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Public Investment Criteria in General Disequilibrium Models with Overlapping Generations*

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Based on a *monetary* overlapping generations model where the nominal money supply is *fixed*, this paper examines public investment criterion which maximizes a sum of generational utilities discounted with a social rate of time preference, in each of the Walrasian equilibrium and Malinvaud's three disequilibrium regimes. It is shown that in stationary states, if public investment has no impacts on the labour employment, then the social discount rates for public investment in the Walrasian equilibrium, Keynesian unemployment, classical unemployment, and repressed inflation regimes should be all equal to a *weighted average* of the social rate of "time preference" and the rate of return on "private investment".

1. Introduction

The problem of what the appropriate shadow prices ought to be for evaluating public projects has been a subject of extensive controversy in benefit-cost analysis. It is well known that this problem arises from the market imperfections and failures, for example, capital income taxation and uncertainty. Recently, such a shadow pricing problem has been studied in a framework of market disequilibria by Johansson (1982), Drèze (1984), Marchand, Mintz and Pestieau (1984 and 1985), Cuddington, Johansson and Ohlsson (1985) and others. In particular, Marchand, Mintz and Pestieau (1985) examined the optimal shadow prices for investment and labour which a public firm should use in various regimes of disequilibria occuring in the capital and labour markets, when both the real wage rate and the real rate of interest are exogenously fixed. Their public investment model was mainly based on a simple two-period general equilibrium model developed by Sandmo and Drèze (1971), whose framework has been utilized by subsequent authors to consider the effects of introducing various complications and distortions.

However, as was pointed out by Yoshida (1986), the two-period or the finite-

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horizon model is not always satisfactory for the analysis of public investment problem. Still less is it, if we hope to tackle this problem in a "choice-theoretic" macroeconomic disequilibrium model containing *money*, which is now familiar through the work of Barro and Grossman (1971), Malinvaud (1977) and others. Hence, I believe that the public investment problem should be explored in a general (dis-)equilibrium economy with overlapping generations.(1)

The model in this paper is based on Rankin (1987) who investigated whether the fiscal policy in the disequilibrium framework which is optimal in the sense of maximizing welfare is one in which public investment should be set higher than is optimal in the Walrasian equilibrium. An important advantage of his model is that it enables us to analyze the public investment problem in a *monetary overlapping generations model*, under either of the polar assumptions: (i) prices adjust instantaneously to clear markets in the Walrasian manner, or (ii) they are exogenously given and quantities adjust instead, in the manner of Barro and Grossman and others. To put it more concretely, the latter assumption is that the price and money wage levels are exogenously fixed at some constant levels over the infinite future, resulting in disequilibria in the goods and labour markets, but the bond price adjusts instantaneously to clear the money market. Thus, in the "current" period, there are Malinvaud's familiar three rationing regimes: (a) Keynesian unemployment regime, being the excess supply in both goods and labour markets, (b) classical unemployment regime, being the excess supply and demand in the labour and goods markets, respectively, and (c) repressed inflation regime, being the excess demand in both goods and labour markets.(2)

This paper examines an optimal public policy which maximizes a sum of generational utilities discounted with a social rate of time preference, in each of the Walrasian equilibrium and Malinvaud's three disequilibrium regimes. In order to focus on the "pure" fiscal policy, as in Rankin (1986 and 1987) we shall assume that the nominal money supply is kept at some "constant" level over time period. Thus, the government can control only the levels of public investment, public debts and lump-sum tax to the younger generation. Although the government could also impose the lump-sum tax on the older generation, it is redundant since the debt policy is equivalent to the use of lump-sum taxes to both generations, as was shown by Atkinson and Sandmo (1980). We shall also abstract from the issues of distortionary taxation, uncertainty and so on, so as to concentrate on the consequences of market disequilibria.

Now, it will be useful to clarify some important differences between Rankin's model and our own. First, although public investment in his model was considered either as "waste" or as a direct argument of consumer's utility, in our model it is treated as a public intermediate goods to the private sector's production function, provided free by the government. Second, our analytical purpose is not to compare

the optimal levels of public investment in the Walrasian equilibrium and general disequilibrium regimes, but to derive the *optimal shadow pricing rules* for public investment and labour in both regimes. Third, Rankin assumed that the government objective is to maximize the "lifetime utility" of the representative individual in stationary states. In this case, as the welfare optimum in the disequilibrium framework lies somewhere on the boundary of the Keynesian regime, indeed it is out of question to confine the analysis to the Keynesian-classical and Keynesian-repressed inflation boundaries. However, as we have *discounted* utilities of future generations with the social rate of time preference, the welfare opitmum will lie either in the interior of a rationing regime or on the boundary between two or three regimes, depending on the levels at which the goods price and money wage are fixed relative to their Walrasian equilibrium levels. Thus, it is reasonable that we proceed regime by regime, assuming that the welfare optimum exists in the interior or boundaries of each disequilibrium regime, which may reduce our analysis to be "local" or "regime-specific".

In this paper, we are particularly concerned with how the optimal shadow price for public investment or equivalently, the so-called *social discount rate*, is related to the social rate of time preference, the rate of return on private investment, and the market rate of interest. Before proceeding, from such a point of view we shall briefly summarize the main results in stationary states.

First, the social discount rates for public investment in the Walrasian equilibrium, Keynesian unemployment and repressed inflation regimes should be all equal to a *weighted average* of the social rate of "time preference" and the rate of return on "private investment"; the weight depends on the private capital demand crowded out-in directly and indirectly by a unit increase in public investment. Second, the social discount rate in the classical unemployment regime should be a *modified* version of the above-mentioned weighted average criterion, containing an extra term which represents the marginal opportunity cost of public investment through its direct and indirect impacts on the "labour employment".

Third, the social discount rate in each regime, irrespective of whether it is in equilibrium or disequilibrium, should be in general neither equal to the social rate of time preference nor the rate of return on private investment. However, there are *two special cases* where we can usually adopt either of these two rates as the social discount rate. The one is that if public investment has no impacts on the labour and private capital demands, then the social discount rates in all regimes should be equal to the social rate of time preference. The other is that if a unit increase in public investment does not have any effect on the labour employment but exactly results in a unit decrease in private investment, then the social discount rates in all regimes should be equal to the rate of return on private investment. In the latter case, it is possible for us to use the *market* rate of interest as the social discount rates in all

regimes except for the Keynesian unemployment regime.

Finally, the market rate of interest does not coincide with the social rate of time preference even in the Walrasian equilibrium as well as in Malinvaud's three disequilibrium regimes. This implies that the government cannot attain the *intergenerational* optimum of "income distribution", even if lump-sum redistributive taxations are feasible and there are no distortions due to indirect taxations, uncertainty and others. The reason is that the nominal money supply has been assumed to be fixed at some constant level over the infinite future. Consequently, the social discount rates obtained in this paper are all *second-best* optimal.

The rest of the paper is organized as follows. Section 2 introduces the basic model. In sections 3 and 4, the social discount rates in the Walrasian equilibrium and general disequilibrium regimes are, respectively, derived and examined. Finally, section 5 contains some concluding remarks.

2. The Basic Model

2. 1. Consumer Behaviour

For the sake of simplicity, it is assumed that in each generation there is only one person or equivalently, a fixed number of homogeneous people. The population growth rate is therefore zero. The representative individual lives for two periods, working in the first and enjoying retirement in the second. There are three assets in this economy: money M, public debt B, and physical capital k (i.e., "shares"). The individual can save for retirement by holding these assets. The last two are "perfect substitutes" and so both pay the single interest rate. Money pays no interest, but provides liquidity services represented by real money balances entering the utility function.[3] Public debt has a *one-period* maturity, selling at price q, and having a redemption value of one unit of money after one period. Therefore, the nominal rate of interest is $(1/q)-1$. Physical capital pays a total return equal to the profits of private firm in the following period.

The individual possesses a utility function whose arguments are consumptions in two periods of his lifetime, and real end-of-period money holdings. Thus, one can write the individual utility function assumed to be "identical" for all generations as:

$$(1) \qquad U_t = U(c_t^1,\ c_{t+1}^2,\ M_t^d/P_t),$$

where c_t^1 is consumption during his working period, c_{t+1}^2 consumption during his retirement, M_t^d end-of-period money holdings, P_t goods price, and index t denotes time period. The utility function displays decreasing positive marginal utilities for these arguments. We shall now assume no utility of leisure, so the labour supply permanently equals the young's exogenous time endowment, N, though his "actual" employment, n, will be less than this when there is the excess supply in the labour market. In either case, the labour employment is parametric, so that the

consumer's demand functions are *independent* of the labour market situation.

The budget constraints for the representative individual in the first and second periods of his lifetime are respectively given by:

$$(2) \qquad (W_t/P_t)\, n_t - \tau_t = c_t^1 + (M_t^d/P_t) + s_t,$$

$$(3) \qquad M_t^d/P_{t+1} + (P_t/q_t P_{t+1})(\mu_t + s_t) = c_{t+1}^2,$$

where W_t is money wage at period t, τ_t lump-sum tax on the *young*, s_t real saving, and μ_t net present value of private firm "ownership" [see eq.(8) below]. Eliminating s_t from (2) and (3), we obtain the lifetime budget constraint:

$$(4) \qquad w_t n_t - \tau_t + \mu_t = c_t^1 + (1 - q_t)\, m_t^d + q_t\,(1 + \pi_{t+1})\, c_{t+1}^2,$$

where $w_t(\equiv W_t/P_t)$ is real wage at period t, $m_t^d(\equiv M_t^d/P_t)$ real money holdings, $\pi_{t+1}(\equiv P_{t+1}/P_t - 1)$ rate of inflation. It may be now noted that the *real* rate of interest is $[1/q\,(1+\pi)] - 1$.

When not rationed in the goods market, which occurs in the Walrasian and Keynesian regimes, the individual maximizes the lifetime utility (1) subject to the budget constraint (4) with respect to c_t^1, c_{t+1}^2 and m_t^d. The first-order conditions are:

$$(5) \qquad U_1/U_2 = 1/q_t\,(1 + \pi_{t+1}) \text{ and } U_m/U_2 = (1 - q_t)/q_t\,(1 + \pi_{t+1}),$$

where U_1, U_2 and U_m are the partial derivatives of the utility function U with respect to c_t^1, c_{t+1}^2 and m_t^d, respectively. From these two conditions and (4), the *effective* demand functions for c_t^1, c_{t+1}^2 and m_t^d can be expressed as functions of w_t, n_t, τ_t, μ_t, q_t and π_{t+1}. Reintroducing these demand functions into (1), one can define an *indirect* utility function: $V(w_t, n_t, \tau_t, \mu_t, q_t, \pi_{t+1})$, where $n_t = N$ in the Walrasian regime, and $n_t < N$, $w_t = w$ (constant) and $\pi_{t+1} = 0$ in the Keynesian. Now, letting λ_t denote the private marginal utility of income $I_t(\equiv w_t n_t - \tau_t + \mu_t)$, then the function V has the following properties by the *envelope theorem*:[4]

$$(6) \qquad \begin{aligned} \lambda &= V_n/w = V_w/n = -V_\tau = V_\mu = V_q/[m^d - (1+\pi)\,c^2] \\ &= -V_\pi/qc^2, \end{aligned}$$

where $\lambda \equiv V_I \equiv \partial V/\partial I$, and similarly for V_n, V_w, V_τ, V_μ, V_q and V_π.

Next, turning to the case of goods market rationing, which occurs in the classical and repressed inflation regimes, the simple rationing rule to be adopted will be that the old and the government have priority, so that only the *young* is rationed. Thus, the old and the government are not rationed at all in any market. Noting that c_t^1 is *rationed*, and $w_t = w$ and $\pi_{t+1} = 0$ in these regimes, then the utility maximization problem for the individual is:

$$\text{maximize} \quad U(c_t^1, c_{t+1}^2, m_t^d)$$
$$\text{subject to} \quad wn_t - \tau_t + \mu_t = c_t^1 + (1 - q_t)\, m_t^d + q_t c_{t+1}^2,$$

The first-order condition for this problem is $U_m/U_2=(1-q_t)/q_t$, which results in the effective demand functions for c_{t+1}^2 and m_t^d as functions of n_t, τ_t, μ_t, q_t and c_t^1. Thus, we obtain an indirect utility function: $v\,(n_t, \tau_t, \mu_t, q_t, c_t^1)$, where $n_t<N$ in the classical regime and $n_t=N$ in the repressed inflation. The partial derivative of the function v with respect to c_t^1 is given by:

$$(7) \qquad V_c \equiv \partial v/\partial c^1 = \partial U/\partial c^1 - \lambda.^{(5)}$$

The partial derivatives of other arguments are the same as (6).

2. 2. Private Firm Behaviour

The technology of this economy is specified by a private sector's production function of the type: $y_t=F(n_t, k_{t-1}, g_{t-1})$, where y_t is real output at period t, k_{t-1} and g_{t-1} are capital stocks in the private and public sectors at period $t-1$, respectively, and n_t is labour employment supplied "actually" at period t. As in Pestieau (1974), the production function is assumed to be homogeneous of degree one in all the arguments, concave, and twice differentiable with positive first partial derivatives. But, now we shall not predetermine the signs of *cross-derivatives*, F_{nk}, F_{ng} and F_{kg}, since they significantly affects public investment decision rules in the following analyses. Further, for the sake of simplicity, it is assumed that both private and public capital only last *one period*, i.e., "immediate depreciation". Thus, the private and public capitals are equal to the private and public investments of the previous period, respectively.

Now, suppose that the government does not recover the full imputed share for the use of public capital by the private firm. Then, the net present value of private ownership, μ_t, becomes:[6]

$$(8) \qquad \mu_t = q_t(1 + \pi_{t+1})\,[F(n_{t+1}, k_t, g_t) - w_{t+1}n_{t+1}] - k_t.$$

In the Walrasian and classical regimes which are not rationed in both goods and labour markets, the private firm decides the labour demand, n_{t+1}, and the private capital demand, k_t, so as to maximize μ_t. Essentially, this is a "two-stage" maximization problem. First, the private firm must hire labour in the spot market at period $t+1$, where the private capital stock has been predetermined. The first-order condition for this *first-stage* problem is

$$(9) \qquad F_n(n_{t+1}, k_t, g_t) = w_{t+1},$$

where $F_n \equiv \partial F/\partial n$. Inverting this condition gives us the labour demand function at period $t+1$, $n_{t+1} = n\,(w_{t+1}, k_t, g_t)$, which has the following partial derivatives:

$$(10) \qquad n_w = 1/F_{nn},\ n_k = -F_{nk}/F_{nn} \text{ and } n_g = -F_{ng}/F_{nn},$$

where $F_{nn} \equiv \partial F_n/\partial n$, and similarly for F_{nk} and F_{ng}. Next, the amount of private capital at period t must be decided upon. Substituting the labour demand function into (8), we obtain the *maximized* net present value of private ownership: $\nu_t(k_t) = \mathrm{Max}_{(n)}\ \mu_t(n_{t+1}, k_t)$. The private firm hires capital so as to maximize $\nu_t(k_t)$. The first-order condition for this *second-stage* problem is $d\nu/dk = 0$. Now, since $d\nu/dk = \partial\mu/\partial k$ by the envelope theorem, we obtain:

$$(11) \qquad F_k[n(w_{t+1}, k_t, g_t), k_t, g_t] = 1/[q_t(1+\pi_{t+1})],$$

where $F_k \equiv \partial F/\partial k$. Solving this condition with respect to k_t, the private capital demand function becomes:

$$(12) \qquad k_t = k(x, w_{t+1}, g_t),$$

where $x \equiv q_t P_{t+1}/P_t$. Using (11), this function has the following partial derivatives:

$$(13) \qquad \begin{aligned} &k_x = -(F_k)^2 F_{nn}/C,\ k_w = -F_{kn}/C \text{ and } k_g = (F_{kn}F_{ng} - F_{nn}F_{kg})/C, \\ &\text{where } C \equiv F_{nn}F_{kk} - (F_{nk})^2. \end{aligned}$$

Secondly, suppose that the private firm is rationed in the *goods* market, which occurs in the Keynesian unemployment regime. With a constraint of y_t on the current output demand, the output-constrained labour demand function is obtained simply by inverting the production function: $n_t = n^r(y_t, k_{t-1}, g_{t-1})$. Note that $F_n > w$, since $n_t < n(w, k_{t-1}, g_{t-1})$. In what follows, let the superscript "r" denote the *constrained* demand functions on labour, private capital, and so on. With an expected constraint y_{t+1} on the future output demand, the problem is as follows:

$$\max \quad \mu_t = q_t(y_{t+1} - wn_{t+1}) - k_t \quad \text{subject to} \quad y_{t+1} = F(n_{t+1}, k_t, g_t).$$

The first-order condition is $F_n(n_{t+1}, k_t, g_t)/F_k(n_{t+1}, k_t, g_t) = wq_t$, so that we obtain the output-constrained capital demand function, or the *accelerator* model of private investment:

$$(14) \qquad k_t = k^r(y_{t+1}, q_t, g_t),$$

which has the following partial drivatives:

$$(15) \qquad \begin{aligned} &k_y^r = [F_{kn} - (F_{nn}/F_n)F_k]/D, \\ &k_q^r = F_n F_k/qD, \\ &k_g^r = \{[(F_k F_{nn}/F_n) - F_{kn}]F_g + F_n F_{kg} - F_k F_{ng}\}/D, \end{aligned}$$

where $D \equiv 2F_k F_{kn} - [F_{nn}(F_k)^2/F_n] - F_{kk}F_n$. In this case, it should be noted that since $F_n > w$ and $F_n/F_k = wq_t$, $F_k > 1/q_t$.(7)

Finally, suppose that the private firm is rationed in the *labour* market, which

occurs in the repressed inflation regime. With a constraint of n_t on the current labour demand, the production function gives the labour-constrained output supply: $y_t = y^r(n_t, k_{t-1}, g_{t-1}) = F(n_t, k_{t-1}, g_{t-1})$. Note that $F_n > w$ in this case. With an expected constraint on the future labour demand n_{t+1}, the problem is to maximize μ_t given by (8) with respect to k_t, where n_{t+1} is *fixed*, $w_{t+1} = w$ and $\pi_{t+1} = 0$. The first-order condition for this problem is $F_k(n_{t+1}, k_t, g_t) = 1/q_t$. Inverting this condition, we obtain the labour-constrained private capital demand function:

$$(16) \qquad k_t = k^r(n_{t+1}, q_t, g_t),$$

which has the following partial derivatives:

$$(17) \qquad k^r_n = -F_{kn}/F_{kk},\ k^r_g = -F_{kg}/F_{kk} \text{ and } k^r_q = -(F_k)^2/F_{kk}.$$

2. 3. Government Behaviour

The government can freely choose the level of public investment without any constraint; it also can impose the lump-sum tax on the young and can issue the one-period public debt, so as to finance public investment; and it, however, cannot recover the full imputed share for use of public capital goods by the private firm. Since we have assumed that the government keeps the nominal money supply in each period "constant", its budget constraint is given by:

$$(18) \qquad g_t + (B_{t-1}/P_t) = \tau_t + (q_t B_t/P_t).$$

In order to speak about a desired public policy, it is necessary to specify an *intergenerational* welfare function. As in Diamond (1973), Pestieau (1974), Atkinson and Sandmo (1980), Yakita (1989) and others, we shall adopt a sum of generational utilities discounted by the rate of social time preference, δ, as the social welfare function:

$$(19) \qquad \sum_{t=0}^{\infty} (1+\delta)^{-t}\, U(c^1_t, c^2_{t+1}, m^d_t),$$

where the welfare of generation -1 living in the second period of his lifetime at period 0 is taken as given. Then, the government's objective is to control public instruments so as to maximize the social welfare (19) subject to its budget constraint and to the demand and supply relations of private sector in each of the Walrasian equilibrium and general disequilibrium regimes.

2. 4. Temporary Equilibrium

There are four temporary equilibrium conditions for goods, labour, money, and bonds in this economy. However, we can ignore any one of them from Walras's Law. In the following analyses, we shall not consider the "capital market" equilibrium condition: $s_t = (q_t/P_t)B_t + k_t$. In the framework of market disequilibria with the arbitrary fixed price and money wage, it is necessary to specify the form of capital

demand function in order to define a temporary equilibrium. Since it depends on the next period's regime, an appropriate expectation hypothesis concerning the *future* regime must be introduced. In this paper, it is assumed that all agents expect that the next period will have the *same constraint regime* as the current period. This appears to be the most "natural" expectational assumption.[8] Then, the temporary equilibrium conditions in the Walrasian equilibrium and Malinvaud's three disequilibrium regimes can be formulated as follows, respectively.

2. 4. 1. Walrasian Equilibrium

The price mechanism now functions perfectly so that all the markets are instantaneously brought to an equilibrium at each period. The temporary equilibrium conditions in the goods, labour and money markets for determining prices (P_t, W_t, q_t) are given by

$$
\begin{aligned}
(W_t/P_t)\,N &= c^1\,[(W_t/P_t)\,N - \tau_t + \mu_t,\ q_t,\ P_{t+1}/P_t\,] \\
&\qquad + (B_{t-1} + M)/P_t + k\,(q_t P_{t+1}/P_t,\ W_{t+1}/P_{t+1},\ g_t) + g_t, \\
N &= n\,(W_t/P_t,\ k_{t-1},\ g_{t-1}), \\
M/P_t &= m^d\,[(W_t/P_t)\,N - \tau_t + \mu_t,\ q_t,\ P_{t+1}/P_t],
\end{aligned}
\tag{20}
$$

where M denotes the constant level of nominal money supply and the first equilibrium condition has been derived from (3), (8), and the goods market identity, $y_t = c_t^1 + c_t^2 + k_t + g_t$.

2. 4. 2. Keynesian Unemployment Equilibrium

In this regime, both price and money wage are fixed at such constant values as to generate the notional excess supply in both goods and labour markets:

$$
\begin{aligned}
&c^1\,(wN + \mu_t - \tau_t,\ q_t) + c_t^2 + k^r\,(y_{t+1},\ q_t,\ g_t) + g_t \\
&\qquad\qquad < F\,[n\,(w,\ k_{t-1},\ g_{t-1}),\ k_{t-1},\ g_{t-1}], \\
&n\,(w,\ k_{t-1},\ g_{t-1}) < N.
\end{aligned}
\tag{21}
$$

In such a case, the young has to make dual decisions on consumptions in two periods of his lifetime and real money holdings, because he is rationed in the current labour market. However, as he is not constrained in the current goods market, his effective labour supply continues to be exogenous time endowment, N. On the contrary, the private firm is constrained in the current goods market. Therefore, it has to make dual decisions on labour demand. Since the next period's regime too has been assumed to be the Keynesian, the private firm must also make dual decisions on capital demand.

Since $F_n > 0$, $\partial c^1/\partial n > 0$ and (21), it can be easily shown that the *effective* demand-supply relations in the goods and labour markets continue to be the excess supply, after dual decisions of the young and the private firm:

$$c^{1r}(wn_t - \tau_t + \mu_t, q_t) + c_t^2 + k^r(y_{t+1}, q_t, g_t) + g_t$$
$$< F[n(w, k_{t-1}, g_{t-1}), k_{t-1}, g_{t-1}],$$
$$n^r(y_t, k_{t-1}, g_{t-1}) < N.$$

Now, assuming that the actually "rationed" output and labour (y_t, n_t) are instantaneously adjusted to the *shorter-side* of effective demand-supply levels in the goods and labour markets, then the temporary equilibrium conditions for determining (y_t, n_t, q_t) are given by:

$$\text{(22)} \quad \begin{aligned} wn_t &= c^{1r}(wn_t - \tau_t + \mu_t, q_t) + m + b_{t-1} + k^r(y_{t+1}, q_t, g_t) + g_t, \\ n_t &= n^r(y_t, k_{t-1}, g_{t-1}), \qquad m = m^r(wn_t - \tau_t + \mu_t, q_t), \end{aligned}$$

where $m(\equiv M/P)$ and $b_t(\equiv B_t/P)$ denote the real constant money supply and the real public debt at period t, respectively.

2. 4. 3. Classical Unemployment Equilibrium

The exogenous price and money wage levels now generate the notional excess supply in the labour market but the notional excess demand in the goods market. Therefore, since the young is rationed in both current markets, he has to make dual decisions on money demand. However, because the private firm is not constrained at all in both current labour and goods markets, it does not have to make any dual decision on labour demand and goods supply. Since it has been assumed that the next period's regime too is the classical, the private firm need not make dual decisions on capital demand.

It is clear that the effective demand-supply relation in the current labour market continues to be the excess supply, after dual decisions of the young. Now, if we shall assume that the rationed first-period consumption of the young, c_t^1, is instantaneously adjusted to its effective level:

$$F[n(w, k_{t-1}, g_{t-1}), k_{t-1}, g_{t-1}] - c_t^2 - k_t - g_t,$$

then the "short-side principle" of the quantity adjustment with respect to the actually rationed labour and first-period consumption (n_t, c_t^1) gives us the temporary equilibrium conditions for determining (n_t, c_t^1, q_t):

$$\text{(23)} \quad \begin{aligned} wn_t &= c_t^1 + m + b_{t-1} + k(q_t, g_t; w) + g_t, \\ n_t &= n(w, k_{t-1}, g_{t-1}), \\ m &= m^r(wn_t - \tau_t + \mu_t - c_t^1, q_t, c_t^1). \end{aligned}$$

2. 4. 4. Repressed Inflation Equilibrium

There are the notional excess demand in both current labour and goods markets in this regime. Because the young is rationed in the goods market, he has

to make dual decisions on money demand. Since the private firm is constrained in the labour market but not in the goods market, it has to make dual decisions on goods supply. It has been assumed that the next period's regime too is the repressed inflation. Therefore, as the private firm is rationed in the future labour market, it must also make dual decisions on capital demand.

Under the short-side principle of the quantity adjustment with respect to the actually rationed labour and first-period consumption (n_t, c_t^1), the temporary equilibrium conditions for determining (n_t, c_t^1, q_t) are given by:

$$
\begin{aligned}
& wN = c_t^1 + m + b_{t-1} + k^r(n_{t+1}, q_t, g_t) + g_t, \\
(24) \qquad & n_t = N, \\
& m = m^r(wN - \tau_t + \mu_t - c_t^1, q_t, c_t^1).
\end{aligned}
$$

3. Public Investment Criterion in the Walrasian Equilibrium

In this section, we shall formulate the welfare maximization problem for the government in the Walrasian equilibrium regime and then derive the decision rule for public investment. Since the temporary equilibrium conditions (20) in this regime contain the expectational variables $(P_{t+1}, W_{t+1}, \mu_t)$, an appropriate expectation hypothesis must be introduced in order that equilibrium can be fully defined. It is now assumed that all agents have *perfect foresight* to the price and money wage levels which will prevail in the future, because the focus of this paper is not on the consequence of incorrect expectation.

To solve out the expectational variables (w_{t+1}, μ_t) under the assumption of perfect foresight, it should be noted that w_{t+1} must equal $F_n(N, k_t, g_t)$. Substituting this relation into the capital demand function (12), we obtain

$$k_t = k\,[q_t P_{t+1}/P_t,\ F_n(N, k_t, g_t),\ g_t].$$

Thus, k_t becomes an implicit function of $q_t P_{t+1}/P_t$ and g_t. Let us express this as:

$$(25) \qquad k_t = k^W(q_t P_{t+1}/P_t, g_t).$$

In what follows, let the superscript "W" (K, C and R) denote the economy to be held at the Walrasian (Keynesian, classical, and repressed inflation) regime. The function k^W has the following partial derivatives from (11) and (13):

$$
\begin{aligned}
& k_g^W = (k_w F_{ng} + k_g)/(1 - k_w F_{nk}) = -F_{kg}/F_{kk}, \\
(26) \qquad & k_q^W = (P_{t+1}/P_t)\,k_x/(1 - k_w F_{nk}) = -(P_{t+1}/P_t)(F_k)^2/F_{kk}, \\
& k_\pi^W = q_t k_x/(1 - \mathrm{k}_w F_{nk}) = -(P_t/P_{t+1})F_k/F_{kk}.
\end{aligned}
$$

Given (25), the net present value of firm ownership μ_t becomes also a function of $q_t P_{t+1}/P_t [= x_t = q_t(1+\pi_{t+1})]$ and g_t:

$$(27) \qquad \mu_t = \mu^W(x_t, g_t) \equiv x_t[F(N, k_t^W, g_t) - F_n(N, k_t^W, g_t)N] - k_t^W.$$

Now, the government objective is to maximize the social welfare function (19) subject to the temporary equilibrium conditions (20), the government budget constraint (18), eq.(25), and the initial conditions ($k_0 = \underline{k}_0$, $g_0 = \underline{g}_0$, $B_0 = \underline{B}_0$, $P_1 = \underline{P}_1$):

$$\text{Maximize} \sum_{t=0}^{\infty} (1+\delta)^{-t} V[(W_t/P_t)N - \tau_t + \mu^W(q_t P_{t+1}/P_t, g_t), q_t, P_{t+1}/P_t]$$

subject to

$$(28) \qquad \begin{aligned} &(W_t/P_t)N = c^1[(W_t/P_t)N - \tau_t + \mu^W(q_t P_{t+1}/P_t, g_t), q_t, P_{t+1}/P_t] \\ &\qquad + (B_{t-1} + M)/P_t + k_t + g_t, \\ &g_t + (B_{t-1}/P_t) = \tau_t + (q_t B_t/P_t), \\ &M/P_t = m^d[(W_t/P_t)N - \tau_t + \mu^W(q_t P_{t+1}/P_t, g_t), q_t, P_{t+1}/P_t], \\ &N = n(W_t/P_t, k_{t-1}, g_{t-1}), \\ &k_t = k^W(q_t P_{t+1}/P_t, g_t), \end{aligned}$$

where the demand functions, c^1, m^d and so on, are the same across generations. We call constraints (28) the *dynamic Walrasian system*, which determines the endogenous variables (k_t, P_{t+1}, W_t, q_t, B_t), given the predetermined variables (k_{t-1}, g_{t-1}, B_{t-1}, P_t) and the "independent" policy variables (g_t, τ_t). It should be noted that since it has been assumed that the nominal money supply is left unchanged at some constant level over period and that public debt has the one-period maturity, stationary states of the dynamic Walrasian system (28) must involve the "constant" prices.

The above-mentioned welfare maximization problem can be solved by applying *dynamic programming*. It is now assumed that an optimal path exists and that it converges to a stationary state. From the first-order conditions for the optimal path, we obtain the following optimality conditions in stationary states (see Appendix A):

$$(29) \qquad 1 + \delta = (1/q) + \frac{(m_q^d|_U)(\rho/q)}{c_q^1|_U + k_q^W},$$

$$(30) \qquad 1 + \delta = (1/q) + \frac{(m_\pi^d|_U)(\rho/q)}{c_\pi^1|_U + k_\pi^W} + \frac{[(\rho/q) - (1-q)/q]}{[(1+\delta) - (1/q)](c_\pi^1|_U + k_\pi^W)} m,$$

$$(31) \qquad F_g = (1+\delta)(1 + k_g^W) + (F_k)(-k_g^W),$$

where ρ is the *shadow price* of "real money balances" in terms of government revenue, it follows from (26) that

$$k_g^W = -F_{kg}/F_{kk}, \qquad k_q^W = -(F_k)^2/F_{kk}, \qquad k_\pi^W = -F_k/F_{kk},$$

and finally the subscript "U" denotes a *compensated* derivative, i.e., "pure substitution effect":[9]

$$(32)\qquad \begin{aligned} c_q^1|_U &= c_q^1 + (c^2 - m^d)\, c_y^1, & c_\pi^1|_U &= c_\pi^1 + qc^2 c_y^1, \\ m_q^d|_U &= m_q^d + (c^2 - m^d)\, m_y^d, & m_\pi^d|_U &= m_\pi^d + qc^2 m_y^d \end{aligned}$$

Note that k_π^W, $c_\pi^1|_U$ and $m_\pi^d|_U$ are all evaluated at $\pi=0$.

We shall now examine the economic implications of optimality conditions (29)-(31). At first, let us interpret (29) and (30). At one time period t, the government bids resources away from the generation t with public borrowing and lump-sum taxation so as to finance public investment. This changes the prices of public debt and goods. The derivatives with respect to the price of public debt in (29), $c_q^1|_U$, k_q^W and $m_q^d|_U$, are those of the demand schedules of that generation for the first-period consumption, private investment and real money holdings. Since $(1/q)$ measures the marginal opportunity cost of transferring a unit of resources from the first-period consumption and private investment, and since (ρ/q) measures the shadow value of real money holdings, the total opportunity costs of funds are $(1/q)[(c_q^1|_U+k_q^W)+\rho\,(m_q^d|_U)]$, in terms of "second-period" consumption of the generation t. We call this the *intragenerational* opportunity costs of funds. On the contrary, it is also possible to measure the total opportunity costs of funds, in terms of "first-period" consumption of the next generation $t+1$. As $1+\delta$ is the value to the generation $t+1$ of one unit of resources of future consumption, such costs of funds are $(1+\delta)(c_q^1|_U+k_q^w)$. We call this the *intergenerational* opportunity costs of funds. It should be now noted that $m_q^d|_U$ need not be considered in this calculation, because money is a "paper asset" but not resources. Under the optimal public policy, these two opportunity costs of funds must coincide:

$$(1+\delta)(c_q^1|_U + k_q^W) = (1/q)[(c_q^1|_U + k_q^W) + \rho\,(m_q^d|_U)],$$

which is the same as (29). Similarly, we can interpret (30) by the "opportunity cost principle", rewriting it as follows:

$$\begin{aligned}(1+\delta)(c_\pi^1|_U + k_\pi^W) = {} & (1/q)[(c_\pi^1|_U + k_\pi^W) + \rho\,(m_\pi^d|_U)] \\ & + [(\rho/q) - (1-q)/q][(1+\delta) - (1/q)]^{-1} m.\end{aligned}$$

Now, we shall show that the Walrasian stationary equilibrium in our monetary overlapping generations model is not the *first-best* optimum. It can be easily shown that an optimal first-best stationary state (c^1, c^2, n, k, g, m) is determined by the equations:

$$(33)\qquad \begin{aligned} &\frac{U_1(c^1, c^2, m)}{U_2(c^1, c^2, m)} = F_k(n, k, g) = F_g(n, k, g) = 1+\delta, \\ &U_m(c^1, c^2, m) = 0, \qquad f(n, k, g) = c^1 + c^2 + k + g, \qquad \text{and } n = N. \end{aligned}$$

It follows from (5), (9) and (11) that in the Walrasian stationary equilibrium,

$$(34) \qquad \frac{U_1}{U_2} = F_k = \frac{1}{q\,(1+\pi)}, \qquad \frac{U_m}{U_2} = \frac{1-q}{q\,(1+\pi)}, \qquad F_n = w.$$

Therefore, if the "nominal" rate of interest, $(1/q)-1$, is zero (i.e., $q=1$) and if the "real" rate of interest, $1/[q\,(1+\pi)]-1$, is equal to the positive rate of social time preference, δ, then the Walrasian stationary equilibrium is the first-best optimum. However, it is not so, because $\pi=0$ in the case of *constant* money supply and $\delta \neq (1/q)-1>0$ by the optimality condition (29) or (30). This means that even if there are no distortions and there are lump-sum redistributive instruments, the government cannot attain the *intergenerational* optimum of "income distribution" even in the Walrasian equilibrium regime. The reason is that the nominal money supply has been fixed over the infinite future. Then, the market rate of interest is imposed on "twofold roles" that must clear not only the capital market, but also the money market where real money balances usually *absorb* a part of "real savings" of the young. If the government can freely control the nominal money supply as well as public investment, public debt and lump-sum taxation, then it goes without saying that the first-best can be attained in the Walrasian equilibrium regime. In this case, the real rate of interest should be equated to δ and the nominal rate of interest to zero, implying a "negative" inflation rate, $-\delta$. This is simply Friedman's *optimum quantity of money rule*: since real balances are costless to produce, they should be held to satiation by the consumer, which requires the nominal rate of interest to be driven to zero.

Next, we shall interpret the optimality condition (31). This gives us the *second-best* criterion for public investment. That is, the social discount rate for public investment in the Walrasian stationary equilibrium, F_g-1, should be a *weighted average* of the social rate of "time preference", δ, and the rate of return on "private investment", F_k-1; the weight depends on the private capital demand, k_g^W, induced by a unit increase in public investment. There is an intuitive rationale behind this result. Suppose that at one time period t, public investment costing a unit of resources is contemplated. This investment project not only yields F_g units of resources at the next period, but also induces the private capital demand by k_g^W at the current period. Thus, one unit of public investment yields total resources, $F_g+F_k(k_g^W)$, to the generation t *at period* $t+1$. Now, consider the alternative scheme that transfers resources available at period t, $1+k_g^W$, to the *next generation* $t+1$. Since the value to the next generation $t+1$ of one unit of future consumption is $1+\delta$, total benefits gained from the alternative become $(1+\delta)(1+k_g^W)$. It is clear that public investment should be undertaken if and only if

$$F_g + F_k\,(k_g^W) \geqq (1+\delta)(1+k_g^W).$$

The weight k_g^W can be interpreted as a sort of *multiplier* of public investment which

impacts on the private capital demand, since

$$k_g^W = \sum_{h=1}^{\infty} (k_w F_{nk})^h (k_g + k_w F_{ng}) = (k_w F_{ng} + k_g)/(1 - k_w F_{nk}) = - F_{kg}/F_{kk},$$

through an interaction between the private capital demand in the current period and the "real wage" in the next period. Thus, the weight k_g^W represents the private capital demand *crowded out-in* directly and indirectly by a unit increase in public investment. Note that k_g^W is not constrained to lie in the interval $(-1, 0)$.

Public investment criterion (31) instructs us that it is not always desirable to use both the social rate of time preference and the rate of return on private investment, as the social rate of discount. However, it should be noted that there are *two special cases* that we can usually adopt either the social rate of time preference or the rate of return on private investment as the social rate of discount. They are as follows:

Case 1: if $F_{kg} = 0$, i.e., $k_g^W = 0$, then $F_g - 1 = \delta$,
Case 2: if $F_{kg} = F_{kk}$, i.e., $k_g^W = -1$, then $F_g - 1 = F_k - 1$.

Thus, if public and private capital are *independent* inputs (*perfect substitutes*) in the aggregate production process, the social rate of discount for public investment should be the social rate of time preference (the rate of return on private investment). It is only in *Case 2* that we can use usually the market rate of interest as the social discount rate, in spite of the "second-best" situation due to the constant money supply.

Finally, it may appear that our weighted average criterion (31) for public investment contradicts that derived by Pestieau (1974) in a *non-monetary* overlapping generations economy. However, it is not so. If we remove money from our model and take goods as numéraire, then our optimality conditions for welfare maximization consist of (29), where the second term of the right-hand side vanishes, and (31). In this case, since the social rate of time preference is usually equal to the market rate of interest, we obtain $F_g - 1 = \delta$. This is the same as Pestieau's criterion.

4. Public Investment Criteria in the General Disequilibria

In section 2, we have introduced the expectation hypothesis of agents that the next period will have the same constraint regime as the current period, so as to specify the form of the capital demand function. In order that equilibrium in each disequilibrium regime can be fully defined, we must further set an appropriate hypothesis to expectation variables which the temporary equilibrium conditions contain. It is assumed that as in the Walrasian equilibrium regime, all agents have perfect foresight: they can foresee not only the levels of wages and prices which will prevail in the future and the constraints which will be binding, but also the *magnitude*

of these constrains. In this section, based on these two expectation hypotheses, we shall derive the second-best public policy which maximizes a discounted sum of generational utilities in each of Malinvaud's three disequilibrium regimes. Before proceeding, however, it will be now useful to make several remarks on such two hypotheses and our approach to the welfare maximization problem.

At first, these expectation hypotheses constrain the fiscal policy so that the economy at every period is held at the "same" regime as that in the initial period. This implies that because a problem of "optimal regime switching" is completely ignored, we can no longer search for *globally* optimal levels of public instruments. Thus, public investment criteria derived in this section are all *regime-specific*. Indeed, such a regime switching problem has been attacked by Marchand, Mintz and Pestieau (1985) in a two-period model with public investment, and has been more exactly analyzed by Cuddignton, Johansson and Ohlsson (1985) in a static model with public production. However, it is very difficult to dispose of the problem fully in our dynamic disequilibrium model with overlapping generations. In the present paper, we cannot but introduce these expectation hypotheses in the following two reasons. The one is because we need to maintain the so-called "stationarity" assumption in dynamic programming applied for solving our social welfare maximization problem. The other is as follows. There will be an *optimal stationary regime* in our model, depending on the levels at which the goods price and money wage are fixed relative to their Walrasian equilibrium levels. If the "initial" conditions are given in the optimal stationary regime, which may reduce our analysis to be local and regime-specific, then these expectation hypotheses will be justified.

Next, let us explain our welfare maximization approach. Instead, we could suppose that the government objective is to maximize the lifetime utility of the representative individual in stationary states. Then, as in Rankin (1986 and 1987), we can show that the welfare optimum in our model also lies somewhere on the *boundary* of the Keynesian unemployment regime. Therefore, it is possible to confine the analysis to the Keynesian-classical and Keynesian-repressed inflation boundaries. Thus, in this case the above-mentioned regime switching problem does not arise. However, in our approach since utilities of future generations have been discounted with the positive rate of social time preference, the welfare optimum will lie either in the interior of a rationing regime or on the boundary between two or three regimes, depending on the fixed levels of the goods price and money wage. Therefore, it is reasonable that we proceed the analysis regime by regime.

We have also assumed that the government can freely choose the desirable levels of the public debts and the lump-sum tax on the young, simultaneously with the desirable level of public investment. This implies that public investment criteria obtained in this manner are appropriate only for evaluating public projects in situations where the economy starts from a welfare "optimum" path. Thus, our

approach may be regarded as the *second-best* approach, as in Sandmo and Drèze (1971), Diamond (1973), Pestieau (1974), Drèze (1982), Marchand, Mint and Pestieau (1984 and 1985), Yoshida (1986), Burgess (1988) and others. However, there is an another effective approach that derives public investment criteria which apply whether or not the economy is on the second-best path and for a wide variety of instruments constraints. This is called the *cost-benefit* approach and has been developed by Boadway (1975 and 1978), Johanson (1982), Cuddington, Johansson and Ohlsson (1985) and others. Indeed, it may be more general for us to take the latter approach in order to study decision rules for public investment. However, because we can not apply directly the dynamic optimization method in this approach, it seems to be very difficult and complicated in our overlapping generations model. I think that the choice between these approaches is a matter of "specification" about the public instruments over which the government can control, as is also pointed out by Drèze and Stern (1987).

4. 1. Keynesian Unemployment Regime

The temporary equilibrium conditions (22) in this regime contain the expectational variables (y_{t+1}, μ_t). Since the next period has been assumed to have the same constraint regime as the current period, it must hold that $y_{t+1}=F(n_{t+1}, k_t, g_t)$ under the assumption of perfect foresight. Substituting this into the capital demand function (14), we obtain

$$k_t = k^r\,[F\,(n_{t+1},\ k_t,\ g_t),\ q_t,\ g_t].$$

Thus, k_t is an implicit function of n_{t+1}, q_t and g_t:

$$k_t = k^K\,(n_{t+1},\ q_t,\ g_t), \tag{35}$$

which has the following partial derivatives from (15):

$$\begin{aligned} k_g^K &= (k_y^r F_g + k_g^r)/(1 - k_y^r F_k) = (F_k F_{ng} - F_n F_{kg})/E, \\ k_q^K &= k_q^r/(1 - k_y^r F_k) = -\,w\,(F_k)^2/E, \\ k_n^K &= k_y^r F_n/(1 - k_y^r F_k) = (F_k F_{nn} - F_n F_{kn})/E, \end{aligned} \tag{36}$$

where $E=F_n F_{kk}-F_k F_{nk}$. Given (36), the net present value of firm ownership μ_t becomes also a function of n_{t+1}, q_t and g_t:

$$\mu_t = \mu^K\,(n_{t+1},\ q_t,\ g^t) \equiv q_t\,[F\,(n_{t+1},\ k_t^K,\ g_t) - wn_{t+1}] - k_t^K. \tag{37}$$

Now, the government objective is to maximize the social welfare function (19) subject to the temporary equilibrium conditions (22), the government budget constraint (18), eq.(35), the initial conditions $(k_0=\underline{k}_0,\ g_0=\underline{g}_0,\ b_0=\underline{b}_0,\ n_1=\underline{n}_1)$, and finally an "inequality" constraint which requires that the economy in every period must be held at the Keynesian unemployment regime by the fiscal policy:

$$F(n_{t+1}, k_t, g_t) \leqq \min\{F(N, k_t, g_t), F[n(w, k_t, g_t), k_t, g_t]\}.$$

It follows from $F_n > 0$ that this inequality constraint can be now resolved into the two inequalities: $n_{t+1} \leqq N$ and $n_t \leqq n(w, k_t, g_t)$. If the former (latter) inequality is binding by expanding effective demand, then the economy is on the *K-R* (*K-C*) boundary.

Thus, the welfare maximization problem can be formulated as follows:

$$\text{Maximize} \quad \sum_{t=1}^{\infty} (1+\delta)^{-t} V[wn_t - \tau_t + \mu^K(n_{t+1}, q_t, g_t), q_t]$$

subject to the *equality* constraints:

$$(38) \quad \begin{aligned} & wn_t = c^{1r}[wn_t - \tau_t + \mu^K(n_{t+1}, q_t, g_t), q_t] + m + b_{t-1} + k_t + g_t, \\ & m = m^r[wn_t - \tau_t + \mu^K(n_{t+1}, q_t, g_t), q_t], \\ & g_t + b_{t-1} = \tau_t + q_t b_t, \qquad n_t = n^r(y_t, k_{t-1}, g_{t-1}), \\ & k_t = k^K(n_{t+1}, q_t, g_t), \end{aligned}$$

and the *inequality* constraints:

$$(39) \quad n_{t+1} \leqq N, \qquad n_{t+1} \leqq n(w, k_t, g_t).$$

It should be noted that since the independent policy variables are only two, it is impossible that both inequality constraints are simultaneously binding. We call the equality constraints (38) the *dynamic Keynesian system*, which determines the endogenous variables (q_t, b_t, n_{t+1}, y_t, k_t), given the predetermined variables (b_{t-1}, n_t, k_{t-1}, g_{t-1}) and the independent policy variables (g_t, τ_t).

From the first-order conditions for the above-mentioned welfare maximization problem, we obtain the following optimality conditions in stationary states (see Appendix B):

$$(40) \quad 1+\delta = \frac{c_q^{1r}|_U (1/q) + k_q^K (F_k)}{c_q^{1r}|_U + k_q^K} + \frac{(\rho/q)\, m_q^r|_U}{c_q^{1r}|_U + k_q^K} + (\theta/q)(n_k k_q^K),$$

$$(41) \quad F_n = (1+\delta)\, k_n^K + (F_k)(-k_n^K) + (\varkappa/q) - (\theta/q)(n_k k_n^K - 1),$$

$$(42) \quad F_g = (1+\delta)(1+k_g^K) + (F_k)(-k_g^K) - (\theta/q)(n_k k_g^K + n_g),$$

$$(43) \quad \varkappa \geqq 0, \qquad \varkappa(N-n) = 0,$$

$$(44) \quad \theta \geqq 0, \qquad \theta[n(w,k,g) - n] = 0,$$

where k_n^K, k_g^K and k_q^K are given by (36), n_k and n_g are given by (10), ρ is the shadow price of real money balances, $\varkappa$ and θ are the *shadow wages* of the exogenous labour "supply" and Walrasian labour "demand" in terms of government revenue, respectively, and finally $c_q^{1r}|_U$ and $m_q^r|_U$ are the compensated derivatives of the constrained

first-period consumption and real money demand with respect to the price of bonds, respectively. In what follows, we shall examine individually the economic implications of these optimality conditions in the interior of the Keynesian stationary equilibrium and on the K-R and K-C boundaries.

At first, let us proceed from the interior of the Keynesian stationary equilibrium. Since $n<N$ and $n<n\ (w,\ k,\ g)$ there, it follows from the so-called "complementarity" conditions, (43) and (44), that $\varkappa=\theta=0$. Therefore, the optimality conditions (40)-(42) become as follows:

$$(45)\qquad 1+\delta=\frac{c_q^{1r}|_U\,(1/q)+k_q^K\,(F_k)}{c_q^{1r}|_U+k_q^K}+\frac{(\rho/q)\,m_q^r|_U}{c_q^{1r}|_U+k_q^K},$$

$$(46)\qquad F_n=(1+\delta)\,k_n^K+F_k\,(-k_n^K),$$

$$(47)\qquad F_g=(1+\delta)(1+k_g^K)+(F_k)(-k_g^K).$$

The optimality condition (45) shows that the inter and intra generational opportunity costs of "funds" must coincide under the optimal public policy, as eq.(29) in the Walrasian regime. However, since $U_1/U_2=1/q<F_k$ as has already been shown in section 2, the marginal opportunity cost of transferring a unit of resources from the first-period consumption, U_1/U_2, is *smaller* than that from the private investment, F_k, meaning that the private capital is underinvested. Such a distortion in the capital market is very similar to that due to the *corporation profits tax*.

The optimality condition (46) gives us the criterion for determining the optimal level of rationed "labour employment". Note that this condition does not contain the marginal utility of labour, since we have assumed no utility of leisure. The marginal productivity of labour, F_n, is now interpreted as the *shadow wage* of rationed labour, which is larger than the fixed real wage, w. Rewriting this condition as $(1+\delta)\,k_n^K=F_n+F_k k_n^K$, we can interpret it by the opportunity cost principle.

The optimality condition (47) shows that as eq.(31) in the Walrasian regime, the social discount rate for public investment should be the *weighted average* of the social rate of "time preference" and the rate of return on "private investment", the weight depending on the private capital demand crowded out-in by a unit increase in public investment. Since the social rate of time preference does not coincide with the rate of return on private investment because of the optimality condition (45), either of these two rates is not always the appropriate discount rate. If the marginal productivity of labour is *independent* of private and public capital, i.e., $F_{nk}=F_{ng}=0$, then because the weight k_g^K becomes $k_g^K=-F_{kg}/F_{kk}$ from (36), we have now also two special cases in the Walrasian regime, i.e., Case 1 and Case 2. Thus, it is only in Case 1 (Case 2) that the social rate of time preference (the rate of return on private investment) becomes usually the social discount rate. However, as the rate of return on private investment is larger than the market rate of interest, note

that we should *not* adopt the market rate of interest as the social discount rate even in Case 2 as well as in Case 1.

Now, substituting $1+\delta$ in the optimality condition (45) into (47), we obtain the alternative presentation of public investment criterion as follows:

$$F_g = \frac{c_q^{1r}|_U (1/q) + k_q^K F_k}{c_q^{1r}|_U + k_q^K} + \frac{[(1/q) - F_k](c_q^{1r}|_U)(k_g^K)}{c_q^{1r}|_U + k_q^K} + \frac{(\rho/q)(m_q^r|_U)(1 + k_g^K)}{c_q^{1r}|_U + k_q^K}. \tag{48}$$

It is interesting that the first and second terms of the right-hand side are the *same* as public investment criterion derived by Burgess (1988) in a simple two-period general equilibrium model distorted with the corporate profits taxation. The first term which represents a weighted average of the rates of interest facing consumers and producers is the so-called *Sandmo-Drèze formula* (1971). Dividing now the numerator and denominator in (48) by $c_q^{1r}|_U$ and taking the limit as it goes to infinity, we obtain:

$$F_g = (1/q)(1 + k_g^K) + F_k(-k_g^K).$$

As is also pointed out by Burgess (1988), this is the same as public investment criterion derived by Ogura and Yohe (1977).

Next, let us briefly sum up the optimality conditions (40)-(42) on the K-R and K-C boundaries. Since $n=N$ and $n<n\ (w,\ k,\ g)$ on the K-R boundary, $\varkappa>0$ and $\theta=0$. As the private firm is not rationed in the goods market, $1/q=F_k$. Further, it follows from $k^K(n,\ q,\ g)=k^R(q,\ g)$ that $k_n^K=0$. Therefore, these optimality conditions on the K-R boundary become as follows:

$$\begin{aligned} 1 + \delta &= (1/q) + (\rho/q)(m_q^r|_U)/(c_q^{1r}|_U + k_q^K), \\ F_n &= (\varkappa/q) < w, \\ F_g &= (1 + \delta)(1 + k_g^K) + (F_k)(-k_g^K). \end{aligned}$$

On the other hand, since $n<N$ and $n=n\ (w,\ k,\ g)$ on the K-C boundary, $\varkappa=0$ and $\theta>0$. As the private firm is not rationed in both labour and goods markets, $1/q=F_k$ and $w=F_n$. Further, it follows from $k^K(n,\ q,\ g)=k^C(q,\ g,\ w)$ that $k_n^K=0$. Therefore, the optimality conditions on the K-C boundary become as follows:

$$\begin{aligned} 1 + \delta &= (1/q) + (\rho/q)(m_q^r|_U)/(c_q^{1r}|_U + k_q^K) + wn_k k_q^K, \\ F_n &= (\theta/q) = w, \\ F_g &= (1 + \delta)(1 + k_g^K) + (F_k)(-k_g^K) + w\ (-n_k k_g^K - n_g). \end{aligned}$$

Now, it is well known that if the government objective is to maximize the *lifetime utility* of the representative individual in stationary states, then the optimality conditions for this problem are equal to these for our welfare maximiza-

tion problem, where the social rate of time preference δ is set to *zero*. Since the welfare optimum in this case lies on either the K-R or K-C boundary as has already been pointed out, second-best criteria for public investment in the disequilibrium framework are then given by:

$$K\text{-}R\ Boundary\text{:}\ F_g - 1 = (F_k - 1)(-k_g^K),$$
$$K\text{-}C\ Boundary\text{:}\ F_g - 1 = (F_k - 1)(-k_g^K) + w\,(-n_k k_g^K - n_g).$$

4. 2. Classical Unemployment Regime

Since $n_{t+1} = n\,(k_t,\ g_t;\ w)$ under the assumption of perfect foresight, it follows from (8) and (12) that the private capital demand k_t and the net present value of firm ownership μ_t become the functions of q_t and g_t as follows:

$$k_t = k^C\,(q_t,\ g_t;\ w) = k\,(q_t,\ g_t;\ w), \tag{49}$$
$$\mu_t = \mu^C\,(q_t,\ g_t) \equiv q_t\,\{F\,[n\,(k_t^C,\ g_t),\ k_t^C,\ g_t] - wn\,(k_t^C,\ g_t)\} - k_t^C, \tag{50}$$

where the function k^C has the following derivatives from (13):

$$k_g^C = \frac{F_{kn}F_{ng} - F_{nn}F_{kg}}{F_{nn}F_{kk} - (F_{nk})^2}, \qquad k_q^C = -\frac{(F_k)^2 F_{nn}}{F_{nn}F_{kk} - (F_{nk})^2}. \tag{51}$$

We shall now ignore the C-K and C-R boundaries, because the former has already been considered in the previous subsection and the latter cannot be attained by expanding effective demand in the classical regime. Then, the government objective is to maximize the social welfare function (19) subject to the temporary equilibrium conditions (23), the government budget constraint (18), eq.(49), and the initial conditions ($b_0 = \underline{b}_0,\ k_0 = \underline{k}_0,\ g_0 = \underline{g}_0$):

$$\text{Maximize}\quad \sum_{t=1}^{\infty} (1+\delta)^{-t}\, v\,[wn_t - \tau_t + \mu^C\,(q_t,\ g_t),\ q_t,\ c_t^1]$$

subject to

$$\begin{aligned} &wn_t = c_t^1 + m + b_{t-1} + k_t + g_t, \\ &m = m^r\,[wn_t - \tau_t + \mu^C\,(q_t,\ g_t) - c_t^1,\ q_t,\ c_t^1], \\ &n_t = n\,(k_{t-1},\ g_{t-1}), \qquad g_t + b_{t-1} = \tau_t + q_t b_t, \qquad k_t = k^C\,(q_t,\ g_t). \end{aligned} \tag{52}$$

We call the constraints (52) the *dynamic classical system*, which determines the endogenous variables ($q_t,\ b_t,\ n_t,\ k_t,\ c_t^1$), given the predetermined variables ($b_{t-1},\ k_{t-1},\ g_{t-1}$) and the independent policy variables ($g_t,\ \tau_t$).

We obtain the following optimality conditions in stationary states (see Appendix C):

(53) $1 + \delta = (1/q) + (\rho/q)(m_q^r|_U)/k_q^C + wn_k,$

(54) $U_1/\gamma = \rho\,(m_c^r|_U) - (1 + \delta)\,q,$

(55) $F_g = (1 + \delta)(1 + k_g^C) + (F_k)(-k_g^C) + w\,(-n_k k_g^C - n_g),$

where k_q^C and k_g^C are given by (51), n_k and n_g are given by (10), and $m_c^r|_U$ is the compensated derivative of constrtained money demand with respect to c^1. The optimality condition (53) can be interpreted by the opportunity cost principle. However, since c^1 is a "parameter" in this regime, this condition does not contain its compensated derivative with respect to the price of bonds. The optimality condition (54) gives us the criterion for determining the optimal level of c^1. The term, U_1/γ, is now interpreted as the *shadow price* of rationed first-period consumption, c^1, in terms of government revenue.

The second-best criterion for public investment (55) shows that the social discount rate for public investment should be the *weighted average* of the social rate of "time preference" and the rate of return on "private investment", containing an *extra term* which represents the marginal opportunity cost of public investment through its direct and indirect impacts on the rationed "labour employment", $-w(n_g+n_k k_g^C)$. The reason is that the labour employment is now a function of private and public capital. If there are no such impacts, then public investment criterion in this regime is formally the same weighted average rule as that in the Walrasian and Keynesian regimes. Since $n_k=-F_{nk}/F_{kk}$ and $n_g=-F_{ng}/F_{nn}$, a sufficient condition for this is that the marginal productivity of labour is *independent* of private and public capital. Then, it follows from (51) that we obtain $k_g^C=-F_{kg}/F_{kk}$. Thus, we have also here two special cases, Case 1 and Case 2. In Case 2, we should adopt the market rate of interest as the social rate of discount.

4. 3. Repressed Inflation Regime

Since $n_{t+1}=N$ under the assumption of perferct foresight, it follows from (8) and (16) that the private capital demand k_t and the net present value of private firm ownership μ_t become the functions of q_t and g_t as follows:

(56) $k_t = k^R\,(q_t,\ g_t;\ N) = k^r\,(q_t,\ g_t;\ N),$

(57) $\mu_t = \mu^R\,(q_t,\ g_t;\ N) \equiv q_t\,[F\,(N,\ k_t^R\ g_t) - wN] - k_t^R,$

where the function k^R has the following derivatives from (17):

(58) $k_g^R = -\,F_{kg}/F_{kk}$ and $k_q^R = -\,(F_k)^2/F_{kk}.$

Now, ignoring the R-C and R-K boundaries which have been already considered, then the government objective is to maximize the social welfare function (19) subject to the temporary equilibrium conditions (24), the government budget constraint (18), eq.(56), and the initial condition, $b_0=\underline{b}_0$:

$$\text{Maximize} \quad \sum_{t=1}^{\infty} (1+\delta)^{-t}\, v\,[wN - \tau_t + \mu^R(q_t, g_t; N), q_t, c_t^1]$$

subject to

$$\begin{aligned} & wN = c_t^1 + m + b_{t-1} + k_t + g_t, \\ (59) \quad & m = m^r\,[wN - \tau_t + \mu^R(q_t, g_t; N) - c_t^1, q_t, c_t^1], \\ & g_t + b_{t-1} = \tau_t + q_t b_t, \qquad k_t = k^R(q_t, g_t; N). \end{aligned}$$

It may be now noted that the market equilibrium condition for labour is subtracted from the above constraints, because the labour endowment, N, is usually full employed. We call the constraints (59) the *dynamic repressed inflation system*, which determines the endogenous variables (q_t, b_t, c_t^1, k_t), given the predetermined variable, b_{t-1}, and the independent policy variables (g_t, τ_t).

We obtain the following optimality conditions in stationary states (see Appendix D):

$$\begin{aligned} (60) \quad & 1+\delta = (1/q) + (\rho/q)(m_q^r|_U)/k_q^R, \\ (61) \quad & U_1/\gamma = \rho\,(m_c^r|_U) - (1+\delta)q, \\ (62) \quad & F_g = (1+\delta)(1+k_g^R) + (F_k)(-k_g^R), \end{aligned}$$

where k_q^R and k_g^R are given by (58). These optimality conditions can be interpreted by the opportunity cost principle, as in the previous regimes. Since there are *no* impacts of the private and public investments on the labour employment, note that none of the above optimality conditions contain such impacts. The weighted average criterion (62) for public investment is formally the same as that in the Walrasian regime. Thus, we have two special cases, Case 1 and Case 2, irrespective of whether the marginal productivity of labour is "independent" of private and public capital or not. Noting that $F_k = (1/q)$ in this regime, in Case 2 we can usually adopt the market rate of interest as the social rate of discount.

Finally, in the above analyses of optimal public policy, the nominal money supply has been fixed over the infinite future. It should be noted that even if it can be freely controled by the government, the first-best optimum can *not* be attained in any disequilibrium regime. In this case, the shadow price of real money balances, ρ, is equal to $1-q$. The weighted average rule for public investment derived in each disequilibrium regime continues to be *effective*.

5. Concluding Remarks

In this paper, based on the choice-theoretic macroeconomic model with overlapping generations developed by Rankin (1986 and 1987), we have investigated the optimal second-best policy of public investment, public debts and lump-sum taxation which maximizes a sum of generational utilities discounted with the social

rate of time preference, in each of the Walrasian equilibrium and Malinvaud's three disequilibrium regimes.

It has been shown that in stationary states, if public investment has no impacts on the "labour employment", then the *second-best* discount rates for public investment in the Walrasian equilibrium, Keynesian unemployment, classical unemployment and repressed inflation regimes, should be the *weighted average* of the social rate of "time preference" and the rate of return on "private investment". As the weight depends on the private capital demand crowded out-in directly and indirectly by a unit increase in public investment, there exist two special cases that either of these two rates can be usually used as the social discount rate. The one is that if the public and private capital are *independent* inputs in the aggregate production process, the social discount rate should be the social rate of time preference. The other is that if they are *perfect substitutes*, it should be the rate of return on private investment. In the latter case, it is possible to use the "market" rate of interest as the social discount rates in all regimes except for the Keynesian.

In order to formulate the public investment problem in the disequilibrium framework as simple as possible, we have ignored several issues raised elsewhere. It will be useful to refer to them briefly. First, we have assumed no utility of leisure so that labour supply is exogenously given. An endogenous labour supply may generate different criteria for public investment. Second, the population growth rate has been assumed to be zero. The introduction of a growing population, however, seems to complicate greatly an already complicated analysis. Third, we have assumed that decisions by the private firm and the government within each period are "separable". For example, the firm is *generation-specific* in that the younger generation owns capital invested in the firm, yielding a return when the generation retires, and that the firm retires with the generation. This implies that some of the interesting issues which could arise in a dynamic model, such as *time consistency* of policy and so on, have been ignored. Fourth, in the disequilibrium framework, it has been assumed that the economy in every period is held at the same constaint regime by the fiscal policy, which implies that the problem of optimal regime switching has been completely ommited. Finally, the exogenous prices of goods and labour have been assumed to be *constant* over the infinite future. However, this assumption can be relaxed so that prices are momentarily inflexible, but vary over time periods responding to market disequilibrium. For example, following Nikaido (1980), we could formulate the "intertemporal" adjustment equations on prices as follows:

$$(P_{t+1}/P_t) - 1 = \sigma_1 \left[F(n_t) - \min\left[F(N),\ F\{n(W_t/P_t)\}\right]\right],$$

$$(W_{t+1}/W_t) - 1 = \sigma_2 \left[\min\left[F(n_t),\ F\{n(W_t/P_t)\}\right] - F(N)\right],$$

where min $[F(N), F\{n(W_t/P_t)\}]$ and min $[F(n_t), F\{n(W_t/P_t)\}]$ are the labour-constrained plan of supply and the demand-constrained plan of supply, respectively, and the adjustment coefficients of speed, σ_i $(i=1, 2)$, are both positive. It can be shown that such an introduction of price adjustment processes does not affect our weighted average rules for public investment derived in the disequilibrium framework.

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Footnotes

1. We may study the problem of public investment criteria in the simple two-period general (dis-)equilibrium model with money but *not* overlapping generations. However, it should be noted that such a two-period model has the following structures which are *different* from those of the monetary overlapping generations model. First, capital and labour must be treated symmetrically, since there is one period lag in production that produces a commodity in the second-period, using capital and labour acquired in the first-period. Second, if we take money as numéraire and ignore the capital market according to Walras's law, then *four* independent market equilibrium conditions for money, labour, first-period goods and second-period goods determine prices and/or rationed quantities in each of the Walrasian equilibrium and possible disequilibrium regimes. Third, the government must recover not only public bonds but also *money* in the second period, both of them being issued so as to finance public investment in the first period. Finally, the exchange of goods for money is done between the consumer and the government alone. Hence, the simple two-period general (dis-)equilibrium model without overlapping generations seems to be "inappropriate" for the analysis of the public investment problem in the *monetary economy*.
2. In the current period where both private and public capital are predetermined, it is infeasible that there is the notional excess supply in the goods market but the excess demand in the labour, because the private firm cannot be rationed *simultaneously* in both markets. Thus, there cannot exist such a disequilibrium regime in the short-run period.
3. This procedure to put real balances in the utility function may be unsatisfactory from a theoretical point of view. Although the various stories which have been told to rationalize the holding of money are well known, it seems to be difficult to explain, in the general equilibrium framework, why the consumer should hold money when he can earn interest by holding his wealth in another form. Therefore, in spite of theoretical objections, we shall now suppose that real money balances provide utility, as has been commonly postulated in many studies on the

monetary growth and the optimal rate of inflation [see, for example, Yakita (1989)].

4. For the envelope theorem, see Varian (1978).
5. Note that since c_t^1 is rationed, $\partial U/\partial c^1 \neq \lambda$, so that $v_c \neq 0$. This implies that in the case of goods market rationing, the *private* rate of time preference, U_1/U_2-1, is not equal to the market rate of interest, $(1/q)-1$.
6. Note that the firm ownership is always in the hands of the *older* generation, as is clear from eq.(3).
7. Note that the inequality, $F_k>(1/q)$, implies that private capital is *underinvested*.
8. This hypothesis concerning the future regime is the point expectation. Although the other more attractive expectational assumption is to rely on an explicit probabilistic von Neumann-Morgenstern formulation [see, for example, Muellbauer and Portes (1978)], it will further complicate the analysis of public investment problem.
9. These compensated derivatives can be derived as follows. Let the price of m be $r(=1-q)$ and that of c^2 be $x[=q(1+\pi)]$. Now, supposing that they can vary independently, then we have

$$c_q^1 = (1+\pi)\, c_x^1 - c_r^1, \quad c_\pi^1 = qc_x^1,$$
$$m_q = (1+\pi)\, m_x - m_r, \quad \text{and} \quad m_\pi = qm_x.$$

Substituting the Slutsky equations:

$$c_x^1 = c_x^1|_U - c^2 c_Y^1, \quad c_r^1 = c_r^1|_U - mc_Y^1,$$
$$m_x = m_x|_U - c^2 m_Y, \quad \text{and} \quad m_r = m_r|_U - mm_Y$$

into the above equations, we derive:

$$c_q^1|_U = [1/(1+\pi)][c_q^1 + \{(1+\pi)c^2 - m\}\, c_Y^1],$$

$$m_q|_U = [1/(1+\pi)][m_q + \{(1+\pi)c^2 - m\}\, m_Y],$$

$$c_\pi^1|_U = c_\pi^1 + qc^2 c_Y^1, \qquad m_\pi|_U = m_\pi + qc^2 m_Y.$$

Thus, setting $\pi = 0$, we obtain the desired derivatives.

Mathematical Appendix

A. Dynamic Optimization in the Walrasian Equilibrium Regime

The government is assumed to have inherited at period t the endogenous variables (k_{t-1}, P_t, W_{t-1}, q_{t-1}) and the policy variables (g_{t-1}, τ_{t-1}, B_{t-1}) in the preceding period. We can then introduce the state valuation function J (k_{t-1}, P_t, g_{t-1}, B_{t-1}) to represent the *maximal* level of social welfare discounted to period t obtainable given these initial conditions, of which we can ignore W_{t-1}, q_{t-1} and τ_{t-1}

since they do not affect the maximum attainable level of welfare:

$$J(k_{t-1}, P_t, g_{t-1}, B_{t-1}) = \text{Max} \sum_{s=t}^{\infty} (1+\delta)^{-s+t} V_t$$

The government maximizes the social welfare by choosing k_t, P_{t+1}, W_t, q_t, g_t, τ_t, and B_t subject to the *dynamic Walrasian system* (28). Letting now the Lagrange multipliers corresponding to the constraints (28) α_t, β_t, γ_t, χ_t and ϕ_t, then we can apply the principle of optimality of dynamic programming. In view of the stationarity of the problem, we have the basic recursion relation:

$$\begin{aligned}
&J(k_{t-1}, P_t, g_{t-1}, B_{t-1}) = \text{Max}\,\big[(1+\delta)^{-1} J(k_t, P_{t+1}, g_t, B_t) \\
&+ V[(W_t/P_t)N - \tau_t + \mu^W(q_tP_{t+1}/P_t, g_t), q_t, P_{t+1}/P_t] \\
&+ \alpha_t[(W_t/P_t)N - c^1\{(W_t/P_t)N - \tau_t + \mu^W(q_tP_{t+1}/P_t, g_t), q_t, P_{t+1}/P_t\} \\
&- (M + B_{t-1})/P_t - k_t - g_t] \\
&+ \beta_t[(M/P_t) - m^d\{(W_t/P_t)N - \tau_t + \mu^W(q_tP_{t+1}/P_t, g_t), q_t, P_{t+1}/P_t\}] \\
&+ \gamma_t[g_t + (B_{t-1}/P_t) - \tau_t - (q_tB_t/P_t)] \\
&+ \chi_t[N - n(W_t/P_t, k_{t-1}, g_{t-1})] + \phi_t[k_t - k^W(q_tP_{t+1}/P_t, g_t)]\big].
\end{aligned}$$

The first set of the first-order conditions for the optimal path are obtained by differentiating the maximand with respect to the control variables (τ_t, q_t, W_t, k_t, g_t, B_t, P_{t+1}) and then equating to zero. In stationary states, they are given by:

$$\text{(A. 1)}\quad \begin{aligned}
&\lambda - \alpha c_Y^1 - \beta m_Y^d = -\gamma, \\
&(\lambda - \alpha c_Y^1 - \beta m_Y^d)\mu_q^W + V_q = \alpha c_q^1 + \beta m_q^d + \gamma b + \phi k_q^W, \\
&N(\lambda - \alpha c_Y^1 - \beta m_Y^d) + \alpha N = \chi n_w, \\
&J_k = (1+\delta)(\alpha - \phi), \\
&(1+\delta)^{-1} J_g + (\lambda - \alpha c_Y^1 - \beta m_Y^d)\mu_g^W + \gamma = \alpha + \phi k_g^W, \\
&PJ_B = (1+\delta)\gamma q, \\
&(\lambda - \alpha c_Y^1 - \beta m_Y^d) q\mu_\pi^W + V_\pi + (1+\delta)^{-1} PJ_P \\
&= \alpha c_\pi^1 + \beta m_\pi^d + \phi q k_\pi^W,
\end{aligned}$$

The second set of the first-order conditions for the optimal path are those obtained by differentiating the above basic recursion relation with respect to the state variables (k_{t-1}, g_{t-1}, B_{t-1}, P_t). In stationary states, they are given by:

$$\text{(A. 2)}\quad \begin{aligned}
&J_k/n_k = J_g/n_g = -\chi, \\
&PJ_B = \gamma - \alpha, \\
&PJ_P = -(wN + q\mu_\pi^W)(\lambda - \alpha c_Y^1 - \beta m_Y^d) - V_\pi - \alpha(wN - m - b \\
&\qquad - \alpha c_\pi^1) - \beta(m - m_\pi^d) - \gamma b(1-q) + \chi w n_w + \phi q k_\pi^W
\end{aligned}$$

Now, substituting J_k, J_g, PJ_B, and PJ_P in (A. 2) into (A. 1), using $V_q = (m^d - c^2)\lambda$ and $V_\pi = -qc^2\lambda$ in eqs.(6), and then eliminating the private marginal utility of income λ and the multipliers α, χ and ϕ, we obtain:

$$qN\,[(n_g/n_w)+(n_k/n_w)\,k_g^W]+(1+\delta)\,q=\mu_g^W+[1-(1+\delta)\,q]\,k_g^W,$$

$$(m-c^2)[\{1-(1+\delta)\,q\}\,c_Y^1+(\beta/\gamma)\,m_Y^d-1]+qN\,(n_k/n_w)\,k_q^W=\mu_q^W$$
$$+\,b+(\beta/\gamma)\,m_q^d+[1-(1+\delta)\,q](c_q^1+k_q^W),$$

(A. 3) $$[1-(1+\delta)\,q](c_\pi^1+qc^2c_Y^1+qk_\pi^W-wN+m+b)+wN+q\mu_\pi^W$$
$$+(\beta/\gamma)(m_\pi^d+qc^2m_Y^d-m)-b\,(1-q)-q^2N\,(n_k/n_w)\,k_\pi^W-qc^2$$
$$-(1+\delta)\,qNw=(1+\delta)\big[q^2N\,(n_k/n_w)\,k_\pi^W-[1-(1+\delta)\,q](c_\pi^1$$
$$+qk_\pi^W)-(\beta/\gamma)\,m_\pi^d-q\mu_\pi^W-qc^2\,[\{1-(1+\delta)\,q\}\,c^1+(\beta/\gamma)$$
$$m_Y^d-1]\big].$$

Now, from (27) we have the following partial derivatives: $\mu_g^W=q(F_g-NF_{ng})-qk_g^WNF_{nk}$, $\mu_q^W=y-NF_n+qk_q^WNF_{nk}$, and $\mu_\pi^W=q(y_t-NF_n-k_\pi^WNF_{nk})$. Substituting these equations into (A. 3), and then using (32), $m=m^d$ and $c^2=y-wN+m^d+b$, (A. 3) become finally:

$$1+\delta=\frac{[(c_q^1|_U+\rho\,(m_q^d|_U)](1/q)+(F_k-\mathrm{X}N)\,k_q^W}{c_q^1|_U+k_q^W},$$

$$1+\delta=\frac{[(c_\pi^1|_U)+\rho\,(m_\pi^d|_U)](1/q)+(F_k-\mathrm{X}N)\,k_\pi^W+[\rho-(1-q)](m/\delta q)}{c_\pi^1|_U+k_\pi^W},$$

$$F_g=(1+\delta)(1+k_g^W)+(F_k-\mathrm{X}N)(-k_g^W)+\Phi N,$$

where $\rho\equiv\beta/\gamma$, $\mathrm{X}\equiv F_{nk}+(n_k/n_w)$ and $\Phi\equiv F_{ng}+(n_g/n_w)$. We can show that X and Φ are both zero. Substituting the profit maximization condition: $w=F_n(N,\,k,\,g)$ into the equilibirum condition for labour market: $N=n(w,\,k,\,g)$, $N=n[F_n(N,\,k,\,g),\,k,\,g]$. Differentiating this with respect to k and g, $n_wF_{nk}+n_k=0$ and $n_wF_{ng}+n_g=0$, so that $\mathrm{X}=\Phi=0$. Thus, noting that $F_k=1/q$, we obtain the optimality conditions (29)-(31).

B. Dynamic Optimization in the Keynesian Unemployment Regime

The government maximizes the social welfare by choosing n_{t+1}, q_t, τ_t, b_t, g_t, k_t and y_t *subject to the dynamic Keynesian system* (38) and the inequality constraints (39). Letting the Lagrange multipliers corresponding to these constraints α_t, β_t, γ_t, χ_t, ϕ_t, η_t and ξ_t, then the basic recursion relation can be given by:

$$J\,(n_t,\,b_{t-1},\,k_{t-1},\,g_{t-1})=\max\big[(1+\delta)^{-1}\,J\,(n_{t+1},\,b_t,\,k_t,\,g_t)$$
$$+\,V\,[wn_t-\tau_t+\mu^K\,(n_{t+1},\,q_t,\,g_t),\,q_t]+\alpha_t\,[wn_t-m-b_{t-1}$$
$$-\,c_r^1\{wn_t-\tau_t+\mu^K\,(n_{t+1},\,q_t,\,g_t),\,q_t\}-k_t-g_t]$$
$$+\,\beta_t\,\{[m-m^r\,[wn_t-\tau_t+\mu^K\,(n_{t+1},\,q_t,\,g_t),\,q_t]\}$$
$$+\,\gamma_t\,(g_t+b_{t-1}-\tau_t-q_tb_t)+\chi_t\,[n_t-n^r\,(y_t,\,k_{t-1},\,g_{t-1})]$$
$$+\,\phi_t\,[k_t-k^K\,(n_{t+1},\,q_t,\,g_t)]+\eta_t\,(N-n_{t+1})+\xi_t\,[n\,(w,\,k_t,\,g_t)-n_{t+1}]\big].$$

The first-order conditions for the optimal path are obtained by differentiating the maximand with respect to n_{t+1}, q_t, τ_t, b_t, g_t, m_t^s, k_t and y_t, and then equating to zero. In stationary states, they are given by:

$$
\text{(B. 1)}\quad
\begin{aligned}
&(1+\delta)^{-1}J_n = (\alpha c_Y^{1r} + \beta m_Y^r - \lambda)\,\mu_n^K + \phi k_n^K + \eta + \xi,\\
&V_q = (\alpha c_Y^{1r} + \beta m_Y^r - \lambda)\,\mu_q^K + \alpha c_q^{1r} + \beta m_q^r + \gamma b + \phi k_q^K,\\
&\lambda - \alpha c_Y^{1r} - \beta m_Y^r = -\gamma, \qquad J_b = (1+\delta)\,\gamma q,\\
&(1+\delta)^{-1}J_g = (\alpha c_Y^{1r} + \beta m_Y^r - \lambda)\,\mu_g^K + \alpha - \gamma + \phi k_g^K - \xi n_g,\\
&\alpha = \phi + (1+\delta)^{-1}J_k + \xi n_k, \qquad \chi n_y^r = 0,
\end{aligned}
$$

where note that $\eta(N-n)=0$ and $\xi\,[n(w, k, g)-\mathrm{n}]=0$. The second set of equations are those obtained by differentiating the basic recursion relation with respect to the state variables n_t, b_{t-1}, k_{t-1} and g_{t-1}. In stationary states, they are given by:

$$
\text{(B. 2)}\quad
\begin{aligned}
&J_n = w\lambda + \alpha w\,(1 - c_Y^{1r}) - \beta w m_Y^r + \chi,\\
&J_b = \gamma - \alpha, \qquad J_k = -\chi n_k^r, \qquad J_g = -\chi n_g^r.
\end{aligned}
$$

It may be noted that $\chi=0$ from the last equation in (B. 1) because $n_y^r \neq 0$. This implies that the labour market equilibrium condition is not binding. It is only used to determine y_t residually. Substituting J_n, J_b ,J_k and J_g in (B. 2) into (B. 1), and then using $V_q = (m - c^2)\,\lambda$, we obtain:

$$
\text{(B. 3)}\quad
\begin{aligned}
&(1+\delta)^{-1}\,[(\lambda - \alpha c_y^{1r} - \beta m_y^r)\,w + \alpha w]\\
&\quad = (\alpha c_y^{1r} + \beta m_y^r - \lambda)\,\mu_n^K + \phi k_n^K + \eta + \xi,\\
&(m - c^2)\,\lambda = (\alpha c_y^{1r} + \beta m_y^r - \lambda)\,\mu_q^K + \alpha c_q^{1r} + \beta m_q^r + \gamma b + \phi k_q^K,\\
&\lambda - \alpha c_y^{1r} - \beta m_y^r = -\gamma, \qquad \gamma - \alpha = (1+\delta)\,\gamma q,\\
&(\lambda - c_y^{1r} - \beta m_y^r)\,\mu_g^K = \alpha - \gamma + \phi k_g^K - \xi n_g, \qquad \alpha = \phi + \xi n_k.
\end{aligned}
$$

Next, using $c_q^{1r}|_U = c_q^{1r} - (m - c^2)c_y^{1r}$ and $m_q^r|_U = m_q^r - (m - c^2)m_y^r$, and then eliminating λ, α and ϕ, we obtain the following equations:

$$
\text{(B. 4)}\quad
\begin{aligned}
&\mu_g^K = (1+\delta)\,q - k_g^K\,[1 - 1 + \delta)\,q + \theta n_k] + \theta n_g,\\
&\mu_n^K = -wq - k_n^K\,[1 - (1+\delta)\,q + \theta n_k] + (\varkappa + \theta),\\
&\mu_q^K = -\,[1 - (1+\delta)\,q](c_q^{1r}|_U) - \rho m_q^r|_U - (m - c^2 + b)\\
&\qquad - k_q^K\,[1 - (1+\delta)\,q + \theta n_k]
\end{aligned}
$$

where $\rho \equiv \beta/\gamma$, $\varkappa \equiv -\eta/\gamma$, and $\theta \equiv -\xi/\gamma$. Now, since (37) we have the partial derivatives: $\mu_g^K = qF_g + (qF_k - 1)k_g^K$, $\mu_q^K = y - wn + (qF_k - 1)k_q^K$, and $\mu_n^K = q(F_n - w) + (qF_k - 1)k_n^K$. Substituting these equations into (B. 4), and then using $c^2 = y - wn + m + b$, we obtain the optimality conditions (40)-(42).

C. Dynamic Optimization in the Classical Unemployment Regime

Letting the Lagrange multipliers corresponding to the *dynamic classical system* (52) α_t, β_t, γ_t, χ_t and ϕ_t, then the basic recursion relation can be given by:

$$
\begin{aligned}
J(b_{t-1}, k_{t-1}, g_{t-1}) = \max \Big[& (1+\delta)^{-1} J(b_t, k_t, g_t) \\
& + v[wn_t - \tau_t + \mu_C(q_t, g_t), q_t, c_t^1] \\
& + \alpha_t (wn_t - c'_1 - m - b_{t-1} - k_t - g_t) \\
& + \beta_t [m - m^r \{wn_t - \tau_t + \mu^C(q_t, g_t) - c_t^1, q_t, c_t^1\}] \\
& + \gamma_t (g_t + b_{t-1} - \tau_t - q_t b_t) \\
& + \chi_t [n_t - n(k_{t-1}, g_{t-1}; w)] + \phi_t [k_t - k^C(q_t, g_t)] \Big].
\end{aligned}
$$

The first-order conditions for the optimal path are obtained by differentiating the maximand with respect to n_t, q_t, τ_t, b_t, g_t, k_t and c_t^1, and then equating to zero:

$$
\text{(C. 1)} \quad
\begin{aligned}
& w\lambda + \alpha w + \chi = \beta w m_Y^r, \\
& \lambda \mu_q^C + v_q = \beta (m_Y^r \mu_q^C + m_q^r) + \gamma b + \phi k_q^C, \\
& \lambda - \beta m_Y^r = -\gamma, \qquad J_b = (1+\delta)\gamma q, \\
& \lambda \mu_g^C = \alpha + \beta m_Y^r \mu_g^C - \gamma + \phi k_g^C - (1+\delta)^{-1} J_g, \\
& \alpha = \phi + (1+\delta)^{-1} J_k, \qquad v_c = \alpha + \beta (m_c^r - m_Y^r),
\end{aligned}
$$

where $m_c^r \equiv \partial m^r / \partial c^1$. From (50) we have the partial derivatives: $\mu_g^C = qF_g$ and $\mu_q^C = y - wn$, since $qF_k = 1$ and $F_n = w$. The second set of equations are those obtained by differentiating the basic recursion relation with respect to the state variables b_{t-1}, k_{t-1} and g_{t-1}:

$$
\text{(C. 2)} \quad J_b = \gamma - \alpha \quad \text{and} \quad J_k / n_k = J_g / n_g = -\chi.
$$

Now, substituting J_b, J_k and J_g in (C. 2), v_p and v_c in (6) and (7), and $\mu_g^C = qF_g$ and $\mu_q^C = y - wn$ into (C. 1), and then eliminating λ, α, β, χ and ϕ, we obtain the optimality conditions (53)-(55).

D. Dynamic Optimization in the Repressed Inflation Regime

Letting the Lagrange multipliers corresponding to the *dynamic repressed inflation system* (59) α_t, β_t and γ_t, then the recursion relation can be given by:

$$
\begin{aligned}
J(b_{t-1}) = \max \Big[& (1+\delta)^{-1} J(b_t) + v[wN - \tau_t + \mu^R(q_t, g_t), q_t, c_t^1] \\
& + \alpha_t [wN - c_t^1 - b_{t-1} - k^R(q_t, g_t) - g_t] \\
& + \beta_t [m - m^r \{wN - \tau_t + \mu^R(q_t, g_t) - c_t^1, q_t, c_t^1\}] \\
& + \gamma_t (g_t + b_{t-1} - \tau_t - q_t b_t) \Big].
\end{aligned}
$$

In stationary states, the first-order conditions for the optimal path are as follows:

$$
\begin{aligned}
& \lambda \mu_q^R + v_q = \beta (m_Y^r \mu_q^R + m_q^r) + \gamma b + \alpha k_q^R, \\
& \lambda - \beta m_Y^r = -\gamma, \qquad J_b = (1+\delta)\gamma q, \\
& \lambda \mu_g^R = \alpha (1 + k_g^R) + \beta m_Y^r \mu_g^R - \gamma, \\
& J_b = \gamma - \alpha, \qquad v_c = \alpha + \beta (m_c^r - m_Y^r).
\end{aligned}
$$

It follows from (57) that we have the following partial derivatives: $\mu_g^R = qF_g$ and $\mu_q^R =$

$y-wN$ since $qF_k=1$. Puting into the same operation, we obtain the optimality conditions (60)-(62).

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