$ policy options as a chamber of a braid arrangement. Similarly, we interpret a social welfare function as the map $\Phi$ and the dictatorship by the $h$-th individual as the projection to the $h$-th component. The requirements (A) (Pareto property) and (B) (pairwise independence) correspond to the conditions (1) $(\varphi^\sigma_j(+, \ldots, +) = +)$ and (2) (commutativity) in Definition 1.1 respectively. So, in our terminology, Arrow’s impossibility theorem can be formulated as:

If $\mathcal{A}$ is a braid arrangement with $\ell \geq 3$, then every admissible map is projective.

Thanks to Theorems 1.4 and 1.5 we have the following necessary and sufficient condition for a nonempty real central arrangement to have the property that every admissible map is projective:

**Corollary 1.7.** Let $\mathcal{A}$ be a nonempty real central arrangement and $m$ be a positive integer. Every admissible map is projective if and only if

- (case 1) $m = 1$, or
- (case 2) $\mathcal{A}$ is indecomposable with $|\mathcal{A}| \geq 3$.

### 2 Proof of Theorem 1.4

Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a nonempty real central arrangement in $\mathbb{R}^\ell$. Let $\mathcal{B}$ be a subarrangement of $\mathcal{A}$, in other words, $\mathcal{B} \subseteq \mathcal{A}$. We say that $\mathcal{B}$ is **dependent** if

$$r(\mathcal{B}) = \text{codim}_{\mathbb{R}^\ell}(\bigcap_{H \in \mathcal{B}} H) < |\mathcal{B}|.$$ 

A subarrangement $\mathcal{B}$ of $\mathcal{A}$ is called **independent** if it is not dependent. If $\mathcal{B}$ is a minimally dependent subset, then $\mathcal{B}$ is called a **circuit**. If $\mathcal{B}$ is a maximally independent subset in $\mathcal{A}$, then $\mathcal{B}$ is called a **basis** for $\mathcal{A}$.
We introduce a graph $\Gamma(A)$ associated with $A$. The set of vertices of $\Gamma(A)$ is $A$. Two vertices $H_{j_1}, H_{j_2} \in A$ ($j_1 \neq j_2$) are connected by an edge if and only if there exists a circuit (in $A$) containing $\{H_{j_1}, H_{j_2}\}$.

**Lemma 2.1.** A nonempty real central arrangement $A$ is indecomposable if and only if the graph $\Gamma(A)$ is connected.

**Proof.** If $\Gamma(A)$ is disconnected, then decompose $A$ as $A = A_1 \cup A_2$ so that $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, and $\{H_{j_1}, H_{j_2}\}$ is not contained in any circuit whenever $H_{j_p} \in A_p$ ($p = 1, 2$). Choose a basis $B_p$ of $A_p$ ($p = 1, 2$).

Then $B_1 \cup B_2$ is also independent because it does not contain any circuit. Thus

$$r(A) = |B_1 \cup B_2| = |B_1| + |B_2| = r(A_1) + r(A_2),$$

which implies $A = A_1 \cup A_2$. So $A$ is decomposable.

Conversely assume that $A = A_1 \cup A_2$ with $A_1 \neq \emptyset$, $A_2 \neq \emptyset$. We may assume, after an appropriate linear coordinate change, that the defining polynomials for $A_1$ and $A_2$ have no common variables. Let $H_{j_p} \in A_p$ ($p = 1, 2$). Suppose that there exists a circuit $B$ containing $H_{j_1}$ and $H_{j_2}$. Then $B \cap A_1$ and $B \cap A_2$ are both independent. This implies that $B$ is also independent, which is a contradiction.

**Example 2.2.** Let $A$ be a braid arrangement in $\mathbb{R}^\ell (\ell \geq 2)$:

$$A = \{H_{ij} \mid 1 \leq i < j \leq \ell\},$$

where $H_{ij} = \ker(x_i - x_j)$. If $\ell = 2$, then $|A| = 1$ and $A$ is indecomposable. Let $\ell \geq 3$. Then $\{H_{ij}, H_{jk}, H_{ik}\}$ for $1 \leq i < j < k \leq \ell$ is a circuit. Thus it is easy to check that $A$ is indecomposable by applying Lemma 2.1.

By Lemma 2.1, we immediately have

**Proposition 2.3.** Any nonempty real central arrangement $A$ can be uniquely (up to order) decomposed into nonempty indecomposable arrangements

$$A = A_1 \cup A_2 \cup \cdots \cup A_r.$$
Proof. An arbitrary element of $B^m$ can be expressed as $S$ for some $S \subseteq \{1, 2, \ldots, m\}$. Suppose that $C$ and $C'$ are chambers such that $C \subseteq H_j^+$ and $C' \subseteq H_j^-$. Define $C = (C_1, C_2, \ldots, C_m) \in \text{Ch}^m$ by

$$C_i = \begin{cases} C & \text{if } i \in S, \\ C' & \text{if } i \notin S. \end{cases}$$

Then we have $e_j^+(C) = S$. Let $-C = (-C_1, -C_2, \ldots, -C_m) \in \text{Ch}^m$, where $-C_i$ denotes the antipodal chamber of $C_i$. Then $e_j^-(-C) = -(S^c)_+ = S_+$. □

**Proposition 2.5.** When $\Phi$ is an admissible map, a family of maps $\varphi_{\sigma,j} (1 \leq j \leq n, \sigma \in B = \{+, -\})$ satisfying the conditions in Definition 1.1 is uniquely determined.

**Proof.** It is obvious because of Proposition 2.4. □

**Proposition 2.6.** When $\Phi$ is an admissible map, $\Phi(C, C, \ldots, C) = C$ for any chamber $C \in \text{Ch}$.

**Proof.** By Definition 1.1, two chambers $\Phi(C, C, \ldots, C)$ and $C$ are on the same side of every $H_j \in A$. Thus $\Phi(C, C, \ldots, C) = C$. □

Suppose that $A = A_1 \uplus A_2$ with $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. We may assume that the defining polynomials for $A_1$ and $A_2$ have no common variables. Then the following lemma is obvious:

**Lemma 2.7.** The map

$$\alpha : \text{Ch}(A_1)^m \times \text{Ch}(A_2)^m \to \text{Ch}(A_1 \uplus A_2)^m,$$

given by

$$\alpha(C_1, \ldots, C_m, D_1, \ldots, D_m) = (C_1 \cap D_1, \ldots, C_m \cap D_m)$$

for $C_i \in \text{Ch}(A_1), D_i \in \text{Ch}(A_2) (i = 1, \ldots, m)$, is bijective.

**Lemma 2.8.** Let $p \in \{1, 2\}$. For $H_j \in A_p$, the diagram

$$\begin{array}{ccc}
\text{Ch}(A_1) \times \text{Ch}(A_2) & \xrightarrow{\alpha} & \text{Ch}(A_1 \uplus A_2) \\
\pi_p \downarrow & & \downarrow e_j^p \\
\text{Ch}(A_p) & \xrightarrow{e_j^p} & B
\end{array}$$

is commutative, where $\pi_p$ is the projection to the $p$-th component, and $e_j^p$ is the map $e_j^p$ for $A_p$. 6
Proof. Let \( p = 1 \) for simplicity. Then
\[
ep^\sigma_j \circ \alpha(C, D) = \ep^\sigma_j(C \cap D) = \ep^\sigma_j,1(C) = \ep^\sigma_j,1 \circ \pi_1(C, D)
\]
for \( C \in \Ch(A_1), \ D \in \Ch(A_2), \) and \( H_j \in \mathcal{A}_1. \)

From now on, identify \( \Ch(A_1)^m \times \Ch(A_2)^m \) and \( \Ch(A_1 \sqcup A_2)^m \) by the bijection \( \alpha \) in Lemma 2.7. Then Lemma 2.8 can be stated as
\[
ep^\sigma_j,p \circ \pi_p = \ep^\sigma_j \quad (p \in \{1, 2\}, \sigma \in B, H_j \in \mathcal{A}_p).
\]

**Proposition 2.9.** There exists a natural bijection between \( AM(A_1 \sqcup A_2) \) and \( AM(A_1) \times AM(A_2). \)

**Proof.** Suppose that \( \Phi \) is an admissible map for \( A_1 \sqcup A_2 \) and that a family of maps \( \varphi^\sigma_j \ (H_j \in A_1 \sqcup A_2, \ \sigma \in B) \) satisfies the conditions in Definition 1.1. Fix \( p \in \{1, 2\} \) and \( H_j \in \mathcal{A}_p. \) Consider the following diagram:

\[
\begin{array}{ccc}
\Ch(A_1)^m \times \Ch(A_2)^m & \xrightarrow{\Phi} & \Ch(A_1) \times \Ch(A_2) \\
\pi_p \downarrow & & \pi_p \downarrow \\
\Ch(A_p)^m & \xrightarrow{\Phi_p} & \Ch(A_p) \\
\ep^\sigma_{j,p} \downarrow & & \ep^\sigma_{j,p} \downarrow \\
B^m & \xrightarrow{\varphi^\sigma_j} & B.
\end{array}
\]

By Lemma 2.8, we have
\[
\ep^\sigma_{j,p} \circ \pi_p \circ \Phi = \ep^\sigma_j \circ \Phi = \varphi^\sigma_j \circ \ep^\sigma_j = \varphi^\sigma_j \circ \ep^\sigma_{j,p} \circ \pi_p \quad (p \in \{1, 2\}, \sigma \in B).
\]

Assume \( p = 1 \) for simplicity. Let \( C_i \in \Ch(A_1), \ D_i \in \Ch(A_2) \) for \( 1 \leq i \leq m. \) Then
\[
\ep^\sigma_{j,1} \circ \pi_1 \circ \Phi(C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_m)
= \varphi^\sigma_j \circ \ep^\sigma_{j,1} \circ \pi_1(C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_m)
= \varphi^\sigma_j \circ \ep^\sigma_{j,1}(C_1, C_2, \ldots, C_m)
\]
for each \( H_j \in \mathcal{A}_1. \) Thus the chamber
\[
\pi_1 \circ \Phi(C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_m) \in \Ch(A_1)
\]
is independent of \( D_1, D_2, \ldots, D_m. \) Therefore we can express
\[
\Phi_1(C_1, C_2, \ldots, C_m) = \pi_1 \circ \Phi(C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_m)
\]
for some map
\[ \Phi_1 : \text{Ch}(A_1)^m \to \text{Ch}(A_1). \]
Then \( \Phi_1 \) is an admissible map for \( A_1 \) because the diagram above, including \( \Phi_1 \), is commutative for each \( H_j \in A_1 \). Similarly we can define
\[ \Phi_2 : \text{Ch}(A_2)^m \to \text{Ch}(A_2) \]
so that \( \Phi_2 \) is an admissible map for \( A_2 \). The construction so far gives a natural map
\[ F : \text{AM}(A_1 \uplus A_2) \to \text{AM}(A_1) \times \text{AM}(A_2). \]
Conversely suppose that \( \Phi_p \) is an admissible map for \( A_p \) and that a family of maps \( \varphi^\sigma_j (H_j \in A_p, \sigma \in B) \) satisfies the conditions in Definition 1.1. Define
\[ \Phi := \Phi_1 \times \Phi_2 : \text{Ch}(A_1)^m \times \text{Ch}(A_2)^m \to \text{Ch}(A_1) \times \text{Ch}(A_2). \]
Then \( \Phi \) is an admissible map for \( A_1 \uplus A_2 \) because the family of maps \( \varphi^\sigma_j \) (\( H_j \in A_1 \uplus A_2, \sigma \in B \)) satisfies the conditions in Definition 1.1. This construction gives a map
\[ G : \text{AM}(A_1) \times \text{AM}(A_2) \to \text{AM}(A_1 \uplus A_2). \]
It is easy to check that \( F \) and \( G \) are inverses of each other. \( \square \)

Now we have proved Theorem 1.4 by applying Propositions 2.3 and 2.9.

3 Proof of Theorem 1.5

In this section we assume that \( A = \{ H_1, H_2, \ldots, H_n \} \) is a nonempty real central indecomposable arrangement. We assume \( n \neq 2 \) because any arrangement \( A = \{ H_1, H_2 \} \) is decomposable:
\[ A = \{ H_1 \} \uplus \{ H_2 \}. \]

Lemma 3.1. Let \( m \) be a positive integer. Suppose \( A \) is an arrangement with only one hyperplane \( H_1 \). Let \( H_1^+ \) and \( H_1^- \) be the two chambers. Then
(1) an arbitrary admissible map is given by
\[ \Phi(C_1, C_2, \ldots, C_m) = \begin{cases} H_1^+ & \text{if } C_i = H_1^+ \text{ for all } i, \\ H_1^- & \text{if } C_i = H_1^- \text{ for all } i, \\ \text{either } H_1^+ \text{ or } H_1^- & \text{otherwise}, \end{cases} \]
(2) the number of admissible maps is equal to \( 2^{2m-2} \), and
(3) every admissible map is projective if and only if \( m = 1 \).

Proof. (1) Note that the map \( \epsilon_1^\sigma : \text{Ch}(A) \to B \) is a bijection. So the commutativity condition in Definition 1.1 can be ignored and we simply consider a map \( \Phi : \text{Ch}^m \to \text{Ch} \) satisfying \( \Phi(H_1^\sigma, H_1^\sigma, \ldots, H_1^\sigma) = H_1^\sigma \) (\( \sigma \in B = \{ +, - \} \)).
(2) We have two choices for each element of the set
\[ \text{Ch}^m \setminus \{(H_1^+, H_2^+, \ldots, H_m^+), (H_1^-, H_2^-, \ldots, H_m^-)\} \]
whose cardinality is equal to \(2^m - 2\).

(3) When \(m = 1\), by Proposition 2.6, the only admissible map is the identity map, which is projective. For \(m \geq 2\), the number of admissible maps, which is equal to \(2^{2^m-2}\), exceeds the number of projective ones, which is \(m\).

Therefore we have proved Theorem 1.5 (1). Let us concentrate on Theorem 1.5 (2).

Assume that \(A = \{H_1, H_2, \ldots, H_n\}\) is indecomposable with \(n = |A| \geq 3\). Let \(m\) be a positive integer. We will show that every admissible map of \(A\) is projective. Suppose that \(\Phi\) is an admissible map and that a family of maps \(\varphi_j^\sigma\) \((H_j \in A, \sigma \in B)\) satisfies the conditions in Definition 1.1.

**Lemma 3.2.** Assume \(1 \leq j \leq n\) and \(S \subseteq \{1, 2, \ldots, m\}\). Then \(\varphi_j^+(S_+) = -\varphi_j^-(S_-)\). In particular, \(\varphi_j^-(\text{--}, \ldots, \text{--}) = -\).

**Proof.** By Proposition 2.4, we may choose \(C \in \text{Ch}^m\) so that \(\epsilon_j^+(C) = S_+\). Then
\[
\varphi_j^+(S_+) = + \iff \epsilon_j^+ \circ \Phi(C) = \varphi_j^+ \circ \epsilon_j^+(C) = + \iff \Phi(C) \subseteq H_j^+
\]
\[
\iff - = \epsilon_j^- \circ \Phi(C) = \varphi_j^- \circ \epsilon_j^-(C) = \varphi_j^(-(S_-)) = \varphi_j^-(S_-).
\]

Define \(\delta_A^\sigma : \text{Ch}(A) \rightarrow B^n\), for \(\sigma \in B = \{+,-\}\), by
\[
\delta_A^\sigma(C) = (\epsilon_1^\sigma(C), \epsilon_2^\sigma(C), \ldots, \epsilon_n^\sigma(C)).
\]

Then \(\delta_A^\sigma\) is injective. We frequently suppress the subscript \(\delta^\sigma_A\) when there is no fear of confusion. Note that \(\delta^+(-C) = -\delta^+(C) = \delta^-(C)\), where \(-C\) is the antipodal chamber of \(C\). Thus \(\delta^- = -\delta^+\).

**Lemma 3.3.** Let \(B = \{H_1, H_2, \ldots, H_n\}\) be a circuit with \(3 \leq \nu \leq n\). Then
(1) \(|\text{Ch}(B)| = 2^\nu - 2\), and
(2) there exists \(\tau = (\tau_1, \tau_2, \ldots, \tau_\nu) \in B^\nu\) such that
\[
\text{im} \delta_B^\tau = B^\nu \setminus \{\tau, -\tau\}.
\]

**Proof.** (1) Since the intersection lattice [OT, Definition 2.1] \(L(B)\) of \(B\) is the same as that of the \(\nu\)-dimensional Boolean arrangement (= the arrangement of the \(\nu\) coordinate hyperplanes) in \(\mathbb{R}^\nu\) up to the rank \(\nu - 1\), the Poincaré polynomial \(\pi(B, t)\) coincides with the Poincaré polynomial of the \(\nu\)-dimensional Boolean arrangement up to degree \(\nu - 1\). The Poincaré polynomial of the \(\nu\)-dimensional Boolean arrangement is equal to \((1 + t)^\nu\) [OT, Example 2.49]. Since \(\deg \pi(B, t) = r(B) = \nu - 1\) and \(\pi(B, -1) = 0\), \(\pi(B, t) = (1 + t)^\nu - t^\nu - t^{\nu-1}\). By [Z] [OT, Theorem 2.68], one has \(|\text{Ch}(B)| = \pi(B, 1) = 2^\nu - 2\).
By (1),

\[ |B'' \setminus \text{im} \delta_B^+| = |B''| - |\text{im} \delta_B^+| = |B''| - |\text{Ch}(B)| = 2'' - (2'' - 2) = 2. \]

Since \( \delta_B^+(C) = -\delta_B^+(C) \) for \( C \in \text{Ch}(B) \), the set \( \text{im} \delta_B^+ \) is closed under the operation \( \tau \mapsto -\tau \). Thus the set \( B'' \setminus \text{im} \delta_B^+ \) is expressed as \( \{\tau, -\tau\} \) for some \( \tau \in B'' \).

Define

\[ K_j^\sigma := \{S \subseteq \{1, 2, \ldots, m\} \mid \varphi_j^\sigma(S_i) = +\} \quad (1 \leq j \leq n, \sigma \in B = \{+, -, \}). \]

**Lemma 3.4.** Suppose that \( A \) is indecomposable and \( n = |A| \geq 3 \). Then the maps \( \varphi_j^\sigma \) do not depend upon \( j \) or \( \sigma \).

*Proof.* Choose a circuit \( B \subseteq A \). We may assume that \( B = \{H_1, H_2, \ldots, H_\nu\} \) and \( 3 \leq \nu \leq n \). By Lemma 3.3, there exists \( \tau = (\tau_1, \tau_2, \ldots, \tau_\nu) \in B'' \) such that

\[ B'' = (\text{im} \delta_B^+ \cup \{\tau, -\tau\}) \quad \text{(disjoint)}. \]

Let \( 1 \leq p \leq \nu \), \( 1 \leq q \leq \nu \), \( p \neq q \). Since neither of \( (\tau_1, \ldots, -\tau_q, \ldots, \tau_\nu) \) nor \( (\tau_1, \ldots, -\tau_p, \ldots, \tau_\nu) \) lies in \( \{\tau, -\tau\} \), they both lie in \( \text{im} \delta_B^+ \). Choose \( C, C' \in \text{Ch}(B) \) such that

\[ \delta_B^+(C) = (\tau_1, \ldots, -\tau_p, \ldots, \tau_\nu), \quad \delta_B^+(C') = (\tau_1, \ldots, -\tau_q, \ldots, \tau_\nu). \]

Choose \( \hat{C} \in \text{Ch}(A) \) and \( \hat{C'} \in \text{Ch}(A) \) so that \( \hat{C} \subseteq C \) and \( \hat{C'} \subseteq C' \). Let \( S \subseteq \{1, 2, \ldots, m\} \). Define \( C = (C_1, C_2, \ldots, C_m) \in \text{Ch}(A)^m \) by

\[ C_i = \begin{cases} \hat{C'} & \text{if } i \in S, \\ \hat{C} & \text{if } i \not\in S. \end{cases} \]

Then

\[ \epsilon_p^{\tau_r}(C) = \epsilon_q^{\tau_r}(C) = S_+, \quad \epsilon_r^{\tau_r}(C) = (+, +, \ldots, +) \quad (1 \leq r \leq \nu, r \not\in \{p, q\}). \]

Suppose \( S \in K_j^\sigma \), i.e., \( \varphi_j^\sigma(S_i) = + \). Then

\[ \epsilon_r^{\tau_r} \circ \Phi(C) = \Phi \circ \epsilon_r^{\tau_r}(C) = \varphi_j^\sigma(S_+) = +. \]

This implies that \( \Phi(C) \subseteq H_p^{\tau_r} \). Similarly we have \( \Phi(C) \subseteq H_r^{\tau_r} \) when \( 1 \leq r \leq \nu, r \not\in \{p, q\} \), because \( \varphi_r^{\tau_r} \circ \epsilon_r^{\tau_r}(C) = \varphi_r^{\tau_r}(+, +, \ldots, +) = + \). Note that

\[ \bigcap_{j=1}^{\nu} H_j^{\tau_j} = \emptyset \]

because \( \tau \not\in \text{im} \delta_B^+ \). Therefore

\[ \Phi(C) \subseteq \bigcap_{j \neq q} H_j^{\tau_j} \subseteq H_q^{-\tau_q}. \]
Proof. \( S \) (1) Lemma 3.5.

\[ \nu \leq 3 \] which implies

Thus

\[ \varphi^{- \tau_0}(S_+) = \varphi_{- \tau_0} \circ \epsilon_{- \tau_0}(C) = \epsilon_{- \tau_0} \circ \Phi(C) = +, \]

which implies \( S \in K_q^{- \tau_0} \). Therefore \( K_p^{- \tau_0} \subseteq K_q^{- \tau_0} \).

Similarly one can show \( K_p^{- \tau_0} \supseteq K_q^{- \tau_0} \), and thus \( K_p^{- \tau_0} = K_q^{- \tau_0} \) if \( p \neq q \). Since \( \nu \geq 3 \), we can conclude that \( K^\nu_1 \) does not depend upon \( j, 1 \leq j \leq \nu \), or \( \sigma \in B \).

So \( \varphi^\nu_1 \) does not depend upon \( j, 1 \leq j \leq \nu \), or \( \sigma \in B \). Apply Lemma 2.1, and we know \( \varphi^j_1 \) does not depend upon \( j, 1 \leq j \leq n \), or \( \sigma \in B \).

Because of Lemma 3.4, write \( \varphi = \varphi^j_1 \) for \( j, 1 \leq j \leq n \), and \( \sigma \in B \). Let

\[ K = \{ S \subseteq \{ 1, 2, \ldots, m \} : \varphi(S+) = + \}. \]

Lemma 3.5. (1) \( \{ 1, \ldots, m \} \in K \), (2) \( S \in K \) if and only if \( S^c \notin K \), (3) \( S_1 \cap S_2 \in K \) if \( S_1 \in K \) and \( S_2 \in K \).

Proof. (1) is obvious because \( \varphi(+, +, \ldots, +) = + \).

(2) By Lemma 3.2

\[ S \in K = K^1_+ \iff \varphi^1_1(S+) = + \iff \varphi^{-1}((S^c)+) = \varphi_1(-S+) = - \varphi^+_1(S+) = - \iff \varphi^1_1((S^c)+) = - \iff S^c \notin K^1_+. \]

(3) Choose a circuit \( B \subseteq A \). We may assume \( B = \{ H_1, H_2, \ldots, H_\nu \} \) with \( 3 \leq \nu \leq n \). By Lemma 3.3, there exists \( \tau = (\tau_1, \tau_2, \ldots, \tau_\nu) \in B^\nu \) such that

\[ B^\nu = (\text{im} \delta^+) \cup \{ \tau, -\tau \} \] (disjoint).

There exist four chambers \( C, C', C'', C''' \in \text{Ch}(B) \) such that

\[ \delta^+_B(C) = (\tau_1, \tau_2, -\tau_3, \tau_4, \ldots, \tau_\nu), \quad \delta^+_B(C') = (\tau_1, -\tau_2, \tau_3, \tau_4, \ldots, \tau_\nu), \]

\[ \delta^+_B(C'') = (-\tau_1, \tau_2, \tau_3, \tau_4, \ldots, \tau_\nu), \quad \delta^+_B(C''') = (-\tau_1, -\tau_2, \tau_3, \tau_4, \ldots, \tau_\nu). \]

Choose four chambers \( \hat{C}, \hat{C}', \hat{C}'', \hat{C}''' \in \text{Ch}(A) \) such that

\[ \hat{C} \subseteq C, \quad \hat{C}' \subseteq C', \quad \hat{C}'' \subseteq C'', \quad \hat{C}''' \subseteq C'''. \]

Assume that \( S_1, S_2 \in K \). Define \( C = (C_1, C_2, \ldots, C_m) \in \text{Ch}(A)^m \) by

\[ C_i = \begin{cases} \hat{C} & \text{if } i \in S_1 \cap S_2, \\ \hat{C}' & \text{if } i \in S_1 \setminus S_2, \\ \hat{C}'' & \text{if } i \in S_2 \setminus S_1, \\ \hat{C}''' & \text{if } i \notin S_1 \cup S_2. \end{cases} \]

Then

\[ \epsilon^1_1(C) = (S_1)_, \quad \epsilon^2_2(C) = (S_2)_, \quad \epsilon^3_3^{-\tau_0}(C) = (S_1 \cap S_2)_, \]

\[ \epsilon^j_1(C) = (+, +, \ldots, +) \quad (4 \leq j \leq \nu). \]
Thus we have
\[ \epsilon^1 \circ \Phi(C) = \varphi \circ \epsilon^1(C) = \varphi((S_1)_+) = +, \]
\[ \epsilon^2 \circ \Phi(C) = \varphi \circ \epsilon^2(C) = \varphi((S_2)_+) = +, \]
\[ \epsilon^j \circ \Phi(C) = \varphi \circ \epsilon^j(C) = \varphi(+, +, \ldots, +) = + \quad (4 \leq j \leq \nu), \]
which implies
\[ \Phi(C) \subseteq H^1 \cap H^2 \cap H^4 \cap \cdots \cap H^\nu \subseteq H^3. \]
Therefore
\[ \varphi((S_1 \cap S_2)_+) = \varphi \circ \epsilon^{-3}(C) = \epsilon^{-3} \circ \Phi(C) = +, \]
and \( S_1 \cap S_2 \in K \).

Now we are ready to prove the following statement, which is Theorem 1.5 (2).

Let \( \mathcal{A} \) be a real central indecomposable arrangement with \( |\mathcal{A}| \geq 3 \).

Then every admissible map is projective.

**Proof.** Define \( S_0 = \bigcap_{S \in K} S \). By Lemma 3.5 (3), \( S_0 \in K \). By Lemma 3.5 (1) and (2), we have \( \emptyset \notin K \). Thus \( S_0 \neq \emptyset \). Let \( h \in S_0 \). Since \( S_0 \setminus \{h\} \notin K \), \( \{1, 2, \ldots, m\} \setminus S_0 \cup \{h\} \in K \) by Lemma 3.5 (2). By Lemma 3.5 (3),
\[ \{h\} = ((\{1, 2, \ldots, m\} \setminus S_0) \cup \{h\}) \cap S_0 \in K. \]
Thus \( S_0 = \{h\} \). Note that, by Lemma 3.5 (2),
\[ S \in K \Rightarrow h \in S \Leftrightarrow h \notin S^c \Rightarrow S^c \notin K \Leftrightarrow S \in K. \]
Therefore, \( S \in K \) if and only if \( h \in S \):
\[ K = \{S \subseteq \{1, 2, \ldots, m\} \mid h \in S\}. \]
This implies that \( \varphi \) is equal to the projection to the \( h \)-th component. Let \( C \in \text{Ch}^m \). Then
\[ \epsilon^j \circ \Phi(C) = \varphi \circ \epsilon^j(C) = \varphi(\epsilon^j(C_1), \epsilon^j(C_2), \ldots, \epsilon^j(C_m)) = \epsilon^j(C_h). \]
Since \( \Phi(C) \) and \( C_h \) lie on the same side of every hyperplane \( H_j \in \mathcal{A}, \Phi(C) = C_h \). Therefore \( \Phi \) is the projection to the \( h \)-th component. \( \square \)

Decompose a nonempty real central arrangement \( \mathcal{A} \) into nonempty indecomposable arrangements as
\[ \mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_a \uplus \mathcal{B}_1 \uplus \mathcal{B}_2 \uplus \cdots \uplus \mathcal{B}_b, \quad (***) \]
where \( |\mathcal{A}_p| = 1 \) (\( 1 \leq p \leq a \)) and \( |\mathcal{B}_q| \geq 3 \) (\( 1 \leq q \leq b \)). Then, by Lemma 3.1, Theorems 1.4 and 1.5, the number of admissible maps for \( \mathcal{A} \) is equal to
\[ \left( \frac{2^{2m-2}}{a} \right)^a \cdot m^b. \]
This proves Corollary 1.6.

Next we will prove Corollary 1.7: If $m = 1$, then, by Proposition 2.6, the only admissible map is the identity map $Ch \to Ch$, which is projective. Assume $m \geq 2$. Then, by Lemma 3.1, Theorems 1.4 and 1.5, every admissible map is projective if and only if $a = 0$ and $b = 1$ in the decomposition (**) above.

Acknowledgement. The author would like to express his gratitude to Professors H. Kamiya and A. Takemura, who introduced him to Arrow’s impossibility theorem and gave him helpful comments for earlier versions of this paper, and to Dr. T. Abe with whom he had stimulating conversations.

References


