Econ. J. of Hokkaido Univ.,
Vol.21 (1992), pp.47～65.

A Review of Basic Models for Public Investment Criteria

Masatoshi Yoshida

1. Introduction

In the evaluation of the benefits and costs which accrue from alternative public investment projects, the choice of the discount rate is crucial. A project which seems to yield substantial benefits when evaluated at a low percentage rate may well appear extremely wasteful if that rate is doubled. An interest in the choice of the social discount rate is not only an issue on the acceptance or the rejection of specific projects, but more importantly, one on the allocation of resources between the private and public sectors of an economy. There have been some controversies over what the correct discount rate ought to be. These controversies have focused on the relationships among the correct discount rate, the market rates of return, and the so-called social rate of time preference. In the theoretical literature, some answers to this problem have been presented and many different public investment criteria have been proposed. However, some of them appear to be conflicting with each other.

We are concerned with the problem of what rate of return ought to be required for public investment given that the rate of return on private investment is not equal to the social rate of time preference. Such a price distortion may arise for one of the following reasons: the existence of the so-called 'isolation paradox', capital income taxation, uncertainty on future returns, capital market imperfections, and disequilibrium in markets. We will now briefly sum up the effects of these sources on the determination of the social discount rate.

First, capital market, even if it is perfectly competitive, may inadequately weigh the benefits of current savings to the future generations. Following the proposition of the isolation paradox advanced by Sen (1961 and 1967) and Marglin (1963-a) and others, this may arise if members of the present generation benefit from the collective well-being of future generation. In this case, the market interest rate could exceed the social rate of time preference.

Second, price distortions may exist on capital markets. The most obvious of these are taxes on capital income which make the gross rate of return on private investment greater than the social rate of time preference. The problem arises as to which, if either, of these rates of return should be required for public investment. This problem has been extensively treated in the context of the general equilibrium models by Sandmo and Drèze (1971), Diamond and Mirrlees (1971), Diamond (1973),

Drèze (1974), Pestieau (1974 and 1975), Boadway (1975), Hägen(1983), Marchand and Pestieau (1984), Auerbach (1987), Hägen (1988) and others. We can roughly classify these models into two types: the Sandmo-Drèze type and the Diamond-Mirrlees type. In the former, it is assumed that there are fixed taxes which can not be modified by the government but it can use lump-sum transfers as well as tax collections to meet any deficit in public production, with the result that the social rate of discount for public investment is appropriately a weighted average of the gross rate of return on private investment and the social rate of time preference. In the latter, the government is assumed to be unable to affect lump-sum transfer but to fully control taxes as well as public production, with the result that the social rate of discount will be equal to the rate of return on private investment.

Third, the price distortion may occur if risk in a world of uncertainty is not efficiently allocated by capital markets. This problem has been treated by Hirshleifer (1966), Arrow and Lind (1970), Sandmo (1972), Bailey and Jensen (1872), Schmalensee (1976), Mayshar (1977), Glinols (1985) and others. One view, which has been advanced by Hirshleifer and then supported by Sandmo (1972) and Diamond (1967), is that some differences between rates of return on private investments reflect differences in their riskiness, so that these can be of normative significance for the allocation of capital in the public sector. Thus, the government should take a rate of return on private investment of similar riskiness as the social rate of discount for a particular type of public investment. Another view, which counts Samuelson (1964) and Vickrey (1964) among its supporters, is that because of the extremely large and diversified investment portofolio held by the public sector, the marginal return from public investment as a whole is practically risk free and should be equated to the market rate of riskless bonds. Arrow and Lind (1970) also come to the same conclusion for a somewhat different reason, that is, the total risk carried by the public sector is shared among many individuals, so many that each individual's risk burden can become negligible.

Fourth, the source of distortion is imperfect capital markets. Unlike the previous three types of distortions, it is difficult to formalize the notion of distortions arising in capital market imperfections. One particularly simple way to characterize market imperfections has been employed in the context of the macroeconomic growth models by Marglin (1963-b), Feldstein (1964), Arrow and Kurz (1969 and 1970), McFadden (1972), Boadway (1978) and others. They assume that the saving behaviour of the economy can be described by a fixed saving ratio as a proportion of disposable income. Perhaps the first attempt to look at the analytical foundations of public investment criteria in this context was the work of Arrow and Kurz (1969). They examined the optimal path of accumulation in a single commodity neoclassical economic growth model, with public and private sector capital and with various institiutional constraints including a limited number of revenue-raising

instruments. Not surprisingly, the criteria for public investment were found to depend crucially both upon the financing instruments available and the underlying structure of the economy. There is no *a priori* reason to suppose that the social rate of discount should be either the social rate of time preference or a rate of return on private sector investment. They found that, when public investment is financed by debt and all interest payments by income tax, the required rate of return on public capital should be the rate of return on private capital all along the second-best optimal path. When all public investment is financed by an income tax but not by debt, the required rate on public capital in the steady state is the social rate of time preference. On the path towards the steady state, the required rate of return on public capital is more complicated because it involves the time rate of change of auxiliarly variables in the optimal program.

On the contrary, using a neo-classical economic growth model similar to that used by Arrow and Kurz, Boadway (1978) derived the criteria for public investment which may apply to a wide variety of institutional constraints of the economy, whether or not it is on the optimal path. Whereas Arrow and Kurz are concerned with the properties of optimal paths of the economy, the analysis presented by Boadway looks into the question of public investment criteria in the economies along any arbitrary (steady state) growth paths. He showed that with income tax financing alone, the required rate of return on public investment must equal the social rate of time preference, while with debt financing of public investment combined with income tax, the required rate of return is the rate of return on private investment.

Finally, the price distortion may arise from disequilibrium in markets. For example, if both the labour and capital markets are in disequilibrium due to the rigidities of the real wage rate and real rate of interest, then there are four possible disequilibrium regimes. That is, Regime A: 'unemployment-undersavings', Regime B: 'unemployment-underinvestment', Regime C: 'excess demand for labour-undersavings', and Regime D: 'excess demand for labour-underinvestment'. Recently, being based on the two-period general equilibrium model formalized by Sandmo and Drèze (1971), Marchand, Mint, and Pestieau (1985) studied the criteria for public investment in such general disequilibrium regimes. They derived the following results on the shadow prices of labour and capital for the public firm. In some cases, the shadow prices are very simple to compute: with Regime A the shadow prices are the marginal disutility of labour and savings for labour and capital, respectively, and with Regime B they are equal to the marginal productivities in the private sector. In other cases (i. e., Regimes B and C), the shadow prices are more difficult to calculate because the producer or consumer surplus associated with the disequilibrium on each market enters in the shadow pricing formulae. It is of interest to notice that the shadow prices of labour and capital may be greater or lower than the corresponding market prices in a way which is sometimes counter-

intuitive.

It will be worth while to review briefly the typical and basic models for public investment criteria in literature. In this paper, as our basic models, we will select Arrow and Kurz (1969) and Boadway (1978) from among the neo-classical economic growth models and Diamond and Mirrlees (1971) and Pestieau (1974) from among the general equilibrium models. These are respectively surveyed in sections 2 and 3. About the Sandmo and Drèze's model (1971) which is another basic general equilibrium model, see Yoshida (1986).

2. Neoclassical Economic Growth Models

2. 1. Arrow and Kurz's Models

To begin with, let us review their model of public investment with general income tax financing alone. Their model in this case can be formulated as follows:

$$\text{Maximize} \int_0^\infty \exp(-\lambda t)\, U(c, k_g)\, dt$$

subject to

$$(1) \qquad c = (1-s)(1-x)\, f(k_p, k_g),$$

$$(2) \qquad dk_p/dt = s(1-x)\, f(k_p, k_g) - \gamma k_p,$$

$$(3) \qquad dk_g/dt = x f(k_p, k_g) - \gamma k_g,$$

where c represents normalized consumption; $c = \exp(-\gamma t)C$; k_p private capital stock (state variable); k_g public capital stock (state variable); s fixed saving ratio; x income tax rate (control variable); γ natural rate of growth; $\lambda = \omega - \gamma$ [ω (social rate of time preference) = ρ (discount rate for utility) + τ (Harrod neutral technological progress rate) $\times$ σ (elasticity of marginal utility of consumption)]; U utility function per capita; f neoclassical production function per capita; and t time. Eq. (1) shows that total consumption is a fixed proportion of disposable income. Eqs. (2) and (3) show the processes of capital accumulation in the private and public sectors, respectively.

Given an appropriate mix of tax/debt instruments, the economy could be completely controlled even with a fixed saving ratio, and the first-best optimal path could be achieved. However, in this case it can not be attained because, to use Tinbergain's terminology, while there are now two targets: the rates of capital accumulation in the two sectors, there is only one instrument: the income tax policy. Therefore, analytic interest is to determine the second-best optimal policy.

The current-value Hamiltonian for the above second-best problem can be given by the following:

(5) $$H = U[(1-s)(1-x)f(k_p, k_g), k_g]$$

$$+ p_p[s(1-x)f(k_p, k_g) - \gamma k_p] + p_g[xf(k_p, k_g) - \gamma k_g],$$

where p_p and p_g are the auxiliarly variables corresponding to the differential equations (2) and (3), respectively. The sole instrument, the rate of income tax x, must be chosen so as to maximize H.

(6) $$\partial H/\partial x = -U_c(1-s)f - p_p sf + p_g f,$$

where U_c is the marginal utility of consumption. If we assume that H is maximized at a finite value of x, we can conclude that

(7) $$(1-s)U_c[(1-s)(1-x)f(k_p, k_g), k_g] = p_g - sp_p,$$

(8) $$x < 1.$$

The concavity of U ensures that eq. (7) has a unique solution for x as a function of k_p, k_g, p_p and p_g.

The differential equations which govern the law of motion of the auxiliary variables are

(9) $$dp_p/dt = \lambda p_p - (\partial H/\partial k_p) = \omega p_p - p_g f_p,$$

(10) $$dp_g/dt = \lambda p_g - (\partial H/\partial k_g) = \omega p_g - p_g f_g - U_g,$$

where U_g is the marginal utility of public capital.

The movement along the second-best optimal path is governed by the differential equations for the state variables, eqs. (2) and (3), and those for the auxiliary variables, eqs. (9) and (10), with the income tax rate x being determined by eq. (7). A stationary solution is found by setting dk_p/dt, dk_g/dt, dp_p/dt and dp_g/dt in eqs. (2), (3), (9), and (10), respectively, equal to zero, and then using the maximum principle, eq. (7). Distinguishing the stationary values of the variables by the superscript ($^\infty$), we can obtain

(11) $$s(1-x^\infty)y^\infty = \gamma k_p^\infty,$$

(12) $$x^\infty y^\infty = \gamma k_g^\infty,$$

(13) $$f_p^\infty = (p_p^\infty/p_g^\infty)\omega,$$

(14) $$f_g^\infty = \omega - (U_g^\infty/p_g^\infty),$$

(15) $(1 - s)\, U_c^\infty = p_g^\infty - s p_p^\infty,$

(16) $c^\infty = (1 - s)(1 - x^\infty)\, y^\infty,$

(17) $y^\infty = f(k_p^\infty, k_g^\infty).$

Eliminating x^∞ from eqs. (11) and (12) and substituting for y^∞ from eq. (17) yield

(18) $sf(k_p^\infty, k_g^\infty) = \gamma (k_p^\infty + s k_g^\infty),$

which is the analog of the Harrod-Domar relation in this model. The magnitude $k_p + s k_g$ is the amount to which private capital would be raised if the government were to liquidate all its capital and turn the proceeds over to the private sector.

One very interesting aspect of the stationary solution deserves to be stressed. The relation governing the rate of return on public investment, eq. (14), has exactly the same form as in the first-best optimal policy. In the special case: $U_g = 0$, the rate of return on public investment at any stationary point of the second-best policy should be the social rate of time preference: $f_g = \omega$. The government choice of discount rate is, in the long run, the same as what would be if it could completely control savings; the second-best features of the solution might be said to disappear in the long run. On the other hand, from eq. (13), it is not necessary that the rate of return on private investment should also equal the social rate of time preference along the balanced growth path.

Next, let us now review their model in which public investment is financed by borrowing while interest on public debt is paid out of income taxation. In this case, since taxes are not a freely controllable instrument but are determined as a particular function of the state variables, after all there is only one instrument, public debt. Thus, the first-best optimal path can not be achieved in general, so that to determine the second-best optimal path is here the objective. Their model in this case can be fomulated as follows:

Maximize $\int_0^\infty \exp(-\lambda t)\, U(c, k_g)\, dt$

subject to

(19) $c = (1 - s) f(k_p, k_g),$

(20) $dk_p/dt = sf(k_p, k_g) - b - \gamma k_p,$

(21) $dk_g/dt = b - \gamma k_g,$

where b is normalized public debt.

The current-value Hamiltonian for this second-best problem is

$$(22) \qquad H = U[(1-s)f(k_p, k_g), k_g] + p_p[sf(k_p, k_g) - b - \gamma k_p] + p_g(b - \gamma k_g).$$

Maximizing H with respect to the control variable b, we can obtain

$$(23) \qquad p_p = p_g.$$

The differential equations for the auxiliarly variables, p_p and p_g are given by

$$(24) \qquad dp_p/dt = \lambda p_p - [U_c(1-s)f_p + p_p sf_p - \gamma p_p],$$

$$(25) \qquad dp_g/dt = \lambda p_g - [U_c(1-s)f_g + U_g + p_p sf_g - \gamma p_g].$$

From eqs. (23), (24) and (25), we get

$$(26) \qquad [U_c(1-s) + p_p s]f_p = [U_c(1-s) + p_p s]f_g + U_g.$$

In particular, if $U_g = 0$, eq. (26) becomes

$$(27) \qquad f_p = f_g.$$

Thus, public investment being financed by debt together with the income tax, the required rate of return on public investment should be the rate of return on private investment all along the second-best optimal path.

2. 2. Boadway's Models

Let us first review his model of public investment financed by general income tax only. His model in this case can be formulated as follows:

$$(28) \qquad C = (1-s)[F(K_p, K_g, L) - (dK_g/dt)],$$

$$(29) \qquad dK_p/dt = s[F(K_p, K_g, L) - (dK_g/dt)],$$

$$(30) \qquad dK_g/dt = xF(K_p, K_g, L),$$

where L is total effective labour supply. Given s, K_g (t), L (t) and K_p (0), the differential equation (29) dictates the path of K_p (t). The path of C (t) is then derived from eq. (28).

Now, suppose that at some point in time the government increases its accumulation of capital by $dI_g(0)\,dt$, where $I_g(0) \equiv (dK_g/dt)|_{t=0}$ for a small period of time dt, but leaves I_g (t) unchanged thereafter. The increment of public investment causes the following three effects:

(i) *Effect 1*

The stock of public capital at all subsequent points along the path is increased

by $dI_g(0)\,dt$ due to such a government policy change. Let us now denote the change in $K_g(t)$ for positive t by $d\theta = dK_g(t) = dI_g(0)\,dt$.

(ii) *Effect 2*

An increase in public investment of $dI_g(0)\,dt$ causes private investment to fall instantaneously by $dK_p(0) = -sdI_g(0)\,dt = -sd\theta$. This follows from eq. (29).

(iii) *Effect 3*

Such a policy change causes the rate of consumption to fall by $dC(0) = -(1-s)\,dI_g(0)\,dt = -(1-s)\,d\theta$, which follows from eq. (28).

The effects 1 and 2 change the accumulation path of private capital $K_p(t)$ in eq. (29) due to both the parametric rise in $K_g(t)$ and the initial fall in $K_p(0)$. These changes in the paths of $K_p(t)$ and $K_g(t)$ will in turn induce a change in the path of $C(t)$ via eq. (28) over and above the initial reduction $C(0)$. This is the effect 3.

The change in the path of $C(t)$ induced by the increment of public investment at time zero is then evaluated by using the social rate of time preference. This evaluation will be used as a decision rule for undertaking the public investment.

Let us now denote the induced change in $K_p(t)$ due to changes in $K_g(t)$ and in $K_p(0)$ as follows:

$$K_{pg}(t) \equiv \partial K_p(t)/\partial K_g(t) \text{ and } K_{pp}(t) \equiv \partial K_p(t)/\partial K_p(0).$$

Then,

$$dK_p(t) = K_{pg}(t)\,dK_g(t) + K_{pp}(t)\,dK_p(0) = K_{pg}(t)\,d\theta - sdK_{pp}(t)\,d\theta. \tag{31}$$

The explicit equations for $K_{pg}(t)$ and $K_{pp}(t)$ can be derived by using the technique of comparative dynamics introduced by Oniki (1973). Differentiating eq. (29) with respect to $K_g(t)$, we can obtain

$$dK_{pg}(t)/dt = s\,[\partial F/\partial K_p(t)]\,K_{pg}(t) + s\,[\partial F/\partial K_g(t)], \quad \text{where } K_{pg}(0) = 0. \tag{32}$$

Now, to begin with, it is convenient to assume that the economy is in a steady state. Then, we can denote $\partial F/\partial K_p(t)$ and $\partial F/\partial K_g(t)$ by $r(t) = r$ and $q(t) = q$, respectively. Therefore, eq. (32) reduces to

$$dK_{pg}(t)/dt = srK_{pg}(t) + sq, \text{ where } K_{pg}(0) = 0. \tag{33}$$

The solution for this differential equation is

$$K_{pg}(t) = (q/r)\,[\exp(srt) - 1]. \tag{34}$$

Similarly, $K_{pp}(t)$ must satisfy

$$dK_{pp}(t)/dt = srK_p(t), \text{ where } K_{pp}(0) = 1. \tag{35}$$

The solution for this differential equation is

(36) $K_{pp}(t) = \exp(srt).$

Substituting eqs. (34) and (36) into eq. (31), we obtain

(37) $dK_p(t) = \{(q/r)[\exp(srt) - 1] - \exp(srt)s\}\, d\theta.$

The consumption stream changes in the initial time by $-(1-s)\,d\theta$ and thereafter by an amount determined by the changes in $K_p(t)$ and $K_g(t)$ through eq. (28). Totally differentiating eq. (28) gives the following expression for the induced changes in C (t) after the initial change:

(38) $dC(t) = (1-s)[rdK_p(t) + qd\theta] = (1-s)(q-sr)\exp(srt)\,d\theta.$

Then, the present value of the change in C (t) per unity of money of public investment, or the change in the social welfare $dW/d\theta$ is given by

(39) $$dW/d\theta = \int_0^\infty [dC(t)/d\theta]\exp(-\omega t)\,dt$$

$$= (1-s) + \int_0^\infty \exp(-\omega t)(1-s)(q-sr)\exp(srt)\,dt$$

$$= -(1-s) + (1-s)(q-sr)/(\omega - sr),$$

where (a) we have assumed that $\omega > sr$, so that the integral will converge and (b) the social welfare W is now assumed to be

(40) $W = \int_0^\infty \exp(-\rho t)\,L(t)\,U[C(t)/L(t)]\,dt.$

Note that the first equality in eq. (39) is derived by using McFadden's Lemma (1972).

Now, two alternative interpretations may be possible on this change in welfare. First, rearrange (39) to give

(41) $dW/d\theta = (1-s)(q-\omega)/(\omega - sr).$

Thus, with $\omega > sr$, a public investment project increases welfare if its rate of return q is at least as great as the social rate of time preference ω.

Second, we relate the decision rule (39) to the opportunity cost of public investment as discussed in Marglin (1963-b). Eq. (39) may be decomposed into three terms:

(42) $dW/d\theta = -(1-s) - (1-s)\,sr/(\omega - sr) + (1-s)\,q/(\omega - sr).$

(a) *The first term*: $-(1-s)$

This is the opportunity cost of the foregone consumption per dollar of public investment from the initial financing of the project.

(b) *The second term*: $-(1-s)\,sr/(\omega - sr)$

This may be interpreted as the present value of consumption foregone discounted at ω as a result of a reduction in $K_p(0)$ by s dollars. An initial investment of $K_p(0)$ with a rate of return r and saving ratio s will generate a stream of capital accumulation given by the differential equation $dK_p(t)/dt = srK_p(t)$ with the initial condition $K_p(0)$. The solution for this is $K_p(t) = K_p(0)\exp(srt)$. If initial $K_p(0)$ is s, then $K_p(t) = \exp(srt)\,s$. This is the capital that would have been accumulated had the s dollars been used for private capital rather than being devoted to public capital. That stream of $K_p(t)$ generates a rate of return of r of which $(1-s)$ is consumed. Thus, the consumption stream generated by s dollars of $K_p(0)$ is $sr(1-s)\exp(srt)$. The present value of this stream is

$$\int_0^\infty \exp(-\omega t)\,sr(1-s)\exp(srt)\,dt = s(1-s)\,r/(\omega - sr). \tag{43}$$

(c) *The third term*: $(1-s)\,q/(\omega - sr)$

This is the present value of the future stream of consumption generated by the perpetual return on one dollar of public investment when all proceeds are reinvested at the rate s.

These results are exactly the same as those obtained in Marglin's model II′.

Next, consider the case where the government can control the public debt and the income tax. His model in this case can be formulated as follows:

$$C = (1-s)(1-x)\,[F(K_p, K_g, L) + rD], \tag{44}$$

$$dK_p(t)/dt = s(1-x)\,[F(K_p, K_g, L) + rD] - B, \tag{45}$$

$$dK_g(t)/dt = x\,[F(K_p, K_g, L) + rD] + B - rD, \tag{46}$$

where $B(t)$ is the quantity of debt issued at time t and $D(t)$ is the cumulated debt. Solving eq. (46) for x and substituting into eqs. (44) and (45), we have

$$C = (1-s)\,[F(K_p, K_g, L) - (dK_p/dt) + B], \tag{47}$$

$$dK_p/dt = s\,[F(K_p, K_g, L) - (dK_g/dt)] - (1-s)\,B. \tag{48}$$

Let us now suppose that the government increases public investment by $dI_g(0)$ for a time interval dt and the issue of public debt by $dB(0) = dI_g(0)$ at the same time, while leaving $I_g(t)$ and $B(t)$ unchanged thereafter. As a result, both public capital and debt will be increased by $dI_g(0)\,dt = d\theta$. This increase in public capital of $d\theta$ all along the path will be an increase in $K_p(t)$ via eq. (48). At the same time, the initial changes in $I_g(0)$ and $B(0)$ will cause $I_p(0)$ to change by $dI_p(0) = -sdI_g(0) -$

$(1 - s)\, dB(0) = -\, dI_g(0)$ from eq. (48). The change in $K_p(0)$ causes the subsequent path of $K_p(t)$ to change. The overall change in $K_p(t)$ may be calculated using the following equation:

$$(49) \qquad dK_p(t) = K_{pg}(t)\, dK_g(t) + K_{pp}(t)\, dK_p(0)$$

$$= [K_{pg}(t) - K_{pp}(t)]\, d\theta$$

$$= \{(q/r)\,[\exp(srt) - 1] - \exp(srt)\}\, d\theta.$$

On the other hand, even if both $I_g(0)$ and $B(0)$ increase by the same amount, there is no change in $C(0)$ in eq. (47). This is an implication of the fixed saving ratio assumption. At a point of time, aggregate saving is given, and debt-financed public investment completely "crowds out" private investment. All changes in the consumption stream are those induced by changes in $K_p(t)$ and $K_g(t)$ rather than by changes in the saving behaviour. Differentiating eq. (47), we obtain

$$(50) \qquad dC(t) = (1 - s)\,[r dK_p(t) + q dK_g(t)].$$

Substituting $dK_p(t)$ in eq. (49) and $dK_g(t) = d\theta$ into eq. (50),

$$(51) \qquad dC(t) = (1 - s)(q - r)\exp(srt)\, d\theta.$$

The change in the social welfare W is obtained by discounting the stream (51) at the social rate of time preference ω. Assuming $\omega > sr$ for the integral to converge, the change in social welfare per dollar of public investment is

$$(52) \qquad dW/d\theta = (1 - s)(q - r)/(\omega - sr).$$

Therefore, any project should be accepted if $q > r$. That is, when the income tax exists, the rate of return required on public investment should be the rate of return on private investment if a project is financed by issuing bonds.

3. General Equilibrium Models

3. 1. Diamond and Mirrlees's Model

Let us now review their model of public investment in one consumer economy. They assume constant returns to scale technology in the private production sector and the presence of competitive conditions in the sector. Therefore, there are no profits in equilibrium. This is a crucial assumption for the efficiency analysis. Moreover, they also assume that there are no lump-sum transfers to the consumer.

Let us denote the vectors of consumer prices, producer prices, commodity tax rates, consumer demands, private producer supplies and government supplies by $q = (1, q_2, \ldots, q_n)$, $p = (1, p_2, \ldots, p_n)$, $t = (0, t_2, \ldots, t_n)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$, respectively. It may be now noted that since both

consumer demand and firm supply are homogeneous of degree zero in their respective prices, it is assumed to be $q_1 = 1$, $p_1 = 1$, and $t_1 = 0$ as normalizations. Furthermore, we denote the indirect utility function, the private production constraint and the government production constraint by $v(q)$, $y_1 = f(y_2, \ldots\ldots y_n)$ and $z_1 = g(z_2, \ldots\ldots, z_n)$, respectively.

Then, their model can be formulated as follows:

$$\underset{(z,\, t)}{\text{Maximize}}\ v(q)$$

subject to

$$y_1 = f(y_2, \ldots\ldots, y_n), \tag{53}$$

$$x_1(q) = y_1 + z_1, \tag{54}$$

$$x_i(q) = y_i + z_i, \text{ for } i = 2, \ldots, n, \tag{55}$$

$$p_i = -f_i(y_2, \ldots\ldots, y_n), \text{ for } i = 2, \ldots, n, \tag{56}$$

$$q_i = p_i + t_i, \text{ for } i = 2, \ldots, n, \tag{57}$$

$$z_1 = g(z_2, \ldots\ldots, z_n), \tag{58}$$

where f_i denotes the derivative of f with respect to y_i. With a government policy z and t given, eqs. (54)−(57) determine a private sector equilibrium with respect to p, q and y. Let us note that budget constraint for the government:

$$\sum_{i=1}^{n} t_i x_i + \sum_{i=1}^{n} p_i z_i = 0 \tag{59}$$

is not explicitly formulated in the above model since it is satisfied automatically by Walras's Law.

However, once optimal p and q vectors are chosen, the optimal tax rates t are determined, so that we can use the two sets of prices, p and q, as the control variables in place of tax rates. Further, once optimal q and z are chosen, an optimal p is determined from eq. (56). Thus, since we can treat z and q as independent variables, the above second-best problem is formulated as follows:

$$\underset{(z,\, q)}{\text{Maximize}}\ v(q)$$

subject to

$$x_i = y_i + z_i, \text{ for } i = 1, \ldots, n, \tag{60}$$

(61) $y_1 = f(y_2, \ldots, y_n)$,

(62) $z_1 = g(z_2, \ldots, z_n)$.

For simplicity, eliminating of z_1 and y from the constraints finally reduces our problem to

$$\underset{(z,\, q)}{\text{Maximize}}\ v(q)$$

subject to

(63) $x_1(q) - f[x_2(q) - z_2, \ldots, x_n(q) - z_n] - g(z_2, \ldots, z_n) = 0.$

The Lagrangian function for this problem is

(64) $L = v(q) - \alpha\,[x_1 - f(x_2 - z_2, \ldots, x_n - z_n) - g(z_2, \ldots, z_n)].$

Differentiating L with respect to q_k, and then making use of eq. (56), the consumer's budget constraint and $\partial x_1/\partial q_k = \partial x_1/\partial t_k$, we can derive the optimal tax structure:

(65) $v_k = -\alpha\,[\partial(\sum t_i x_i)/\partial t_k]$, for $k = 2, \ldots, n$.

We also differentiate L with respect to z_k:

(66) $\alpha(f_k - g_k) = 0$, for $k = 2, \ldots, n$.

Provided that α is unequal to zero, eq. (66) implies the equal marginal rates of transformation in public and private productions and therefore aggregate production efficiency. Consider now the one good-two period model (i. e., $n = 2$) in order to explain the implication of aggregate production efficiency in the context of the social rate of discount for public investment. Let us denote the first (current) and second (future) periods by subscripts 2 and 1, respectively. Thus, x_1 is the future consumption (*numeraire*), x_2 is the saving, y_1 is the future output from private investment y_2, z_1 is the future output from public investment z_2, f is a private production function, and g is a public production function. Then, the aggregate production efficiency implies that the social rate of discount for public investment should be the marginal rate of return on private investment: $g'(z_2) = f'(y_2)$. We can illustrate the second-best optimum in this case as in Fig. 1 when there is a decreasing returns to scale technology in the public production.

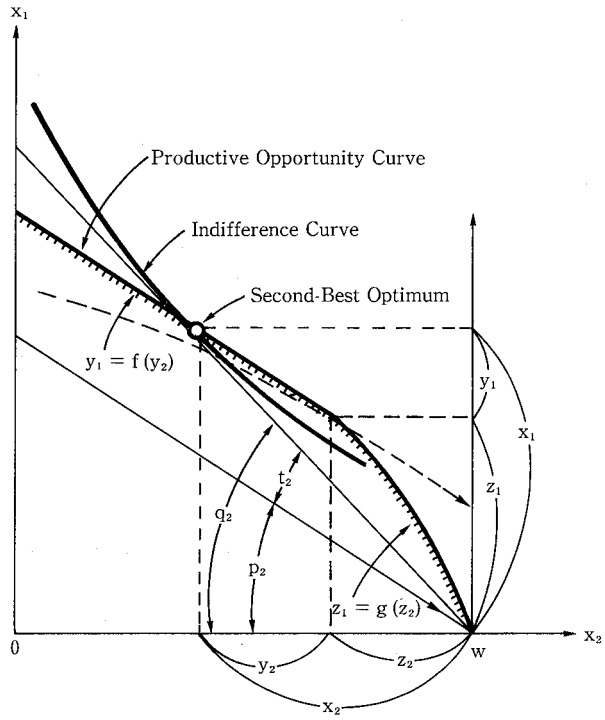

Fig. 1 Second-best optimum in the Diamond-Mirrlees model

3. 2. Pestieau's Model

In the neo-classical economic growth model of public investment, the Keynesian assumption of a constant saving ratio was employed as a way to characterize the implication of a capital market. Such a framework condems the consumer and the producer to passive roles. Their behaviour is unchanged whatever policy instruments the government uses. This assumption is significantly restrictive because it does not allow for an interaction between the investment and the financing behaviour of the public sector and the maximizing behaviour of the private sector.

On the contrary, Pestieau adopts a life-cycle saving approach developed by Diamond (1965) instead of employing the above restrictive assumption. The economy considered here is assumed to have an infinite future. Each individual lives for two periods, working in the first and retiring in the second. His only source of income is his work and he provides for his retirement by lending to a profit maximizing entrepreneur. At each period of time the government controls the level of public investment, taxes on wages and interest income, and the public borrowing, while its decisions are subject to a budget constraint: public investment must equal the sum of tax revenue, the public borrowing, and returns on public capital, less interest on the accumulated public debt. For given tax rates on wage and interest

levels, the consumer chooses an optimal combination of income, leisure, consumption, and saving. Private sector investment is accordingly equal to private savings minus public debt. The government thus strives to obtain an efficient intertemporal allocation of resources subject to its budget constraint and to the demand and supply relations of the private sector. The welfare function is the sum of discounted generational utilities and its maximization problem is solved by dynamic programming. From the first-order conditions so derived, it appears that an optimal policy of taxation and public capital accumulation is the one which sets the tax rates according to Ramsey's optimal taxation structure and which equates the rate of return on public investment to the social rate of time preference.

Pestieau's Model of public investment with the population growth rate being zero can be formulated as follows:

$$\underset{(g_{t+1},\ \theta^r_t,\ \theta^w_t)}{\text{Maximize}} \quad \sum_{t=0}^{\infty} (1+\omega)^{-t}\, U(c^1_t,\ c^2_t,\ L_t)$$

subject to

$$c^2_t = (w_t L_t - c^1_t)(1 + r_t), \tag{67}$$

$$U^1_t = U^2_t (1 + r_t), \tag{68}$$

$$U^3_t = -\, U^1_t w_t, \tag{69}$$

$$F^3_t = \theta^w_t + w_t, \tag{70}$$

$$F^1_t = \theta^r_t + r_{t-1} + 1, \tag{71}$$

$$Y_t = F(k_t,\ g_t,\ N_t), \tag{72}$$

$$N_t = L_t, \tag{73}$$

$$k_{t+1} = c^2_t/(1 + r_t), \tag{74}$$

$$Y_t = c^1_t + c^2_{t-1} + k_{t+1} + g_{t+1}, \tag{75}$$

where c^1 represents consumption in the first period of the lifetime of the representative consumer; c^2 consumption in the second period; L labour supply; N labour demand; k private capital; g public capital; Y level of output; r rate of interest; w wage rate; θ^r interest tax rate; θ^w wage tax rate; U a utility function; F a neoclassical production function; ω social rate of time preference; U^i partial derivative with respect to the argument i; F^i partial derivative with respect to the argument i;

and finally the subscript t denotes the generation or time.

Eq. (67) represents the consumer's budget constraint. Eqs. (68) and (69) represent the utility maximization conditions for the consumer. Eqs. (70) and (71) express the profit maximization conditions for the producer. Eq. (72) is the neo-classical production function. Finally, eqs. (73)−(75) stand for the equilibrium conditions in the labour, capital and output markets. Given the four exogenous variables (c^2_{t-1}, r_{t-1}, k_t, g_t) and the three control variables (g_{t+1}, θ^r_t, θ^w_t), eqs. (67)-(74) determine the eight endogenous variables (c^1_t, c^2_t, L_t, N_t, w_t, r_t, k_{t+1}, Y_t). Note that since the government budget constraint

$$(76) \qquad g_{t+1} = \theta^w_t L_t + \theta^r_t k_t + F^2_t g_t$$

is automatically derived from the above constraints, it is not explicitly introduced here.

Since the above formulation is not necessarily convenient, we will replace the set of control variables (g_{t+1}, θ^r_t, θ^w_t) by the set (g_{t+1}, r_t, w_t) and eliminate the eight variables (c^1_t, c^2_t, c^2_{t-1}, k_t, k_{t+1}, L_t, N_t, Y_t). The above problem can now be re-formulated as follows:

$$\underset{(g_{t+1},\, w_t,\, r_t)}{\text{Maximize}} \qquad \sum_{t=0}^{\infty} V(w_t, r_t)(1+\omega)^{-t}$$

subject to

$$(77) \qquad g_{t+1} = F[c^2_{t-1}(w_{t-1}, r_{t-1})/r_{t-1}, g_t, L_t(w_t, r_t)] - c^2_{t-1}(w_{t-1}, r_{t-1}) - w_t L_t(w_t, r_t),$$

where V is the indirect utility function.

From the first-order conditions for this dynamic programming problem, we obtain finally the following equations:

$$(78) \qquad V^w_t - \lambda_{t+1}(1+\omega)^{-1}[\partial(\theta^w_t L_t)/\partial\theta^w_t] - \lambda_{t+2}(1+\omega)^{-2}[\partial(\theta^r_{t+1} k_{t+1})/\partial\theta^w_t] = 0,$$

$$(79) \qquad V^r_t - \lambda_{t+1}(1+\omega)^{-1}[\partial(\theta^w_t L_t)/\partial\theta^r_{t+1}], - \lambda_{t+2}(1+\omega)^{-2}[\partial(\theta^r_{t+1} k_{t+1})/\partial\theta^r_{t+1}] = 0,$$

$$(80) \qquad \lambda_t - \lambda_{t+1} F^2_t (1+\omega)^{-1} = 0,$$

where $\lambda_t = \partial J/\partial g_t$ and

$$(81) \qquad J(w_{t-1}, r_{t-1}, g_t) = \text{Max}\,[V_t + (1+\omega)^{-1} J(w_t, r_t, g_{t+1})].$$

Eq. (78) [(79)] asserts that the marginal utility of a change in the wage [interest] rate is proportional to the change in tax revenue resulting from a change in the corre-

sponding tax rate [see also Diamond and Mirrlees (1971), p. 16, eq. (22)]. Since a change in the tax rate has an immediate impact on the labour supply, and a delayed impact on the next period investment through a change in the immediate demand for savings, changes in the tax revenue thus must be weighted by different social costs: the utility price of the resource, $\lambda_{t+1}(1+\omega)^{-1}$ for the wage bill, and $\lambda_{t+2}(1+\omega)^{-2}$ for the interest income.

In a steady state, eq. (80) yields

$$F^2 = 1 + \omega, \tag{82}$$

that is, the rate of return on public investment (capital) must be equal to the social rate of time preference. Thus, we can conclude that the government's choice of a discount rate is, in the long run, the same as it would be if the government were able to control the economy as in the first-best situation. Further rearrangement of the first-order conditions (78), (79) and (80) yields

$$F^2 = \frac{F^1(1-\varepsilon_r) + r\varepsilon_r}{1-(\theta^w/w)\,\varepsilon_w} \tag{83}$$

where $\varepsilon_w = (w/L)(dL/dw)|_V$ is the total conpensating variation (TCV) elasticity of labour and ε_r is TCV elasticity of the second period consumption. Here, when $\theta^w = 0$, eq. (83) reduces to

$$F^2 = F^1(1-\varepsilon_r) + r\varepsilon_r. \tag{84}$$

Therefore, we obtain another conclusion that the rate of return on public investment can be also expressed as a weighted average of the market rates of return on private investment in the traditional opportunity cost approach to the social rate of discount [see also Sandmo and Drèze (1971)].

Finally, the public borrowing as an instrument to the above second-best problem being added, the relation $F^1 = F^2$ holds as well as the first-best conditions (78)-(80). Then, the required rate of return on public investment is equal to the rate of return on private investment.

REFERENCES

1. Arrow, K. J. and Kurz, M. A. (1969). "Optimal public investment policy with fixed private savings ratio." *Journal of Economic Theory*, Vol.1, pp.141-177.
2. Arrow, K. J. and Kurz, M. A. (1970). *Public Investment, the Rate of Return, and Optimal Fiscal Policy*. Baltimore: Johns Hopkins University Press.
3. Arrow, K. J. and Lind, R. C. (1970). "Uncertainty and the evaluation of public investment decisions." *American Economic Review*, Vol.60, pp.364-378.
4. Auerbach, A. J. (1987). "Weighted-average discount rates in public expenditure analysis: A generalization", in M. J. Boskin ,ed., *Modern Developments in Public Finance*, Basil Blackwell, pp.40-60.
5. Bailey, M. J. and Jensen, M. C. (1972). Risk and the discount rate for public investment, in Jensen, M. C., ed., *Studies in the Theory of Capital Markets*, Praeger (New York), pp.269-293.
6. Boadway, R. W. (1975). Cost-benefit rules in general equilibrium. *Review of Economic Studies*, Vol.42, pp.361-374.
7. Boadway, R. W. (1978). Public investment decision rules in a neoclassical growing economy. *International Economic Review*, Vol.19, pp.265-286.
8. Diamond, P. A. (1965). National debt in a neoclassical growth model. *American Economic Review*, Vol.55, pp. 1126−1150.
9. Diamond, P. A. (1967). The role of a stock market in a general equilibrium model with technological uncertainty. *American Economic Review*, Vol.57, pp.759-776.
10. Diamond, P. A. (1973). Taxation and public production in a growth setting, in Mirrlees and Stern, eds., *Models of Economic Growth*, the International Economic Association, pp.215-235.
11. Diamond, P. A. and Mirrlees, J. A. (1971). Optimal taxation and public production I and II. *American Economic Review*, Vol.61, pp. 8−27 and pp.261-278.
12. Drèze, J. H. (1974). Discount rate and public investment: A postscriptum. *Economica*, vol. 41, pp.52-61.
13. Feldstein, M. (1964). Net social benefit calculations and the public investment decision. *Oxford Economic Papers*, Vol.16, pp.114-131.
14. Glinols, E. L. (1985). Public investment and social risk sharing. *European Economic Review*, Vol.29, pp.303-321.
15. Hägen, K. P. (1983). Optimal shadow prices and discount rate for budget constrained public firms. *Journal of Public Economics*, Vol.22, pp.27-48.
16. Hägen, K. P. (1988). Optimal shadow prices for public production. *Journal of Public Economics*, Vol.35, pp.119-127.
17. Hirshleifer, J. (1966). Investment decision under uncertainty: Applications of the state-preference approach. *Quarterly Journal of Economics*, Vol.80, pp.252

-277.

18. Marchand, M., Mintz, J. and Pestieau, P. (1985). Public production and shadow pricing in a model of disequilibrium in labour and capital markets. *Journal of Economic Theory*, Vol.36, pp.237-256.
19. Marglin, S. A. (1963-a). Social rate of discount and the optimal rate of investment. *Quarterly Journal of Economics*, Vol.77, pp.95-111.
20. Marglin, S. A. (1963-b). The opportunity costs of public investment. *Quarterly Journal of Economics*, Vol.77,pp.274-289.
21. Mayshar, J. (1977). "Should government subsidize risky private projects?" *American Economic Review*, Vol.67, pp.20-28.
22. McFadden, D. (1972). Criteria for public investment: Comment. *Journal of Political Economy*, Vol.80, pp.1295-1313.
23. Oniki, H. (1973). Comparative dynamics (sensitivity analysis) in optimal control theory. *Journal of Economic Theory*, Vol.6, pp.265-283.
24. Pestieau, P. M. (1974). Optimal taxation and discount rate for public investment in a growh setting. *Journal of Public Economics*, Vol.3, pp.217-235.
25. Pestieau, P. M. (1975). The role of taxation in the determination of the social discount rate. *International Economic Review*, Vol.16, pp 362-368.
26. Samuelson, P. A. (1964). Principle of efficiency: Discussion. *American Economic Review*, Vol.54, pp.93-96.
27. Sandmo, A. (1972). Discount rate for public investment under uncertainty. *International Economic Review*, Vol.13, pp.287-302.
28. Sandmo, A. and Drèze, J. H. (1971). Discount rate for public investment in closed and open economies. *Economica*, Vol.38, pp.395-412.
29. Schmalensee, R. (1976). Public investment criteria, insurance markets, and income taxes. *Journal of Public Economics*, Vol.6, pp.425-445.
30. Sen, A. K. (1961). On optimizing the rate of saving. *Economic Journal*, Vol. 71, pp.479-496.
31. Sen, A. K. (1967). "Isolation, assurance and the social rate of discount." *Quarterly Journal of Economics*, Vol.81, pp.112-124.
32. Vickrey, W. (1964). "Principle of efficiency: Discussion." *American Economic Review*, Vol.54,pp.93-96.
33. Yoshida, M. (1986). Public investment criterion in an overlapping generations economy. *Economica*, Vol.53, pp.247-263.